4.2 The Dot Product and Projections

1. In $\mathbf{R}^3$ the dot product is defined by

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos\theta.$$

2. For $\mathbf{u} = (x_1, y_1, z_1)$ and $\mathbf{v} = (x_2, y_2, z_2)$, we have

$$\mathbf{u} \cdot \mathbf{v} = x_1 x_2 + y_1 y_2 + z_1 z_2.$$

3.

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|},$$

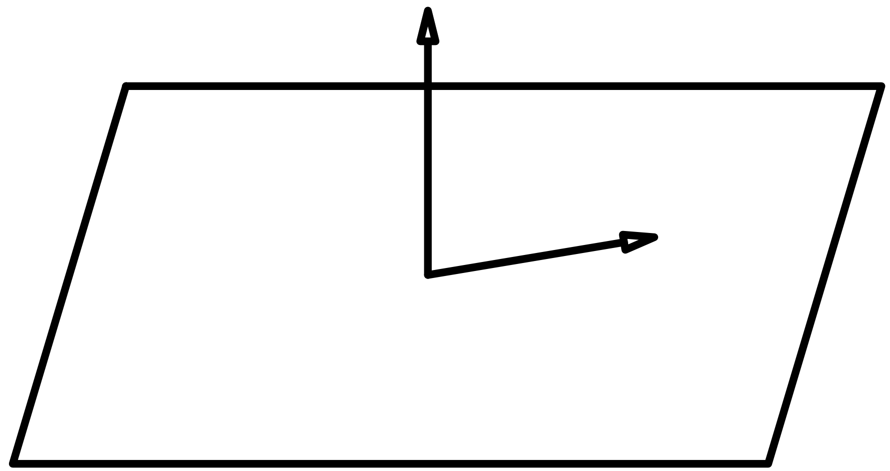
and $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if $\mathbf{u} \cdot \mathbf{v} = 0$.

4.3 Planes

A nonzero vector $\mathbf{n}$ is called a normal to a plane if it is orthogonal to every vector in the plane.

A point P is on the plane with normal $\mathbf{n}$ through the point P_0 if and only if

$$\mathbf{n} \cdot (\overrightarrow{P_0P}) = 0.$$



If $\mathbf{n} = (a, b, c)$, $P_0(x_0, y_0, z_0)$ and $P(x, y, z)$, then

$$\overrightarrow{P_0P} = \overrightarrow{OP} - \overrightarrow{OP_0} = (x - x_0, y - y_0, z - z_0)$$

and

$$(a, b, c) \cdot (x - x_0, y - y_0, z - z_0) = 0.$$

Hence, the plane through $P_0(x_0, y_0, z_0)$ with normal $\mathbf{n} = (a, b, c)$ is given by

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

Eg. *An equation of the plane through* $P_0(1, -1, 3)$ *with normal* $\mathbf{n} = (3, -1, 2)$ *is* $3(x - 1) - (y + 1) + 2(z - 3) = 0$ *.* *This simplifies to* $3x - y + 2z = 10$*.*

7.1 Orthogonality in $\mathbf{R}^n$

Given $X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $Y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$ in $\mathbf{R}^n$, the dot product of X and Y is defined by

$$X \cdot Y = X^T Y = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

The length $\|X\|$ of X is defined by

$$\|X\| = \sqrt{X \cdot X} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

Eg. *If* $X = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 2 \end{bmatrix}$ *and* $Y = \begin{bmatrix} -1 \\ 2 \\ 2 \\ 1 \end{bmatrix}$, *then*

$$X \cdot Y = \begin{bmatrix} 1 & -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 2 \\ 1 \end{bmatrix} = -1 - 2 + 2 + 2 = 1,$$

and $\|X\|^2 = 1 + 1 + 1 + 4 = 7$ *and* $\|X\| = \sqrt{7}$.

Thm.

1. $X \cdot Y = Y \cdot X$

2. $X \cdot (Y + Z) = X \cdot Y + X \cdot Z, \quad (X + Y) \cdot Z = X \cdot Z + Y \cdot Z.$

3. $(kX) \cdot Y = X \cdot (kY) = k(X \cdot Y)$, $k \in \mathbf{R}$.

4. $\|X\| \geq 0$, *and* $\|X\| = 0 \Leftrightarrow X = O$.

5. $\|kX\| = |k|\|X\|$, $k \in \mathbf{R}$.

Eg.

$$\begin{aligned}\|X+Y\|^2 &= (X+Y)\cdot(X+Y)\\ &= X\cdot X + X\cdot Y + Y\cdot X + Y\cdot Y\\ &= \|X\|^2 + \|Y\|^2 + 2(X\cdot Y).\end{aligned}$$

$\dfrac{X}{\|X\|}$ *has length* 1. *Indeed,*

$$\left\|\frac{X}{\|X\|}\right\| = \frac{1}{\|X\|}\|X\| = 1.$$

Def. *1. Two vectors X and Y are orthogonal if $X \cdot Y = 0$.*

2. A set $\{X_1, X_2, \cdots X_m\}$ of nonzero vectors in $\mathbf{R}^n$ is called an orthogonal set if $X_i \cdot X_j = 0$ for $i \neq j$.

3. An orthogonal set $\{X_1, X_2, \cdots, X_m\}$ is orthonormal if $\|X_i\| = 1$ for all i.

Rmk. *If $\{X_1, X_2, \cdots, X_m\}$ is orthogonal, then*

$$\left\{ \frac{X_1}{\|X_1\|}, \frac{X_2}{\|X_2\|}, \dots, \frac{X_m}{\|X_m\|} \right\}$$

is orthonormal.

Eg.

$$X_1 = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \ X_2 = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix} \ \textit{and} \ X_3 = \begin{bmatrix} -4 \\ 0 \\ 4 \end{bmatrix}$$

form an orthogonal set and

$$\frac{X_1}{2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ \frac{X_2}{3\sqrt{2}} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \ \textit{and} \ \frac{X_3}{4\sqrt{2}} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

form an orthonormal set.

Thm. *Every orthogonal set of vectors in $\mathbf{R}^n$ is linearly independent.*

Proof. Let $\{X_1, X_2, \cdots, X_m\}$ be orthogonal. Consider

$$r_1X_1 + r_2X_2 + \cdots + r_mX_m = O.$$

$$\begin{aligned} 0 &= X_i \cdot O = X_i \cdot (r_1X_1 + r_2X_2 + \cdots + r_mX_m) \\ &= r_1(X_i \cdot X_1) + r_2(X_i \cdot X_2) + \cdots + r_m(X_i \cdot X_m) \\ &= r_i\|X_i\|^2. \end{aligned}$$

Hence $r_i = 0$ for each i. □

Thm. *If* $\{X_1, X_2, \cdots, X_n\}$ *is an orthogonal basis of* $\mathbf{R}^n$*, then*

$$X = \frac{X \cdot X_1}{\|X_1\|^2} X_1 + \frac{X \cdot X_2}{\|X_2\|^2} X_2 + \cdots + \frac{X \cdot X_n}{\|X_n\|^2} X_n$$

for every X *in* $\mathbf{R}^n$*.*

Proof. If $X = r_1 X_1 + r_2 X_2 + \cdots + r_n X_n$, then

$$X \cdot X_i = r_i (X_i \cdot X_i) = r_i \|X_i\|^2.$$

Therefore,

$$r_i = \frac{X \cdot X_i}{\|X_i\|^2}.$$

□

Eg. $X_1 = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$, $X_2 = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}$ *and* $X_3 = \begin{bmatrix} -4 \\ 0 \\ 4 \end{bmatrix}$ *form an orthogonal basis of* $\mathbf{R}^3$.

$$X = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = r_1 X_1 + r_2 X_2 + r_3 X_3.$$

$$r_1 = \frac{X \cdot X_1}{\|X_1\|^2} = \frac{1}{2}, \quad r_2 = \frac{X \cdot X_2}{\|X_2\|^2} = \frac{1}{3}, \quad r_3 = \frac{X \cdot X_3}{\|X_3\|^2} = 0.$$

Hence,

$$X = \frac{1}{2} X_1 + \frac{1}{3} X_2.$$

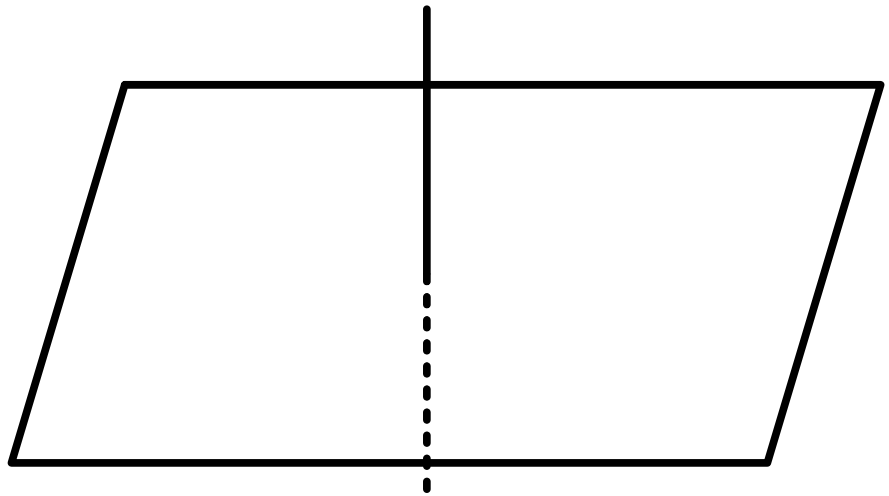
- Projections

Def. *If U is a subspace of $\mathbf{R}^n$, we define the orthogonal complement $U^\perp$ of U by*

$$U^\perp = \{X \in \mathbf{R}^n | X \cdot Y = 0 \text{ for all } Y \in U\}.$$

Observe that if $U = \operatorname{span}\{X_1, \cdots, X_m\}$, then

$$U^\perp = \{X \in \mathbf{R}^n | X \cdot X_i = 0 \text{ for all } i\}.$$



Eg. *Find* $U^\perp$ *if* $U = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 3 \end{bmatrix} \right\}$ *in* $\mathbf{R}^4$.

Solution. *Let* $X = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \in U^\perp$.

Then $X \cdot \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix} = 0$ *and* $X \cdot \begin{bmatrix} 1 \\ 0 \\ -2 \\ 3 \end{bmatrix} = 0$ *yield*

$$x - y + 2z = 0, \quad x - 2z + 3w = 0.$$

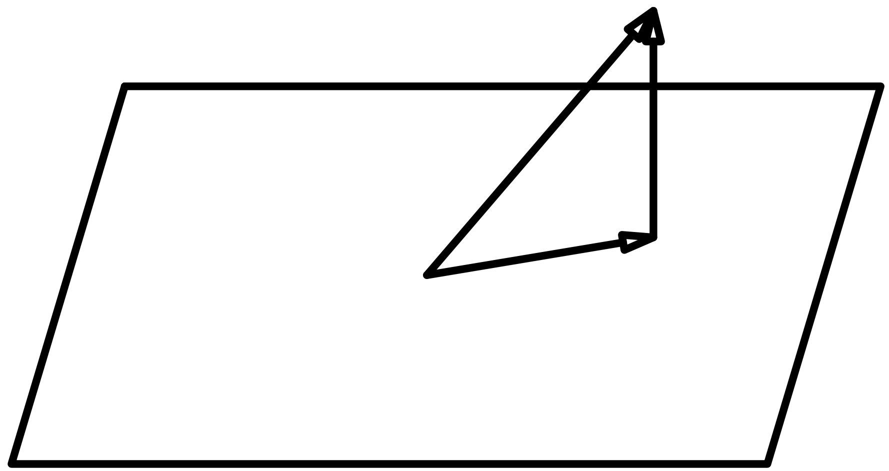
$$\begin{bmatrix} 1 & -1 & 2 & 0 \\ 1 & 0 & -2 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & -4 & 3 \end{bmatrix}, \quad \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = s \begin{bmatrix} 2 \\ 4 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ -3 \\ 0 \\ 1 \end{bmatrix}.$$

$$U^{\perp} = \operatorname{span} \left\{ \begin{bmatrix} 2 \\ 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Def. *Let $\{X_1, X_2, \cdots, X_m\}$ be an orthogonal basis of a subspace U of $\mathbf{R}^n$. Given X in $\mathbf{R}^n$, we define*

$$\mathrm{proj}_U(X) = \frac{X \cdot X_1}{\|X_1\|^2} X_1 + \frac{X \cdot X_2}{\|X_2\|^2} X_2 + \cdots + \frac{X \cdot X_m}{\|X_m\|^2} X_m$$

and call it the orthogonal projection of X on U.



Thm. *If U is a subspace of $\mathbf{R}^n$ and $X \in \mathbf{R}^n$, write $P = \mathrm{proj}_U(X)$. Then*

1. $P \in U$ *and* $X - P \in U^\perp$.

2. $\|X - P\| \leq \|X - Y\|$ *for all* $Y \in U$.

3. $\dim U + \dim U^\perp = n$.

Proof. 1. Clearly, $P \in U$.

$$\begin{aligned} (X - P) \cdot X_i &= X \cdot X_i - P \cdot X_i \\ &= X \cdot X_i - \frac{X \cdot X_i}{\|X_i\|^2} X_i \cdot X_i = 0 \end{aligned}$$

Thus $X - P \in U^\perp$.

2. Write $X - Y = (X - P) + (P - Y)$. Then $P - Y$ is in U and $X - P \in U^{\perp}$.

$$\begin{aligned} \|X - Y\|^2 &= \|(X - P) + (P - Y)\|^2 \\ &= \|X - P\|^2 + \|P - Y\|^2 + 2(X - P) \cdot (P - Y) \\ &= \|X - P\|^2 + \|P - Y\|^2 \geq \|X - P\|^2. \end{aligned}$$

3. Let $\{X_1, \cdots, X_m\}$ and $\{Y_1, \cdots, Y_k\}$ be orthogonal basis of U and $U^{\perp}$, respectively. Then $\{X_1, \cdots, X_m, Y_1, \cdots, Y_n\}$ is orthogonal, so linearly independent. If $X \in \mathbf{R}^n$, then $X = P + (X - P)$. Thus $\{X_1, \cdots, X_m, Y_1, \cdots, Y_n\}$ spans $\mathbf{R}^n$. $\square$

Eg. *Let* $U = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$. *If* $X = \begin{bmatrix} 3 \\ -1 \\ 0 \\ 2 \end{bmatrix}$, *find the vector in* U *closest to* X *and express* X *as the sum of a vector in* U *and a vector in* $U^\perp$.

Solution. *Note that* $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ *is orthogonal.*

$$\begin{aligned} P = \operatorname{proj}_U(X) &= \frac{X \cdot X_1}{\|X_1\|^2} X_1 + \frac{X \cdot X_2}{\|X_2\|^2} X_2 \\ &= \frac{4}{3} X_1 + \frac{-1}{3} X_2 = \frac{4}{3} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + \frac{-1}{3} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 5 \\ 4 \\ -1 \\ 3 \end{bmatrix}. \end{aligned}$$

$$X = P + (X - P) = \frac{1}{3} \begin{bmatrix} 5 \\ 4 \\ -1 \\ 3 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 4 \\ -7 \\ 1 \\ 3 \end{bmatrix}.$$

- Gram-Schmidt Orthogonalization Algorithm

Question : Given a basis $B = \{Y_1, \cdots, Y_m\}$ of U, how can we obtain an orthogonal basis from B?

Answer : Construct $X_1, \cdots, X_m$ in U as follows.

$$X_1 = Y_1,$$
$$X_2 = Y_2 - \frac{Y_2 \cdot X_1}{\|X_1\|^2} X_1,$$
$$X_3 = Y_3 - \frac{Y_3 \cdot X_1}{\|X_1\|^2} X_1 - \frac{Y_3 \cdot X_2}{\|X_2\|^2} X_2,$$
$$\cdots$$
$$X_m = Y_m - \frac{Y_m \cdot X_1}{\|X_1\|^2} X_1 - \frac{Y_m \cdot X_2}{\|X_2\|^2} X_2 - \cdots - \frac{Y_m \cdot X_{m-1}}{\|X_{m-1}\|^2} X_{m-1}.$$

Then $\{X_1, \cdots, X_m\}$ is an orthogonal basis of U.

Proof. Let $U_1 = \text{span}\{X_1\}$, $U_2 = \text{span}\{X_1, X_2\}$, $\cdots$, $U_{m-1} = \text{span}\{X_1, \cdots, X_{m-1}\}$.

$\{X_1\}$ is orthogonal.

$X_2 = Y_2 - \text{proj}_{U_1}(Y_2)$, $X_2 \in U_1^\perp \Rightarrow \{X_1, X_2\}$ is orthogonal.

$X_3 = Y_3 - \text{proj}_{U_2}(Y_3)$, $X_3 \in U_2^\perp \Rightarrow \{X_1, X_2, X_3\}$ is orthogonal.

Continue the process.

$X_m = Y_m - \text{proj}_{U_{m-1}}(Y_m)$, $X_m \in U_{m-1}^\perp \Rightarrow \{X_1, \cdots, X_m\}$ is orthogonal. $\square$

Eg. *Let U be the subspace of $\mathbf{R}^4$ with basis $\{Y_1, Y_2, Y_3\}$, where*

$$Y_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, Y_2 = \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, Y_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \end{bmatrix}.$$

Find an orthogonal basis.

Solution.

$$X_1 = Y_1,$$

$$X_2 = Y_2 - \frac{Y_2 \cdot X_1}{\|X_1\|^2} X_1 = \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} - (-\frac{2}{3}) \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 \\ 2 \\ -1 \\ 3 \end{bmatrix},$$

$$\begin{aligned} X_3 &= Y_3 - \frac{Y_3 \cdot X_1}{\|X_1\|^2} X_1 - \frac{Y_3 \cdot X_2}{\|X_2\|^2} X_2 \\ &= \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \end{bmatrix} - (-\frac{-1}{3}) \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} - (-\frac{-2}{15}) \begin{bmatrix} -1 \\ 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -\frac{4}{5} \\ \frac{3}{5} \\ \frac{1}{5} \\ -\frac{3}{5} \end{bmatrix} \Rightarrow \begin{bmatrix} -4 \\ 3 \\ 1 \\ -3 \end{bmatrix}. \end{aligned}$$

Thus $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -4 \\ 3 \\ 1 \\ -3 \end{bmatrix} \right\}$ *is an orthogonal basis.*

How can we get an orthonormal basis?

$$\left\{ \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{15}} \begin{bmatrix} -1 \\ 2 \\ -1 \\ 3 \end{bmatrix}, \frac{1}{\sqrt{35}} \begin{bmatrix} -4 \\ 3 \\ 1 \\ -3 \end{bmatrix} \right\}$$