1. Systems of Linear Equations

- Motivation : the most frequently recurring problem in math, physics, chemistry, biology, engineering, economics, ...

Eg.

$$\begin{cases} x + y + z = 3 \\ 2x + y + 3 = 1 \end{cases}$$

1.1 Solutions and Elementary Operations

- Equivalence

Def. *Two systems are equivalent if each has the same set of solutions.*

Eg.

$$\begin{cases} x - 3y = -3 \\ 2x + y = 8 \end{cases}, \quad \begin{cases} x - 3y = -3 \\ 7y = 14 \end{cases}, \quad \begin{cases} x = 3 \\ y = 2 \end{cases}.$$

- Scheme : to develop a new linear system that is equivalent to the original system but is much simpler to solve.

– Elementary operations

I. Interchange two equations.

II. Multiply one equation by a nonzero number.

III. Add a multiple of one equation to a different equation.

Thm. *Each elementary operation produces an equivalent system.*

Proof. The idea is that each operation is reversible. □

– Essence

A linear system $\rightsquigarrow$ an augmented matrix

$$\begin{cases} x + y + z = 3 \\ x + 2y - 3z = -1 \end{cases} \quad \rightsquigarrow \quad \begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & 2 & -3 & -1 \end{bmatrix}$$

– Elementary row operation

I. Interchange two rows.

II. Multiply one row by a nonzero number.

III. Add a multiple of one row to a different row.

– Scheme refined

A linear system $\leftrightsquigarrow$ an augmented matrix

$\downarrow$ e. r. operation

the solution $\leftrightsquigarrow$ the simplest matrix

Question : What's the simplest matrix corresponding to the given augmented matrix?

1.2 Gaussian Elimination

Def. *A matrix is in reduced row-echelon form if it satisfies the following conditions:*

1. *All zero rows are at the bottom.*
2. *The first nonzero entry in a row is* 1*, called the leading* 1*.*
3. *Each leading* 1 *is to the right of the preceding leading* 1*.*
4. *Each leading* 1 *is the only nonzero entry in its column.*

Eg.

$$\begin{bmatrix} 1 & 0 & * & 0 \\ 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Thm. *Every matrix can be brought reduced row-echelon form by a series of elementary row operations.*

In other words, every linear system can be solved by a series of elementary operations.

– Reduced row-echelon form ≈ the solution

1.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & 6 \\ 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 & 8 \end{bmatrix} \sim \begin{cases} x_1 = 5 \\ x_2 = 6 \\ x_3 = 7 \\ x_4 = 8 \end{cases}$$

2.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{cases} x_1 + 2x_5 = 0 \\ x_3 + 3x_5 = 0 \\ x_4 + 4x_5 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = -2x_5 \\ x_3 = -3x_5 \\ x_4 = -4x_5 \end{cases}$$

x_1, x_3, x_4 : leading variables corresponding to leading 1's
x_2, x_5 : parameters $\Rightarrow$ let $x_2 = r$ and $x_5 = s$.

$$x_1 = -2s,\ x_2 = r,\ x_3 = -3s,\ x_4 = -4s,\ x_5 = s.$$

3.

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 1 & 1 & 3 \end{bmatrix} \sim \begin{cases} x_1 + x_4 = 1 \\ x_2 - x_4 = 2 \\ x_3 + x_4 = 3 \end{cases} \Rightarrow \begin{cases} x_1 = 1 - x_4 \\ x_2 = 2 + x_4 \\ x_3 = 3 - x_4 \end{cases}$$

x_1, x_2, x_3 : leading variables
x_4 : parameter $\Rightarrow$ let $x_4 = r$.

$$x_1 = 1 - r,\ x_2 = 2 + r,\ x_3 = 3 - r,\ x_4 = r.$$

4.

$$\begin{bmatrix} & & \cdots & & \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{cases} \cdots \\ 0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 = 1 \end{cases}$$

No solution! The system is inconsistent.

– Algorithm

1. Find out the first column with a nonzero entry and the first nonzero entry, say a, in that column.

2. Interchange rows to move a to the first row. (Type I)

3. Multiply the first row by $\frac{1}{a}$. (Type II)

4. Make all the entries below the leading 1 equal to zero. (type III)

5. Repeat 1-4 on the matrix consisting of the remaining rows.

6. Make all the entries above a leading 1 equal to zero from the last leading 1 to the first. (Type III)

Eg.

$$\begin{bmatrix} 0 & 2 & 3 & -4 & 1 \\ 0 & 0 & 2 & 3 & 4 \\ 2 & 2 & -5 & 2 & 4 \\ 2 & 0 & -6 & 9 & 7 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 2 & -5 & 2 & 4 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 2 & 3 & -4 & 1 \\ 2 & 0 & -6 & 9 & 7 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & -\frac{5}{2} & 1 & 2 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 2 & 3 & -4 & 1 \\ 2 & 0 & -6 & 9 & 7 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & -\frac{5}{2} & 1 & 2 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 2 & 3 & -4 & 1 \\ 0 & -2 & -1 & 7 & 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & -\frac{5}{2} & 1 & 2 \\ 0 & 2 & 3 & -4 & 1 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & -2 & -1 & 7 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & -\frac{5}{2} & 1 & 2 \\ 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \\ 0 & 0 & 2 & 3 & 4 \\ 0 & -2 & -1 & 7 & 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & -\frac{5}{2} & 1 & 2 \\ 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 0 & 2 & 3 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & -\frac{5}{2} & 1 & 2 \\ 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{3}{2} & 2 \\ 0 & 0 & 2 & 3 & 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & -\frac{5}{2} & 1 & 2 \\ 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{3}{2} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & -\frac{5}{2} & 1 & 2 \\ 0 & 1 & 0 & -\frac{17}{4} & -\frac{5}{2} \\ 0 & 0 & 1 & \frac{3}{2} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 & \frac{19}{4} & 7 \\ 0 & 1 & 0 & -\frac{17}{4} & -\frac{5}{2} \\ 0 & 0 & 1 & \frac{3}{2} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 9 & \frac{19}{2} \\ 0 & 1 & 0 & -\frac{17}{4} & -\frac{5}{2} \\ 0 & 0 & 1 & \frac{3}{2} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Def. *The rank of A is the number of the leading 1's in the reduced row-echelon form of A.*

Thm. *Given a system in n variables whose augmented matrix is of rank r, the number of parameters is $n - r$.*

Rmk. *Exactly three possibilities:*

1. *No solution.*
2. *Unique solution.* $(n - r = 0)$
3. *Infinitely many solutions.* $(n - r > 0)$

1.3 Homogeneous Equations

Def. *A system is homogeneous if all the constant terms are zero.*

Eg.

$$\begin{cases} 3x_1 - x_2 - x_3 = 0 \\ 2x_1 + x_2 + x_3 = 0 \end{cases}$$

Always there exists the trivial solution : $x_1 = x_2 = x_3 = 0$.

Question : is there any nontrivial solution?

One easy case is the following proposition.

Prop. *Suppose that a homogeneous system in* n *variables has* m *equations. If* $n > m$*, then a nontrivial solution exists. In fact, there are infinitely many solutions.*

Proof. Consider the reduced row-echelon form of the augmented matrix.

$$\# \text{ of leading 1's } = r \leq m < n.$$

Thus $\#$ of parameters $= n - r > 0$. $\square$