RIGID REFLECTIONS OF RANK 3 COXETER GROUPS AND REDUCED ROOTS OF RANK 2 KAC–MOODY ALGEBRAS

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ABSTRACT. In a recent paper by K.-H. Lee and K. Lee, rigid reflections are defined for any Coxeter group via non-self-intersecting curves on a Riemann surface with labeled curves. When the Coxeter group arises from an acyclic quiver, the rigid reflections are related to the rigid representations of the quiver. For a family of rank 3 Coxeter groups, it was conjectured in the same paper that there is a natural bijection from the set of reduced positive roots of a symmetric rank 2 Kac–Moody algebra onto the set of rigid reflections of the corresponding rank 3 Coxeter group. In this paper, we prove the conjecture.

1. INTRODUCTION

Let Q be an acyclic quiver of rank n, and $\mathrm{mod}(Q)$ be the category of finite dimensional representations of Q. In order to understand the category $\mathrm{mod}(Q)$, one needs to consider the indecomposable representations without self-extensions, called *rigid* representations. Their dimension vectors form a special subset of the set of positive real roots of the Kac–Moody algebra $\mathfrak{g}(Q)$ associated to Q, and are called *real Schur roots*. These roots also appear in the denominators of cluster variables, or as the c-vectors of the cluster algebra associated to Q, and can be described combinatorially in terms of non-crossing partitions. See [1, 5, 6, 7, 9, 11, 20, 21, 22] for more details on these connections.

As a new geometric/combinatorial approach to describe rigid representations and real Schur roots, K.-H. Lee and K. Lee conjectured in their paper [16] a correspondence between rigid representations in $\mathrm{mod}(Q)$ and the set of certain non-self-intersecting curves on a Riemann surface Σ with n labeled curves. The conjecture is now proven by A. Felikson and P. Tumarkin [8] for acyclic quivers with multiple edges between every pair of vertices. Very recently, S. D. Nguyen [19] informed us that he proved the conjecture for an arbitrary acyclic quiver.

The conjecture actually characterizes the family of reflections in the Weyl group of $\mathfrak{g}(Q)$ which are associated to real Schur roots via non-self-intersecting curves in Σ. Since reflections make sense for any Coxeter groups, the geometric characterization can be carried over. Indeed, the *rigid reflections* are defined in [17] for any Coxeter group W to be those

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corresponding to non-self-intersecting curves in Σ. Unexpectedly, an interesting phenomenon was observed that the rigid reflections of W are parametrized by the positive roots of a seemingly unrelated Kac–Moody algebra $\mathcal{H}$, and the phenomenon was investigated in detail for a family of rank 3 Coxeter groups.

To be precise, for each positive integer $m \geq 2$, consider the following Coxeter group

$$W(m) = \langle s_1, s_2, s_3 \ : \ s_1^2 = s_2^2 = s_3^2 = (s_1 s_2)^m = (s_2 s_3)^m = e \rangle.$$

Let $\mathcal{H}(m)$ be the rank 2 Kac–Moody algebra associated with the Cartan matrix $\left(\begin{smallmatrix} 2 & -m \\ -m & 2 \end{smallmatrix}\right)$. Denote an element of the root lattice of $\mathcal{H}(m)$ by $[a, b]$, $a, b \in \mathbb{Z}$, where $[1, 0]$ and $[0, 1]$ are the positive simple roots. A root $[a, b]$ of $\mathcal{H}(m)$ is called *reduced* if $\gcd(a, b) = 1$ and $ab \neq 0$. A reduced root determines a non-self-intersecting curve η on the torus Σ with triangulation by three labeled curves.

Now define a function, $[a, b] \mapsto s([a, b]) \in W(m)$, by reading off the labels of the intersection points of η with the labeled curves on Σ and by writing down the products of simple reflections accordingly. See (3.1) for an example. In [17], it was conjectured that this function $[a, b] \mapsto s([a, b])$ is a bijection from the set of reduced roots of $\mathcal{H}(m)$ onto the set of rigid reflections of $W(m)$. If established, it would show that the set of rigid reflections in $W(m)$ has a structure coming from the set of reduced roots of $\mathcal{H}(m)$. Most importantly, the Weyl group action on the set of roots of $\mathcal{H}(m)$ would be transported to the set of rigid reflections on $W(m)$. In the same paper [17], as a main result, it was shown that the function is surjective; however, injectivity was checked only for $m = 2$.

One of the main difficulties in showing injectivity is directly related to the word problem for $W(m)$. Since $s([a, b])$ are given as words in simple reflections, one needs to determine when such two words represent the same (or different) elements in $W(m)$. A solution to this problem may be given by an algorithm to write $s([a, b])$ into a canonical form or a standard word.

In this paper, we obtain such a reduction algorithm and prove the conjecture of [17].

Theorem 1.1. *For $m \geq 2$, the function, $[a, b] \mapsto s([a, b])$, is a bijection from the set of reduced positive roots of $\mathcal{H}(m)$ onto the set of rigid reflections of $W(m)$.*

The canonical forms or standard words of the elements in $W(m)$ are determined by applying Gröbner–Shirshov basis theory. In the first substantial step, *canonical sequences* of positive integers are assigned to each $[a, b]$ and a reduction is accomplished accordingly. The result is described in Corollary 4.17. This reduction through canonical sequences can be visualized naturally in terms of associated curves on the torus Σ and are related to the aforementioned Weyl group action on the set of rigid reflections in $W(m)$. Moreover, we note that the canonical sequences was used in a study of 2-bridge link groups [18] in a slightly different form.

However, this reduction through canonical sequences are not sufficient for our purpose, and we need to perform further reduction until we obtain standard words for the elements in $W(m)$ to distinguish them explicitly and to show injectivity of the map $[a,b] \mapsto s([a,b])$.

As is well known, the word problem for a group is intractable, in general. Surprisingly, Gröbner–Shirshov bases for $W(m)$ are not much different from the original set of defining relations, though $W(m)$ is infinite, and our reduction process becomes feasible. It would be interesting to see if there are other families of infinite Coxeter groups with relatively simple Gröbner–Shirshov bases. To such families, the method of this paper will generalize to reveal precise connections between rigid reflections and roots of Kac–Moody algebras.

The organization of this paper is as follows. In Section 2, we recall the definition of rigid reflections. In the next section, we consider the rank 3 Coxeter groups $W(m)$ and collect known results from [17] about the rigid reflections of $W(m)$. In Section 4, the canonical sequence and level of a reduced root $[a,b]$ are defined and their properties are studied. Using the canonical sequence, we achieve a substantial reduction of $s([a,b])$. In Section 5, we complete the reduction process to obtain the standard words for the elements of $W(m)$ and prove Theorem 1.1, up to computation of Gröbner–Shirshov bases for $W(m)$ which is accomplished in the last section.

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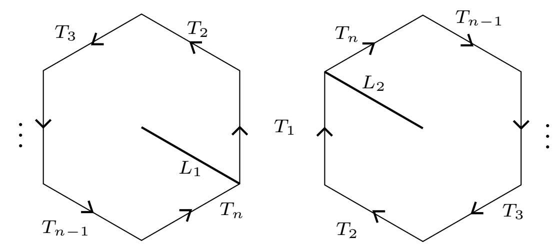
2. Rigid reflections

In this section we recall the definition of a rigid reflection from [17].

Let

$$W = \langle s_1, s_2, ..., s_n \ : \ s_1^2 = \cdots = s_n^2 = e, \ (s_i s_j)^{m_{ij}} = e \rangle$$

be a Coxeter group with $m_{ij} \in \{2, 3, 4, ...\} \cup \{\infty\}$. In order to define the rigid reflections of W, we introduce a Riemann surface Σ equipped with n labeled curves as below. Let G_1 and G_2 be two identical copies of a regular n-gon. Label the edges of each of the two n-gons by $T_1, T_2, \ldots, T_n$ counter-clockwise. On G_i $(i = 1, 2)$, let L_i be the line segment from the center of G_i to the common endpoint of T_n and T_1. Later, these line segments will only be used to designate the end points of admissible curves and will not be used elsewhere. Fix the orientation of every edge of G_1 (resp. G_2) to be counter-clockwise (resp. clockwise) as in the following picture.



Let Σ be the Riemann surface of genus $\lfloor \frac{n-1}{2} \rfloor$ obtained by gluing together the two n-gons with all the edges of the same label identified according to their orientations. The edges of the n-gons become n different curves in Σ. If n is odd, all the vertices of the two n-gons are identified to become one point in Σ and the curves obtained from the edges become loops. If n is even, two distinct vertices are shared by all curves. Let $\mathcal{T} = T_1 \cup \cdots T_n \subset \Sigma$, and V be the set of the vertex (or vertices) on $\mathcal{T}$.

Let $\mathfrak{W}$ be the set of words from the alphabet $\{1, 2, ..., n\}$, and let $\mathfrak{R} \subset \mathfrak{W}$ be the subset of words $\mathfrak{w} = i_1 i_2 \cdots i_k$ such that k is an odd integer and $i_j = i_{k+1-j}$ for all $j \in \{1, ..., k\}$, in other words, $s_{i_1} s_{i_2} \cdots s_{i_k}$ is a reflection in W. For $\mathfrak{w} = i_1 i_2 \cdots i_k \in \mathfrak{W}$, denote $s_{i_1} s_{i_2} ... s_{i_k} \in W$ by $s(\mathfrak{w})$.

Definition 2.1. An *admissible* curve is a continuous function $\eta : [0, 1] \longrightarrow \Sigma$ such that

1) $\eta(x) \in V$ if and only if $x \in \{0, 1\}$;

2) η starts and ends at the common end point of T_1 and T_n. More precisely, there exists $\epsilon > 0$ such that $\eta([0, \epsilon]) \subset L_1$ and $\eta([1 - \epsilon, 1]) \subset L_2$;

3) if $\eta(x) \in \mathcal{T} \setminus V$ then $\eta([x - \epsilon, x + \epsilon])$ meets $\mathcal{T}$ transversally for sufficiently small $\epsilon > 0$.

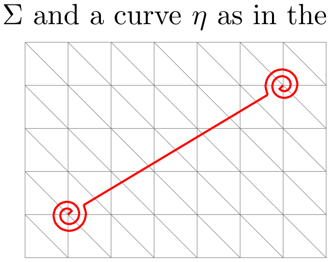
If η is admissible, then we obtain $\upsilon(\eta) := i_1 \cdots i_k \in \mathfrak{W}$ given by

$$\{x \in (0, 1) \ : \ \eta(x) \in \mathcal{T}\} = \{x_1 < \cdots < x_k\} \quad \text{and} \quad \eta(x_\ell) \in T_{i_\ell} \text{ for } \ell \in \{1, ..., k\}.$$

Conversely, note that for every $\mathfrak{w} \in \mathfrak{W}$, there is an admissible curve η with $\upsilon(\eta) = \mathfrak{w}$. Hence, every element in W can be represented by some admissible curve(s). For brevity, let $s(\eta) := s(\upsilon(\eta))$.

Definition 2.2. An element $w \in W$ is called a *rigid reflection* if there exist an expression $w = s_{i_1} s_{i_2} \cdots s_{i_k}$ and a non-self-crossing admissible curve η such that $\upsilon(\eta) = i_1 ... i_k \in \mathfrak{R}$.

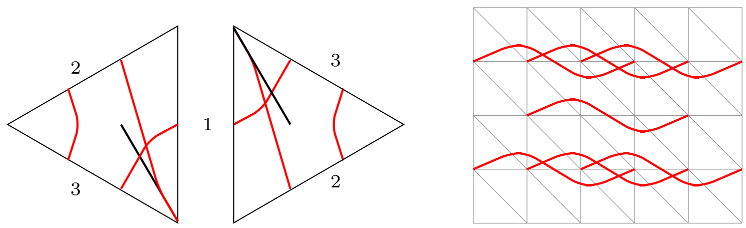
Example 2.3. Let $n = 3$, and $W = \langle s_1, s_2, s_3 \ : \ s_1^2 = s_2^2 = s_3^2 = e \rangle$, i.e., $m_{ij} = \infty$ for $i \neq j$. Consider the universal cover of Σ and a curve η as in the following picture.



Here each horizontal line segment represents T_1, vertical T_3, and diagonal T_2. One sees that η has no self-intersection in Σ. Thus we obtain the corresponding rigid reflection

$$s(\eta) = (s_3s_2s_1)^4 s_2s_3s_2s_1s_2s_3s_2s_3s_2s_1s_2s_3s_2(s_1s_2s_3)^4.$$

On the other hand, the reflection $s_2s_3s_1s_3s_2$ comes from the following curve η' which has a self-intersection. The picture on the right shows several copies of η' on the universal cover.



Consequently, the reflection $s(\eta') = s_2s_3s_1s_3s_2$ is *not* rigid.

Let Φ be the root system of W, realized in the real vector space $\mathbf{E}$ with basis $\{\alpha_1, \dots, \alpha_n\}$ with the symmetric bilinear form B defined by

$$B(\alpha_i, \alpha_j) = -\cos(\pi/m_{ij}) \text{ for } 1 \le i, j \le n.$$

For each $i \in \{1, \dots, n\}$, define the action of s_i on $\mathbf{E}$ by

$$s_i(\lambda) = \lambda - 2B(\lambda, \alpha_i)\alpha_i, \quad \lambda \in \mathbf{E},$$

and extend it to the action of W on $\mathbf{E}$. Then each root $\alpha \in \Phi$ determines a reflection $s_\alpha \in W$. (See [10] for more details.)

Definition 2.4. A positive root $\alpha \in \Phi$ of W is called *rigid* if the corresponding reflection $s_\alpha \in W$ is rigid.

Example 2.5. In Example 2.3, we obtained the rigid reflection

$$(s_3s_2s_1)^4 s_2s_3s_2s_1s_2s_3s_2s_3s_2s_1s_2s_3s_2(s_1s_2s_3)^4.$$

It give rises to a rigid root

$$1662490\alpha_1 + 4352663\alpha_2 + 11395212\alpha_3 = (s_3s_2s_1)^4 s_2s_3s_2s_1s_2s_3\alpha_2.$$

3. A family of rank 3 Coxeter groups

In this section we focus our attention to the rank 3 groups $W(m)$ and collect known results from [17] about the rigid reflections of $W(m)$.

As in the introduction, fix a positive integer $m \ge 2$ and set

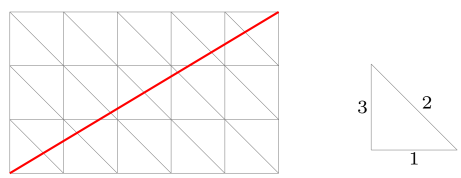
$$W(m) = \langle s_1, s_2, s_3 \ : \ s_1^2 = s_2^2 = s_3^2 = (s_1s_2)^m = (s_2s_3)^m = e \rangle.$$

Note that we put, in particular, $m_{13} = m_{31} = \infty$. Let $\mathcal{H}(m)$ be the rank 2 hyperbolic Kac–Moody algebra associated with the Cartan matrix $\left(\begin{smallmatrix} 2 & -m \\ -m & 2 \end{smallmatrix}\right)$. We denote an element of the root lattice of $\mathcal{H}(m)$ by $[a,b]$, $a,b \in \mathbb{Z}$, where $[1,0]$ and $[0,1]$ are the positive simple roots. A root $[a,b]$ of $\mathcal{H}(m)$ is called *reduced* if $\gcd(a,b) = 1$ and $ab \neq 0$. One can see that every non-simple real root is reduced.

Let $\mathcal{P}^+ = \{[a,b] \,:\, a,b \in \mathbb{Z}_{>0},\ \gcd(a,b) = 1\}$. For every $[a,b] \in \mathcal{P}^+$, let $\eta([a,b])$ be the line segment from $(0,0)$ to (a,b) on the universal cover of the torus, which automatically has no self-intersections. Write $s([a,b]) := s(\eta([a,b])) \in W(m)$ for the corresponding rigid reflection. For example, we have

$$s([5,3]) = s_2s_3s_2s_1s_2s_3s_2s_3s_2s_1s_2s_3s_2 \tag{3.1}$$

as one can check in the following picture.



Recall from [12, 15] that

$$[a,b] \text{ is a root of } \mathcal{H}(m) \quad \text{if and only if} \quad a^2 + b^2 - mab \leq 1. \tag{3.2}$$

Define a sequence $\{F_n\}$ recursively by $F_0 = 0$, $F_1 = 1$, and $F_n = mF_{n-1} - F_{n-2}$. Note that $[a,b]$ is a *real* root if and only if $[a,b]$ is either $[F_n, F_{n+1}]$ or $[F_{n+1}, F_n]$, $n \geq 0$. (See [12, 15].) A non-real root is called *imaginary*. Define another sequence $\{E_n\}$ by $E_0 = E_1 = 1$ and $E_n = mE_{n-1} - E_{n-2}$.

Definition 3.1. Let (a_1, a_2) be a pair of positive integers with $a_1 \geq a_2$.

(1) A *maximal Dyck path* of type $a_1 \times a_2$, denoted by $\mathcal{D}^{a_1 \times a_2}$, is a lattice path from $(0,0)$ to (a_1, a_2) that is as close as possible to the diagonal joining $(0,0)$ and (a_1, a_2) without ever going above it.

(2) Assign $s_2s_3 \in W(m)$ to each horizontal edge of $\mathcal{D}^{a_1 \times a_2}$, and $s_2s_1 \in W(m)$ to each vertical edge. Read these elements in the order of edges along $\mathcal{D}^{a_1 \times a_2}$, then we get a product of copies of s_2s_3 and s_2s_1. Denote the product by $s^{a_1 \times a_2}$.

Lemma 3.2 ([17]). *Assume that $[a,b] \in \mathcal{P}^+$ with $a \geq b$. Then we have the following formulas.*

(1) $\quad s([a,b]) = s_3s_2s^{a \times b}s_1$.

(2) $\qquad s^{F_2\times F_1} = s_2 s_1 \quad$ *and* $\quad s^{E_2\times E_1} = s_3 s_1.$

$$s^{F_n\times F_{n-1}} = \begin{cases} s_1(s_3s_2s_1)^{(n-3)/2}s_2s_3(s_1s_2s_3)^{(n-3)/2}s_1, & \text{for } n\ge 3 \text{ odd;}\\ s_1(s_3s_2s_1)^{(n-4)/2}s_3s_1s_2s_3(s_1s_2s_3)^{(n-4)/2}s_1, & \text{for } n\ge 4 \text{ even.}\end{cases}$$

$$s^{E_n\times E_{n-1}} = \begin{cases} s_1(s_3s_2s_1)^{(n-3)/2}s_2s_1s_2s_3(s_1s_2s_3)^{(n-3)/2}s_1, & \text{for } n\ge 3 \text{ odd;}\\ s_1(s_3s_2s_1)^{(n-4)/2}s_3s_2s_3s_1s_2s_3(s_1s_2s_3)^{(n-4)/2}s_1, & \text{for } n\ge 4 \text{ even.}\end{cases}$$

(3) $\qquad (s_3s_2s_1s([F_n,F_{n-1}]))^m = (s_1s_2s_3s([F_n,F_{n-1}]))^m = e \qquad$ *for all* $n\ge 1$.

We now state the main theorem of this paper.

Theorem 3.3. *The function, $[a,b]\mapsto s^{a\times b}$, is an injection from the set of reduced positive roots of $\mathcal{H}(m)$ into $W(m)$.*

Combining this theorem with Lemma 3.2 (1) above and Theorem 1.2 in [17], we obtain Theorem 1.1 which we state again:

Corollary 3.4. *For $m\ge 2$, the function, $[a,b]\mapsto s([a,b])$, is a bijection from the set of reduced positive roots of $\mathcal{H}(m)$ to the set of rigid reflections of $W(m)$.*

Equivalently, if we let $\beta([a,b])$ be the rigid root determined by the rigid reflection $s([a,b])$, the function, $[a,b]\mapsto\beta([a,b])$, is a bijection from the set of reduced positive roots of $\mathcal{H}(m)$ to the set of rigid roots of $W(m)$.

The rest of this paper is concerned with a proof of Theorem 3.3.

4. Canonical sequences and levels

In this section we introduce canonical sequences and levels attached to a reduced root of $\mathcal{H}(m)$ which will play an important role in classifying reduced roots and rigid reflections. Recall that a positive real root $[a,b]$ ($a\ge b$) of $\mathcal{H}(m)$ is of the form $[F_n,F_{n-1}]$. Since we already have $s^{F_n\times F_{n-1}}$ in Lemma 3.2, we only consider imaginary roots of $\mathcal{H}(m)$ whenever simplicity is attained.

We start with some definitions.

Definition 4.1. Let N be a positive integer. Suppose that $\boldsymbol{c}=(a_1,a_2,\dots,a_d)$ is a finite sequence such that $a_i=N$ or $N+1$ for all i.

(1) If $d>1$ and $(a_i,a_{i+1})\ne(N,N)$ for any i, then $\boldsymbol{c}$ is called *type* $+$.
(2) If $d>1$ and $(a_i,a_{i+1})\ne(N+1,N+1)$ for any i, then $\boldsymbol{c}$ is called *type* $-$.
(3) If $d>1$ and $a_i\ne a_{i+1}$ for any i, then $\boldsymbol{c}$ is called *type* $=$.
(4) If $d=1$, then $\boldsymbol{c}$ is called *type* 0.

Throughout this section, let $[a,b]$ be a reduced positive root of $\mathcal{H}(m)$ with $a\ge b$.

Definition 4.2. Define a sequence $\boldsymbol{c}_1 = (a_{1,1}, a_{1,2}, \ldots, a_{1,b})$ of positive integers to be such that

$$\mathcal{D}^{a\times b} = h^{a_{1,1}} v h^{a_{1,2}} v \cdots h^{a_{1,b}} v, \tag{4.1}$$

where h is a horizontal edge and v is a vertical edge and the product means concatenation.

Lemma 4.3. *We have*

$$a_{1,i} = \left\lceil \frac{ai}{b} \right\rceil - \left\lceil \frac{a(i-1)}{b} \right\rceil \quad (1 \le i \le b). \tag{4.2}$$

Proof. Since the slope of the line $\eta([a,b])$ is $\frac{b}{a}$, the number $a_{1,1}$ is the smallest positive integer such that $\frac{b}{a} a_{1,1} \ge 1$. Thus, we obtain $a_{1,1} = \lceil \frac{a}{b} \rceil$. Since the y-coordinate of the point on the line with the x-coordinate $a_{1,2}$ is greater than or equal to 2, the number $a_{1,2}$ is the smallest positive integer such that $\frac{b}{a}(a_{1,1}+a_{1,2}) \ge 2$. Thus, we obtain $a_{1,2} = \lceil \frac{2a}{b} \rceil - \lceil \frac{a}{b} \rceil$. Now assume that $a_{1,i} = \lceil \frac{ai}{b} \rceil - \lceil \frac{a(i-1)}{b} \rceil$ for $1 \le i \le k$. By a similar argument, $a_{1,k+1}$ is the smallest positive integer such that $\frac{b}{a}(a_{1,1} + \cdots + a_{1,k+1}) \ge k+1$. Since $(a_{1,1} + \cdots + a_{1,k}) = \lceil \frac{ak}{b} \rceil$, we obtain $a_{1,k+1} = \lceil \frac{a(k+1)}{b} \rceil - \lceil \frac{ak}{b} \rceil$. By induction, we are done. $\square$

Lemma 4.4. *The function $[a,b] \mapsto (a_{1,1}, \ldots, a_{1,b})$ is an injection from the set of reduced positive roots into the set of finite sequences in $\mathbb{Z}_{>0}$.*

Proof. It follows directly from (4.2). $\square$

Recall the assumption that $[a,b]$ is a reduced positive root of $\mathcal{H}(m)$ with $a \ge b$.

Lemma 4.5. *Assume that $\frac{a}{b} \ne m$. Let*

$$\frac{a}{b} = N_1 + \rho_1 \quad \textit{with } N_1 = \left\lfloor \frac{a}{b} \right\rfloor. \tag{4.3}$$

Then we have

(1) $1 \le N_1 \le m-1$;
(2) $a_{1,i} = N_1$ *or* $N_1 + 1$ *for all* $1 \le i \le b$;
(3) $a_{1,1} = N_1 + 1$ *and* $a_{1,b} = N_1$;
(4) $\boldsymbol{c}_1 = (a_{1,i})$ *is of type* $\begin{cases} + & \textit{if } \rho_1 > \frac{1}{2}, \\ - & \textit{if } 0 < \rho_1 < \frac{1}{2}, \\ = & \textit{if } \rho_1 = \frac{1}{2}. \end{cases}$

Proof. (1) Since $[a,b]$ is a reduced positive root of $\mathcal{H}(m)$, we have $a^2 + b^2 - mab \le 1$ from (3.2). Thus we have $1 \le \frac{a}{b} \le m$. Since we assume that $\frac{a}{b} \ne m$, we obtain the desired result.

(2), (3) It is clear from the fact that $a_{1,i} = N_1 + \lceil \rho_1 i \rceil - \lceil \rho_1 (i-1) \rceil$.

(4) Suppose $\rho_1 > \frac{1}{2}$. We will show that $(a_{1,i}, a_{1,i+1}) \ne (N_1, N_1)$ for all i. Note that $a_{1,i} = N_1$ if and only if $\lceil \rho_1 i \rceil - \lceil \rho_1 (i-1) \rceil = 0$. If $(a_{1,i}, a_{1,i+1}) = (N_1, N_1)$ for some i, then

$$\lceil \rho_1 (i+1) \rceil - \lceil \rho_1 i \rceil = \lceil \rho_1 i \rceil - \lceil \rho_1 (i-1) \rceil = 0 \iff \lceil \rho_1 (i+1) \rceil = \lceil \rho_1 i \rceil = \lceil \rho_1 (i-1) \rceil$$

which implies that there exists an integer t such that $t < \rho_1(i-1) < \rho_1 i < \rho_1(i+1) < t+1$. This contradicts to $\rho_1 > \frac{1}{2}$. The other cases can be proved similarly. $\square$

Definition 4.6. Let $[a,b]$ be a reduced positive root of $\mathcal{H}(m)$ with $a \geq b$. For $n \geq 1$, define inductively ρ_n, N_n and $\boldsymbol{c}_n = (a_{n,1}, a_{n,2}, \cdots, a_{n,d_n})$ as follows.

(0) Note that ρ_1, N_1 and $\boldsymbol{c}_1 = (a_{1,1}, \ldots, a_{1,d_1})$ with $d_1 = b$ are already defined in (4.1) and (4.3).

(1) If $\boldsymbol{c}_{n-1}$ is of type $=$ or 0, stop the process. Otherwise, $\boldsymbol{c}_n = (a_{n,i})_{1 \leq i \leq d_n}$ are defined to be the sequence recording the numbers of consecutive occurrences of

$$\begin{cases} (N_{n-1}+1)\text{'s in } \boldsymbol{c}_{n-1} & \text{if } \boldsymbol{c}_{n-1} \text{ is of type } +, \\ N_{n-1}\text{'s in } \boldsymbol{c}_{n-1} & \text{if } \boldsymbol{c}_{n-1} \text{ is of type } -, \end{cases}$$

where d_n is the number of N_{n-1} (resp. $N_{n-1}+1$) in $\boldsymbol{c}_{n-1}$ if it is of type $+$ (resp. type $-$).

(2) ρ_n is defined to be a rational number with $0 \leq \rho_n < 1$ and N_n is to be a positive integer such that

$$N_n + \rho_n = \begin{cases} \frac{\rho_{n-1}}{1-\rho_{n-1}} & \text{if } \rho_{n-1} \geq \frac{1}{2}, \\ \frac{1-\rho_{n-1}}{\rho_{n-1}} & \text{if } \rho_{n-1} < \frac{1}{2}. \end{cases}$$

The sequences $\boldsymbol{c}_n$, $n = 1, 2, \ldots$, are called the *canonical sequences* of $[a,b]$.

Example 4.7. (1) Let $m = 3$ and $[a,b] = [5,3]$. Then $N_1 = 1$, $\rho_1 = \frac{2}{3}$ and $d_1 = b = 3$. From the definition or by Lemma 4.3, the sequence $\boldsymbol{c}_1$ is given by

$$\boldsymbol{c}_1 = (a_{1,1}, a_{1,2}, a_{1,3}) = (2, 2, 1),$$

which is of type $+$. Since

$$\frac{\rho_1}{1-\rho_1} = \frac{\frac{2}{3}}{1-\frac{2}{3}} = 2 = N_2,$$

we have $\boldsymbol{c}_2 = (a_{2,1}) = (2)$, which is of type 0.

(2) Let $m = 3$ and $[a,b] = [8,5]$. In this case, $N_1 = 1, \rho_1 = \frac{3}{5}, d_1 = b = 5$ and

$$\boldsymbol{c}_1 = (a_{1,1}, a_{1,2}, a_{1,3}, a_{1,4}, a_{1,5}) = (2, 2, 1, 2, 1),$$

which is of type $+$. Then $N_2 = 1, \rho_2 = \frac{1}{2}, d_2 = 2$ and

$$\boldsymbol{c}_2 = (a_{2,1}, a_{2,2}) = (2, 1),$$

which is of type $=$.

(3) Assume $m = 3$ and $[a, b] = [59, 23]$. Then we have

$$\begin{aligned}
&\boldsymbol{c}_1 = (3,3,2,3,2,3,2,3,3,2,3,2,3,2,3,3,2,3,2,3,2,3,2), && N_1 = 2, && \rho_1 = 13/23, && \text{type } +,\\
&\boldsymbol{c}_2 = (2,1,1,2,1,1,2,1,1,1), && N_2 = 1, && \rho_2 = 3/10, && \text{type } -,\\
&\boldsymbol{c}_3 = (2,2,3), && N_3 = 2, && \rho_3 = 1/3, && \text{type } -,\\
&\boldsymbol{c}_4 = (2), && N_4 = 2, && \rho_4 = 0, && \text{type } 0.
\end{aligned}$$

(4) Suppose $m = 5$ and $[a, b] = [62, 13]$. Then we obtain

$$\begin{aligned}
&\boldsymbol{c}_1 = (5,5,5,5,4,5,5,5,4,5,5,5,4), && N_1 = 4, && \rho_1 = 10/13, && \text{type } +,\\
&\boldsymbol{c}_2 = (4,3,3), && N_2 = 3, && \rho_2 = 1/3, && \text{type } -,\\
&\boldsymbol{c}_3 = (2), && N_3 = 2, && \rho_3 = 0, && \text{type } 0.
\end{aligned}$$

For a positive rational number r, let $D(r) = q$ when $r = \frac{p}{q}$ and p, q are relatively prime integers.

Lemma 4.8. *The following holds for $n \geq 2$.*

(1) $\boldsymbol{c}_n = (a_{n,i})_{1 \leq i \leq d_n}$ *is given by*

$$a_{n,i} = \begin{cases} \left\lceil \frac{\rho_{n-1} i}{1-\rho_{n-1}} \right\rceil - \left\lceil \frac{\rho_{n-1}(i-1)}{1-\rho_{n-1}} \right\rceil, & \text{when } \boldsymbol{c}_{n-1} \text{ is of type } +,\\ \left\lfloor \frac{(1-\rho_{n-1}) i}{\rho_{n-1}} \right\rfloor - \left\lfloor \frac{(1-\rho_{n-1})(i-1)}{\rho_{n-1}} \right\rfloor, & \text{when } \boldsymbol{c}_{n-1} \text{ is of type } -, \end{cases}$$

for $1 \leq i \leq d_n$, and we get $d_n = D(\rho_n)$.

(2) *We have $a_{n,i} = N_n$ or $N_n + 1$ for $1 \leq i \leq d_n$.*

(3) $a_{n,1} = N_n + 1$ *(resp. N_n) and* $a_{1,d_n} = N_n$ *(resp. $N_n + 1$) if $\rho_n \neq 0$ and $\boldsymbol{c}_{n-1}$ is of type $+$ (resp. type $-$).*

(4) $\boldsymbol{c}_n = (a_{n,i})$ *is of type* $\begin{cases} + & \text{if } \rho_n > \frac{1}{2},\\ - & \text{if } 0 < \rho_n < \frac{1}{2},\\ = & \text{if } \rho_n = \frac{1}{2},\\ 0 & \text{if } \rho_n = 0. \end{cases}$

Proof. (1) Suppose that $\boldsymbol{c}_{n-1}$ is of type $+$. We have

$$\begin{aligned} a_{n-1,i} &= \lceil (N_{n-1} + \rho_{n-1}) i \rceil - \lceil (N_{n-1} + \rho_{n-1})(i-1) \rceil \\ &= N_{n-1} + \lceil \rho_{n-1} i \rceil - \lceil \rho_{n-1}(i-1) \rceil. \end{aligned}$$

Since $a_{n,1}$ is the number of first successive $N_{n-1} + 1$ in the sequence $\boldsymbol{c}_{n-1}$, we have

$$\lceil \rho_{n-1} i \rceil - \lceil \rho_{n-1}(i-1) \rceil = \begin{cases} 1 & \text{if } i = 1, 2, \cdots, a_{n,1},\\ 0 & \text{if } i = a_{n,1} + 1. \end{cases}$$

This implies that

$$(a_{n,1} - 1)\rho_{n-1} < a_{n,1} - 1 < a_{n,1}\rho_{n-1} < (a_{n,1} + 1)\rho_{n-1} \leq a_{n,1}.$$

So we obtain

$$a_{n,1} \geq 1, \qquad \frac{\rho_{n-1}}{1-\rho_{n-1}} \leq a_{n,1} < \frac{1}{1-\rho_{n-1}},$$

and hence $a_{n,1} = \lceil \frac{\rho_{n-1}}{1-\rho_{n-1}} \rceil$.

Since $a_{n,2}$ is the number of successive $N_{n-1}+1$ in $\boldsymbol{c}_{n-1}$ between the first and the second N_{n-1}, we have

$$\lceil \rho_{n-1} i \rceil - \lceil \rho_{n-1}(i-1) \rceil = \begin{cases} 1 & \text{if } i = a_{n,1}+2, a_{n,1}+3, \cdots, a_{n,1}+a_{n,2}+1, \\ 0 & \text{if } i = a_{n,1}+a_{n,2}+2. \end{cases}$$

This implies that

$$(a_{n,1}+a_{n,2})\rho_{n-1} < a_{n,1}+a_{n,2}-1 < (a_{n,1}+a_{n,2}+1)\rho_{n-1} < (a_{n,1}+a_{n,2}+2)\rho_{n-1} \leq a_{n,1}+a_{n,2}.$$

Hence, we have

$$\frac{2\rho_{n-1}}{1-\rho_{n-1}} - a_{n,1} \leq a_{n,2} < -a_{n,1} + \frac{\rho_{n-1}+1}{1-\rho_{n-1}}.$$

Since $a_{n,2}$ is an integer and $\frac{\rho_{n-1}+1}{1-\rho_{n-1}} - \frac{2\rho_{n-1}}{1-\rho_{n-1}} = 1$, we obtain

$$a_{n,2} = \left\lceil \frac{2\rho_{n-1}}{1-\rho_{n-1}} \right\rceil - \left\lceil \frac{\rho_{n-1}}{1-\rho_{n-1}} \right\rceil.$$

By a similar argument, we have

$$a_{n,i} = \left\lceil \frac{\rho_{n-1} i}{1-\rho_{n-1}} \right\rceil - \left\lceil \frac{\rho_{n-1}(i-1)}{1-\rho_{n-1}} \right\rceil \qquad \text{for } i \geq 3.$$

Next, we show that $d_n = D(\rho_n)$. By the definition of $\boldsymbol{c}_n = (a_{n,i})_{1 \leq i \leq d_n}$, we have

$$d_{n-1} = \sum_{1 \leq i \leq d_n} a_{n,i} + d_n = \left\lceil \frac{\rho_{n-1} d_n}{1-\rho_{n-1}} \right\rceil + d_n.$$

Suppose that $\rho_{n-1} = \frac{e_{n-1}}{d_{n-1}}$, where e_{n-1} and d_{n-1} are relatively prime. Then we have

$$N_n + \rho_n = \frac{\rho_{n-1}}{1-\rho_{n-1}} = \frac{e_{n-1}}{d_{n-1}-e_{n-1}}.$$

Since d_{n-1} and e_{n-1} are relatively prime, $D(\rho_n) = D(\frac{e_{n-1}}{d_{n-1}-e_{n-1}}) = d_{n-1} - e_{n-1}$. Hence, we obtain

$$d_{n-1} = \left\lceil \frac{e_{n-1} d_n}{d_{n-1}-e_{n-1}} \right\rceil + d_n \iff d_n = d_{n-1} - e_{n-1} = D(\rho_n).$$

The proof for the case when $\boldsymbol{c}_{n-1}$ is of type $-$ is similar, and we omit the details.

(2) Suppose that $\boldsymbol{c}_{n-1}$ is of type $+$. Since $\frac{\rho_{n-1}}{1-\rho_{n-1}} = N_n + \rho_n$, we have

$$a_{n,i} = \lceil N_n i + \rho_n i \rceil - \lceil N_n(i-1) + \rho_n(i-1) \rceil = N_n + \lceil \rho_n i \rceil - \lceil \rho_n(i-1) \rceil.$$

Since $0 \leq \rho_n < 1$, we obtain $a_{n,i} = N_n$ or $N_n + 1$ for $1 \leq i \leq d_n$. The case $\boldsymbol{c}_{n-1}$ is of type $-$ is similar.

(3) Suppose that $\rho_n \neq 0$ and $\boldsymbol{c}_{n-1}$ is of type $+$. Write $\rho_{n-1} = \frac{e_{n-1}}{d_{n-1}}$ with e_{n-1} and d_{n-1} relatively prime. Then $N_n + \rho_n = \frac{e_{n-1}}{d_{n-1}-e_{n-1}}$. Since $\rho_n \neq 0$, we have

$$\begin{aligned} a_{n,1} &= N_n + \lceil \rho_n \rceil = N_n + 1, \\ a_{n,d_n} &= \lceil (N_n+\rho_n)d_n \rceil - \lceil (N_n+\rho_n)(d_n-1) \rceil \\ &= \left\lceil \frac{e_{n-1}}{d_{n-1}-e_{n-1}} d_n \right\rceil - \left\lceil \frac{e_{n-1}}{d_{n-1}-e_{n-1}}(d_n-1) \right\rceil \\ &= \lceil e_{n-1} \rceil - \lceil e_{n-1} - N_n - \rho_n \rceil = e_{n-1} - e_{n-1} + N_n = N_n. \end{aligned}$$

The case $\boldsymbol{c}_{n-1}$ is of type $-$ is similar.

(4) The proof is similar to that of Lemma 4.5 (4), and we omit the details. $\square$

Lemma 4.9. *Let $[a,b]$ be a reduced positive root of $\mathcal{H}(m)$ with $a \geq b$, and N_k and $\boldsymbol{c}_k$ be defined as in Definition 4.6. Denote the type of $\boldsymbol{c}_k$ by ϵ_k. For each $k \geq 0$, the data*

$$(N_1, \epsilon_1, N_2, \epsilon_2, \ldots, N_k, \epsilon_k, \boldsymbol{c}_{k+1})$$

determines $[a,b]$ uniquely.

Proof. By Lemma 4.8 and Definition 4.6, we obtain $\boldsymbol{c}_k$ from $(N_k, \epsilon_k, \boldsymbol{c}_{k+1})$ and continue the process to obtain $\boldsymbol{c}_1$. Now the assertion follows from Lemma 4.4. $\square$

After establishing another lemma below, we will define the level of $[a,b]$.

Lemma 4.10. *Assume that $[a,b]$ is an imaginary reduced positive root of $\mathcal{H}(m)$ with $a \geq b$. For $n \geq 2$, if $N_k = m-2+\delta_{1,k}$ and $\boldsymbol{c}_k$ is of type $+$ for $1 \leq k \leq n-1$, then*

$$1 < N_n + \rho_n < \gamma - 1,$$

where we set $\gamma := \frac{m+\sqrt{m^2-4}}{2}$. In particular, $1 \leq N_n \leq m-2$ for $n \geq 2$.

Proof. We use induction on n. It follows from the assumptions and Lemma 4.5 that $\frac{a}{b} = N_1 + \rho_1 = m - 1 + \rho_1$ with $\frac{1}{2} < \rho_1 < 1$ and $N_2 + \rho_2 = \frac{\rho_1}{1-\rho_1}$. Since $[a,b]$ is a reduced positive root of $\mathcal{H}(m)$ and $[a,b] \neq [F_i, F_{i-1}]$ for any $i = 2, 3, \ldots$, we have $1 \leq \frac{a}{b} < \gamma$ and $\frac{1}{2} < \rho_1 < \gamma - (m-1)$. Note that $\gamma(m-\gamma) = 1$. Since $y = \frac{x}{1-x}$ is an increasing function for $0 \leq x < 1$, we obtain

$$\frac{\frac{1}{2}}{1-\frac{1}{2}} = 1 < N_2 + \rho_2 = \frac{\rho_1}{1-\rho_1} < \frac{\gamma - m + 1}{-\gamma + m} = \gamma - 1 < m - 1.$$

Hence, $1 \leq N_2 \leq m-2$.

Now assume that we have $1 < N_{n-1} + \rho_{n-1} < \gamma - 1$. Since $N_{n-1} = m-2$ and $\boldsymbol{c}_{n-1}$ is of type $+$, we have $\frac{1}{2} < \rho_{n-1} < \gamma - (m-1)$. By the same argument as in the case $n = 2$, we have

$$1 < N_n + \rho_n < \gamma - 1 \quad \text{and} \quad 1 \leq N_n \leq m-2.$$

$\square$

Let us consider the sequence $\{\gamma_n\}$ given by

$$\gamma_0 = 0, \qquad \gamma_1 = m - \frac{1}{2}, \qquad \gamma_n = m - \frac{1}{\gamma_{n-1}} \quad (n \geq 2).$$

It is straightforward to check that $\gamma_n < \gamma_{n+1} < \gamma$ for $n \geq 1$, and

$$\gamma_n \to \gamma \qquad \text{as} \;\; n \to \infty,$$

where $\gamma = \frac{m+\sqrt{m^2-4}}{2}$ as before.

Definition 4.11. Let $[a, b]$ be a reduced positive root of $\mathcal{H}(m)$ with $a \geq b$. Then the *level* L of $[a, b]$ is defined to be the positive integer uniquely determined by the inequalities

$$\gamma_{L-1} < \frac{a}{b} \leq \gamma_L.$$

Example 4.12. Let $m = 3$ and $[a, b] = [339, 130]$. Then we have $\gamma_1 = \frac{5}{2}, \gamma_2 = \frac{13}{5}, \gamma_3 = \frac{34}{13}$ and $\gamma_2 < \frac{a}{b} < \gamma_3$. Thus the level of $[a, b]$ is 3.

Proposition 4.13. *Let $[a, b]$ be an imaginary reduced positive root of $\mathcal{H}(m)$ with $a \geq b$. Assume that the level of $[a, b]$ is L. Then the following statements hold:*

(i) $N_k = m - 2 + \delta_{1,k}$ *for* $1 \leq k \leq L - 1$, *and hence*

$$(N_1, N_2, \ldots, N_L) = (m - 1, m - 2, m - 2, \ldots, m - 2, c)$$

for $1 \leq c \leq m - 2 + \delta_{1,L}$, *where* $\delta_{i,j}$ *is the Kronecker's delta.*

(ii) $\boldsymbol{c}_k$ *is of type* $+$ *for* $1 \leq k \leq L - 1$,

(iii) $N_L + \rho_L \leq m - 2 + \delta_{1,L} + \frac{1}{2}$.

Proof. Suppose the level of $[a, b]$ is 1. Then we obtain $N_1 + \rho_1 = \frac{a}{b} \leq \gamma_1 = m - \frac{1}{2}$. If the level of $[a, b]$ is 2. Then $(m - 1) + \frac{1}{2} < N_1 + \rho_1 \leq \gamma_2 = (m - 1) + \frac{2m-3}{2m-1}$. Thus we obtain $\frac{1}{2} < \rho_1 \leq \frac{2m-3}{2m-1}$. It implies that $1 < N_2 + \rho_2 = \frac{\rho_1}{1-\rho_1} \leq (m - 2) + \frac{1}{2}$. Hence, $N_1 = m - 1$, $\boldsymbol{c}_1$ is of type $+$ and $N_L + \rho_L \leq m - 2 + \frac{1}{2}$.

Now suppose the level of $[a, b]$ is $L \geq 3$. Then

$$(m - 1) + 1 - \frac{1}{\gamma_{L-2}} = \gamma_{L-1} < \frac{a}{b} \leq \gamma_L = (m - 1) + 1 - \frac{1}{\gamma_{L-1}}.$$

Thus we obtain $1 - \frac{1}{\gamma_{L-2}} < \rho_1 \leq 1 - \frac{1}{\gamma_{L-1}}$. It implies that $\gamma_{L-2} - 1 < N_2 + \rho_2 = \frac{\rho_1}{1-\rho_1} \leq \gamma_{L-1} - 1$. Since $m-2 < \gamma_{L-2}-1$, we have $N_2 = m-2$ and $\gamma_{L-2}-m+2 < \rho_2 \leq \gamma_{L-1}-1-m+2$.

By the recursive relation of $\{\gamma_n\}$, we obtain

$$\gamma_{L-3} - 1 < N_3 + \rho_3 = \frac{\rho_2}{1 - \rho_2} \leq \gamma_{L-2} - 1.$$

Repeating this argument yields

$$m - 2 + \frac{1}{2} = \gamma_1 - 1 < N_{L-1} + \rho_{L-1} = \frac{\rho_{L-2}}{1 - \rho_{L-2}} \leq \gamma_2 - 1.$$

Hence, $1 < N_L + \rho_L = \frac{\rho_{L-1}}{1-\rho_{L-1}} \leq \gamma_1 - 1 = m - 2 + \frac{1}{2}$. It implies that $N_k = m - 2 + \delta_{1,k}$ for $1 \leq k \leq L - 1$. Moreover, $\boldsymbol{c}_k$ is of type $+$ for $1 \leq k \leq L - 1$ because $\rho_k > \frac{1}{2}$. Since $N_L + \rho_L \leq m - 2 + \frac{1}{2}$, we obtain the desired result. $\square$

Corollary 4.14. *Assume that L is the level of $[a,b]$.*

(1) *If $N_L = m-2+\delta_{1,L}$ then $\boldsymbol{c}_L$ cannot be of type $+$.*
(2) *If $L \geq 2$ and $N_L = 1$ then $\boldsymbol{c}_L$ cannot be of type 0.*

Proof. Part (1) is an immediate consequence of Proposition 4.13 (iii) and Lemma 4.8 (4). For part (2), assume that $N_L = 1$ and $\boldsymbol{c}_L$ of type 0. Then $\rho_L = 0$ by Lemma 4.8 (4) and $\rho_{L-1} = \frac{1}{2}$ by the definition of ρ_n. However, $\rho_{L-1} > \frac{1}{2}$ by Proposition 4.13 (ii) and Lemma 4.8 (4), which is a contradiction. ☐

4-A. **Reduction according to canonical sequences.** In this subsection we show that $s^{a\times b}$ can be written through the canonical sequences $\boldsymbol{c}_k$. This result will be used in the next section and has its own interest. In what follows, we write only the subscripts of simple reflections when we express elements in $W(m)$. For example, we write $23 = s_2s_3$ and $21 = s_2s_1$.

Proposition 4.15. *Let $[a,b]$ be an imaginary positive reduced root of $\mathcal{H}(m)$ with $a \geq b$, and $\boldsymbol{c}_k = (a_{k,1}, a_{k,2}, \ldots, a_{k,d_k})$ its canonical sequences. Suppose the level of $[a,b]$ is L. According to the values of k, define $H_k, V_k \in W(m)$ by*

k	H_k	V_k
1	23	21
2	21	31
$2l+1$ $(l \geq 1)$	$1(321)^{l-1}23(123)^{l-1}1$	$1(321)^{l-1}2123(123)^{l-1}1$
$2l+2$ $(l \geq 1)$	$13(213)^{l-1}12(312)^{l-1}31$	$13(213)^{l-1}2312(312)^{l-1}31$

Then, for $1 \leq k \leq L$, we have

$$s^{a\times b} = H_k^{a_{k,1}} V_k H_k^{a_{k,2}} V_k \cdots H_k^{a_{k,d_k}} V_k. \tag{4.4}$$

Example 4.16. Continuing Example 4.7 (4), suppose $m = 5$ and $[a,b] = [62,13]$. Since $\gamma_1 = \frac{9}{2} < \frac{62}{13} < \gamma_2 = \frac{43}{9}$, the level of $[62,13]$ is 2. Since $\boldsymbol{c}_2 = (4,3,3)$, we obtain

$$s^{62\times 13} = (21)^4(31)(21)^3(31)(21)^3(31) \in W(5).$$

Proof of Proposition 4.15. If $k=1$, then (4.4) follows from the definition of $s^{a\times b}$ and (4.1). Suppose $k=2$. By Proposition 4.13, $N_1 = m-1$ and $\boldsymbol{c}_1$ is of type $+$. Then, by Lemma 4.5 and the definition of $\boldsymbol{c}_2$, we have

$$\boldsymbol{c}_1 = (a_{1,1}, a_{1,2}, \ldots, a_{1,d_1}) = (m^{a_{2,1}}, m-1, m^{a_{2,2}}, m-1, \ldots, m^{a_{2,d_2}}, m-1),$$

where we write $m^s = \underbrace{m, m, \ldots, m}_{s\text{-times}}$. Since $(23)^m = e$ and $(23)^{m-1}21 = 31$, we obtain

$$\begin{aligned} s^{a\times b} =& H_1^{a_{1,1}} V_1 H_1^{a_{1,2}} V_1 \cdots H_1^{a_{1,d_1}} V_1 \\ =& (21)^{a_{2,1}}(31)(21)^{a_{2,2}}(31)\cdots(21)^{a_{2,d_2}}(31) = H_2^{a_{2,1}} V_2 H_2^{a_{2,2}} V_2 \cdots H_2^{a_{2,d_2}} V_2. \end{aligned}$$

Suppose that $k = 3$. By Proposition 4.13, $N_2 = m - 2$ and $\boldsymbol{c}_2$ is of type $+$. By Lemma 4.8, we have $a_{2,i} = m - 2$ or $m - 1$ for $1 \le i \le d_2$, $a_{2,1} = m - 1$ and $a_{2,d_2} = m - 2$. Then, by the definition of $\boldsymbol{c}_3$, we have

$$\begin{aligned} \boldsymbol{c}_2 &= (a_{2,1}, a_{2,2}, \ldots, a_{2,d_2}) \\ &= ((m-1)^{a_{3,1}}, m-2, (m-1)^{a_{3,2}}, m-2, \ldots, (m-1)^{a_{3,d_3}}, m-2), \end{aligned}$$

where we write $(m-1)^s = \underbrace{m-1, m-1, \ldots, m-1}_{s\text{-times}}$. Since $(21)^{m-1}(31) = 1231$ and $(21)^{m-2}(31) = 121231$, we have

$$\begin{aligned} s^{a\times b} =& H_2^{a_{2,1}} V_2 H_2^{a_{2,2}} V_2 \cdots H_2^{a_{2,d_2}} V_2 \\ =& (1231)^{a_{3,1}}(121231)(1231)^{a_{3,2}}(121231)\cdots(1231)^{a_{3,d_3}}(121231) \\ =& H_3^{a_{3,1}} V_3 H_3^{a_{3,2}} V_3 \cdots H_3^{a_{3,d_3}} V_3. \end{aligned}$$

The proof for $k = 4$ is similar to the case $k = 3$ with

$$\begin{aligned} (1231)^{m-1}(121231) &= 1(3123)1 = 13(12)31, \\ (1231)^{m-2}(121231) &= 1(323123)1 = 13(2312)31. \end{aligned}$$

Now let us use induction on k. Suppose (4.4) is true for $k = 2l + 2$ and consider $k+1 = 2l+3 \le L$. Then $N_k = m-2$ and $\boldsymbol{c}_k$ is of type $+$. By Lemma 4.8 and the definition of $\boldsymbol{c}_{k+1}$, we have $a_{k,1} = m - 1$, $a_{k,d_k} = m - 2$ and

$$\begin{aligned} \boldsymbol{c}_k &= (a_{k,1}, a_{k,2}, \ldots, a_{k,d_k}) \\ &= ((m-1)^{a_{k+1,1}}, m-2, (m-1)^{a_{k+1,2}}, m-2, \ldots, (m-1)^{a_{k+1,d_{k+1}}}, m-2). \end{aligned} \tag{4.5}$$

Since we have

$$\begin{aligned} H_k^{m-1} V_k &= 13(213)^{l-1}(12)^{m-1}2312(312)^{l-1}31 = 1(321)^l 23(123)^l 1 = H_{k+1}, \\ H_k^{m-2} V_k &= 13(213)^{l-1}(12)^{m-2}2312(312)^{l-1}31 = 1(321)^l 2123(123)^l 1 = V_{k+1}, \end{aligned}$$

it follows from (4.5) and the induction hypothesis that

$$\begin{aligned} s^{a\times b} =& H_k^{a_{k,1}} V_k H_k^{a_{k,2}} V_k \cdots H_k^{a_{k,d_k}} V_k \\ =& H_{k+1}^{a_{k+1,1}} V_{k+1} H_{k+1}^{a_{k+1,2}} V_{k+1} \cdots H_{k+1}^{a_{k+1,d_{k+1}}} V_{k+1}. \end{aligned}$$

The proof for the next step (i.e. $k = 2l + 3$) is similar, and we omit the details. □

The following corollary will play an important role in the next section and has interest in its own right.

Corollary 4.17. *Let $[a, b]$ $(a \ge b)$ be an imaginary positive reduced root of $\mathcal{H}(m)$ with level $L \ge 2$, and $\boldsymbol{c}_k = (a_{k,1}, a_{k,2}, \ldots, a_{k,d_k})$ its canonical sequences for $2 \le k \le L$.*

(1) If $k = 2l + 2$, then

$$s^{a\times b} = (132)^l (21)^{a_{k,1}}(31)(21)^{a_{k,2}}(31)\cdots(21)^{a_{k,d_k}}(31)(231)^l. \tag{4.6}$$

(2) If $k=2l+1$, then

$$s^{a\times b}=(132)^l(23)^{a_{k,1}+1}(21)(23)^{a_{k,2}+1}(21)\cdots(23)^{a_{k,d_k}+1}(21)(231)^l. \tag{4.7}$$

Moreover, we have

$$(231)^l s^{a\times b}(132)^l=s^{\tilde a\times\tilde b}$$

for $[\tilde a,\tilde b]$ whose first canonical sequence is $(a_{k,1}+1,\dots,a_{k,d_k}+1)$ with level $L-k+1$.

Remark 4.18. The part (2) is related to Lemma 3.3 (2) of [17], which connects the Weyl group action on the set of roots of $\mathcal{H}(m)$ with the set of rigid reflections. Though the above corollary is not enough to prove injectivity, one may find that the expressions look more natural than those obtained in the next section after the reduction is completed.

Proof. Assume that $k=2l+2$. If $k=2$ then (4.6) is nothing but (4.4). If $k\geq 4$, we obtain from (4.4)

$$\begin{aligned}
s^{a\times b}&=13(213)^{l-1}(12)^{a_{k,1}}(2312)\cdots(12)^{a_{k,d_k}}(2312)(312)^{l-1}31\\
&=13(213)^{l-1}21(12)^{a_{k,1}}12(23)\cdots(12)^{a_{k,d_k}}12(2312)(312)^{l-1}31\\
&=(132)^l1(12)^{a_{k,1}}(13)\cdots(12)^{a_{k,d_k}}(13)1(231)^l\\
&=(132)^l(21)^{a_{k,1}}(31)(21)^{a_{k,2}}(31)\cdots(21)^{a_{k,d_k}}(31)(231)^l.
\end{aligned}$$

The case $k=2l+1$ is similar and we omit the details. The last assertion is clear from the definitions. ☐

Though Proposition 4.15 and Corollary 4.17 provide reductions of the initial expression of $s^{a\times b}$, it is not sufficient to prove injectivity of the map $[a,b]\mapsto s^{a\times b}$. In the next section, we will further reduce $s^{a\times b}$ to its standard word to show injectivity.

5. Reduction to standard words

In this section, we reduce each $s^{a\times b}$ to its standard word in $W(m)$, starting with an expression in Proposition 4.15, and prove Theorem 3.3 by showing all the standard words are distinct.

For $k\in\mathbb{Z}_{\geq 2}$, define

$$\begin{aligned}
\mathcal{S}(2k-1)=&\{s_1^2-e,s_2^2-e,s_3^2-e,(s_1s_2)^{k-1}s_1-(s_2s_1)^{k-1}s_2,(s_2s_3)^{k-1}s_2-(s_3s_2)^{k-1}s_3\},\\
\mathcal{S}(2k)=&\{s_1^2-e,s_2^2-e,s_3^2-e,(s_1s_2)^k-(s_2s_1)^k,\\
&(s_2s_3)^k-(s_3s_2)^k,(s_1s_2)^{k-1}s_1(s_3s_2)^k-(s_2s_1)^ks_3(s_2s_3)^{k-1}\}.
\end{aligned}$$

It will be shown in Propositions 6.7 and 6.8 that $\mathcal{S}(m)$ is a Gröbner–Shirshov basis of $W(m)$ for $m\geq 3$. Thus we take $\mathcal{S}(m)$-standard words or monomials (see Definition 6.1) as standard expressions of the elements of $W(m)$.

In this section, as in Proposition 4.15, we write only subscripts of simple reflections when we express elements in $W(m)$, and the identity element of $W(m)$ will be denoted by e. Before delving into general cases, let us look at a simple example.

Example 5.1. Suppose $[a,b]=[5,3]$ and $m=3$. By definition an $\mathcal{S}(3)$-standard word cannot have any of $11, 22, 33, 121, 232$ as a subword. Clearly, the level of $[5,3]$ is 1. By Example 4.7 (1) and Proposition 4.15, we have

$$s^{5\times 3}=(23)^2(21)(23)^2(21)(23)^1(21),$$

which is not $\mathcal{S}(3)$-standard because 232 is a subword. Using the relations $232=323, 22=e, 33=e$, we obtain

$$s^{a\times b}=(31)(31)(3231),$$

which is $\mathcal{S}(3)$-standard.

In what follows we obtain $\mathcal{S}(m)$-standard words for $s^{a\times b}$, where $[a,b]$ is a positive imaginary reduced root of $\mathcal{H}(m)$. Let L be the level of $[a,b]$. The canonical sequences $\boldsymbol{c}_k=(a_{k,1},a_{k,2},\dots,a_{k,d_k})$ and the numbers N_k are defined in Definition 4.6.

Define $N:=N_L-\delta_{L,1}$ and $\ell:=\lfloor m/2\rfloor$, where $\delta_{i,j}$ is Kronecker's delta. According to the values of N, let w_1 and w_2 be the elements of $W(m)$ defined by the following table.

(5.1)

N	w_1	w_2
$N\le \ell-3$	$(23)^{N+2}(21)$	$(23)^{N+1}(21)$
$\ell-2$	$(32)^{m-\ell-1}(31)$	$(23)^{\ell-1}(21)$
$\ell-1\le N\le m-3$	$(32)^{m-N-3}(31)$	$(32)^{m-N-2}(31)$
$m-2$	21	31

Lemma 5.2 (level $L=1$)**.** *Assume that the level L of $[a,b]$ is 1. Unless $m=3$, $N_1=2$ and $\boldsymbol{c}_1$ is of type $-$, the following expression of $s^{a\times b}$ is $\mathcal{S}(m)$-standard:*

$$s^{a\times b}=\begin{cases} w_1^{a_{2,1}}w_2\cdots w_1^{a_{2,d_2}}w_2 & \text{if } \boldsymbol{c}_1 \text{ is of type } +,\\ w_1w_2^{a_{2,1}}\cdots w_1w_2^{a_{2,d_2}} & \text{if } \boldsymbol{c}_1 \text{ is of type } -,\\ w_1w_2 & \text{if } \boldsymbol{c}_1 \text{ is of type } =,\\ w_2, & \text{if } \boldsymbol{c}_1 \text{ is of type } 0.\end{cases}$$

The case when $m=3$, $N_1=2$ and $\boldsymbol{c}_1$ is of type $-$ is covered in Lemma 5.6.

Example 5.3. Continuing with Example 5.1, assume $[a,b]=[5,3]$ and $m=3$. Then $N=N_1-1=0$, $\ell=1$, $\rho_1=\frac{2}{3}$ and $\boldsymbol{c}_2=(2)$ from Example 4.7 (1). Thus we have $w_1=31$ and $w_2=3231$, and

$$s^{a\times b}=w_1^2w_2=(31)(31)(3231).$$

This coincides with the standard word in Example 5.1.

Proof of Lemma 5.2. By Proposition 4.15, we obtain

$$s^{a\times b} = (23)^{a_{1,1}}(21)(23)^{a_{1,2}}(21)\cdots(23)^{a_{1,d_1}}(21). \tag{5.2}$$

Note that the $\mathcal{S}(m)$-standard word of $(23)^s(21)$ is equal to itself

$$\begin{aligned} &(23)^s(21), \quad \text{if } s \le \ell-1, \text{ or} \\ &(32)^{m-s-1}(31), \quad \text{if } \ell \le s < m. \end{aligned} \tag{5.3}$$

Since $N = N_1 - 1$, we have four different cases according to (5.1).

Case 1: $1 \le N_1 \le \ell-2$. Since $a_{1,i} = N_1$ or N_1+1, the $\mathcal{S}(m)$-standard word of $(23)^{a_{1,i}}(21)$ is equal to itself for all $1 \le i \le d_1$. Since there are no relations involving $w_1 = (23)^{N_1+1}(21)$ and $w_2 = (23)^{N_1}(21)$ in the expression (5.2), it is already standard. Thus, if $\boldsymbol{c}_1$ is of type $+$ (resp. $-$), we obtain

$$s^{a\times b} = w_1^{a_{2,1}}w_2\cdots w_1^{a_{2,d_2}}w_2 \quad (\text{resp. } w_1w_2^{a_{2,1}}\cdots w_1w_2^{a_{2,d_2}})$$

from the definition of $\boldsymbol{c}_2$.

Suppose $\boldsymbol{c}_1$ is of type $=$. Since $\frac{a}{b} = N_1 + \frac{1}{2}$, we obtain $b = 2$ and

$$s^{a\times b} = (23)^{N_1+1}(21)(23)^{N_1}(21) = w_1w_2.$$

Assume $\boldsymbol{c}_1$ is of type 0. Then $\frac{a}{b} = N_1$. Since a and b are coprime, $a = b = 1$ and $N_1 = 1$. By the definition of $s^{a\times b}$, we obtain

$$s^{a\times b} = (23)(21) = w_2.$$

Case 2: $N_1 = \ell - 1$. Recall $a_{1,i} = N_1$ or $N_1 + 1$. The word $(23)^{N_1}(21)$ is $\mathcal{S}(m)$-standard, while $(23)^{N_1+1}(21)$ is reduced to $(32)^{m-\ell-1}(31)$. These words are w_2 and w_1 respectively. Moreover, there are no additional relations between $(23)^{N_1}(21)$ and $(32)^{m-\ell-1}(31)$. Hence, we obtain the desired expressions of $s^{a\times b}$ similarly to *Case 1.*

Case 3: $\ell \le N_1 \le m-2$. The $\mathcal{S}(m)$-standard words of $(23)^{N_1}(21)$ and $(23)^{N_1+1}(21)$ are $w_2 = (32)^{m-N_1-1}(31)$ and $w_1 = (32)^{m-N_1-2}(31)$ respectively. Moreover, there are no additional relations between w_1 and w_2. Hence, we obtain the desired expressions of $s^{a\times b}$ similarly to *Case 1.*

Case 4: $N_1 = m - 1$. Note that

$$\begin{aligned} &(23)^{m-1}(21) = (32)(21) = 31, \\ &(23)^{m}(21) = 21. \end{aligned}$$

As in *Case 1*, if $\boldsymbol{c}_1$ is of type 0, then $s^{a\times b}$ is equal to $w_2 = 31$, and if $\boldsymbol{c}_1$ is of type $=$, then $s^{a\times b}$ is equal to $w_1w_2 = (21)(31)$. If $\boldsymbol{c}_1$ is of type $-$, then

$$\boldsymbol{c}_1 = (m, (m-1)^{a_{2,1}}, m, (m-1)^{a_{2,2}}, \ldots, m, (m-1)^{a_{2,d_2}}),$$

where we write $(m-1)^s = \underbrace{m-1, m-1, \ldots, m-1}_{s\text{-times}}$, and $s^{a\times b}$ in (5.2) becomes equal to

$$(21)(31)^{a_{2,1}}(21)(31)^{a_{2,2}}\cdots(21)(31)^{a_{2,d_2}} = w_1w_2^{a_{2,1}}\cdots w_1w_2^{a_{2,d_2}}. \tag{5.4}$$

If $m > 3$, then this expression is standard and we obtain the desired form. If $m = 3$, then it is not standard because of the subword 121 and this case is covered in Lemma 5.6. Finally, $\boldsymbol{c}_1$ cannot be of type $+$ by Corollary 4.14 (1). It completes the proof. $\square$

Now we move on to higher levels. As before, define $\ell := \lfloor m/2 \rfloor$. According to the values of N_L, let v_1 and v_2 be defined by the following table.

(5.5)

N_L	v_1	v_2
$N_L \le \ell - 2$	$(12)^{N_L+1}(13)$	$(12)^{N_L}(13)$
$\ell - 1$	$(21)^{m-\ell-1}(23)$	$(12)^{\ell-1}(13)$
$\ell \le N_L \le m-2$	$(21)^{m-N_L-2}(23)$	$(21)^{m-N_L-1}(23)$

Define

$$x = (132)^{\lfloor \frac{L-2}{2} \rfloor} 1, \quad x^{-1} = 1(231)^{\lfloor \frac{L-2}{2} \rfloor} \quad \text{and} \quad y = \begin{cases} (132)^{\frac{L-4}{2}} 13 & \text{if } L \ge 4, \\ \hat{2} & \text{if } L = 2, \end{cases}$$

where $\hat{2}$ means 2 if the following letter is different from 2, or removing the following letter 2 otherwise. For example, $\hat{2}13 = 213$ and $\hat{2}23 = 3$. If $L = 3$, we do not need to define y.

Lemma 5.4 (level $L \ge 2$)**.** *Assume that the level of $[a, b]$ is $L \ge 2$.*

(1) *Suppose that L is even. Unless $m = 3, 4, 5$ and $N_L = m - 2$, the $\mathcal{S}(m)$-standard word of $s^{a \times b}$ is equal to*

$$\begin{cases} y v_2 v_1^{a_{L+1,1}-1} v_2 [v_1^{a_{L+1,2}} v_2 \cdots v_1^{a_{L+1,d_{L+1}}} v_2] x^{-1} & \text{if } \boldsymbol{c}_L \text{ is of type } +, \\ y v_2^{a_{L+1,1}+1} [v_1 v_2^{a_{L+1,2}} \cdots v_1 v_2^{a_{L+1,d_{L+1}}}] x^{-1} & \text{if } \boldsymbol{c}_L \text{ is of type } -, \\ y v_2^2 x^{-1} & \text{if } \boldsymbol{c}_L \text{ is of type } =, \\ y (12)^{N_L - 1} 13 x^{-1} & \text{if } \boldsymbol{c}_L \text{ is of type } 0 \text{ and } N_L \le \ell, \\ x v_2 x^{-1} & \text{if } \boldsymbol{c}_L \text{ is of type } 0 \text{ and } N_L \ge \ell + 1. \end{cases}$$

Here the expression inside [ ] is void if $d_{L+1} = 1$. The case when $m = 3, 4, 5$ and $N_L = m - 2$ is considered in Lemma 5.9.

(2) *Suppose that L is odd. Unless $m = 3$ and $\boldsymbol{c}_L$ is of type $-$, the $\mathcal{S}(m)$-standard word of $s^{a \times b}$ is equal to*

$$\begin{cases} x w_2 w_1^{a_{L+1,1}-1} w_2 [w_1^{a_{L+1,2}} w_2 \cdots w_1^{a_{L+1,d_{L+1}}} w_2] 23 x^{-1} & \text{if } \boldsymbol{c}_L \text{ is of type } +, \\ x w_2^{a_{L+1,1}+1} [w_1 w_2^{a_{L+1,2}} \cdots w_1 w_2^{a_{L+1,d_{L+1}}}] 23 x^{-1} & \text{if } \boldsymbol{c}_L \text{ is of type } -, \\ x w_2^2 23 x^{-1} & \text{if } \boldsymbol{c}_L \text{ is of type } =, \\ x (23)^{N_L} 2123 x^{-1} & \text{if } \boldsymbol{c}_L \text{ is of type } 0 \text{ and } N_L \le \ell - 1, \\ x 32 w_2 23 x^{-1} & \text{if } \boldsymbol{c}_L \text{ is of type } 0 \text{ and } N_L \ge \ell. \end{cases}$$

Here w_1 and w_2 are given by (5.1) as before and the expression inside [ ] is void if $d_{L+1} = 1$. The case when $m = 3$ and $\boldsymbol{c}_L$ is of type $-$ is dealt with in Lemma 5.8.

Example 5.5. Let $m = 6$ and $[a, b] = [73, 13]$. Then

$$\boldsymbol{c}_1 = (6, 6, 5, 6, 6, 5, 6, 5, 6, 6, 5, 6, 5), \quad \boldsymbol{c}_2 = (2, 2, 1, 2, 1), \quad \boldsymbol{c}_3 = (2, 1),$$

and $N_1 = 5, N_2 = 1, \rho_1 = \frac{8}{13}, \rho_2 = \frac{3}{5}$. The level L is equal to 2 and $\boldsymbol{c}_2$ is of type $+$. Note that $v_1 = 121213$ and $v_2 = 12134$ from (5.5). By Lemma 5.4 (1), we obtain

$$s^{73\times 13} = yv_2v_1v_2v_1v_2x^{-1} = 2(1213)(121213)(1213)(121213)(1213)1.$$

Proof of Lemma 5.4. Suppose $L = 2g + 2 \geq 2$. By Corollary 4.17 (1), we have

$$\begin{aligned} s^{a\times b} &= (132)^g(21)^{a_{L,1}}(31)(21)^{a_{L,2}}(31)\cdots(21)^{a_{L,d_L}}(31)(231)^g \\ &= (132)^g1(12)^{a_{L,1}}(13)\cdots(12)^{a_{L,d_L}}(13)1(231)^g \\ &= x(12)^{a_{L,1}}(13)\cdots(12)^{a_{L,d_L}}(13)x^{-1}. \end{aligned}$$

We have $1 \leq a_{L,i} \leq m-1$ for $1 \leq i \leq d_L$ by Lemma 4.8 (2) and Lemma 4.10. Note that the $\mathcal{S}(m)$-standard word of $(12)^s(13)$ for $1 \leq s \leq m-1$ is

$$\begin{cases} (12)^s(13) & \text{if } s \leq \ell - 1, \\ (21)^{m-s-1}(23) & \text{if } \ell \leq s \leq m-1. \end{cases} \tag{5.6}$$

We apply the same argument as in the proof of Lemma 5.2 to

$$x^{-1}s^{a\times b}x = (12)^{a_{L,1}}(13)\cdots(12)^{a_{L,d_L}}(13) \tag{5.7}$$

with (5.3) replaced by (5.6) and obtain

$$x^{-1}s^{a\times b}x = \begin{cases} v_1^{a_{L+1,1}}v_2\cdots v_1^{a_{L+1,d_{L+1}}}v_2 & \text{if } \boldsymbol{c}_L \text{ is of type } +, \\ v_1v_2^{a_{L+1,1}}\cdots v_1v_2^{a_{L+1,d_{L+1}}} & \text{if } \boldsymbol{c}_L \text{ is of type } -, \\ v_1v_2 & \text{if } \boldsymbol{c}_L \text{ is of type } =, \\ v_2, & \text{if } \boldsymbol{c}_L \text{ is of type } 0. \end{cases}$$

After conjugating both sides by x, we obtain an expression of $s^{a\times b}$. Assume it is not the case that $m = 3, 4, 5$ and $N_L = m - 2$. If $\boldsymbol{c}_L$ is of type $+$, $-$ or $=$, we apply $xv_1 = yv_2$ to the leftmost part of the expression and obtain the desired standard word. For $\boldsymbol{c}_L$ of type 0, the word xv_2x^{-1} is standard if $N_L \geq \ell + 1$; otherwise xv_2x^{-1} reduces to $y(12)^{N_L-1}13x^{-1}$, which is standard.

In the case when $m = 3, 4, 5$ and $N_L = m - 2$, the standard word of $(12)^{a_{L,i}}(13)$ in (5.7) is 2123 or 23 since $a_{L,i} = m - 2$ or $m - 1$ for $i = 1, 2, \ldots, d_L$. For example, if $\boldsymbol{c}_L$ is of type $-$, an expression of $s^{a\times b}$ is equal to

$$x(23)(2123)^{a_{L+1,1}}(23)(2123)^{a_{L+1,2}}\cdots(23)(2123)^{a_{L,d_{L+1}}}x^{-1}.$$

Since this expression has a subword 23232, it is not standard exactly when $m = 3, 4, 5$. This case will be handled in Lemma 5.9.

Now suppose $L = 2g+1 \geq 3$. By Corollary 4.17 (2), we have

$$\begin{aligned} s^{a\times b} &= (132)^g (23)^{a_{L,1}+1}(21)(23)^{a_{L,2}+1}(21)\cdots(23)^{a_{L,d_L}+1}(21)(231)^g \\ &= x(32)(23)^{a_{L,1}+1}(21)(23)^{a_{L,2}+1}(21)\cdots(23)^{a_{L,d_L}+1}(21)(23)x^{-1}. \end{aligned}$$

We apply the same argument as in the proof of Lemma 5.2 to

$$(23)x^{-1}s^{a\times b}x(32) = (23)^{a_{L,1}+1}(21)(23)^{a_{L,2}+1}(21)\cdots(23)^{a_{L,d_L}+1}(21) \tag{5.8}$$

and obtain

$$(23)x^{-1}s^{a\times b}x(32) = \begin{cases} w_1^{a_{L+1,1}}w_2\cdots w_1^{a_{L+1,d_{L+1}}}w_2 & \text{if } \boldsymbol{c}_L \text{ is of type } +, \\ w_1 w_2^{a_{L+1,1}}\cdots w_1 w_2^{a_{L+1,d_{L+1}}} & \text{if } \boldsymbol{c}_L \text{ is of type } -, \\ w_1 w_2 & \text{if } \boldsymbol{c}_L \text{ is of type } =, \\ w_2 & \text{if } \boldsymbol{c}_L \text{ is of type } 0. \end{cases} \tag{5.9}$$

Here the shift of $\boldsymbol{c}_L = (a_{L,1},\ldots,a_{L,d_L})$ by 1 in the exponents of (5.8) is reflected in the definition of $N = N_L - \delta_{L,1}$, and we still get $\boldsymbol{c}_{L+1} = (a_{L+1,1},\ldots,a_{L+1,d_{L+1}})$ in the exponents of (5.9) since the shift of $\boldsymbol{c}_L$ by 1 does not change how many times a number repeats in the sequence.

We see from (5.1) that

$$(32)w_1 = w_2 \qquad \text{for } 1 \leq N_L \leq m-2.$$

For example, when $N_L = \ell - 2$, we have

$$(32)w_1 = (32)^{m-\ell}(31) = (23)^{\ell}(31) = (23)^{\ell-1}(23)(31) = (23)^{\ell-1}(21) = w_2.$$

Assume it is not the case that $m=3$ and $\boldsymbol{c}_L$ is of type $-$. After conjugating both sides of (5.9) by $x(32)$, we obtain an expression of $s^{a\times b}$. If $\boldsymbol{c}_L$ is of type $+$, $-$ or $=$, we apply $x(32)w_1 = xw_2$ to the leftmost part of the expression and obtain the desired standard word. For $\boldsymbol{c}_L$ of type 0, the word $x(32)w_2(23)x^{-1}$ is standard if $N_L \geq \ell$; otherwise $x(32)w_2(23)x^{-1}$ reduces to the standard word $x(23)^{N_L}2123x^{-1}$.

In the case $m=3$, the standard word of $(23)^{a_{L,i}+1}(21)$ in (5.8) is equal to $w_2 = 31$ or $w_1 = 21$ since $a_{L,i} = 1$ or 2, and the word w_2w_1 is not standard. The sequence $\boldsymbol{c}_L$ cannot be of type $+$ or 0 by Corollary 4.14. If $\boldsymbol{c}_L$ is of type $=$, the expression in the lemma does not have w_2w_1 as a subword (and the argument in the previous paragraph is valid). If $\boldsymbol{c}_L$ is of type $-$, the expression in the lemma is not standard since it has w_2w_1 as a subword. This case will be considered in Lemma 5.8.

□

5-A. **Exceptional cases.** In this subsection, we deal with exceptional cases in which the expressions in Lemmas 5.2 and 5.4 are not standard. These cases are restricted to specific conditions with $m = 3, 4$ or 5.

Lemma 5.6 ($m = 3$; level 1)**.** *Assume that the level of $[a, b]$ is 1 and suppose that $m = 3$, $N_1 = 2$ and $\boldsymbol{c}_1$ is of type $-$.*

(1) If $N_2 > 1$, then the following expression of $s^{a\times b}$ is $\mathcal{S}(3)$-standard:

$$w_1[(w_2)^{a_{2,1}-1}w_3 \cdots (w_2)^{a_{2,d_2-1}-1}w_3](w_2)^{a_{2,d_2}},$$

where $w_1 = 21, w_2 = 31, w_3 = 3212$ and the expression inside [ ] is void if $d_2 = 1$.

(2) If $N_2 = 1$, then the following expression of $s^{a\times b}$ is $\mathcal{S}(3)$-standard:

$$\begin{cases} w_1w_3(w_2w_3)^{a_{3,1}-1}[w_6(w_2w_3)^{a_{3,2}-1} \cdots w_6(w_2w_3)^{a_{3,d_3}-1}]w_2^2 & \text{for } \boldsymbol{c}_2 \text{ of type } +, \\ 21321(w_4)^{a_{3,1}-1}[w_5(w_4)^{a_{3,2}} \cdots w_5(w_4)^{a_{3,d_3}}]23131 & \text{for } \boldsymbol{c}_2 \text{ of type } -, \\ w_1w_3w_2^2 & \text{for } \boldsymbol{c}_2 \text{ of type } =, \end{cases}$$

where $w_1 = 21, w_2 = 31, w_3 = 3212, w_4 = 3231, w_5 = 231321$, $w_6 = 3132132312$ and the expression inside [ ] is void if $d_3 = 1$.

Example 5.7. Suppose $m = 3$ and $[a, b] = [17, 7]$. Then the level of $[a, b]$ is 1. Since $\frac{17}{7} = 2 + \frac{3}{7} < 2 + \frac{1}{2}$, we get $N_1 = 2$ and $\boldsymbol{c}_1$ of type $-$. One can check $\boldsymbol{c}_1 = (3, 2, 3, 2, 3, 2, 2)$. By Proposition 4.15 or by definition of $s^{a\times b}$, we obtain

$$\begin{aligned} s^{a\times b} =& (23)^3(21)(23)^2(21)(23)^3(21)(23)^2(21)(23)^3(21)(23)^2(21)(23)^2(21) \\ =& 213\,\underline{121}\,3\,\underline{121}\,3131 = 21321\,\underline{232}\,123131 = 21321323123131, \end{aligned}$$

where the underlined subwords are replaced using relations $121 = 212$ and $232 = 323$.

On the other hand, we have $\boldsymbol{c}_2 = (1, 1, 2)$, $N_2 = 1$ and $\boldsymbol{c}_2$ of type $-$. It is clear that $\boldsymbol{c}_3 = (2)$. By Lemma 5.6 (2),

$$s^{a\times b} = 21321(w_4)^1 23131 = 21321323123131.$$

Thus we get the same standard word.

Proof of Lemma 5.6. Since

$$(31)^{a_{2,i}}(21) = (31)^{a_{2,i}-1}3121 = (31)^{a_{2,i}-1}3212 \quad \text{for } 1 \le i \le d_2,$$

we obtain from (5.4)

$$\begin{aligned} s^{a\times b} =& (21)(31)^{a_{2,1}}(21)(31)^{a_{2,2}} \cdots (21)(31)^{a_{2,d_2}} \\ =& (21)(31)^{a_{2,1}-1}(3212)(31)^{a_{2,2}-1}(3212) \cdots (31)^{a_{2,d_2}-1}(3212)(31)^{a_{2,d_2}} \\ =& w_1(w_2)^{a_{2,1}-1}w_3 \cdots (w_2)^{a_{2,d_2-1}-1}w_3(w_2)^{a_{2,d_2}}. \end{aligned} \tag{5.10}$$

Suppose $N_2 > 1$. It implies that $a_{2,i} > 1$ for all $1 \le i \le d_2$, and the expression (5.10) is standard.

Now suppose $N_2 = 1$. If $\boldsymbol{c}_2$ is of type $+$, we have

$$\boldsymbol{c}_2 = (1, \underbrace{2, \ldots, 2}_{a_{3,1}\text{-times}}, 1, \underbrace{2, \ldots, 2}_{a_{3,2}\text{-times}}, \ldots, 1, \underbrace{2, \ldots, 2}_{a_{3,d_3}\text{-times}})$$

by Lemma 4.8 (3). We start from (5.10) to see that $s^{a\times b}$ is equal to

$$
\begin{aligned}
&(21)(31)^{a_{2,1}-1}(3212)(31)^{a_{2,2}-1}(3212)\cdots(31)^{a_{2,d_2-1}-1}(3212)(31)^{a_{2,d_2}-1}(31)\\
=&(21)(321231)^{a_{3,1}}(3212)(321231)^{a_{3,2}}(3212)\cdots(321231)^{a_{3,d_3}}(31)\\
=&(213212)[(313212)^{a_{3,1}-1}(31321\underline{232}12)(313212)^{a_{3,2}-1}(31321\underline{232}12)\cdots(313212)^{a_{3,d_3}-1}](3131)\\
=&(213212)[(313212)^{a_{3,1}-1}(3132132312)(313212)^{a_{3,2}-1}(3132132312)\cdots(313212)^{a_{3,d_3}-1}](3131)\\
=&w_1w_3(w_2w_3)^{a_{3,1}-1}w_6(w_2w_3)^{a_{3,2}-1}\cdots w_6(w_2w_3)^{a_{3,d_3}-1}(w_2)^2,
\end{aligned}
$$

where the underlines are put to indicate the replacements $232=323$. The remaining cases can be proven similarly. $\square$

Recall that we set

$$x=(132)^{\lfloor\frac{L-2}{2}\rfloor}1 \quad\text{and}\quad x^{-1}=1(231)^{\lfloor\frac{L-2}{2}\rfloor}.$$

Now we present the remaining cases in Lemmas 5.8 and 5.9 below. Since the proofs of these lemmas are similar to that of Lemma 5.6, we omit the proofs.

Lemma 5.8 ($m=3$; odd $L\geq 3$)**.** *Assume that the level L of $[a,b]$ is ≥ 3 and odd. Suppose $m=3$, $N_L=1$ and $\boldsymbol{c}_L$ is of type $-$.*

(1) If $N_{L+1}\neq 1$, the $\mathcal{S}(3)$-standard word of $s^{a\times b}$ is equal to

$$x31[{u_3}^{a_{L+1,1}-1}u_4\cdots {u_3}^{a_{L+1,d_{L+1}-1}-1}u_4]{u_3}^{a_{L+1,d_{L+1}}}23x^{-1},$$

where $u_3=31$, $u_4=3212$ and the expression inside [ ] is void if $d_{L+1}=1$.

(2) If $N_{L+1}=1$, the $\mathcal{S}(3)$-standard word of $s^{a\times b}$ is equal to

$$
\begin{cases}
x3132{u_5}^{a_{L+2,1}-1}[u_6{u_5}^{a_{L+2,2}}\cdots u_6{u_5}^{a_{L+2,d_{L+2}}}]u_723x^{-1} & \text{for } \boldsymbol{c}_{L+1} \text{ of type } -,\\
x3132[{u_6}^{a_{L+2,1}}u_5{u_6}^{a_{L+2,2}}\cdots u_5]{u_6}^{a_{L+2,d_{L+2}}-1}u_723x^{-1} & \text{for } \boldsymbol{c}_{L+1} \text{ of type } +,\\
x3132u_723x^{-1} & \text{for } \boldsymbol{c}_{L+1} \text{ of type } =,
\end{cases}
$$

where $u_5=1323$, $u_6=123132$, $u_7=123131$ and the expression inside [ ] is void if $d_{L+2}=1$.

Lemma 5.9 ($m=3,4,5$; even $L\geq 2$)**.** *Assume that the level L of $[a,b]$ is ≥ 2 and even. Suppose $m=3,4,5$ and $N_L=m-2$. Let*

$$v_3=31,\quad v_4=3231,\quad v_5=2321\quad\text{and}\quad v_6=323231.$$

Then the $\mathcal{S}(m)$-standard word of $s^{a\times b}$ is given by the following:

$$
\text{for } m=3,\qquad
\begin{cases}
xv_4^{a_{L+1,1}}[v_3v_4^{a_{L+1,2}-1}\cdots v_3v_4^{a_{L+1,d_{L+1}}-1}]23x^{-1} & \text{if } \boldsymbol{c}_L \text{ is of type } -,\\
xv_423x^{-1} & \text{if } \boldsymbol{c}_L \text{ is of type } =;
\end{cases}
$$

$$
\text{for } m=4,\qquad
\begin{cases}
xv_5^{a_{L+1,1}}[v_4v_5^{a_{L+1,2}-1}\cdots v_4v_5^{a_{L+1,d_{L+1}}-1}]23x^{-1} & \text{if } \boldsymbol{c}_L \text{ is of type } -,\\
xv_523x^{-1} & \text{if } \boldsymbol{c}_L \text{ is of type } =,\\
y1213x^{-1} & \text{if } \boldsymbol{c}_L \text{ is of type } 0;
\end{cases}
$$

$$\text{for } m=5, \qquad \begin{cases} xv_5^{a_{L+1,1}}[v_6v_5^{a_{L+1,2}-1}\cdots v_6v_5^{a_{L+1,d_{L+1}}-1}]23x^{-1} & \text{if } \boldsymbol{c}_L \text{ is of type } -, \\ xv_5 23x^{-1} & \text{if } \boldsymbol{c}_L \text{ is of type } =, \\ x2123x^{-1} & \text{if } \boldsymbol{c}_L \text{ is of type } 0. \end{cases}$$

Here the expression inside [ ] is void if $d_{L+1}=1$.

Remark 5.10. By Corollary 4.14 (1) the sequences $\boldsymbol{c}_L$ cannot be of type $+$ in Lemma 5.9. Moreover, when $m=3$, the type of $\boldsymbol{c}_L$ cannot be 0 by Corollary 4.14 (2). Thus all the possible cases are covered.

Example 5.11. (1) Suppose $m=3$ and $[a,b]=[13,5]$. Since $\boldsymbol{c}_1=(3,3,2,3,2)$ and $\boldsymbol{c}_2=(2,1)$, we obtain from Proposition 4.15

$$\begin{aligned} s^{a\times b} =&(23)^3(21)(23)^3(21)(23)^2(21)(23)^3(21)(23)^2(21) \\ =&(21)^2(31)(21)(31)=12321231=13231231, \end{aligned}$$

where we use the relations $121=212$, $232=323$ and $22=e$. On the other hand, $L=2$, $N_2=1$, $\rho_2=\frac{1}{2}$ and $\boldsymbol{c}_2$ is of type $=$. By Lemma 5.9, we have

$$s^{a\times b}=xv_4 23x^{-1}=13231231,$$

which is the same standard word.

(2) Let $m=4$ and $[a,b]=[85,23]$. Then

$$\begin{aligned} &\boldsymbol{c}_1=(4,4,4,3,4,4,3,4,4,3,4,4,4,3,4,4,3,4,4,3,4,4,3), \\ &\boldsymbol{c}_2=(3,2,2,3,2,2,2), \quad \boldsymbol{c}_3=(2,3), \end{aligned}$$

and $N_1=3, N_2=2, \rho_1=\frac{16}{23}, \rho_2=\frac{2}{7}$. The level L is equal to 2 and the sequence $\boldsymbol{c}_2$ is of type $-$. By Lemma 5.9, we obtain

$$s^{a\times b}=xv_5^2v_4v_5^2 23x^{-1}=1(2321)^2(3231)(2321)^2 231. \tag{5.11}$$

One can check that the initial word of $s^{a\times b}$ indeed reduces to the standard word in (5.11).

5-B. **Proof of Theorem 3.3.**

Proof. Suppose $[a,b]\neq[c,d]$. We will show that $s^{a\times b}\neq s^{c\times d}$ by comparing $\mathcal{S}(m)$-standard words given in Lemmas 5.2, 5.4, 5.6, 5.8 and 5.9.

Let the levels of $[a,b]$ and $[c,d]$ be L and L', respectively. Without loss of generality, we may assume $L\leq L'$. Write $g=\lceil\frac{L-2}{2}\rceil$. It is enough to show $(231)^g s^{a\times b}(132)^g\neq(231)^g s^{c\times b}(132)^g$. By Corollary 4.17 (2), an expression of $(231)^g s^{a\times b}(132)^g$ is equal to $s^{\tilde{a}\times\tilde{b}}$ where $[\tilde{a},\tilde{b}]$ has level $L-2g=1$ or 2. Similarly, $(231)^g s^{c\times d}(132)^g$ is equal to $s^{\tilde{c}\times\tilde{d}}$ where $[\tilde{c},\tilde{d}]$ has level $L'-2g$. Consequently, we may assume that $[a,b]$ has level $L=1$ or 2 and $[c,d]$ has level $L'\geq L$, and it is sufficient to prove $s^{a\times b}\neq s^{c\times d}$.

We summarize consequences of Lemmas 5.2, 5.4, 5.6, 5.8 and 5.9 in what follows. Let $[e,f]$ be an arbitrary positive reduced root of level L''.

- If $L''=1$, the standard word of $s^{e\times f}$ starts with one of

$$2131,\quad 2132,\quad 2321,\quad 2323,\quad 31,\quad 3231,\quad 3232, \tag{5.12}$$

as one can see from Lemmas 5.2 and 5.6.
- If $L''=2$, the standard word of $s^{e\times f}$ starts with one of

$$12121,\quad 12123,\quad 12321,\quad 13231,\quad 21212,\quad 21213 \tag{5.13}$$

from Lemmas 5.4 and 5.9. Note that 2131 cannot occur as having $\boldsymbol{c}_2$ of type 0 and $N_2=1$ is impossible by Corollary 4.14 (2). Likewise 12323 cannot occur because having $\boldsymbol{c}_2$ of type $+$ and $N_L=m-2$ is impossible by Corollary 4.14 (1).
- If $L''=2g+1\geq 3$, the standard word of $s^{e\times f}$ starts with one of

$$x2323,\quad x31,\quad x3231,\quad x3232 \tag{5.14}$$

from Lemmas 5.4 and 5.8, where $x=(132)^{\lfloor\frac{L''-2}{2}\rfloor}1$. Note that $x2321$ cannot occur since having $N_L=1$ and $\boldsymbol{c}_L$ of type 0 is impossible by Corollary 4.14 (2).
- If $L''=2g+2\geq 4$, the standard word of $s^{e\times f}$ starts with one of

$$y1212,\quad y1213,\quad y13,\quad y2121,\quad y2123,\quad x2121,\quad x2123,\quad x2321,\quad x3231, \tag{5.15}$$

from Lemmas 5.4 and 5.9, where $x=(132)^{\lfloor\frac{L''-2}{2}\rfloor}1$ and $y=(132)^{\frac{L''-4}{2}}13$.

First assume $L<L'$. If $L=1$, then none of the words in (5.12) appears as a starting word in (5.13), (5.14) and (5.15). Thus the standard word of $s^{a\times b}$ must be different from that of $s^{c\times d}$, and hence $s^{a\times b}\neq s^{c\times d}$. If $L=2$ and $L'\geq 4$, then none of the words in (5.13) appears as a starting word in (5.14) and (5.15), and we obtain $s^{a\times b}\neq s^{c\times d}$. If $L=2$ and $L'=3$, then 13231 is common in (5.13) and (5.14). However, if $L=2$, a standard word of $s^{a\times b}$ starts with 13231 only when $m=3$; now, if $L'=3$ and $m=3$, no standard word actually starts with 13231 by Lemmas 5.4 and 5.8. Thus the standard word $s^{a\times b}$ is different from that $s^{c\times d}$ in this case, and we have $s^{a\times b}\neq s^{c\times d}$.

Next assume that $L=L'=1$. Let N_k, $\boldsymbol{c}_k$ and ϵ_k be defined for $[a,b]$ as in Definition 4.6, where ϵ_k denotes the type of $\boldsymbol{c}_k$, and use notations N'_k, $\boldsymbol{c}'_k$ and ϵ'_k for $[c,d]$. One can check that the standard words in Lemmas 5.2 and 5.6 are all different for each m. If $m\neq 3$, the standard words of $s^{a\times b}$ and $s^{c\times d}$ are determined by $(N_1,\epsilon_1,\boldsymbol{c}_2)$ and $(N'_1,\epsilon'_1,\boldsymbol{c}'_2)$ respectively by Lemma 5.2. Since $[a,b]\neq[c,d]$, we have $(N_1,\epsilon_1,\boldsymbol{c}_2)\neq(N'_1,\epsilon'_1,\boldsymbol{c}'_2)$ by Lemma 4.9 and the corresponding standard words are different. Thus $s^{a\times b}\neq s^{c\times d}$. If $m=3$, the standard words are determined either by $(N_1,\epsilon_1,\boldsymbol{c}_2)$ and $(N'_1,\epsilon'_1,\boldsymbol{c}'_2)$, or by $(N_1,\epsilon_1,N_2,\epsilon_2,\boldsymbol{c}_3)$ and $(N'_1,\epsilon'_1,N'_2,\epsilon'_2,\boldsymbol{c}'_3)$. Since $[a,b]\neq[c,d]$, we have $s^{a\times b}\neq s^{c\times d}$ by Lemmas 4.9, 5.2 and 5.6.

Finally assume that $L=L'=2$. Similarly, as in the case that $L=L'=1$, one can check that the standard words in Lemmas 5.4 (1) and 5.9 are all different for each m. The standard words of $s^{a\times b}$ and $s^{c\times d}$ are determined by $(N_2,\epsilon_2,\boldsymbol{c}_3)$ and $(N'_2,\epsilon'_2,\boldsymbol{c}'_3)$ respectively, and note that $N_1=N'_1=m-1$ and $\epsilon_1=\epsilon'_1=+$. Since $[a,b]\neq[c,d]$, we have $(N_2, \epsilon_2, \boldsymbol{c}_3) \neq (N_2', \epsilon_2', \boldsymbol{c}_3')$ by Lemma 4.9 and the corresponding standard words are different. Thus $s^{a\times b} \neq s^{c\times d}$. This completes the proof. $\square$

6. Gröbner–Shirshov basis for $W(m)$

In this section we determine Gröbner–Shirshov bases for $W(m)$, which are used in the previous sections.

6-A. **Gröbner–Shirshov basis theory.** We briefly recall the Gröbner–Shirshov basis theory (or Diamond Lemma). See [2, 3, 4, 13, 14] for more details. Let $X = \{x_1, x_2, \cdots\}$ be the set of alphabets and let X^* be the free monoid of associative monomials on X. We denote the empty monomial by e and the *length* of a monomial u by $l(u)$. Thus we have $l(e) = 0$. A well-ordering $\prec$ on X^* is called a *monomial order* if $x \prec y$ implies $axb \prec ayb$ for all $a, b \in X^*$. For two monomials

$$u = x_{i_1}x_{i_2}\cdots x_{i_k}, \quad v = x_{j_1}x_{j_2}\cdots x_{j_l} \in X^*,$$

define $u \prec_{\text{deg-lex}} v$ if and only if $k < l$ or $k = l$ and $i_r > j_r$ for the first r such that $i_r \neq j_r$; it is a monomial order on X^* called the *degree lexicographic order*. We denote the degree lexicographic order on X^* simply by $\prec$. In particular, we have $x_1 \succ x_2 \succ \ldots$.

Let $\mathcal{A}_X$ be the free associative algebra generated by X over a field $\mathbb{F}$. Given a nonzero element $p \in \mathcal{A}_X$, we denote by $\overline{p}$ the maximal monomial appearing in p under the ordering $\prec$. Thus $p = \alpha\overline{p} + \sum \beta_i w_i$ with $\alpha, \beta_i \in \mathbb{F}$, $w_i \in X^*$, $\alpha \neq 0$ and $w_i \prec \overline{p}$. If $\alpha = 1$, p is said to be *monic*.

Let S be a subset of monic elements of $\mathcal{A}_X$, let J be the two-sided ideal of $\mathcal{A}_X$ generated by S. Then we say that the algebra $A = \mathcal{A}_X/J$ is *defined by* S. The images of $p \in \mathcal{A}_X$ in A will also be denoted by p.

Definition 6.1. Given a subset S of monic elements of $\mathcal{A}_X$, a monomial $u \in X^*$ is said to be *S-standard* if $u \neq a\overline{s}b$ for any $s \in S$ and $a, b \in X^*$. Otherwise, the monomial u is said to be *S-reducible*.

Through inductive steps, every $p \in \mathcal{A}_X$ can be expressed as

$$p = \sum \alpha_i a_i s_i b_i + \sum \gamma_k u_k, \tag{6.1}$$

where $\alpha_i, \gamma_k \in \mathbb{F}$, $a_i, b_i, u_k \in X^*$, $s_i \in S$, $a_i\overline{s_i}b_i \preceq \overline{p}$, $u_k \preceq \overline{p}$ and u_k are S-standard. The term $\sum \gamma_k u_k$ in the expression (6.1) is called a *standard* (or *normal*) form of p with respect to the pair S (and with respect to the monomial order $\prec$). In general, a standard word is not unique. Nonetheless, it is clear that the set of S-standard monomials linearly spans the algebra A defined by S.

Definition 6.2. A subset S of monic elements of $\mathcal{A}_X$ is a *Gröbner–Shirshov basis* if the set of S-standard monomials forms a linear basis of the algebra A defined by S.

Let p and q be monic elements of $\mathcal{A}_X$ with leading terms $\overline{p}$ and $\overline{q}$. We define the *composition* of p and q as follows.

Definition 6.3. (a) If there exist a and b in X^* such that $\overline{p}a = b\overline{q} = w$ with $l(\overline{p}) > l(b)$, then the *composition of intersection* is defined to be $(p,q)_w = pa - bq$.

(b) If there exist a and b in X^* such that $b \neq e$, $\overline{p} = a\overline{q}b = w$, then the *composition of inclusion* is defined to be $(p,q)_w = p - aqb$.

For $p, q \in \mathcal{A}_X$ and $w \in X^*$, we define a *congruence relation* on $\mathcal{A}_X$ as follows: $p \equiv q \mod (S;w)$ if and only if $p - q = \sum \alpha_i a_i s_i b_i$, where $\alpha_i \in \mathbb{F}$, $a_i, b_i \in X^*$, $s_i \in S$, and $a_i \overline{s_i} b_i \prec w$.

Definition 6.4. A subset S of monic elements in $\mathcal{A}_X$ is said to be *closed under composition* if $(p,q)_w \equiv 0 \mod (S;w)$ for all $p, q \in S$, $w \in X^*$ whenever the composition $(p,q)_w$ is defined.

The following is Shirshov's Composition Lemma.

Lemma 6.5 ([3])**.** *Let S be a subset of monic elements of $\mathcal{A}_X$, and let $A = \mathcal{A}_X/J$ be the associative algebra defined by S. Assume that S is closed under composition. If the image of $p \in \mathcal{A}_X$ is trivial in A, then the word $\overline{p}$ is S-reducible.*

As a consequence, we obtain:

Theorem 6.6 ([2, 3])**.** *Let $\mathscr{S}$ be a subset of monic elements in $\mathcal{A}_X$. Then the following are equivalent :*

(a) *$\mathscr{S}$ is a Gröbner–Shirshov basis;*

(b) *$\mathscr{S}$ is closed under composition;*

(c) *For each $p \in \mathcal{A}_X$, the standard word of p is unique.*

6-B. **Coxeter groups.** Consider a Coxeter group

$$W = \langle s_1, s_2, ..., s_n \ : \ s_1^2 = \cdots = s_n^2 = e, \ (s_i s_j)^{m_{ij}} = e \ (i \neq j) \rangle,$$

where $m_{ij} \in \{2, 3, 4, ...\} \cup \{\infty\}$. Let $X = \{s_1, s_2, \ldots, s_n\}$. Then X^* has the degree lexicographic order $\prec$ defined in the previous subsection. In particular, we have

$$s_1 \succ s_2 \succ \cdots \succ s_n.$$

Let S be the set of relations:

$$\begin{aligned} & s_i^2 - e && \text{for } i = 1, 2, \ldots, n, \\ & (s_i s_j)^{m_{ij}/2} - (s_j s_i)^{m_{ij}/2} && \text{if } s_i \succ s_j \text{ and } m_{ij} \text{ is even,} \\ & (s_i s_j)^{\lfloor m_{ij}/2 \rfloor} s_i - (s_j s_i)^{\lfloor m_{ij}/2 \rfloor} s_j && \text{if } s_i \succ s_j \text{ and } m_{ij} \text{ is odd.} \end{aligned}$$

As in the previous subsection, let J be the ideal of $\mathcal{A}_X$ generated by S, and $A = \mathcal{A}_X/J$ be the algebra defined by S. Then A is nothing but the group algebra $\mathbb{F}[W]$ of W.

If S is not a Gröbner–Shirshov basis, we extend S by putting all nontrivial compositions into S, and denote the resulting set of relations by $S^{(1)}$. If it is a Gröbner–Shirshov basis, we stop; otherwise, we extend $S^{(1)}$ in the same way to obtain $S^{(2)}$ and continue the process. If this process terminates at $S^{(N)}$ for some N, we obtain a Gröbner–Shirshov basis $\mathscr{S} := S^{(N)}$ for $A = \mathbb{F}[W]$. By abusing language, we also call $\mathscr{S}$ a *Gröbner–Shirshov basis* for W.

It follows from the construction that every element in $\mathscr{S}$ is of the form $u - v$ with $u, v \in X^*$, and the identity $u = v$ is valid in the group W. Consequently, an element in $w \in W$ can be written uniquely into an $\mathscr{S}$-standard monomial using the identities $u = v$ in W for $u - v \in \mathscr{S}$.

6-C. **The groups $W(m)$.** In this subsection, for each $m \geq 3$, we will compute a Gröbner–Shirshov basis for $W(m)$. We need to separate two cases according to the parity of m. In the proofs, we simply write $u = v$ for $u \equiv v \mod (S'; w)$ where S' and w are clear from the context.

(i) Assume that $m = 2k - 1$, $k \geq 2$. Let $X = \{s_1, s_2, s_3\}$. The set S of the defining relations are given by

$$s_1^2 - e, \tag{6.2}$$

$$s_2^2 - e, \tag{6.3}$$

$$s_3^2 - e, \tag{6.4}$$

$$(s_1s_2)^{k-1}s_1 - (s_2s_1)^{k-1}s_2, \tag{6.5}$$

$$(s_2s_3)^{k-1}s_2 - (s_3s_2)^{k-1}s_3. \tag{6.6}$$

Proposition 6.7. *Let $m = 2k - 1$ for $k \geq 2$. The set S of defining relations* (6.2)-(6.6) *is a Gröbner–Shirshov basis $\mathscr{S}$ for $W(m)$. That is, we have $\mathscr{S} = S$ in this case.*

Proof. There are no possible compositions among (6.2), (6.3) and (6.4). The composition of (6.2) and (6.5) is

$$\begin{aligned}(6.2) \times s_2(s_1s_2)^{k-2}s_1 - s_1 \times (6.5) &= s_1(s_2s_1)^{k-1}s_2 - s_2(s_1s_2)^{k-2}s_1 \\ &= (s_1s_2)^{k-1}s_1s_2 - s_2(s_1s_2)^{k-2}s_1 = (s_2s_1)^{k-1}s_2s_2 - (s_2s_1)^{k-1} = 0,\end{aligned}$$

where we used (6.5) and (6.3). The composition of (6.5) and (6.2) is

$$\begin{aligned}(6.5) \times s_1 - (s_1s_2)^{k-1} \times (6.2) &= -(s_2s_1)^k + (s_1s_2)^{k-1} = -s_2(s_2s_1)^{k-1}s_2 + (s_1s_2)^{k-1} \\ &= -(s_1s_2)^{k-1} + (s_1s_2)^{k-1} = 0,\end{aligned}$$

where we used (6.5) and (6.3). Similarly, the composition of (6.3) and (6.6) and that of (6.6) and (6.3) are all trivial.

The compositions between (6.5) and (6.5) are

$$\begin{aligned}(6.5) \times s_2(s_1s_2)^{\ell-1}s_1 - (s_1s_2)^{\ell} \times (6.5) &= -(s_2s_1)^{k-1}s_2s_2(s_1s_2)^{\ell-1}s_1 + (s_1s_2)^{\ell}(s_2s_1)^{k-1}s_2 \\ &= -(s_2s_1)^{k-\ell}s_1 + (s_2s_1)^{k-\ell-1}s_2 = 0\end{aligned}$$

for $1 \le \ell \le k-1$. Similarly, the compositions between (6.6) and (6.6) are trivial.

There is no more possible composition. Thus the set of defining relations (6.2)-(6.6) is closed under composition, and it is a Gröbner–Shirshov basis by Theorem 6.6. $\square$

(ii) Assume that $m = 2k$, $k \ge 2$. Let $X = \{s_1, s_2, s_3\}$. The set S of the defining relations are given by

$$s_1^2 - e, \tag{6.7}$$

$$s_2^2 - e, \tag{6.8}$$

$$s_3^2 - e, \tag{6.9}$$

$$(s_1s_2)^k - (s_2s_1)^k, \tag{6.10}$$

$$(s_2s_3)^k - (s_3s_2)^k. \tag{6.11}$$

Proposition 6.8. *Let $m = 2k$ for $k \ge 2$. A Gröbner–Shirshov basis $\mathscr{S}$ for $W(m)$ is given by the set consisting of defining relations* (6.7)-(6.11) *and one additional relation*

$$(s_1s_2)^{k-1}s_1(s_3s_2)^k - (s_2s_1)^k s_3(s_2s_3)^{k-1}. \tag{6.12}$$

Proof. There are no non-trivial compositions among (6.7), (6.8) and (6.9). The composition of (6.7) and (6.10) is

$$\begin{aligned}(6.7) \times s_2(s_1s_2)^{k-1} - s_1 \times (6.10) &= s_1(s_2s_1)^k - s_2(s_1s_2)^{k-1} = (s_1s_2)^k s_1 - s_2(s_1s_2)^{k-1} \\ &= (s_2s_1)^k s_1 - s_2(s_1s_2)^{k-1} = 0,\end{aligned}$$

where we used (6.10) and (6.7). The composition of (6.10) and (6.8) is

$$\begin{aligned}(6.10) \times s_2 - (s_1s_2)^{k-1}s_1 \times (6.8) &= -(s_2s_1)^k s_2 + (s_1s_2)^{k-1}s_1 = -s_2(s_1s_2)^k + (s_1s_2)^{k-1}s_1 \\ &= -s_2(s_2s_1)^k + (s_1s_2)^{k-1}s_1 = 0,\end{aligned}$$

where we used (6.10) and (6.8). Similarly, the composition of (6.8) and (6.11) and that of (6.11) and (6.9) are all trivial.

The composition between (6.10) and (6.11) is

$$(6.10) \times s_3(s_2s_3)^{k-1} - (s_1s_2)^{k-1}s_1 \times (6.11) = (s_1s_2)^{k-1}s_1(s_3s_2)^k - (s_2s_1)^k s_3(s_2s_3)^{k-1}.$$

Thus we have obtained a new relation

$$(s_1s_2)^{k-1}s_1(s_3s_2)^k - (s_2s_1)^k s_3(s_2s_3)^{k-1},$$

which is the relation (6.12).

The composition between (6.7) and (6.12) is

$$\begin{aligned}&(6.7) \times s_2(s_1s_2)^{k-2}s_1(s_3s_2)^k - s_1 \times (6.12) = s_1(s_2s_1)^k s_3(s_2s_3)^{k-1} - s_2(s_1s_2)^{k-2}s_1(s_3s_2)^k \\ &= (s_1s_2)^k s_1s_3(s_2s_3)^{k-1} - (s_2s_1)^{k-1}(s_3s_2)^k = (s_2s_1)^k s_1s_3(s_2s_3)^{k-1} - (s_2s_1)^{k-1}(s_3s_2)^k \\ &= (s_2s_1)^{k-1}(s_2s_3)^k - (s_2s_1)^{k-1}(s_3s_2)^k = 0,\end{aligned}$$

where we use (6.10), (6.7) and (6.11). The composition between (6.12) and (6.8) is

$$\begin{aligned}
&(6.12) \times s_2 - (s_1s_2)^{k-1}s_1(s_3s_2)^{k-1}s_3 \times (6.8)\\
=&-(s_2s_1)^k s_3(s_2s_3)^{k-1}s_2 + (s_1s_2)^{k-1}s_1(s_3s_2)^{k-1}s_3\\
=&-s_2(s_1s_2)^{k-1}s_1(s_3s_2)^k + (s_1s_2)^{k-1}s_1(s_3s_2)^{k-1}s_3\\
=&-s_2(s_2s_1)^k s_3(s_2s_3)^{k-1} + (s_1s_2)^{k-1}s_1(s_3s_2)^{k-1}s_3\\
=&-(s_1s_2)^{k-1}s_1s_3(s_2s_3)^{k-1} + (s_1s_2)^{k-1}s_1(s_3s_2)^{k-1}s_3 = 0,
\end{aligned}$$

where we use (6.12) and (6.8).

The compositions between (6.10) and (6.12) are, for $1 \le \ell \le k-1$,

$$\begin{aligned}
&(6.10) \times (s_1s_2)^{\ell-1}s_1(s_3s_2)^k - (s_1s_2)^\ell \times (6.12)\\
=&-(s_2s_1)^k(s_1s_2)^{\ell-1}s_1(s_3s_2)^k + (s_1s_2)^\ell(s_2s_1)^k s_3(s_2s_3)^{k-1}\\
=&-(s_2s_1)^{k-\ell+1}s_1(s_3s_2)^k + (s_2s_1)^{k-\ell}s_3(s_2s_3)^{k-1}\\
=&-(s_2s_1)^{k-\ell}(s_2s_3)^k s_2 + (s_2s_1)^{k-\ell}(s_3s_2)^{k-1}s_3\\
=&-(s_2s_1)^{k-\ell}(s_3s_2)^k s_2 + (s_2s_1)^{k-\ell}(s_3s_2)^{k-1}s_3 = 0,
\end{aligned}$$

where we use (6.11). The compositions between (6.12) and (6.11) are, for $1 \le \ell \le k$,

$$\begin{aligned}
&(6.12) \times (s_3s_2)^{\ell-1}s_3 - (s_1s_2)^{k-1}s_1s_3(s_2s_3)^{\ell-1} \times (6.11)\\
=&(s_1s_2)^{k-1}s_1s_3(s_2s_3)^{\ell-1}(s_3s_2)^k - (s_2s_1)^k s_3(s_2s_3)^{k-1}(s_3s_2)^{\ell-1}s_3\\
=&(s_1s_2)^{k-1}s_1s_3(s_3s_2)^{k-\ell+1} - (s_2s_1)^k s_3(s_2s_3)^{k-\ell}s_3\\
=&(s_1s_2)^{k-1}s_1(s_2s_3)^{k-\ell}s_2 - (s_2s_1)^k(s_3s_2)^{k-\ell}\\
=&(s_1s_2)^k(s_3s_2)^{k-\ell} - (s_2s_1)^k(s_3s_2)^{k-\ell} = 0,
\end{aligned}$$

where we use (6.10).

There is no more possible composition. Thus the set consisting of relations (6.7)-(6.12) is closed under composition, and it is a Gröbner–Shirshov basis by Theorem 6.6. □

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