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Dominant maximal weights of highest weight modules and Young tableaux

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Abstract. We study the multiplicities of dominant maximal weights of integrable highest weight modules $V(\Lambda)$ with highest weights Λ, including all fundamental weights, over affine Kac–Moody algebras of types $B_n^{(1)}$, $D_n^{(1)}$, $A_{2n-1}^{(2)}$, $A_{2n}^{(2)}$ and $D_{n+1}^{(2)}$. We introduce new families of Young tableaux, called the almost even tableaux and (spin) rigid tableaux, and prove that they enumerate the crystal basis elements of dominant maximal weight spaces. By applying inductive insertion schemes for tableaux, in some special cases we prove that the weight multiplicities of maximal weights form the Pascal, Motzkin, Riordan and Bessel triangles.

Keywords: Dominant maximal weight, Young tableaux, Crystal basis, Pascal triangle, Motzkin triangle, Riordan triangle

1 Introduction

Representations of affine Kac–Moody algebras have been studied extensively for more than four decades as their applications have been found throughout mathematics and mathematical physics. In particular, weight multiplicities of an integrable highest weight representation $V(\Lambda)$ of an affine Kac–Moody algebra are of great interest as they can be interpreted in several different ways such as generalized partition numbers, Fourier coefficients of certain modular forms and numbers of irreducible modules of Hecke-type algebras. However, our understanding of weight multiplicities is, in general, very limited.

The set of weights of $V(\Lambda)$ can be divided into so-called δ-strings and the first weight of each string is called a *maximal weight*. Maximal weights and their multiplicities are fundamental in understanding the structure of $V(\Lambda)$. Since weight multiplicities are invariant under the Weyl group action, it is enough to consider *dominant* maximal weights, and it is well-known that the set of dominant maximal weights for each highest weight

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Λ is finite. Nonetheless, we do not have any explicit description of dominant maximal weights and their multiplicities in most cases. Except for trivial cases, only level 2 maximal weights of type $A_{n-1}^{(1)}$ and their multiplicities are completely known [9], and recently, some maximal weights of $V(k\Lambda_0 + \Lambda_s)$, $k \in \mathbb{Z}_{>0}$, $s = 0, 1, \ldots, n-1$, of type $A_{n-1}^{(1)}$ have been studied [3, 2, 10], where Λ_s are the fundamental weights. Virtually, there has been no systematic description of dominant maximal weights and their multiplicities for affine types other than type $A_n^{(1)}$.

In this work, we evaluate multiplicities of dominant maximal weights for types $B_n^{(1)}$, $D_n^{(1)}$, $A_{2n-1}^{(2)}$, $A_{2n}^{(2)}$ and $D_{n+1}^{(2)}$ by adopting new strategies. First, we introduce new classes of tableaux that realize crystal basis elements of dominant maximal weights in tensor products of crystals of level 1 fundamental representations.

Second, we consider a family of highest weights Λ at the same time and form a triangular array of numbers out of multiplicities of maximal weights as Λ varies in the family. The resulting triangular arrays are the Pascal, Motzkin, Riordan and Bessel triangles, respectively, for various families of highest weights Λ. We connect these triangular arrays with combinatorics of the new classes of tableaux through developing insertion schemes, and prove that they enumerate the multiplicities of the highest weights.

Third, we classify fundamental weights into two types: type $\mathfrak{D}$ and type $\mathfrak{B}$, and show that the weight multiplicities of maximal weights are governed by the types of fundamental weights, which make up the highest weight Λ, without regard to the affine types.

As consequences of our approach, we obtain explicit descriptions of maximal weights and their multiplicities of level 2 and level 3 highest weights for affine types $B_n^{(1)}$, $D_n^{(1)}$, $A_{2n-1}^{(2)}$, $A_{2n}^{(2)}$ and $D_{n+1}^{(2)}$. Some maximal weights of higher levels are also considered. In this extended abstract, we only present our results for type $B_n^{(1)}$ for simplicity.

The multiplicities of maximal weights turn out to have intriguing connections to several combinatorial and representation theoretic constructions. In particular, they are related to weight multiplicities of finite types, Schur–Weyl type dualities, pattern-avoiding permutations and random matrices. We plan to investigate these connections more closely in subsequent papers.

2 Quantum affine algebras and crystals

2.1 Quantum affine algebras and dominant maximal weights

Let $I = \{0, 1, ..., n\}$ be an index set. The *affine Cartan datum* $(A, P^\vee, P, \Pi^\vee, \Pi)$ consists of (a) an affine Cartan matrix $A = (a_{ij})_{i,j\in I}$ of corank 1, (b) a dual weight lattice $P^\vee = \bigoplus_{i=0}^{n} \mathbb{Z}h_i \oplus \mathbb{Z}d$ with $\mathfrak{h} := \mathbb{C} \otimes_{\mathbb{Z}} P^\vee$, (c) a weight lattice $P = \bigoplus_{i=0}^{n} \mathbb{Z}\Lambda_i \oplus \mathbb{Z}\delta \subset \mathfrak{h}^*$, (d) the set of *simple coroots* $\Pi^\vee = \{h_i \mid i \in I\} \subset P^\vee$, (e) the set of *simple roots* $\Pi = \{\alpha_i \mid i \in I\} \subset P$. Here Λ_i is the *i-th fundamental weight* ($i \in I$), $\delta = \sum_{i \in I} a_i \alpha_i$ is the *null root*, and d is the *degree derivation*. Let $c = \sum_{i \in I} a_i^\vee h_i$ be the center of the affine Cartan datum.

We say that a weight $\Lambda \in P$ is of *level* k if $\Lambda(c) = k$. There exists a non-degenerate symmetric bilinear form $(\ |\ )$ on $\mathfrak{h}^*$ ([4, (6.2.2)]). We denote by $P^+ := \{\Lambda \in P \mid \langle h_i, \Lambda \rangle \in \mathbb{Z}_{\geqslant 0},\ i \in I\}$ the set of *dominant integral weights*. The free abelian group $\mathsf{Q} := \oplus_{i \in I} \mathbb{Z}\alpha_i$ is called the *root lattice* and we set $\mathsf{Q}^+ := \oplus_{i \in I} \mathbb{Z}_{\geqslant 0}\alpha_i$.

Let us denote by $U_q(\mathfrak{g})$ the quantum affine algebra associated to an affine Cartan datum $(A, P^\vee, P, \Pi^\vee, \Pi)$, which is generated by e_i, f_i ($i \in I$) and q^h ($h \in P^\vee$) subject to the usual defining relations. We also denote by $\mathcal{O}_{\rm int}$ the category consisting of integrable weight-admissible modules over $U_q(\mathfrak{g})$. It is well-known that the category $\mathcal{O}_{\rm int}$ is a semisimple tensor category with its irreducible objects being isomorphic to *the highest weight modules* $V(\Lambda)$ ($\Lambda \in P^+$), each of which is generated by a *highest weight vector* v_Λ. Recall, e.g. from [4, Chapter 10], that if $M, N \in \mathcal{O}_{\rm int}$, then, for $\Lambda \in P^+$ and $t \in \mathbb{Z}$,

$$\text{(a) } M \simeq N \text{ if and only if } \mathrm{ch}(M) = \mathrm{ch}(N) \text{ and (b) } \mathrm{ch}(V(\Lambda)) = e^{-t\delta}\mathrm{ch}(V(\Lambda + t\delta)), \quad (2.1)$$

where $M = \oplus_{\mu \in P} M_\mu$ and $\mathrm{ch}(M) := \sum_{\mu \in P} (\dim_{\mathbb{C}} M_\mu) e^\mu$ is the *character* of M.

The dimension of the μ-weight space $V(\Lambda)_\mu$ is called the *multiplicity* of μ in $V(\Lambda)$ and we denote it by $m_\mu(\Lambda)$. A weight μ is *maximal* if $\mu + \delta$ is not a weight of $V(\Lambda)$. The set of all maximal weights of $V(\Lambda)$ with Λ of level k is denoted by $\max(\Lambda|k)$. We denote by $\max^+(\Lambda|k) := \max(\Lambda|k) \cap P^+$ the set of all dominant maximal weights of $V(\Lambda)$. Then we have $\max(\Lambda|k) = W \cdot \max^+(\Lambda|k)$ where W is the affine Weyl group.

Proposition 2.1. *[4, Proposition 12.6] The orthogonal projection $\mu \longmapsto \overline{\mu}$ induces a bijection from $\max^+(\Lambda|k)$ onto $k\mathcal{C}_{\rm af} \cap (\overline{\Lambda} + \overline{\mathsf{Q}})$ where Λ is of level k. In particular, $|\max^+(\Lambda|k)| < \infty$.*

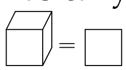
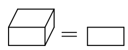
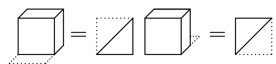
Fundamental weights Λ_i of quantum affine algebras of types $B_n^{(1)}$, $D_n^{(1)}$, $A_{2n-1}^{(2)}$, $A_{2n}^{(2)}$ and $D_{n+1}^{(2)}$ can be classified according to their levels $\Lambda_i(c)$. It is also well-known that $\Lambda_i(c) = 1$ or 2. We denote arbitrary fundamental weights of level 1 by $\boldsymbol{\Lambda}$ to distinguish them from other (fundamental) weights.

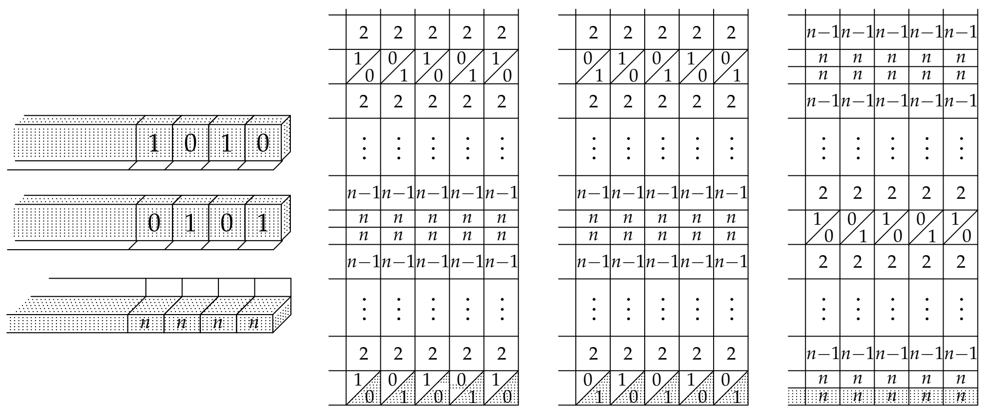
2.2 Crystals and Young walls

In [6, 7], Kashiwara proved that each $M \in \mathcal{O}_{\rm int}$ has a *crystal basis* (L, B) and the crystal bases behave well with respect to tensor products of modules in $\mathcal{O}_{\rm int}$. Furthermore, he proved that B has a colored oriented graph structure induced by the Kashiwara operators $\tilde{e}_i$, $\tilde{f}_i$ ($i \in I$). That is, we have $b \xrightarrow{i} b' \iff \tilde{f}_i b = b'$ for $b, b' \in B$.

The graph structure encodes information on the algebraic structure of M. For example, (i) $B = \coprod_{\mu \in P} B_\mu$ and $|B_\mu| = \dim_{\mathbb{Q}(q)} M_\mu$ for $\mu \in \mathrm{wt}(M)$, (ii) the graph of B is connected if and only if M is irreducible. In particular, we have $\operatorname{ch} V(\Lambda) = \sum_{\mu \in P} |B(\Lambda)_\mu| e^\mu$ and $|B(\Lambda)_\mu| = |B(\Lambda + k\delta)_{\mu+k\delta}|$, where $B(\Lambda)$ is the crystal basis for $V(\Lambda)$.

In [5], Kang constructed Young wall realizations of *level* 1 highest weight crystals $B(\mathbf{\Lambda})$ for all classical quantum affine algebras except $C_n^{(1)}$. In this extended abstract, we focus on the affine type $B_n^{(1)}$ for simplicity, and similar results are available for the affine types $A_{2n}^{(2)}$, $A_{2n-1}^{(2)}$, $D_n^{(1)}$ and $D_{n+1}^{(2)}$.

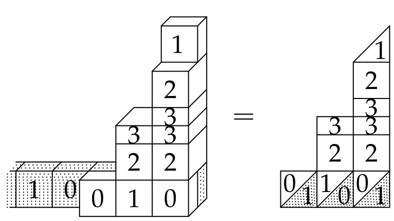
Basically, Young walls are built from colored blocks. There are three types of blocks: (a)  (regular), (b)  (half-height), (c)  (half-thickness). These blocks stack on the ground-state Young wall $\boxed{\mathbf{\Lambda}}$, which is given below as the shaded part in (2.2), by the following rules: (a) blocks should be placed in the pattern given below in (2.2), (b) no block can be placed on top of a column of half-unit thickness, (c) there should be no free space to the right of any block except the rightmost column.



$B_n^{(1)}$, $\mathbf{\Lambda}_1$ $\quad$ $B_n^{(1)}$, $\mathbf{\Lambda}_0$ $\quad$ $B_n^{(1)}$, $\mathbf{\Lambda}_n$ (2.2)

According to the ground-state Young walls in (2.2), we call $\mathbf{\Lambda}_i$ $(i = 0, 1)$ *type* $\mathfrak{D}$ and $\mathbf{\Lambda}_n$ as *type* $\mathfrak{B}$, since the vertex i (respectively n) in the affine Dynkin diagram of type $B_n^{(1)}$ corresponds to the vertex n in the finite Dynkin diagram D_n (respectively B_n) ([4, Chapter 4]). We denote by $Y_{\mathbf{\Lambda}}$ a Young wall stacked on $\boxed{\mathbf{\Lambda}}$. For a Young wall $Y_{\mathbf{\Lambda}}$, we write $Y_{\mathbf{\Lambda}} = (y_k)_{k=1}^\infty = (\ldots, y_2, y_1)$ as a sequence of its columns from the right.

Example 2.2. *For* $\mathfrak{g} = B_3^{(1)}$ *and* $\mathbf{\Lambda}_0$*, the following is an example of a Young wall* $Y_{\mathbf{\Lambda}_0}$:



(2.3)

A column of a Young wall is called a *full column* if its height is a multiple of the unit length and its top is of unit thickness. A Young wall is said to be *proper* if none of the full columns have the same heights.

For a given proper Young wall $\mathsf{Y}_{\mathbf{\Lambda}} = (y_i)_{i=1}^{\infty}$, define a partition $|\mathsf{Y}_{\mathbf{\Lambda}}| = (|y_1|, |y_2|, \ldots)$, where the $|y_i|$ is the number of blocks in the i-th column of $\mathsf{Y}_{\mathbf{\Lambda}}$ above $\boxed{\mathbf{\Lambda}}$, and call $|\mathsf{Y}_{\mathbf{\Lambda}}|$ the *partition associated* to $\mathsf{Y}_{\mathbf{\Lambda}}$. Conversely, for a partition λ and $\mathbf{\Lambda}$, we can build a proper Young wall $\mathsf{Y}_{\mathbf{\Lambda}}^{\lambda}$ so that its associated partition $|\mathsf{Y}_{\mathbf{\Lambda}}^{\lambda}|$ is equal to λ, if the Young wall $\mathsf{Y}_{\mathbf{\Lambda}}^{\lambda}$ is uniquely determined.

For $\mathsf{Y}_{\mathbf{\Lambda}}$, we define $\mathrm{cont}(\mathsf{Y}_{\mathbf{\Lambda}}) = \sum_{i \in I} m_i \alpha_i$ and $\mathrm{wt}(\mathsf{Y}_{\mathbf{\Lambda}}) = \mathbf{\Lambda} - \mathrm{cont}(\mathsf{Y}_{\mathbf{\Lambda}})$, where m_i is the number of i-blocks that have been added to the ground-state wall $\boxed{\mathbf{\Lambda}}$. We call them the *content* and *weight* of $\mathsf{Y}_{\mathbf{\Lambda}}$, respectively.

Theorem 2.3. [5] *Let $\mathcal{Z}(\mathbf{\Lambda})$ be the set of all proper Young walls on $\boxed{\mathbf{\Lambda}}$. Then there exist combinatorial Kashiwara operators $\tilde{e}_i$ and $\tilde{f}_i$ on $\mathcal{Z}(\mathbf{\Lambda})$ such that $\mathcal{Z}(\mathbf{\Lambda})$ becomes an affine crystal. Furthermore,* (i) *we can characterize the subset $\mathcal{Y}(\mathbf{\Lambda})$ of those proper Young walls which are contained in the connected component of the crystal graph originated from the empty Young wall $\boxed{\mathbf{\Lambda}}$ in $\mathcal{Z}(\mathbf{\Lambda})$,* (ii) *$\mathcal{Y}(\mathbf{\Lambda})$ is isomorphic to $B(\mathbf{\Lambda})$ as crystals.*

2.3 Higher level crystals

In this subsection, we will realize the crystal $B(\Lambda)$ for $\Lambda(c) \geqslant 2$ in terms of tensor products of Young walls. To begin with, we consider the crystal $B(k\mathbf{\Lambda})$ of level k and see that $B(k\mathbf{\Lambda})$ is realized as *the subcrystal of $\mathcal{Z}(\mathbf{\Lambda})^{\otimes k}$ whose graph is the connected component of the k-tuple of ground-state Young walls, denoted by $\boxed{k\mathbf{\Lambda}} := \boxed{\mathbf{\Lambda}} \otimes \cdots \otimes \boxed{\mathbf{\Lambda}}$ (k times).*

Lemma 2.4. *For each fundamental weight Λ_u of level 2, we have following equations by direct computations: For $\epsilon \in \{0, 1\}$ satisfying $\epsilon \equiv_2 u$,*

$$\Lambda_u - \left\lfloor \frac{u}{2} \right\rfloor \delta = \mathbf{\Lambda}_0 + \mathbf{\Lambda}_\epsilon - \mathrm{cont}(\mathsf{Y}_{\mathbf{\Lambda}_\epsilon}^{\lambda(u)}) \ (\textit{type } \mathfrak{D}), \quad \textit{and} \quad \Lambda_u = 2\mathbf{\Lambda}_n - \mathrm{cont}(\mathsf{Y}_{\mathbf{\Lambda}_n}^{\lambda(n-u)}) \ (\textit{type } \mathfrak{B})$$

where $\lambda(m) = (m, \ldots, 2, 1)$ is the staircase partition starting with m ($\lambda(-t) := (0)$ for $t \in \mathbb{Z}_{\geqslant 0}$).

By defining 2-fold tensor product Young walls

$$\boxed{\Lambda_{2u}^{0,0}} := \boxed{\mathbf{\Lambda}_0} \otimes \mathsf{Y}_{\mathbf{\Lambda}_0}^{\lambda(2u-1)}, \quad \boxed{\Lambda_{2u+1}^{0,1}} := \boxed{\mathbf{\Lambda}_0} \otimes \mathsf{Y}_{\mathbf{\Lambda}_1}^{\lambda(2u)} \text{ and } \quad \boxed{\Lambda_{n-2u}^{n,n}} := \boxed{\mathbf{\Lambda}_n} \otimes \mathsf{Y}_{\mathbf{\Lambda}_n}^{\lambda(n-u)},$$

the crystal $B((k-2)\mathbf{\Lambda} + \Lambda_u)$ of level k is realized as *the subcrystal of $\mathcal{Z}(\mathbf{\Lambda})^{\otimes k-2} \otimes \mathcal{Z}(\mathbf{\Lambda}) \otimes \mathcal{Z}(\mathbf{\Lambda}')$ generated by the highest weight crystal $\boxed{(k-2)\mathbf{\Lambda}} \otimes \boxed{\Lambda_u}$ whose weight is $(k-2)\mathbf{\Lambda} + \Lambda_u$ up to $\mathbb{Z}\delta$ by* (2.1)*, since $B((k-2)\mathbf{\Lambda}) \hookrightarrow \mathcal{Z}(\mathbf{\Lambda})^{\otimes k-2}$ and $B(\Lambda_u) \hookrightarrow \mathcal{Z}(\mathbf{\Lambda}) \otimes \mathcal{Z}(\mathbf{\Lambda}')$.*

Note that, for each Λ_u, there are two realizations of $B((k-2)\mathbf{\Lambda} + \Lambda_u)$ depending on the choice of $\boxed{\Lambda_u}$.

3 Young tableaux and lattice paths

3.1 Standard Young tableaux at most k rows

We say that a Young tableau T is *row-strict* (respectively *column-strict*) if its entries in each row (respectively column) are strictly decreasing. We say that a Young tableau is *standard* if it is row and column-strict, simultaneously.

Let $\mathsf{S}_m^{(k)}$ denote the set of standard Young tableaux with m cells and at most k rows. An explicit enumeration of $|\mathsf{S}_m^{(k)}|$ is difficult in general and only known for $k \leqslant 5$ (see [1, 8]). In this extended abstract, we always deal with standard Young tableaux.

Note that each element in $\mathsf{S}_m^{(k)}$ can be expressed in terms of a sequence of strict partitions as follows:

$$\mathsf{S}_m^{(k)} = \left\{ \underline{\lambda} = (\lambda^{(1)}, \cdots, \lambda^{(\ell)}) \mid \ell \leqslant k,\ \lambda^{(i)} \supset \lambda^{(i+1)}\ (1 \leqslant i < \ell) \text{ and } \lambda^{(1)} * \cdots * \lambda^{(\ell)} = \lambda(m) \right\},$$

where $\lambda^{(1)} * \lambda^{(2)}$ is a partition obtained by rearranging parts of $\lambda^{(1)}$ and $\lambda^{(2)}$ in a weakly decreasing way and $\lambda(m) := (m, m-1, \ldots, 2, 1)$ is the staircase partition. Now we shall denote by $\underline{\lambda}_T$ the sequence of strict partitions corresponding to a Young tableau T.

Example 3.1. *For partitions $\lambda^{(1)} = (7,3,1)$, $\lambda^{(2)} = (8,6,6,3)$ and $\lambda^{(3)} = (7,5,4,1)$, we have*

$$\mathop{*}_{t=1}^{3} \lambda^{(t)} = (8,7,7,6,6,5,4,3,3,1,1).$$

For a sequence of fundamental weights $\underline{\mathbf{\Lambda}} = (\mathbf{\Lambda}_{i_1}, \mathbf{\Lambda}_{i_2}, \cdots, \mathbf{\Lambda}_{i_k})$ of level 1 and a Young tableau T of shape $\mu \backslash \lambda$ of row length k, we define a k-fold tensor product of Young walls,

$$\mathbb{Y}_{\underline{\mathbf{\Lambda}}}^{T} \text{ or } \mathbb{Y}_{\underline{\mathbf{\Lambda}}}^{\underline{\lambda}} := \mathsf{Y}_{\mathbf{\Lambda}_{i_1}}^{\lambda^{(1)}} \otimes \mathsf{Y}_{\mathbf{\Lambda}_{i_2}}^{\lambda^{(2)}} \otimes \cdots \otimes \mathsf{Y}_{\mathbf{\Lambda}_{i_k}}^{\lambda^{(k)}} \quad \text{with } \underline{\lambda} = \underline{\lambda}_T.$$

Now we shall introduce special families of Young tableaux.

Definition 3.2. *We define the subset $\mathsf{S}_m^{(k,t)}$ of $\mathsf{S}_m^{(k)}$ in the following way: $T \in \mathsf{S}_m^{(k,t)}$ if the shape $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ $(\ell \leqslant k)$ of T has exactly t odd parts.*

For each $0 \leqslant t \leqslant k \leqslant 5$, we prove closed-form formulas for $|\mathsf{S}_m^{(k,t)}|$. For example,

$$\left|\mathsf{S}_{2m}^{(5,2)}\right| = \sum_{i=0}^{m} \frac{2i}{i+3} \binom{2m}{2i} C_i C_{i+1} - \sum_{i=0}^{m-1} \frac{2i}{i+3} \binom{2m}{2i+1} C_{i+1}^2.$$

Here $C_i = \frac{1}{i+1}\binom{2i}{i}$ denotes the ith Catalan number. Furthermore, we can prove

$$\left|\mathsf{S}_m^{(k)}\right| = \int_{O(k+1)} (1 - \det(X))(1 - \mathrm{Tr}(X))^m d\mu(X),$$

where the integral is taken with respect to the normalized Haar measure μ over the orthogonal group $O(k+1)$. In particular, $|\mathsf{S}_m^{(6)}|$ is given as follows: For $h_{i,j,k} = \binom{i}{\lfloor \frac{i}{2} \rfloor}\binom{j}{\lfloor \frac{j}{2} \rfloor}\binom{k}{\lfloor \frac{k}{2} \rfloor}$,

$$|\mathsf{S}_m^{(6)}| = \sum_{i+j+k=m} \binom{m}{i,j,k} \left(h_{i+4,j+2,k} - h_{i+3,j+3,k} - h_{i+4,j+1,k+1} - h_{i+2,i+2,k+2} + 2h_{i+3,j+2,k+1} \right).$$

In this note, we will mainly deal with $\mathsf{AE}_{2m}^{(k)} := \mathsf{S}_{2m}^{(k,2)}$, $\mathsf{AE}_{2m-1}^{(k)} := \mathsf{S}_{2m-1}^{(k,1)}$ and ${}^0\mathsf{P}_m^{(k)} := \mathsf{S}_m^{(k,0)}$, ${}^1\mathsf{P}_m^{(k)} := \mathsf{S}_m^{(k,k)}$, and call them *almost even tableaux* and *parity tableaux*, respectively.

3.2 Lattice paths

Definition 3.3. (1) (i) *A* Motzkin path *is a path on the lattice* $\mathbb{Z}^2$ *starting from* $(0,0)$*, using steps* $(1,1)$*,* $(1,0)$*,* $(1,-1)$ *without going below the x-axis.* (ii) *A* generalized Motzkin number $\mathsf{M}_{(m,k)}$ *for* $m \geqslant k \geqslant 0$ *is the number of all Motzkin paths ending at the lattice point* (m,k)*. In particular, we write* $\mathsf{M}_m = \mathsf{M}_{(m,0)}$ *and call it the* m-th Motzkin number. *We call the triangular array consisting of* $\mathsf{M}_{(m,s)}$ *the* Motzkin triangle.

(2) (i) *A* Riordan path *is a Motzkin path which has no horizontal step on the x-axis.* (ii) *A* generalized Riordan number $\mathsf{R}_{(m,k)}$ *for* $m \geqslant k \geqslant 0$ *is the number of all Riordan paths ending at the lattice point* (m,k)*. In particular, we write* $\mathsf{R}_m = \mathsf{R}_{(m,0)}$ *and call it the* m-th Riordan number. *We call the triangular array consisting of* $\mathsf{R}_{(m,k)}$ *the* Riordan triangle.

$$\mathsf{M}_{(m,k)} = \begin{array}{cccccccc} & & & & & & & \cdots \\ & & & & & & 1 & \cdots \\ & & & & & 1 & 6 & \cdots \\ & & & & 1 & 5 & 20 & \cdots \\ & & & 1 & 4 & 14 & 44 & \cdots \\ & & 1 & 3 & 9 & 25 & 69 & \cdots \\ & 1 & 2 & 5 & 12 & 30 & 76 & \cdots \\ 1 & 1 & 2 & 4 & 9 & 21 & 51 & \cdots \end{array} \qquad \mathsf{R}_{(m,k)} = \begin{array}{cccccccc} & & & & & & & \cdots \\ & & & & & & 1 & \cdots \\ & & & & & 1 & 5 & \cdots \\ & & & & 1 & 4 & 15 & \cdots \\ & & & 1 & 3 & 10 & 29 & \cdots \\ & & 1 & 2 & 6 & 15 & 40 & \cdots \\ & 1 & 1 & 3 & 6 & 15 & 36 & \cdots \\ 1 & 0 & 1 & 1 & 3 & 6 & 15 & \cdots \end{array} \tag{3.1}$$

Interestingly, the number M_m is also equal to the number of all Young tableaux with m cells and at most 3 rows.

The following theorem shows that parity tableaux in $\mathsf{P}_m^{(3)} := {}^0\mathsf{P}_m^{(k)} \sqcup {}^1\mathsf{P}_m^{(k)}$ and almost even tableaux in $\mathsf{AE}_{m-1}^{(3)}$ can be taken as tableaux models for Riordan numbers R_m.

Theorem 3.4. $\mathsf{R}_m = \left|\mathsf{AE}_{m-1}^{(3)}\right| = \left|\mathsf{P}_m^{(3)}\right|$ *and* $\mathsf{R}_{(m,k)} = \sum_{i=0}^{m-k} (-1)^i (\mathsf{M}_{(m-1-i,k)} + \mathsf{M}_{(m-1-i,k-1)})$.

4 Dominant maximal weights and (spin) rigid-tableaux

For simplicity, we continue to assume $\mathfrak{g} = B_n^{(1)}$. We have similar results for other types.

4.1 Dominant maximal weights and pair of staircase partitions

Lemma 4.1 (Level 2). ($\mathfrak{D}$) *For $\Lambda = (\delta_{s,0} + \delta_{s,1})\mathbf{\Lambda}_0 + \Lambda_s$ $(0 \leqslant s \leqslant n-1)$ of level 2, we set*

$$\mathrm{smax}^+_{\mathfrak{D}}(\Lambda|2) := \left\{\Lambda - \mathrm{cont}\left(\mathsf{Y}^{\lambda(2u-1+s)}_{\mathbf{\Lambda}_0}\right) + \mathrm{cont}\left(\mathsf{Y}^{\lambda(s-1)}_{\mathbf{\Lambda}_0}\right)\right\} \quad (1 \leqslant u \leqslant \lfloor n-s+1/2 \rfloor).$$

Then we have $\mathrm{smax}^+_{\mathfrak{D}}(\Lambda|2) \subset \max^+(\Lambda|2)$.

($\mathfrak{B}$) *For $(1+\delta_{s,n})\Lambda_s + \delta_{s,1}\mathbf{\Lambda}_0$ $(1 \leqslant s \leqslant n)$ of level 2, we set*

$$\mathrm{smax}^+_{\mathfrak{B}}(\Lambda|2) := \left\{\Lambda - \mathrm{cont}\left(\mathsf{Y}^{\lambda(n-u)}_{\mathbf{\Lambda}_n}\right) + \mathrm{cont}\left(\mathsf{Y}^{\lambda(n-s)}_{\mathbf{\Lambda}_n}\right)\right\} \quad (0 \leqslant u \leqslant s).$$

Then we have $\mathrm{smax}^+_{\mathfrak{B}}(\Lambda|2) \subset \max^+(\Lambda|2)$.

Lemma 4.2 (Level 3, $\mathfrak{D}$). *For $\Lambda = (1 + \delta_{s,0} + \delta_{s,1})\mathbf{\Lambda}_0 + \Lambda_s$ $(0 \leqslant s \leqslant n-1)$, we set*

$$\mathrm{smax}^+_{\mathfrak{D},\mathrm{i}}(\mathbf{\Lambda}_0 + \Lambda|3) := \left\{\Lambda - \mathrm{cont}\left(\mathsf{Y}^{\lambda(2u+s)}_{\mathbf{\Lambda}_1}\right) + (\alpha_1 - \alpha_0) + \mathrm{cont}\left(\mathsf{Y}^{\lambda(s-1)}_{\mathbf{\Lambda}_0}\right)\right\}.$$

Then we have $\mathrm{smax}^+_{\mathfrak{D},\mathrm{i}}(\mathbf{\Lambda}_0 + \Lambda|3) \subset \max^+(\Lambda|3)$.

Definition 4.3. (1) *For $\mathbf{\Lambda}_0 + \Lambda$ and $\mathbf{\Lambda}_n + \Lambda$ of level 3, we define*

- $\mathrm{smax}^+_{\mathfrak{D}}(\mathbf{\Lambda}_0 + \Lambda|3) := \mathbf{\Lambda}_0 + \mathrm{smax}^+_{\mathfrak{D}}(\Lambda|2) \bigsqcup \mathrm{smax}^+_{\mathfrak{D},\mathrm{i}}(\mathbf{\Lambda}_0 + \Lambda|3) \subset \max^+(\mathbf{\Lambda}_0 + \Lambda|3)$,
- $\mathrm{smax}^+_{\mathfrak{B}}(\mathbf{\Lambda}_n + \Lambda|3) := \mathbf{\Lambda}_n + \mathrm{smax}^+_{\mathfrak{B}}(\Lambda|2) \subset \max^+(\mathbf{\Lambda}_n + \Lambda|3)$.

(2) *For $\Lambda = (k-2)\mathbf{\Lambda}_0 + \Lambda$ and $\Lambda' = (k-2)\mathbf{\Lambda}_n + \Lambda$ $(k \geqslant 4)$ of level k, we define*

- $\mathrm{smax}^+_{\mathfrak{D}}(\Lambda|k) := (k-3)\mathbf{\Lambda}_0 + \mathrm{smax}^+_{\mathfrak{D}}(\mathbf{\Lambda}_0 + \Lambda|3) \subset \max^+(\Lambda|k)$,
- $\mathrm{smax}^+_{\mathfrak{B}}(\Lambda'|k) := (k-3)\mathbf{\Lambda}_n + \mathrm{smax}^+_{\mathfrak{B}}(\mathbf{\Lambda}_n + \Lambda|3) \subset \max^+(\Lambda|k)$.

Note that, to each pair $(\lambda(m), \lambda(s))$ $(m \geqslant s)$ of staircase partitions and $\mathbf{\Lambda}_0$ (respectively $\mathbf{\Lambda}_n$), we have a unique $\mathfrak{D}$-type (respectively $\mathfrak{B}$-type) dominant maximal weight η of $V(\Lambda)$ of level k.

Example 4.4. *For $n = 7$,$(\lambda(5), \lambda(1))$ and $k \geqslant 2$, we have the dominant maximal weights $\eta \in \mathrm{smax}^+_{\mathfrak{D}}(\Lambda := (k-2)\mathbf{\Lambda}_0 + \Lambda_2|k)$ and $\mu \in \mathrm{smax}^+_{\mathfrak{B}}(\Lambda' := (k-2)\mathbf{\Lambda}_7 + \Lambda_6|k)$, where*

$$\eta = \Lambda - \mathrm{cont}\left(\mathsf{Y}^{\lambda(5)}_{\mathbf{\Lambda}_0}\right) + \mathrm{cont}\left(\mathsf{Y}^{\lambda(1)}_{\mathbf{\Lambda}_0}\right) \quad \text{and} \quad \mu = \Lambda' - \mathrm{cont}\left(\mathsf{Y}^{\lambda(5)}_{\mathbf{\Lambda}_n}\right) + \mathrm{cont}\left(\mathsf{Y}^{\lambda(1)}_{\mathbf{\Lambda}_n}\right).$$

4.2 Weight multiplicities and (spin-)rigid Young tableaux

In this subsection, we will investigate the multiplicities of dominant maximal weights $\eta \in \mathrm{smax}^+(\Lambda) := \mathrm{smax}^+_{\mathfrak{D}}(\Lambda|k) \bigsqcup \mathrm{smax}^+_{\mathfrak{B}}(\Lambda|k)$ and describe the corresponding crystal basis elements by introducing *rigid Young tableaux*.

Theorem 4.5. *For a sequence of strict partitions $\underline{\lambda} = \lambda^{(1)} \supset \cdots \supset \lambda^{(k)}$ such that $\lambda^{(1)}_1 \leqslant n$,*

$$\mathbb{Y}^{\underline{\lambda}}_{k\mathbf{\Lambda}} = \mathsf{Y}^{\lambda^{(1)}}_{\mathbf{\Lambda}} \otimes \mathsf{Y}^{\lambda^{(2)}}_{\mathbf{\Lambda}} \otimes \cdots \otimes \mathsf{Y}^{\lambda^{(k)}}_{\mathbf{\Lambda}} \text{ is connected to highest weight vector } \boxed{k\mathbf{\Lambda}} \text{ in } \mathcal{Z}(\mathbf{\Lambda})^{\otimes k}.$$

For a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ and $1 \leqslant u \leqslant k$, we define partitions $\lambda_{>u}$ and $\lambda_{\geqslant u}$:

$$\lambda_{>u} = (\lambda_{u+1}, \lambda_{u+2}, \ldots, \lambda_k) \quad \text{and} \quad \lambda_{\geqslant u} = (\lambda_u, \lambda_{u+1}, \ldots, \lambda_k).$$

Definition 4.6. *For $s, m \in \mathbb{Z}_{\geqslant 0}$ with $m \leqslant n$, a sequence of strict partitions $(\lambda^{(1)}, \ldots, \lambda^{(k-1)}, \lambda^{(k)})$ is called a* rigid Young tableau of (s, m) with k rows *if it satisfies the following conditions:*

(a) $\ell(\lambda^{(k)}) \geqslant s$,

(b) $\lambda^{(1)} * \lambda^{(2)} * \cdots * \lambda^{(k-1)} * \lambda^{(k)} = \uplambda(m)$,

(c) $\lambda^{(i)} \supset \lambda^{(i+1)}$ *for* $1 \leqslant i \leqslant k-2$,

(d) $\lambda^{(k-1)} \supset \lambda^{(k)}_{>s}$ *but* $\lambda^{(k-1)} \not\supset \lambda^{(k)}_{\geqslant s}$ *if* $s \geqslant 1$.

We denote by ${}_s\mathfrak{B}^{(k)}_m$ the set of all rigid Young tableaux of (s, m) with k rows. In particular ${}_0\mathfrak{B}^{(k)}_m = \mathsf{S}^{(k)}_m$.

An element T in ${}_s\mathfrak{B}^{(k)}_m$ can be described in terms of skew-Young tableaux:

Example 4.7. *$T = ((432), (51)) \in {}_1\mathfrak{B}^{(2)}_5$, since $\begin{array}{|c|c|c|c|}\hline * & 4 & 3 & 2 \\ \hline 5 & 1 \\ \cline{1-2}\end{array}$ is a skew-Young tableau and $\begin{array}{|c|c|c|}\hline 4 & 3 & 2 \\ \hline 5 & 1 \\ \cline{1-2}\end{array} \notin \mathfrak{B}^{(2)}_5$. On the other hand, $((532), (41)) \notin {}_1\mathfrak{B}^{(2)}_5$, since $\begin{array}{|c|c|c|}\hline 5 & 3 & 2 \\ \hline 4 & 1 \\ \cline{1-2}\end{array} \in \mathfrak{B}^{(2)}_5$.*

Theorem 4.8. *For $\eta \in \mathrm{smax}^+_{\mathfrak{B}}(\Lambda|k)$ $(k \geqslant 2)$ corresponding to $(\uplambda(m), \uplambda(s))$, we have*

$$\dim(V(\Lambda)_\eta) = |{}_s\mathfrak{B}^{(k)}_m|.$$

Corollary 4.9. *The numbers $|\mathsf{S}^{(k)}_m|$ for $m \leqslant n$ appear as weight multiplicities of $V(k\mathbf{\Lambda}_n)$. In particular, the central binomial coefficients $\binom{m}{\lfloor \frac{m}{2} \rfloor} = |\mathsf{S}^{(2)}_m|$ appear as weight multiplicities of $V(2\mathbf{\Lambda}_n)$ and the Motzkin numbers $\mathsf{M}_m = |\mathsf{S}^{(3)}_m|$ appear as weight multiplicities of $V(3\mathbf{\Lambda}_n)$.*

Definition 4.10. *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a partition. We write $\lambda \Vdash_0 m$ if λ is a shape of some $T \in \mathsf{AE}^{(k)}_m$, and write $\lambda \Vdash_1 m$ if $\sum_{i=1}^{k-1} \left\lceil \frac{\lambda_i}{2} \right\rceil + \left\lfloor \frac{\lambda_k}{2} \right\rfloor = \lceil \frac{m}{2} \rceil$.*

Now we define the *spin-rigid* Young tableaux of (s, m) with k rows.

Definition 4.11. *For $s, m \in \mathbb{Z}_{\geqslant 0}$ with $n \geqslant m \geqslant s - 1 \geqslant 0$, a sequence of strict partitions*

$$\underline{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(k-1)}, \lambda^{(k)})$$

is called a spin-rigid Young tableau of (s, m) with k rows *if it satisfies the following conditions:*

(a) $\overset{k}{\underset{s=1}{*}} \lambda^{(s)} = \lambda(m)$, (b) $\overset{k}{\underset{s=1}{*}} \ell(\lambda^{(s)}) \Vdash_{\epsilon} m$ *where* $\epsilon = \begin{cases} 0 & \text{if } m \equiv_2 0 \text{ and } k > 2, \\ 1 & \text{otherwise,} \end{cases}$

(c) $\ell(\lambda^{(k)}) \geqslant s - 1$, (d) $\lambda^{(i)} \supset \lambda^{(i+1)}$ *for* $1 \leqslant i \leqslant k-2$,

(e) $\lambda^{(k-1)} \supset \lambda^{(k)}_{\geqslant s+1}$ *but* $\lambda^{(k-1)} \not\supset \lambda^{(k)}_{\geqslant s-1}$ *if* $s > 1$.

We denote by ${}_s\mathfrak{D}^{(k)}_m$ *the set of all spin-rigid Young tableaux of* (s, m) *with* k *rows. In particular* ${}_0\mathfrak{D}^{(k)}_m = \mathsf{AE}^{(k)}_m$ *and hence* ${}_0\mathfrak{D}^{(2)}_{2m-1} = \mathsf{S}^{(2,1)}_{2m-1}$.

Theorem 4.12. *For* $\eta \in \mathrm{smax}^+_{\mathfrak{D}}(\Lambda|k)$ $(k \geqslant 2)$ *corresponding to* $(\lambda(m), \lambda(s))$, *we have*

$$\dim(V(\Lambda)_\eta) = \left| {}_{s+1}\mathfrak{D}^{(k)}_m \right|.$$

Corollary 4.13. *The numbers* $|\mathsf{AE}^{(k)}_m|$ *for* $m \leqslant n$ *appear as weight multiplicities of* $V(k\mathbf{\Lambda}_0)$ *and* $V((k-1)\mathbf{\Lambda}_0 + \mathbf{\Lambda}_1)$. *In particular, the binomial coefficients* $\binom{2m-1}{m} = |\mathsf{AE}^{(2)}_{2m-1}|$ *appear as weight multiplicities of* $V(2\mathbf{\Lambda}_0)$ *and* $V(\mathbf{\Lambda}_0 + \mathbf{\Lambda}_1)$, *and the Riordan numbers* $\mathsf{R}_m = |\mathsf{AE}^{(3)}_{m-1}|$ *appear as weight multiplicities of* $V(3\mathbf{\Lambda}_0)$ *and* $V(2\mathbf{\Lambda}_0 + \mathbf{\Lambda}_1)$.

5 Triangular arrays and multiplicities of $V(\Lambda)$ of level k

In this section, we compute $|{}_s\mathfrak{D}^{(k)}_m|$ and $|{}_s\mathfrak{B}^{(k)}_m|$ using triangular arrays of numbers for $k = 2, 3$, and thus obtain explicit formulas for weight multiplicities.

Definition 5.1. *For a sequence of strict partitions* $\underline{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(k)})$ *with* $\lambda^{(1)} * \cdots * \lambda^{(k)} = \lambda(m-1)$ *and a positive integer* $1 \leqslant u \leqslant k$, *we define*

$$\underline{\lambda} \underset{u}{*} m = (\lambda'^{(1)}, \dots, \lambda'^{(k)}), \quad \text{where} \quad \begin{cases} \lambda'^{(j)} = \lambda^{(j)} & \text{if } j \neq u, \\ \lambda'^{(u)} = (m) * \lambda^{(u)} & \text{otherwise.} \end{cases}$$

5.1 Pascal triangle at level 2

Lemma 5.2. *For* $T = (\lambda, \mu) \in {}_s\mathfrak{B}^{(2)}_{m-1}$, *we have* $T \underset{1}{*} m \in {}_{s-1}\mathfrak{B}^{(2)}_m$ *and* $T \underset{2}{*} m \in {}_{s+1}\mathfrak{B}^{(2)}_m$.

Theorem 5.3. *For every* $s \leqslant m$ *and* $\eta \in \mathrm{smax}^+_{\mathfrak{B}}(\Lambda|2)$ *corresponding to* $(\lambda(m), \lambda(s))$, *we have*

$$\dim V(\Lambda_s)_\eta = \left| {}_s\mathfrak{B}^{(2)}_m \right| = \binom{m}{\lfloor (m-s)/2 \rfloor}.$$

Corollary 5.4. *For $m \geqslant s \geqslant 0$, set $a = \lfloor (m-s)/2 \rfloor$ and $b = m - a$. We have a bijection between ${}_s\mathfrak{B}^{(2)}_m$ and $L(a,b)$, the lattice paths from $(0,0)$ to (a,b) according to the vectors $(1,0)$ and $(0,1)$.*

Theorem 5.5. *For $s, k \geqslant 0$ and $\eta \in \mathrm{smax}^+_{\mathfrak{D}}(\Lambda|2)$ corresponding to $(\lambda(2k+s-1), \lambda(s-1))$,*

$$\dim V(\Lambda_s)_\eta = \left| {}_s\mathfrak{D}^{(2)}_{2k+s-1} \right| = \binom{2k+s-\delta_{s,0}}{k}.$$

As a summary, *the triangular array consisting of weight multiplicities of $V(\Lambda)$, where Λ varies over level 2 weights, is equal to the Pascal triangle for n sufficiently large.*

5.2 Motzkin triangle and Riordan triangle at level 3

Lemma 5.6. *For $T = (\lambda, \mu, \nu) \in {}_s\mathfrak{B}^{(3)}_{m-1}$, we have $T \underset{1}{*} m \in {}_s\mathfrak{B}^{(3)}_m$ and $T \underset{3}{*} m \in {}_{s+1}\mathfrak{B}^{(3)}_m$.*

Theorem 5.7. *For $\eta \in \mathrm{smax}^+_{\mathfrak{B}}(\Lambda|k)$ corresponding to $(\lambda(m), \lambda(s))$, we have*

$$\dim(V(\Lambda)_\eta) = \left| {}_s\mathfrak{B}^{(3)}_m \right| = \mathsf{M}_{(m,s)}.$$

Proof. We develop an algorithm, named as the *rigid-type jeu de taquin*, to give a bijection between ${}_{s+1}\mathfrak{B}^{(3)}_{m-1}$ and ${}_s\mathfrak{B}^{(3)}_m \backslash \left({}_s\mathfrak{B}^{(3)}_{m-1} \underset{1}{*} m \bigsqcup {}_{s-1}\mathfrak{B}^{(3)}_{m-1} \underset{3}{*} m \right)$. □

Corollary 5.8. *We have a bijective map between ${}_s\mathfrak{B}^{(3)}_m$ and $M_{(m,s)}$ where $M_{(m,s)}$ is the set of Motzkin paths ending at (m,s).*

Theorem 5.9. *For $m \geqslant s \geqslant 0$ and $\eta \in \mathrm{smax}^+_{\mathfrak{D}}(\Lambda|3)$ corresponding to $(\lambda(m), \lambda(s-1))$, we have*

$$\dim(V(\Lambda)_\eta) = \left| {}_s\mathfrak{D}^{(3)}_m \right| = \mathsf{R}_{(m+1,s)}.$$

Proof. We apply the Robinson-Schensted algorithm to prove our assertion. □

As a summary, *the triangular array consisting of weight multiplicities of $V(\Lambda)$, where Λ varies over a family of level 3 weights of type $\mathfrak{B}$ (respectively $\mathfrak{D}$), is equal to the Motzkin triangle (respectively the Riordan triangle) for n sufficiently large.*

5.3 At Level ∞

It is well-known that, for a fixed positive integer m,

$$\lim_{k\to\infty} |\mathsf{S}^{(k)}_m| \text{ converges to } \mathcal{S}^{(\infty)}_m := \sum_{s=0}^{\lfloor m/2 \rfloor} \frac{m!}{(m-2s)! \times 2^s \times s!}.$$

Theorem 5.10. *We have $\lim_{k\to\infty} \left| \mathsf{AE}^{(k)}_{2m-1} \right| = (2m-1)!!$ and $\lim_{k\to\infty} \left| \mathsf{AE}^{(k)}_{2m} \right| = m(2m-1)!!$.*

Note that $\mathsf{S}_m^{(k)}$ and $\mathsf{AE}_m^{(k)}$ can be understood as special cases of ${}_s\mathfrak{B}_m^{(k)}$ and ${}_s\mathfrak{D}_m^{(k)}$ respectively.

Theorem 5.11. (1) $\lim_{k\to\infty} |{}_s\mathfrak{B}_m^{(k)}| = \binom{m}{s} \times \mathcal{S}_{m-s}^{(\infty)}$,

(2) $\lim_{k\to\infty} |{}_s\mathfrak{D}_m^{(k)}| = \binom{m+1}{s} \times (m-s)!!$ *if* $s \not\equiv_2 m$,

(3) $\lim_{k\to\infty} |{}_s\mathfrak{D}_m^{(k)}| = \binom{m}{s} \times (m-s)!! + \binom{m}{s-1} \times (m-s+1)!!$ *if* $s \equiv_2 m$.

The second formula in the above theorem is a closed-form formula for the triangular array consisting of coefficients of Bessel polynomials ([OEIS:A001497]). Thus we have: *For sufficiently large n and k, the multiplicities of dominant maximal weights of $V(\Lambda)$, as Λ varies over a family of level k weights, form the triangular arrays whose entries are given by the closed-form formulas (1), (2) and (3) in Theorem 5.11, respectively.*

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