ENTIRETY OF CUSPIDAL EISENSTEIN SERIES ON KAC–MOODY GROUPS

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Abstract. Let G be an infinite-dimensional representation-theoretic Kac–Moody group associated to a nonsingular symmetrizable generalized Cartan matrix. We consider Eisenstein series on G induced from unramified cusp forms on finite-dimensional Levi subgroups of maximal parabolic subgroups. Under a natural condition on maximal parabolic subgroups, we prove that the cuspidal Eisenstein series are entire on the full complex plane.

1. Introduction

Meromorphic continuation of Eisenstein series on reductive groups, established in the seminal work of Langlands [La], has been a foundational basis for other developments in Langlands' program. Most notably, the Langlands–Shahidi method (e.g., [KimSh, Kim]) in the study of L–functions originates from the idea of exploiting analytic properties of Eisenstein series together with the fact that L-functions appear in the Fourier coefficients of Eisenstein series.

The extension of the theory of Eisenstein series to Kac–Moody groups has been developed with anticipation of its potential roles in some of the central problems in number theory [BFH, Sh]. Indeed, starting with Garland's pioneering work [G99, G04, G06], there has been a significant body of work [GMS1, GMS2, GMS3, GMS4, Li, GMP, G11] on Eisenstein series on affine Kac–Moody groups, including the function field case [BK, Ka, P, LL]. Beyond affine Kac–Moody groups, Eisenstein series on rank 2 symmetric hyperbolic Kac–Moody groups were studied in [CLL]. For a more general class of Kac–Moody groups, in the paper [CGLLM], the absolute convergence of Eisenstein series is established for spectral parameters in the Godement range.

In the process of these developments, a striking difference between the finite-dimensional case and the affine case was observed in [BK, GMP], where the affine Eisenstein series induced from cusp forms on finite-dimensional Levi subgroups were shown to be *entire* and not just meromorphic as they are in the finite-dimensional case. This phenomenon is not restricted to the affine case, and it was shown in [CLL] that cuspidal Eisenstein series on rank 2 symmetric hyperbolic Kac–Moody groups are also entire.

To understand the analytic properties of cuspidal Eisenstein series on Kac–Moody groups, one may naturally ask: *For which parabolic subgroups are cuspidal Eisenstein series on Kac–Moody groups entire?* For a general maximal parabolic subgroup, even in the finite dimensional case

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cuspidal Eisenstein series will not be entire, as is readily seen for the real analytic Eisenstein series on SL_n.

In this paper, we give a natural condition on parabolic subgroups (Property RD, below and Section 4) and show that cuspidal Eisenstein series attached to a parabolic subgroup satisfying Property RD is holomorphic on the full complex plane.

More precisely, let G be a representation-theoretic Kac–Moody group, and let $\mathfrak{g}$ be the corresponding real Kac–Moody algebra with a fixed Cartan subalgebra $\mathfrak{h}$, Weyl group W and the set $\{\alpha_i\}_{i\in I}$ of simple roots, where I is the set of indices for the simple roots. We assume that $\mathfrak{g}$ is infinite-dimensional and non-affine since the other cases have been studied well as mentioned above.

The real group $G_{\mathbb{R}}$ has the Iwasawa decomposition $G_{\mathbb{R}} = UA^+K$ ([DGH]), where U is a maximal pro-unipotent subgroup, A^+ is the connected component of a maximal torus, and K is a subgroup of $G_{\mathbb{R}}$ analogous to the maximal compact subgroup in the finite-dimensional theory. Denote by $\Gamma = G_{\mathbb{Z}}$ the arithmetic subgroup of $G_{\mathbb{R}}$ (Subsection 2.1), and by $\mathcal{C} \subset \mathfrak{h}$ the open fundamental chamber. Using the exponential map $\exp : \mathfrak{h} \to A^+$, set $A_{\mathcal{C}} = \exp \mathcal{C}$.

Let P be a maximal parabolic subgroup of $G_{\mathbb{R}}$ with fixed Levi decomposition and associated finite-dimensional Levi subgroup M, which is associated with a subset $\theta \subseteq I$. We denote by α_P the simple root associated to the one element index in $I\backslash\theta$. Let L be the derived subgroup of M. For a cusp form f on $(L\cap\Gamma)\backslash L$, we recall that f is unramified if f is right invariant under the action of $L\cap K$.

For such an f, we define the Eisenstein series $E_f(s,g)$, $s \in \mathbb{C}$, $g \in G_{\mathbb{R}}$, in analogy with the classical case (see (3.5)). In Proposition 3.1, we use the reduction mechanism of [Bo, MW] to obtain absolute convergence of the cuspidal Eisenstein series $E_f(s,g)$ from that of Borel Eisenstein series established in [CGLLM].

Denote by ϖ_P the fundamental weight associated to α_P, and by ρ_M the Weyl vector of M. Let

$$W^\theta = \{w \in W : w^{-1}\alpha_i > 0, i \in \theta\}.$$

Then P is said to satisfy *Property RD* if there exists a constant $D > 0$, such that for every nontrivial element $w \in W^\theta$ it holds that

$$\langle D\varpi_P + \rho_M, \alpha^\vee\rangle \le 0$$

for any positive root α such that $w^{-1}\alpha < 0$. Property RD states that the coefficient of the simple root α_P grows faster than the coefficients of the simple roots in the subset θ. This property allows us to make use of the rapid decay of cusp forms on parabolic subgroups.

Now we state our main theorem.

Theorem 1.1. *Let f be an unramified cusp form on $(L\cap\Gamma)\backslash L$. If the maximal parabolic subgroup P satisfies Property RD, then for any compact subset $\mathfrak{S}$ of $A_{\mathcal{C}}$, there exists a measure zero subset S_0 of $(\Gamma\cap U)\backslash U\mathfrak{S}$ such that $E_f(s,g)$ is an entire function of $s \in \mathbb{C}$ for $g \in (\Gamma\cap U)\backslash U\mathfrak{S}K - S_0K$.*

Here a measure zero set appears because absolute convergence is only established almost everywhere for the Eisenstein series in general. In the setting of everywhere convergence (as in [CGLLM]), the measure zero set is not needed. See Theorem 4.8 for more precise statements.

The main idea of the proof is to exploit rapid decay of a cusp form on the maximal parabolic subgroup, guaranteed by Property RD. This property may be considered as extraction of the essential properties of maximal parabolic subgroups used in the earlier results on affine Kac–Moody groups [GMP] and on rank 2 hyperbolic groups [CLL].

In Section 5 we show that a large class of Kac–Moody groups have parabolic subgroups satisfying Property RD, including the Kac–Moody group G associated with the Feingold–Frenkel rank 3 hyperbolic Kac–Moody algebra [FF]. We note that the corresponding Kac–Moody group did not satisfy the conditions in [CGLLM] that allowed the authors to prove convergence of Eisenstein series.

It would be an interesting question for future investigation to characterize the full class of Kac–Moody groups that admit parabolic subgroups satisfying Property RD.

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2. Kac–Moody groups and Borel Eisenstein series

In this section we recall the construction of Kac–Moody groups and then state the convergence results for Borel Eisenstein series proven in [CGLLM] and used in this work. We try to keep the exposition self-contained as much as possible, and refer other details to *loc. cit.* and the references therein.

2.1. **Kac-Moody groups.**

Let $I = \{1, 2, \ldots, r\}$, $\mathsf{A} = (a_{ij})_{i,j\in I}$ be an $r \times r$ symmetrizable generalized Cartan matrix, and $(\mathfrak{h}_{\mathbb{C}}, \Delta, \Delta^\vee)$ be a realization of A, where $\Delta = \{\alpha_1, ..., \alpha_r\} \subset \mathfrak{h}^*_{\mathbb{C}}$ and $\Delta^\vee = \{\alpha_1^\vee, ..., \alpha_r^\vee\} \subset \mathfrak{h}_{\mathbb{C}}$ are the set of simple roots and set of simple coroots, respectively.

Following [CGLLM], we assume that A is nonsingular, so that $\mathfrak{h}_{\mathbb{C}}$ and $\mathfrak{h}^*_{\mathbb{C}}$ are spanned by the simple roots α_i and simple coroots $\alpha_i^\vee$, respectively. Recall that $\langle \alpha_j, \alpha_i^\vee \rangle = a_{ij}$ for $i, j \in I$, where $\langle \cdot, \cdot \rangle$ is the natural pairing between $\mathfrak{h}^*_{\mathbb{C}}$ and $\mathfrak{h}_{\mathbb{C}}$. Denote the fundamental weights by $\varpi_i \in \mathfrak{h}^*_{\mathbb{C}}$, $i \in I$, which form the basis of $\mathfrak{h}^*_{\mathbb{C}}$ dual to the $\alpha_i^\vee$.

Let $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}}(\mathsf{A})$ be the Kac–Moody algebra associated to $(\mathfrak{h}, \Delta, \Delta^\vee)$, which we assume to be infinite-dimensional throughout this paper. We denote by Φ the set of roots of $\mathfrak{g}_{\mathbb{C}}$ and have $\Phi = \Phi_+ \sqcup \Phi_-$, where Φ_+ (resp. Φ_-) is the set of positive (resp. negative) roots corresponding to the choice of Δ. Let $w_i := w_{\alpha_i}$ denote the simple Weyl reflection associated to the simple root α_i. The w_i for $i \in I$ generate the Weyl group W of $\mathfrak{g}_{\mathbb{C}}$.

Let $\mathfrak{g}$ be the real Lie subalgebra generated by the Chevalley generators e_i and f_i for $i \in I$, so that $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$. As is standard (see [CG, §4]), we can define the $\mathbb{Z}$-form $\mathcal{U}_{\mathbb{Z}}$ of the universal enveloping algebra $\mathcal{U}_{\mathbb{C}}$ of $\mathfrak{g}$ using the Chevalley generators.

We now review the representation theoretic Kac–Moody groups G_F associated to $\mathfrak{g}$ and a field F of characteristic zero, following [CG]. Let (π, V) denote the unique irreducible highest weight module for $\mathfrak{g}$ corresponding to a choice of some dominant integral weight, and fix a nonzero highest weight vector $v \in V$. Then we have a lattice

$$V_{\mathbb{Z}} \;=\; \mathcal{U}_{\mathbb{Z}} \cdot v,$$

and we put $V_F = F \otimes_{\mathbb{Z}} V_{\mathbb{Z}}$. Since V is integrable, it makes sense to define for $s, t \in F$ and $i \in I$ the operators

$$\begin{aligned} u_{\alpha_i}(s) \;&=\; \exp(\pi(se_i)) \\ \text{and } u_{-\alpha_i}(t) \;&=\; \exp(\pi(tf_i)), \end{aligned}$$

which act locally finitely on V_F and thus define elements of $\mathrm{Aut}(V_F)$.

For $t \in F^\times$ and $i \in I$ we set

$$w_{\alpha_i}(t) \;=\; u_{\alpha_i}(t) u_{-\alpha_i}(-t^{-1}) u_{\alpha_i}(t)$$

and define

$$h_{\alpha_i}(t) \;=\; w_{\alpha_i}(t) w_{\alpha_i}(1)^{-1}.$$

Each simple root α_j defines a character on $\{h_{\alpha_i}(t) \;:\; t \in F^\times\}$ by

$$h_{\alpha_i}(t)^{\alpha_j} \;=\; t^{\langle \alpha_j, \alpha_i^\vee \rangle}. \tag{2.1}$$

The subgroup $\langle w_{\alpha_i}(1) : i \in I\rangle$ of $\mathrm{Aut}(V_F)$ contains a full set of Weyl group representatives. For a real root α given by $\alpha = w_{\beta_1} \cdots w_{\beta_\ell} \alpha_i$ for some $i \in I$ and $\beta_1, \ldots, \beta_\ell \in \Delta$, let $\{u_\alpha(s) :\; s \in F\}$, denote a choice of corresponding one-parameter subgroup such that

$$u_\alpha(s) \quad = \quad w u_{\alpha_i}(\pm s) w^{-1} \in \mathrm{Aut}(V_F) \qquad (s \in F)$$

for $w = w_{\beta_1}(1) \cdots w_{\beta_\ell}(1)$.

Define

$$G_F^0 \;=\; \langle u_{\alpha_i}(s), u_{-\alpha_i}(t) : s, t \in F,\; i \in I \rangle \;\subset\; \mathrm{Aut}(V_F).$$

Following [CG, §5], we choose a coherently ordered basis Ψ of $V_{\mathbb{Z}}$, and denote by B_F^0 the subgroup of G_F^0 consisting of elements which act upper-triangularly with respect to Ψ. The basis Ψ induces certain topologies on B_F^0 and G_F^0, and we let B_F and G_F denote their completions respectively. When the field F is unspecified, B should be interpreted as $B_{\mathbb{R}}$.

The following subgroups of G_F are of particular importance to us:

- $A_F = \langle h_{\alpha_i}(s) : s \in F^\times, i \in I\rangle$; and
- $U_F \subset B_F$ is defined exactly as B_F, but with the additional stipulation that elements act unipotently upper-triangularly with respect to Ψ. It contains all subgroups parameterized by the $u_\alpha(\cdot)$, where $\alpha \in \Phi_+$ is a real root. Then $B_F = U_F A_F = A_F U_F$.

When no subscript is given, A and U should be interpreted as $A_{\mathbb{R}}$ and $U_{\mathbb{R}}$. We also have the following subgroups specific to the situation $F = \mathbb{R}$:

- K is the subgroup of $G_{\mathbb{R}}$ generated by all $\exp(t(e_i - f_i))$, $t \in \mathbb{R}$ and $i \in I$ (see [KP]); and
- $A^+ = \langle h_{\alpha_i}(s) : s \in \mathbb{R}_{>0}, i \in I\rangle$. In fact, $(\mathbb{R}_{>0})^r$ can be identified with A^+ via the isomorphism $(x_1, \ldots, x_r) \mapsto h_{\alpha_1}(x_1) \cdots h_{\alpha_r}(x_r)$, under which A^+ has the Haar measure da corresponding to $\prod_{i=1}^r \frac{dx_i}{x_i}$.

Theorem 2.1 ([DGH]). *We have the Iwasawa decomposition*

$$G_{\mathbb{R}} \;=\; UA^+K \tag{2.2}$$

with uniqueness of expression.

Let $u(g)$, $a(g)$, and $k(g)$ denote the projections from $G_{\mathbb{R}}$ onto each of the respective factors in (2.2). We define the discrete group $\Gamma = G_{\mathbb{Z}}$ as $G_{\mathbb{R}} \cap \mathrm{Aut}(V_{\mathbb{Z}}) = \{\gamma \in G_{\mathbb{R}} : \gamma \cdot V_{\mathbb{Z}} = V_{\mathbb{Z}}\}$. As in [G04], it can be shown that $(\Gamma \cap U)\backslash U$ is the projective limit of a projective family of finite-dimensional compact nil-manifolds and thus admits a projective limit measure du which is a right U-invariant probability measure.

2.2. **Borel Eisenstein series.** Using the identification of A^+ with $(\mathbb{R}_{>0})^r$, each element of $\mathfrak{h}^*_{\mathbb{C}}$ gives rise to a quasicharacter of A^+. Let $\lambda \in \mathfrak{h}^*_{\mathbb{C}}$ and let $\rho \in \mathfrak{h}^*_{\mathbb{C}}$ be the Weyl vector, which is characterized by $\langle \rho, \alpha_i^\vee \rangle = 1$, $i \in I$. We set

$$\begin{aligned} \Phi_\lambda : G_{\mathbb{R}} &\to \mathbb{C}^\times, \\ g &\mapsto a(g)^{\lambda+\rho}, \end{aligned} \tag{2.3}$$

which is well-defined by the uniqueness of the Iwasawa decomposition.

Let B and Γ be as defined in Section 2.1. Define the Borel Eisenstein series on $G_{\mathbb{R}}$ to be the infinite formal sum

$$E_\lambda(g) \quad := \sum_{\gamma \in (\Gamma \cap B)\backslash \Gamma} \Phi_\lambda(\gamma g).$$

We define for all $g \in G_{\mathbb{R}}$ the so-called "upper triangular" constant term[1]

$$E^\sharp_\lambda(g) \;:=\; \int_{(\Gamma \cap U)\backslash U} E_\lambda(ug)du,$$

which is left U-invariant and right K-invariant. It is immediate from the definition that $E^\sharp_\lambda(g)$ is determined by the A^+-component $a(g)$ of g in the Iwasawa decomposition.

We now state the convergence results for the constant term $E^\sharp_\lambda(g)$ as well as the Eisenstein series $E_\lambda(g)$ itself obtained in [CGLLM]. We will use these results later in this paper.

Let $\mathcal{C} \subset \mathfrak{h}$ be the open fundamental chamber

$$\mathcal{C} \;=\; \{x \in \mathfrak{h} : \langle \alpha_i, x \rangle > 0,\; i \in I\}.$$

Let $\mathfrak{C}$ denote the interior of the Tits cone $\bigcup_{w \in W} w\overline{\mathcal{C}}$ corresponding to $\mathcal{C}$ [K, §3.12]. Using the exponential map $\exp : \mathfrak{h} \to A^+$, set $A_{\mathcal{C}} = \exp \mathcal{C}$ and $A_{\mathfrak{C}} = \exp \mathfrak{C}$. Let

$$\mathcal{C}^* \;=\; \{\lambda \in \mathfrak{h}^* : \langle \lambda, \alpha_i^\vee \rangle > 0,\; i \in I\}. \tag{2.4}$$

Based on a crucial lemma due to Looijenga [Lo] together with some elementary estimate on the Riemann ζ-function, the following theorem is proven in [CGLLM].

[1] This is to be distinguished from the proposal in [BK] to consider constant terms in uncompleted, or so-called "lower triangular", parabolics.

Theorem 2.2. [CGLLM] *If $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$ satisfies $\mathrm{Re}(\lambda-\rho) \in \mathcal{C}^*$, then $E_\lambda^\sharp(g)$ converges absolutely for $g \in UA_{\mathfrak{C}}K$, and in fact uniformly for $a(g)$ lying in any fixed compact subset of $A_{\mathfrak{C}}$.*

Consequently, a simple application of Tonelli's theorem as in [G04, §9] yields the following result.

Corollary 2.3. [CGLLM] *For $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$ with $\mathrm{Re}(\lambda-\rho) \in \mathcal{C}^*$ and any compact subset $\mathfrak{S}$ of $A_{\mathfrak{C}}$, there exists a measure zero subset S_0 of $(\Gamma \cap U)\backslash U\mathfrak{S}$ such that the series $E_\lambda(g)$ converges absolutely for $g \in (\Gamma \cap U)\backslash U\mathfrak{S}K - S_0K$.*

Moreover, the everywhere convergence of $E_\lambda(g)$ over $\Gamma UA_{\mathfrak{C}}K$ was established under the following combinatorial property, by exploring the idea that the Borel Eisenstein series can be nearly bounded by its constant term. We set

$$\Phi_w \;:=\; \Phi_+ \cap w^{-1}\Phi_-, \quad w \in W. \tag{2.5}$$

The following property was introduced in [CGLLM] and will be used in Theorem 4.8.

Property 2.4. *Every nontrivial $w \in W$ can be written as $w = vw_\beta$, where β is a positive simple root, $\ell(v) < \ell(w)$, and $\alpha - \beta$ is never a real root for any $\alpha \in \Phi_v$.*

In [CGLLM] it is shown that Property 2.4 holds when the Cartan matrix $\mathsf{A} = (a_{ij})$ is symmetric with $|a_{ij}| \geq 2$ for all i, j; for rank 2 hyperbolic groups, the condition $|a_{ij}| \geq 2$ is sufficient and the matrix A does not need to be symmetric. The main result of *loc. cit.* is the following.

Theorem 2.5. [CGLLM] *Assume that $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$ satisfies $\mathrm{Re}(\lambda-\rho) \in \mathcal{C}^*$, and that Property 2.4 holds. Then the Kac–Moody Eisenstein series $E_\lambda(g)$ converges absolutely for $g \in \Gamma UA_{\mathfrak{C}}K$.*

3. Cuspidal Kac–Moody Eisenstein series

3.1. **Definition of Eisenstein series.** For any subset $\theta \subset I$, denote by W_θ the subgroup of W generated by the reflections w_i for $i \in \theta$. Assume that we choose θ such that W_θ is a finite group. Define the parabolic subgroup of $G_{\mathbb{R}}$ associated to θ by

$$P_\theta \;:=\; BW_\theta B\,,$$

where each $w \in W_\theta$ is identified with one of its representatives in $G_{\mathbb{R}}$ as in Section 2.1. For the rest of this section, we will fix θ and write $P = P_\theta$.

Let N be the pro-unipotent radical of P (see [Ku]). Let L be the subgroup of P generated by $u_{\pm\alpha_i}(t)$, $i \in \theta$, $t \in \mathbb{R}$. Then L is a finite dimensional semisimple group. Let $A_1 := A \cap L$ be the subgroup of A generated by $h_{\alpha_i}(t)$, $t \in \mathbb{R}^\times$, $i \in \theta$; it is a maximal split torus of L. Set

$$H := \{a \in A : a^{\alpha_i} = \pm 1, i \in \theta\} \quad \text{ and } \quad M := LH,$$

so that $A = A_1H$, and $P = MN = NM$ is a Levi decomposition.

It will be convenient to rewrite $a(g)$ as

$$\mathrm{Iw}_{A^+} : G_{\mathbb{R}} \to A^+, \quad g \mapsto a(g)$$

from the Iwasawa decomposition $G_{\mathbb{R}} = UA^+K$ in (2.2). Let H^+ be the identity component of H, so that $H^+ \cong H/(H \cap K)$. We also introduce

$$A_1^+ \;=\; \langle h_{\alpha_i}(t) : t \in \mathbb{R}_{>0}, i \in \theta \rangle \;\cong\; A_1/(A_1 \cap K)\,.$$

Then

$$A^+ \;\cong\; A_1^+ \times H^+.$$

The parabolic subgroup P gives rise to the decomposition $G_{\mathbb{R}} = NMK$, which unlike the Iwasawa decomposition (2.2) is not unique since $M \cap K$ is nontrivial. However the projection

$$\mathrm{Iw}_M : G_{\mathbb{R}} \to M/(M \cap K)$$

is well-defined. Although $M = LH$ is not a direct product, the fact $L \cap H \subset K$ implies that the maps

$$\mathrm{Iw}_L : G_{\mathbb{R}} \to L/(L \cap K) \quad \text{ and } \quad \mathrm{Iw}_{H^+} : G_{\mathbb{R}} \to H^+,$$

which are induced by the projections onto the factors L and H of M, respectively, are also well-defined.

Similarly, using the Iwasawa decomposition for L, we may define the map

$$\mathrm{Iw}_{A_1^+} : L/(L \cap K) \to A_1^+.$$

Then it is clear that

$$\mathrm{Iw}_{A^+} = (\mathrm{Iw}_{A_1^+} \circ \mathrm{Iw}_L) \times \mathrm{Iw}_{H^+} : G_{\mathbb{R}} \longrightarrow A^+ \cong A_1^+ \times H^+. \tag{3.1}$$

Consider $\lambda_P = \sum_{i \notin \theta} s_i \varpi_i \in \mathfrak{h}_{\mathbb{C}}^*$, $s_i \in \mathbb{C}$, where ϖ_i are the fundamental weights. As before, let $\rho \in \mathfrak{h}_{\mathbb{C}}^*$ be the Weyl vector, i.e., $\langle \rho, \alpha_i^\vee \rangle = 1$, $i \in I$. Likewise, let ρ_M be half the sum of the positive roots of M, and set $\rho_P := \rho - \rho_M$. Since λ_P is clearly trivial on $A_1 = A \cap L$, the decomposition (3.1) implies that

$$\mathrm{Iw}_{A^+}(\cdot)^{\lambda_P} = \mathrm{Iw}_{H^+}(\cdot)^{\lambda_P}. \tag{3.2}$$

Similarly, every root β of L is trivial on H^+, hence

$$\mathrm{Iw}_{A^+}(\cdot)^{\beta} = \Big(\mathrm{Iw}_{A_1^+} \circ \mathrm{Iw}_L(\cdot)\Big)^{\beta}. \tag{3.3}$$

For any $\lambda_P = \sum_{i \notin \theta} s_i \varpi_i \in \mathfrak{h}_{\mathbb{C}}^*$, define the auxiliary Eisenstein series

$$E(\lambda_P, g) := \sum_{\gamma \in (\Gamma \cap P)\backslash \Gamma} \mathrm{Iw}_{H^+}(\gamma g)^{\lambda_P + \rho_P}, \quad g \in G_{\mathbb{R}}. \tag{3.4}$$

More generally, for a bounded function f defined on $M/M \cap K$, define the Eisenstein series $E_f(\lambda_P, g)$ by

$$E_f(\lambda_P, g) := \sum_{\gamma \in (\Gamma \cap P)\backslash \Gamma} \mathrm{Iw}_{H^+}(\gamma g)^{\lambda_P + \rho_P}\, f\left(\mathrm{Iw}_M(\gamma g)\right), \quad g \in G_{\mathbb{R}}. \tag{3.5}$$

Then (3.4) corresponds to the special case that $f \equiv 1$.

3.2. **Convergence by the reduction mechanism.** The idea of a reduction mechanism is to bound general Eisenstein series by Borel ones. It was first suggested by Bernstein to Borel, and used by Garland in [G11] for the affine case, who was inspired by some results in [GMRV]. Following this idea, we prove the absolute convergence of parabolic Eisenstein series $E_f(\lambda_P, g)$ in the Godement range. Similar proofs for the finite-dimensional case can be found in [Bo, Proposition 12.6] and [MW, Proposition II.1.5], which also use Borel Eisenstein series to bound parabolic ones, but the arguments are slightly different.

Proposition 3.1. *Let f be a bounded function on $M/M\cap K$. For $\lambda_P=\sum_{i\notin\theta} s_i\varpi_i\in\mathfrak{h}^*_{\mathbb{C}}$, with $\mathrm{Re}(s_i)>\langle\rho_P,\alpha_i^\vee\rangle$, $i\notin\theta$, and any compact subset $\mathfrak{S}$ of $A_{\mathfrak{C}}$, there exists a measure zero subset S_0 of $(\Gamma\cap U)\backslash U\mathfrak{S}$ such that the series $E_f(\lambda_P,g)$ converges absolutely for $g\in(\Gamma\cap U)\backslash U\mathfrak{S}K-S_0K$. Moreover, S_0 can be chosen to work uniformly for all s_i, $i\notin\theta$, in right half-spaces of the form $\mathrm{Re}(s_i)>\sigma_i$, where $\sigma_i>\langle\rho_P,\alpha_i^\vee\rangle$ are fixed.*

Proof. Since the function f is assumed to be bounded, it suffices to prove the almost everywhere absolute convergence of the auxiliary Eisenstein series $E(\lambda_P,g)$, and we may assume that $s_i>\langle\rho_P,\alpha_i^\vee\rangle$ are real numbers for $i\notin\theta$. Let $\varepsilon>0$ be sufficiently small such that

$$s_i>(1+\varepsilon)\langle\rho_P,\alpha_i^\vee\rangle$$

for $i\notin\theta$. We put

$$\lambda=(1+\varepsilon)\rho_M+\lambda_P.$$

Then $\lambda-\rho\in\mathcal{C}^*$, where $\mathcal{C}^*$ is defined in (2.4). Indeed, $\langle\lambda,\alpha_i^\vee\rangle=1+\varepsilon$ for $i\in\theta$, and

$$\langle\lambda,\alpha_i^\vee\rangle=(1+\varepsilon)\langle\rho_M,\alpha_i^\vee\rangle+s_i>(1+\varepsilon)\langle\rho_M+\rho_P,\alpha_i^\vee\rangle=1+\varepsilon$$

for $i\notin\theta$.

Consider the Borel Eisenstein series

$$E_\lambda(g)=\sum_{\gamma\in(\Gamma\cap B)\backslash\Gamma}\Phi_\lambda(\gamma g)=\sum_{\gamma\in(\Gamma\cap P)\backslash\Gamma}\ \sum_{\delta\in(\Gamma\cap B)\backslash\Gamma\cap P}\Phi_\lambda(\delta\gamma g).$$

Using the Iwasawa decomposition $G_{\mathbb{R}}=NMK$, the inner sum is written as

$$\sum_{\delta\in(\Gamma_M\cap B_M)\backslash\Gamma_M}\Phi_\lambda(\delta\cdot\mathrm{Iw}_M(\gamma g)),$$

where $\Gamma_M:=\Gamma\cap M$ and $B_M:=B\cap M$. By our choice of λ, it can be further manipulated to give

$$\begin{aligned}&\mathrm{Iw}_{H^+}(\gamma g)^{\lambda_P+\rho_P}\sum_{\delta\in(\Gamma_M\cap B_M)\backslash\Gamma_M}\Phi_\lambda(\delta\cdot\mathrm{Iw}_L(\gamma g))\\ =&\mathrm{Iw}_{H^+}(\gamma g)^{\lambda_P+\rho_P}\sum_{\delta\in(\Gamma_M\cap B_M)\backslash\Gamma_M}\mathrm{Iw}_{A_1^+}(\delta\cdot\mathrm{Iw}_L(\gamma g))^{\lambda+\rho}.\end{aligned}$$

Since λ and ρ restrict to $(1+\varepsilon)\rho_M$ and ρ_M on A_1^+ respectively, the last sum is a Borel Eisenstein series on M with spectral parameter $(1+\varepsilon)\rho_M$, which is in the Godement range. Since the spectral parameter is real, the series is a sum of positive terms, and hence is never zero. Moreover, it has a positive lower bound. This fact should be well-known, but it does not seem to be widely available in the literature. We supply a proof below for completeness.

Recall that a Siegel domain for M is of the form

$$\mathfrak{S}_{t,\omega} := \omega A_t K_M,$$

where ω is a compact subset of B_M, $K_M = K \cap M$ and

$$A_t := \{a \in A^+ : a^{\alpha_i} > t, i \in \theta\}$$

for $t > 0$. By classical reduction theory (see e.g. [MW, I.2.1]), we have $M = \Gamma_M \cdot \mathfrak{S}_{t,\omega}$ for ω sufficiently large and t sufficiently small. Write $m \in M$ accordingly as

$$m = \gamma x a k,$$

where $\gamma \in \Gamma_M$, $x \in \omega$, $a \in A_t$ and $k \in K_M$. The Borel Eisenstein series $E_{(1+\varepsilon)\rho_M}(\cdot)$ on M takes value

$$\begin{aligned} E_{(1+\varepsilon)\rho_M}(m) &= E_{(1+\varepsilon)\rho_M}(xa) \geq \Phi_{(1+\varepsilon)\rho_M}(xa) = a^{(2+\varepsilon)\rho_M}\Phi_{(1+\varepsilon)\rho_M}(x) \\ &\geq \kappa := t^{(2+\varepsilon)C_M} \cdot h_\omega > 0, \end{aligned}$$

where C_M is the sum of the coefficients of α_i in ρ_M and h_ω is the minimum value of the positive continuous function $\Phi_{(1+\varepsilon)\rho_M}(\cdot)$ on the compact set ω.

With this lower bound established, it follows that

$$E(\lambda_P, g) = \sum_{\gamma \in (\Gamma \cap P)\backslash\Gamma} \mathrm{Iw}_{H^+}(\gamma g)^{\lambda_P + \rho_P} \leq \kappa^{-1} E_\lambda(g).$$

The proposition follows from this comparison and Corollary 2.3. $\square$

4. Holomorphy of cuspidal Eisenstein series

In this section, we prove our main result. We keep the assumption that $\mathfrak{g}$ is infinite-dimensional with nonsingular Cartan matrix A. Let $P = P_\theta$ be a maximal parabolic subgroup of $G_\mathbb{R}$ with fixed Levi decomposition and associated finite-dimensional split reductive Levi subgroup M as in Section 3. We set $\{i_P\} := I \setminus \theta$ and recall that $\alpha_P = \alpha_{i_P}$, and ϖ_P is the fundamental weight for α_P. Let W^θ be the set of minimal length representatives of $W_\theta \backslash W$, i.e.

$$W^\theta := \{w \in W : w^{-1}\alpha_i > 0, i \in \theta\}.$$

Definition 4.1. A maximal parabolic subgroup $P = P_\theta$ whose Levi subgroup M is a finite-dimensional split reductive group, is said to satisfy *Property RD* if there exists a constant $D > 0$, such that for every nontrivial element $w \in W^\theta$ we have $\langle D\varpi_P + \rho_M, \alpha^\vee \rangle \leq 0$ for any $\alpha \in \Phi_{w^{-1}}$.

Remark 4.2. Note that $\langle \varpi_P, \alpha^\vee \rangle > 0$ for any $\alpha \in \Phi_{w^{-1}}$ with $w \in W^\theta$. Hence if there exists D that satisfies the condition in Definition 4.1, then any positive constant smaller than D will also satisfy the condition.

The fact $L \cap H \subset K$ allows us to view a function f on $L/L \cap K$ as defined on $M/M \cap K$. In particular an unramified cusp form f on L can be regarded as a bounded function on $M/M \cap K$, hence one may define the Eisenstein series $E_f(\lambda_P, g)$ as in Section 3. In the sequel we shall adopt this interpretation without further remark. Moreover, since $\lambda_P = s\varpi_P$ for some $s \in \mathbb{C}$, we write $E_f(s, g) = E_f(\lambda_P, g)$ in the rest of the paper.

We state the main theorem again.

Theorem 4.3 (Theorem 1.1)**.** *Let f be an unramified cusp form on $L \cap \Gamma \backslash L$. If the maximal parabolic subgroup P satisfies Property RD, then for any compact subset $\mathfrak{S}$ of $A_{\mathcal{C}}$, there exists a measure zero subset S_0 of $(\Gamma \cap U)\backslash U\mathfrak{S}$ such that $E_f(s,g)$ is an entire function of $s \in \mathbb{C}$ for $g \in (\Gamma \cap U)\backslash U\mathfrak{S}K - S_0K$.*

The rest of this section is devoted to the proof of Theorem 4.3. We shall follow the strategy of [GMP] and [CLL] with Property RD at hand. We prove an inequality in Lemma 4.7 which is crucial in our proof of the main theorem. This inequality enables us to deduce the entirety of Eisenstein series from the rapid decay of cusp forms. Along the way, we note some interesting properties of the infinite-dimensional Kac-Moody algebras that are considered here.

The following lemma is essentially the same as [GMP, Lemma 3.2]. Write $x^y = yxy^{-1}$ for $x, y \in G_{\mathbb{R}}$, and set $U_w = \prod_{\alpha \in \Phi_w} U_\alpha$ for $w \in W$ where U_α is the one-parameter subgroup corresponding to α.

Lemma 4.4. *Let $w \in W$. If $\gamma \in \Gamma \cap BwB$, then*

$$\mathrm{Iw}_{A^+}(\gamma g) = \mathrm{Iw}_{A^+}(g)^w \cdot \mathrm{Iw}_{A^+}(wu_w)$$

for some $u_w \in U_w$ depending on γ and g.

We also recall that one has, for $u_w \in U_w$,

$$\log\left(\mathrm{Iw}_{A^+}(wu_w)\right) = \sum_{\alpha \in \Phi_{w^{-1}}} c_\alpha \alpha^\vee \quad \text{with} \quad c_\alpha \le 0 \tag{4.1}$$

by [GMS2, Lemma 6.1] or [GMP, (3.25)].

To utilize Property RD, we need the property of the coefficient of α_P in the fundamental weight ϖ_P, given by the following lemma.

Lemma 4.5. *The coefficient n_{i_P} of α_P in the expansion $\varpi_P = \sum_{i \in I} n_i \alpha_i$ is negative.*

Proof. By relabeling the simple roots, we may assume that $\alpha_P = \alpha_r$ and the upper-left $(r-1)\times(r-1)$ corner of A is the Cartan matrix of the root system Φ_M of M. Let $\mathsf{D} = \mathrm{diag}(d_1, \dots, d_r)$ be a symmetrizer of A and write

$$\mathsf{DA} = \begin{pmatrix} U & \beta \\ \beta^T & c \end{pmatrix}, \quad (\mathsf{DA})^{-1} = \begin{pmatrix} V & \gamma \\ \gamma^T & d \end{pmatrix},$$

where U, V are $(r-1)\times(r-1)$ matrices and c, d are scalars. Then $U = U^T$ is positive definite because Φ_M is of finite type. If $\gamma^T = (\gamma_1, \dots, \gamma_{r-1})$, then

$$\varpi_r = \sum_{i=1}^{r-1} d_r \gamma_i \alpha_i + d_r d \alpha_r.$$

Hence we only need to show that $d < 0$.

The matrix DA has the same signature as

$$\begin{pmatrix} U & 0 \\ 0 & c - \beta^T U^{-1} \beta \end{pmatrix}.$$

Since we assume that $\mathfrak{g}$ is of infinite type and A is nonsingular, we must have $c - \beta^T U^{-1}\beta < 0$. We have the relations

$$U\gamma + \beta d = 0, \quad \beta^T \gamma + cd = 1.$$

It follows that

$$\beta^T\gamma + \beta^T U^{-1}\beta d = \beta^T U^{-1}(U\gamma + \beta d) = 0,$$

which implies that $(c - \beta^T U^{-1}\beta)d = 1$. Hence $d < 0$ as desired. □

As a corollary, we obtain the following result.

Lemma 4.6. *For $D > 0$ sufficiently small, $w^{-1}(D\varpi_P + \rho_M)$ is a positive linear combination of simple roots for any nontrivial $w \in W^\theta$.*

Proof. Recall that ρ_M is the half sum of the positive roots of M. Write $\varpi_P = \sum_{i\in I} n_i\alpha_i$ as in Lemma 4.5. Then we have

$$D\varpi_P + \rho_M = \sum_{i\in\theta} m_i\alpha_i + m_{i_P}\alpha_P,$$

where

$$\sum_{i\in\theta} m_i\alpha_i = \rho_M + D\sum_{i\in\theta} n_i\alpha_i,$$

and $m_{i_P} = Dn_{i_P} < 0$ by Lemma 4.5. Clearly for $D > 0$ sufficiently small we have $m_i > 0$ for $i \in \theta$. Consider a nontrivial element $w \in W^\theta$. Since $w^{-1}\alpha_i > 0$ for $i \in \theta$, we must have $w^{-1}\alpha_P < 0$. The lemma follows immediately. □

With the above results, we can now state and prove the following inequality.

Lemma 4.7. *If P satisfies Property RD, then for $D > 0$ sufficiently small,*

$$\mathrm{Iw}_{A^+}(\gamma g)^{-\rho_M} \le \mathrm{Iw}_{A^+}(\gamma g)^{D\varpi_P}$$

for any $g \in UA_{\mathcal{C}}K$, $w \in W^\theta$ and $\gamma \in \Gamma \cap BwB$.

Proof. From Definition 4.1, there exists $D > 0$ such that

$$\langle D\varpi_P + \rho_M, \alpha^\vee\rangle \le 0 \quad \text{for } \alpha \in \Phi_{w^{-1}}. \tag{4.2}$$

By Remark 4.2, we may shrink the constant D, if necessary, and assume that the property in Lemma 4.6 holds with this constant D. By Lemma 4.4 and (4.1), in order to prove Lemma 4.7, it suffices to establish its logarithmic form

$$\langle w^{-1}(D\varpi_P + \rho_M), \log(\mathrm{Iw}_{A^+}(g))\rangle + \sum_{\alpha\in\Phi_{w^{-1}}} c_\alpha \langle D\varpi_P + \rho_M, \alpha^\vee\rangle \ge 0, \tag{4.3}$$

with $c_\alpha \le 0$, $\alpha \in \Phi_{w^{-1}}$. The first term is positive by Lemma 4.6 and the assumption that $\log(\mathrm{Iw}_{A^+}(g)) \in \mathcal{C}$. The summation in (4.3) is nonnegative by Property RD (4.2). Hence (4.3) holds. □

Proof of Theorem 4.3. Fix any real number $s_0 > \langle \rho_P, \alpha_P^\vee \rangle$, and assume that $\mathrm{Re}(s) < s_0$. By the rapid decay of cuspidal automorphic forms (e.g. [MiSch]), for any $n > 0$ there exists a constant $C_1 > 0$ depending on n such that

$$|f(g)| \leq C_1 \, \mathrm{Iw}_{A_1^+}(g)^{-n\rho_M}. \tag{4.4}$$

We take

$$n := \frac{s_0 - \mathrm{Re}(s)}{D} > 0,$$

with D the constant in Lemma 4.7.

Recall that we have the Bruhat decomposition ([CG])

$$G_{\mathbb{R}} = \bigcup_{w \in W^\theta} PwB.$$

We claim that for any $\gamma \in (\Gamma \cap P)\backslash(\Gamma \cap PwB)$ with $w \in W^\theta$, we can choose a representative of γ which lies in $\Gamma \cap BwB$. Recall the Levi decomposition $P = MN$. By the Iwasawa decomposition over $\mathbb{Q}$, we have a representative of γ of the form

$$\tilde{\gamma} = mnwb \in \Gamma \cap PwB,$$

where $m \in M_{\mathbb{Q}}$, $n \in N_{\mathbb{Q}}$ and $b \in B_{\mathbb{Q}}$. Since M is finite dimensional, we have $M_{\mathbb{Q}} = \Gamma_M B_{M,\mathbb{Q}}$ (see [Go]), where $\Gamma_M := \Gamma \cap M$ and $B_{M,\mathbb{Q}} := B_{\mathbb{Q}} \cap M$. Write $m = \gamma_M \, b_M$ accordingly, where $\gamma_M \in \Gamma_M$ and $b_M \in B_{M,\mathbb{Q}}$. Then

$$\gamma_M^{-1} \tilde{\gamma} = b_M nwb$$

is a representative of γ, which clearly lies in $\Gamma \cap BwB$.

With the claim established, it follows from (3.2), (3.3), Lemma 4.7 and (4.4) that

$$\begin{aligned}
\left| \mathrm{Iw}_{H^+}(\gamma g)^{s\varpi_P} f\big(\mathrm{Iw}_M(\gamma g)\big) \right| &\leq C_1 \, \mathrm{Iw}_{H^+}(\gamma g)^{(\mathrm{Re}\, s)\varpi_P} \mathrm{Iw}_{A_1^+} \circ \mathrm{Iw}_L(\gamma g)^{-n\rho_M} \\
&\leq C_1 \, \mathrm{Iw}_{H^+}(\gamma g)^{(\mathrm{Re}\, s)\varpi_P} \mathrm{Iw}_{H^+}(\gamma g)^{nD\varpi_P} \\
&= C_1 \, \mathrm{Iw}_{H^+}(\gamma g)^{s_0 \varpi_P}.
\end{aligned}$$

Note that the constants C_1 and D are independent of w. Multiplying by $\mathrm{Iw}_{H^+}(\gamma g)^{\rho_P}$ and taking the summation over γ, it follows that $E_f(s,g)$ is bounded by $C_1 E(s_0, g)$. Hence by Proposition 3.1 the series $E_f(s,g)$ is absolutely convergent on $U\mathfrak{S}K$ off $S_0 K$, with S_0 a measure zero subset of $(\Gamma \cap U)\backslash U\mathfrak{S}$. Note that this subset S_0 depends on s_0. To conclude, we may take a sequence of real numbers $s_0^{(i)} > \langle \rho_P, \alpha_P^\vee \rangle$ which goes to infinity, and take the countable union of the corresponding measure zero subsets $S_0^{(i)}$. $\square$

Assuming Property 2.4, we have everywhere convergence of cuspidal Eisenstein series over the full Tits cone by the reduction mechanism. Thus in this case Proposition 3.1 and Theorem 4.3 can be strengthened. We summarize this as the following result.

Theorem 4.8. *Assume Property 2.4, and let $P = MN$ be a maximal parabolic subgroup of $G_{\mathbb{R}}$ with finite-dimensional Levi subgroup M. Then the following hold.*

(i) For a bounded function f on $M \cap \Gamma \backslash M / M \cap K$, the Eisenstein series $E_f(s,g)$ converges absolutely for $g \in \Gamma U A_{\mathfrak{C}} K$ and $\mathrm{Re}(s) > 1$.

(ii) If P satisfies Property RD and f is an unramified cusp form on $L\cap\Gamma\backslash L$, then the Eisenstein series $E_f(s,g)$ is an entire function of $s\in\mathbb{C}$ for $g\in\Gamma UA_{\mathcal{C}}K$.

5. Parabolic subgroups satisfying Property RD

In this section we investigate a sufficient condition which implies that every maximal parabolic subgroup of $G_{\mathbb{R}}$ with finite-dimensional Levi subgroup satisfies Property RD, and then discuss some examples, including the rank 3 Feingold–Frenkel hyperbolic algebra $\mathcal{F}$.

We keep the notations of the previous section. In particular, $P=P_\theta$ stands for a maximal parabolic subgroup of $G_{\mathbb{R}}$, whose Levi subgroup M is a finite-dimensional split reductive group.

Proposition 5.1. *For any $w\in W$ and $\alpha\in\Phi_w$, suppose that we have $\langle\alpha_i,\alpha^\vee\rangle\le 0$ for any α_i such that $w\alpha_i>0$. Then every maximal parabolic subgroup P with finite dimensional Levi subgroup satisfies Property RD.*

Proof. As in the proof of Lemma 4.6, for small $D>0$ we have

$$D\varpi_P+\rho_M=\sum_{i\in\theta}m_i\alpha_i+m_{i_P}\alpha_P$$

where $m_i>0$ for $i\in\theta$ and $m_{i_P}<0$. Take $w\in W^\theta$, $w\neq\mathrm{id}$, and $\alpha\in\Phi_{w^{-1}}$. Then by assumption

$$\langle\alpha_i,\alpha^\vee\rangle\le 0\quad\text{for}\quad i\in\theta. \tag{5.1}$$

Since $w_{\alpha_P}\alpha\in\Phi_{w^{-1}w_{\alpha_P}}$ and $w^{-1}w_{\alpha_P}\alpha_P=-w^{-1}\alpha_P>0$, we have by assumption

$$\langle\alpha_P,\alpha^\vee\rangle=-\langle\alpha_P,w_{\alpha_P}\alpha^\vee\rangle\ge 0. \tag{5.2}$$

From (5.1) and (5.2) it follows that for any $\alpha\in\Phi_{w^{-1}}$ one has

$$\langle D\varpi_P+\rho_M,\alpha^\vee\rangle=\sum_{i\in I}m_i\langle\alpha_i,\alpha^\vee\rangle\le 0.$$

This proves that P satisfies Property RD. ☐

Now we list some families of groups which have parabolic subgroups satisfying Property RD. Recall that $\mathsf{A}=(a_{ij})_{i,j\in I}$ is the generalized Cartan matrix of $\mathfrak{g}$.

Proposition 5.2.

(1) *If $\mathfrak{g}$ is a rank 2 hyperbolic Kac–Moody algebra with generalized Cartan matrix $\mathsf{A}=\begin{pmatrix}2&-b\\-a&2\end{pmatrix}$ where $a,b\ge 2$ and $ab\ge 5$ then every maximal parabolic subgroup of $G_{\mathbb{R}}$ satisfies Property RD.*

(2) *If $\mathfrak{g}$ is associated with symmetric $\mathsf{A}=(a_{ij})$ such that $|a_{ij}|\ge 2$ for all $i,j\in I$, then every maximal parabolic subgroup of $G_{\mathbb{R}}$ with finite dimensional Levi satisfies Property RD.*

Proof. For part (1), first recall that W is the infinite dihedral group generated by w_1 and w_2. Without loss of generality, assume that $w=(w_2w_1)^m$ or $w_1(w_2w_1)^m$ for some $m\ge 0$. Then each

$\alpha \in \Phi_w$ has the form either $(w_1w_2)^n\alpha_1$ or $(w_1w_2)^n w_1\alpha_2$ for some $n \geq 0$. As in the proof of [CGLLM, Proposition 4.4], define

$$\mu = \frac{\sqrt{ab} + \sqrt{ab-4}}{2} \quad \text{and} \quad h_n = \frac{1}{\mu - \mu^{-1}}(\mu^n - \mu^{-n}).$$

Then we get

$$(w_1w_2)^n\alpha_1 = h_{2n+1}\alpha_1 + \tfrac{\sqrt{a}}{\sqrt{b}}h_{2n}\alpha_2, \tag{5.3}$$

$$(w_1w_2)^n w_1\alpha_2 = \tfrac{\sqrt{a}}{\sqrt{b}}h_{2n+2}\alpha_1 + h_{2n+2}\alpha_2. \tag{5.4}$$

Since $w\alpha_2 > 0$ and $w\alpha_1 < 0$, we need only to show $\langle \alpha_2, \alpha^\vee \rangle < 0$ for $\alpha \in \Phi_w$ in order to apply Proposition 5.1. Because $\langle \alpha_2, \alpha^\vee \rangle$ and $\langle \alpha, \alpha_2^\vee \rangle$ have the same sign, we consider the latter for convenience. When $\alpha = (w_1w_2)^n\alpha_1$, we obtain from (5.3) and [CGLLM, (4.10)]

$$\begin{aligned} \langle \alpha, \alpha_2^\vee \rangle &= \langle h_{2n+1}\alpha_1 + \tfrac{\sqrt{a}}{\sqrt{b}}h_{2n}\alpha_2, \alpha_2^\vee \rangle \\ &= -\tfrac{\sqrt{a}}{\sqrt{b}}(\sqrt{ab}h_{2n+1} - 2h_{2n}) \\ &< 0. \end{aligned}$$

Similarly, when $\alpha = (w_1w_2)^n w_1\alpha_2$, we obtain from (5.4) and [CGLLM, (4.10)]

$$\begin{aligned} \langle \alpha, \alpha_2^\vee \rangle &= \langle \tfrac{\sqrt{a}}{\sqrt{b}}h_{2n+2}\alpha_1 + h_{2n+1}\alpha_2, \alpha_2^\vee \rangle \\ &= -\sqrt{ab}h_{2n+2} + 2h_{2n+1} \\ &< 0. \end{aligned}$$

Now it follows from Proposition 5.1 that parabolic subgroups satisfy Property RD.

For part (2), let $v \in W$ be nontrivial. If $\alpha \in \Phi_v$ and $v\alpha_i > 0$ for some $\alpha_i \in \Delta$, then the reduced word of v cannot have w_i as the rightmost letter and vw_i is a reduced word since W has no braid relations in the case $|a_{ij}| \geq 2$. It was shown in [CGLLM, (4.5)] that $\langle \alpha, \alpha_i^\vee \rangle < 0$ for any $\alpha \in \Phi_v$. Since $\langle \alpha, \alpha_i^\vee \rangle$ and $\langle \alpha_i, \alpha^\vee \rangle$ have the same sign, we have $\langle \alpha_i, \alpha^\vee \rangle < 0$. Now that v is an arbitrary element of W, it follows from Proposition 5.1 that every maximal parabolic subgroup satisfies Property RD.

□

In the rest of this section, we consider the rank 3 hyperbolic Kac–Moody algebra $\mathcal{F}$ studied by Feingold and Frenkel [FF], whose Cartan matrix is

$$\mathsf{A} = (a_{ij}) = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

We will see below that the condition of Proposition 5.1 fails for the algebra $\mathcal{F}$, but there are only finitely many exceptions and we can still show that every maximal parabolic subgroup with finite-dimensional Levi satisfies Property RD.

Since A is symmetric, we may identify $\langle\cdot,\cdot\rangle$ with the unique bilinear form $(\cdot|\cdot)$ on $\mathfrak{h}^*$ satisfying $(\alpha_i|\alpha_j) = a_{ij}$, $1 \le i,j \le 3$. Define the linear isomorphism of vector spaces $\psi : \mathfrak{h}^* \xrightarrow{\cong} S_2(\mathbb{C})$, the vector space of 2×2 symmetric complex matrices, by

$$\psi(x\alpha_1 + y\alpha_2 + z\alpha_3) = \begin{pmatrix} y-z & y-x \\ y-x & z \end{pmatrix}, \tag{5.5}$$

so that

$$\psi(\alpha_1) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad \psi(\alpha_2) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad \psi(\alpha_3) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$(\lambda|\lambda) \ = \ -2\ \det\psi(\lambda),\ \ \lambda\in\mathfrak{h}^*.$$

For convenience we shall henceforth identify $\mathfrak{h}^*$ with $S_2(\mathbb{C})$ using ψ. Then, for example, the inner product $(\cdot|\cdot)$ is given by

$$\left(\begin{pmatrix} \nu_1 & \nu_2 \\ \nu_2 & \nu_3 \end{pmatrix} \middle| \begin{pmatrix} \mu_1 & \mu_2 \\ \mu_2 & \mu_3 \end{pmatrix}\right) = -\nu_3\mu_1 + 2\nu_2\mu_2 - \nu_1\mu_3.$$

The Weyl group W of $\mathcal{F}$ is generated by

$$w_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad w_2 = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \quad w_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{5.6}$$

and acts on $S_2(\mathbb{C})$ by $g\cdot S = gSg^t$; it is isomorphic to $PGL_2(\mathbb{Z})$.

The set of real roots is given by

$$\Phi_{\mathrm{re}} = \left\{ \begin{pmatrix} n_1 & n_2 \\ n_2 & n_3 \end{pmatrix} \in S_2(\mathbb{Z}) \ \mid\ n_1n_3 - n_2^2 = -1 \right\},$$

and the set of positive real roots is determined to be

$$\Phi_{\mathrm{re}}^+ = \left\{ \begin{pmatrix} n_1 & n_2 \\ n_2 & n_3 \end{pmatrix} \in S_2(\mathbb{Z}) \ \mid\ n_1n_3 - n_2^2 = -1,\ n_1+n_3 \ge n_2,\ n_1+n_3\ge 0,\ n_3 \ge 0 \right\}. \tag{5.7}$$

For the rest of this section we write

$$\alpha = \begin{pmatrix} n_1 & n_2 \\ n_2 & n_3 \end{pmatrix} \in \Phi_{\mathrm{re}}^+ \quad \text{and} \quad v = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in W, \tag{5.8}$$

and frequently use the computation

$$v\cdot\alpha = \begin{pmatrix} a^2n_1 + 2abn_2 + b^2n_3 & acn_1 + (ad+bc)n_2 + bdn_3 \\ acn_1 + (ad+bc)n_2 + bdn_3 & c^2n_1 + 2cdn_2 + d^2n_3 \end{pmatrix}. \tag{5.9}$$

In particular,

$$v \cdot \alpha_1 = \begin{pmatrix} -2ab & -bc-ad \\ -bc-ad & -2cd \end{pmatrix}, \tag{5.10}$$

$$v \cdot \alpha_2 = \begin{pmatrix} a^2+2ab & ac+bc+ad \\ ac+bc+ad & c^2+2cd \end{pmatrix}, \tag{5.11}$$

$$\text{and} \quad v \cdot \alpha_3 = \begin{pmatrix} b^2-a^2 & bd-ac \\ bd-ac & d^2-c^2 \end{pmatrix}. \tag{5.12}$$

Since the elements of Φ_v are precisely the positive real roots flipped by v, a root $\alpha \in \Phi_v$ is constrained by the inequalities

$$c^2 n_1 + 2cdn_2 + d^2 n_3 \leq 0, \tag{5.13}$$

$$(a^2+c^2)n_1 + 2(ab+cd)n_2 + (b^2+d^2)n_3 \leq 0, \quad \text{and} \tag{5.14}$$

$$(a^2+c^2)n_1 + 2(ab+cd)n_2 + (b^2+d^2)n_3 \leq acn_1 + (ad+bc)n_2 + bdn_3, \tag{5.15}$$

in addition to (5.7).

We have two standard maximal parabolic subgroups P_1 and P_2 with finite dimensional Levi subgroups M_1 and M_2, which correspond to simple roots α_1 and α_2 respectively. Note that in terms of previous notations, they have semisimple subgroups $L_1 \cong SL(3)$ and $L_2 \cong SL(2) \times SL(2)$ respectively.

We will use the following lemma to prove that the parabolic subgroups P_1 and P_2 satisfy Property RD (Proposition 5.4). The proof of this lemma is by direct calculation and will be omitted.

Lemma 5.3. *If $w \in W$, $w \neq \mathrm{id}$ and $w^{-1}\alpha_i > 0$, then $\langle \alpha_i, \alpha^\vee \rangle \leq 0$ for any $\alpha \in \Phi_{w^{-1}}$ unless*

(i) $i = 2$ and α equals

$$\beta_1 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{or} \quad \beta_2 = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix},$$

or (ii) $i = 3$ and α equals β_1 as above or

$$\beta_3 = \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}.$$

Moreover in these exceptional cases one has $\langle \alpha_i, \alpha^\vee \rangle = 1$.

Proposition 5.4. *The parabolic subgroups P_1 and P_2 both satisfy Property RD.*

Proof. Recall that for $j = 1, 2$ and $D > 0$ sufficiently small, one has

$$D\varpi_j + \rho_{M_j} = \sum_i m_i \alpha_i$$

where $m_j < 0$ and $m_i > 0$ for $i \in \theta_j := \{1,2,3\} \setminus \{j\}$. Then we need to show that taking smaller D if necessary, one has

$$\langle D\varpi_j + \rho_{M_j}, \alpha^\vee \rangle = \sum_i m_i \langle \alpha_i, \alpha^\vee \rangle < 0$$

for any $w \in W^{\theta_j}$ and $\alpha \in \Phi_{w^{-1}}$. Note that $\langle \alpha_i, \alpha^\vee \rangle$, $i = 1, 2, 3$, cannot be all zero. We shall consider P_1 and P_2 separately.

(1) Assume that $j = 1$. Then $w^{-1}\alpha_1 < 0$ and $w^{-1}\alpha_i > 0$ for $i = 2, 3$. By Lemma 5.3 and (5.2), one always has $\langle \alpha_1, \alpha^\vee \rangle \geq 0$ for $\alpha \in \Phi_{w^{-1}}$, and it suffices to consider the exceptional cases when $\alpha = \beta_i$ for some $i = 1, 2, 3$.

Since $\beta_1 = \alpha_2 + \alpha_3$, we have $w^{-1}\beta_1 > 0$, hence $\alpha = \beta_1$ cannot happen. Since $\langle \alpha_3, \beta_2^\vee \rangle = 2$, it follows from Lemma 5.3 that $\alpha = \beta_2$ cannot happen either. Finally if $\alpha = \beta_3$, then $\langle \rho_{M_1}, \beta_3^\vee \rangle = -2 < 0$, hence $\langle D\varpi_1 + \rho_{M_1}, \beta_3^\vee \rangle = D - 2 < 0$ for $D < 2$.

(2) Assume that $j = 2$. Then $w^{-1}\alpha_2 < 0$ and $w^{-1}\alpha_i > 0$ for $i = 1, 3$. By Lemma 5.3 and (5.2), one has $\langle \alpha_2, \alpha^\vee \rangle \geq 0$ for $\alpha \in \Phi_{w^{-1}}$ unless $\alpha = w_2\beta_1$ or $w_2\beta_2$. However, we find that $w_2\beta_1 = \alpha_3$ and $\langle \alpha_3, w_2\beta_2^\vee \rangle = 3$, hence $\alpha \neq w_2\beta_1, w_2\beta_2$, thanks to $w^{-1}\alpha < 0$ and Lemma 5.3 respectively. Thus we have proved that $\langle \alpha_2, \alpha^\vee \rangle \geq 0$ always holds.

By Lemma 5.3 again, it suffices to consider the exceptional cases $\alpha = \beta_1$ or β_3. For $\alpha = \beta_1$ we have $\langle \rho_{M_2}, \beta_1^\vee \rangle = -1/2 < 0$, hence $\langle D\varpi_2 + \rho_{M_2}, \beta_1^\vee \rangle = D - 1/2 < 0$ for $D < 1/2$. The case $\alpha = \beta_3$ cannot happen, because $\langle \alpha_1, \beta_3^\vee \rangle = 2$ which violates Lemma 5.3. $\square$

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