GEOMETRIC DESCRIPTION OF C-VECTORS AND REAL LÖSUNGEN

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Abstract. We introduce real Lösungen as an analogue of real roots. For each mutation sequence of an arbitrary skew-symmetrizable matrix, we define a family of reflections along with associated vectors which are real Lösungen and a set of curves on a Riemann surface. The matrix consisting of these vectors is called *L-matrix*. We explain how the L-matrix naturally arises in connection with the C-matrix. Then we conjecture that the L-matrix depends (up to signs of row vectors) only on the seed, and that the curves can be drawn without self-intersections, providing a new combinatorial/geometric description of c-vectors.

1. Introduction

Let Q be a quiver with n vertices and no oriented cycles of length ≤ 2. The most basic invariant of a representation of Q is its dimension vector. By Kac's Theorem [16], the dimension vectors of indecomposable representations of Q are positive roots of the Kac–Moody algebra $\mathfrak{g}_Q$ associated to the quiver Q.

When Q is acyclic, a representation M of Q is called *rigid* if $\mathrm{Ext}^1(M, M) = 0$, and the dimension vectors of indecomposable rigid representations are called *real Schur roots* as they are indeed real roots of $\mathfrak{g}_Q$. In the category of representations of Q, rigid objects are foundational. Therefore an explicit description of real Schur roots is essential for the study of the category, and there have been various results related to description of real Schur roots of an acyclic quiver ([4, 14, 15, 22, 24, 28]).

In a previous paper [17], we conjectured a correspondence between real Schur roots of an acyclic quiver and non-self-crossing curves on a marked Riemann surface and hence proposed a new combinatorial/geometric description. Recently, Felikson and Tumarkin [10] proved our conjecture for all 2-complete acyclic quivers. (An acyclic quiver is called *2-complete* if it has multiple edges between any pair of vertices.)

Now, when Q is general, it is natural to consider the *c-vectors* of Q as dimension vectors of rigid objects. Indeed, when Q is acyclic, the set of positive c-vectors is identical with the set of

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real Schur roots [19]. For an arbitrary quiver Q, a positive c-vector is the dimension vector of a rigid indecomposable representation of a quotient of the completed path algebra. This quotient was introduced by Derksen, Weyman and Zelevinksy [7], and is called a *Jacobian algebra*. Thus c-vectors naturally generalize real Schur roots in this sense, though they are not necessarily real roots of the corresponding Kac–Moody algebra.

Originally, c-vectors (and C-matrices) were defined in the theory of cluster algebras [11], and together with their companions, g-vectors (and G-matrices), played fundamental roles in the study of cluster algebras (for instance, see [7, 12, 13, 18, 20]). As a cluster algebra is defined not only for a skew-symmetric matrix (i.e. a quiver) but also for an arbitrary skew-symmetrizable matrix, one can ask:

> *Can we have a combinatorial/geometric description of the c-vectors (and C-matrices) of a cluster algebra associated with an arbitrary skew-symmetrizable matrix?*

In this paper, we propose a conjectural, combinatorial/geometric model for C-matrices associated to an arbitrary skew-symmetrizable matrix, which extends our model from the acyclic case [17].

For this purpose, we introduce the notion of *real Lösungen* as an analogue of real roots, and define a family of reflections along with associated vectors which are real Lösungen for each mutation sequence of an arbitrary skew-symmetrizable matrix. The matrix consisting of these real Lösungen is called *L-matrix*. We show that the L-matrix comes from certain leading terms when the C-matrix is presented using reflections. We conjecture that the L-matrices (up to signs of row vectors) depend only on seeds, i.e., do not depend on mutation sequences leading to the same seed. We believe that understanding these new matrices is a key to generalizing Coxeter groups and their quotients arising from cluster algebras, in particular, generalizing Felikson–Tumarkin's result [9].

When a skew-symmetrizable matrix is acyclic, it is natural to consider the corresponding symmetrizable generalized Cartan matrix. For a general skew-symmetrizable matrix, we consider *generalized intersection matrices* (GIMs)[1] introduced by Slodowy [27, 26]. A GIM is a square matrix $A = [a_{ij}]$ with integral entries such that

(1) for diagonal entries, $a_{ii} = 2$;
(2) $a_{ij} > 0$ if and only if $a_{ji} > 0$;
(3) $a_{ij} < 0$ if and only if $a_{ji} < 0$.

Since we are more interested in cluster algebras associated with skew-symmetrizable matrices, we restrict ourselves to the class of symmetrizable GIMs. This class contains the collection of all symmetrizable generalized Cartan matrices as a special subclass.

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[1]Some authors call them *quasi-Cartan matrices*. For example, see [2].

Let $\mathcal{A}$ be the (unital) $\mathbb{Z}$-algebra generated by s_i, e_i, $i = 1, 2, \ldots, n$, subject to the following relations:

$$s_i^2 = 1, \quad \sum_{i=1}^{n} e_i = 1, \quad s_i e_i = -e_i, \quad e_i s_j = \begin{cases} s_i + e_i - 1 & \text{if } i = j, \\ e_i & \text{if } i \neq j, \end{cases} \quad e_i e_j = \begin{cases} e_i & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Let $\mathcal{W}$ be the subgroup of the units of $\mathcal{A}$ generated by s_i, $i = 1, \ldots, n$. Note that $\mathcal{W}$ is (isomorphic to) the universal Coxeter group. Thus the algebra $\mathcal{A}$ can be considered as the algebra generated by the reflections and projections of the universal Coxeter group. Keeping computations at the level of $\mathcal{A}$ will reveal some important features of mutations.

Definition 1.1. Let $A = [a_{ij}]$ be an $n \times n$ symmetrizable GIM, and $D = \mathrm{diag}(d_1, \ldots, d_n)$ be the *symmetrizer*, i.e. the diagonal matrix such that $d_i \in \mathbb{Z}_{>0}$, $\gcd(d_1, \ldots, d_n) = 1$ and AD is symmetric. Let $\Gamma = \sum_{i=1}^{n} \mathbb{Z}\alpha_i$ be the lattice generated by the formal symbols $\alpha_1, \cdots, \alpha_n$.

(1) An element $\gamma = \sum m_i \alpha_i \in \Gamma$ is called a *Lösung* if

$$\sum_{1 \le i,j \le n} d_j a_{ij} m_i m_j = 2d_k \quad \text{for some } k = 1, \ldots, n. \tag{1.2}$$

A Lösung is *positive* if $m_i \geq 0$ for all i. Each α_i is called a *simple Lösung*.

(2) Define a representation $\pi : \mathcal{A} \to \mathrm{End}(\Gamma)$ by

$$\pi(s_i)(\alpha_j) = \alpha_j - a_{ji}\alpha_i \quad \text{and} \quad \pi(e_i)(\alpha_j) = \delta_{ij}\alpha_i, \quad i, j = 1, \ldots, n.$$

We suppress π when we write the action of an element of $\mathcal{A}$ on Γ. A Lösung γ is *real* if $\gamma = s_{i_1} s_{i_2} \cdots s_{i_k}(\alpha_i)$ for some $i = 1, \ldots, n$ and $k \geq 0$.

If A is a generalized Cartan matrix, then real Lösungen are the same as real roots of the Kac–Moody algebra associated with A. We expect that, for each symmetrizable GIM, there exists a Lie algebra for which real roots can be defined and are compatible with real Lösungen, but we do not yet know which Lie algebra would be adequate. Some related works can be found in [2, 3, 5, 6, 21, 26, 27, 29].

Fix an $n \times n$ skew-symmetrizable matrix $B = [b_{ij}]$ and let $D = \mathrm{diag}(d_1, \ldots, d_n)$ be its symmetrizer such that BD is symmetric, $d_i \in \mathbb{Z}_{>0}$ and $\gcd(d_1, \ldots, d_n) = 1$. Consider the $n \times 2n$ matrix $\begin{bmatrix} B & I \end{bmatrix}$. After a sequence $\boldsymbol{w}$ of mutations, we obtain $\begin{bmatrix} B^{\boldsymbol{w}} & C^{\boldsymbol{w}} \end{bmatrix}$. The matrix $C^{\boldsymbol{w}}$ is called the *C-matrix* and its row vectors the *c-vectors*. Write their entries as

$$B^{\boldsymbol{w}} = \begin{bmatrix} b_{ij}^{\boldsymbol{w}} \end{bmatrix}, \qquad C^{\boldsymbol{w}} = \begin{bmatrix} c_{ij}^{\boldsymbol{w}} \end{bmatrix} = \begin{bmatrix} c_1^{\boldsymbol{w}} \\ \vdots \\ c_n^{\boldsymbol{w}} \end{bmatrix}, \tag{1.3}$$

where $c_i^{\boldsymbol{w}}$ are the c-vectors. For a mutation sequence $\boldsymbol{w} = [i_i, i_2, \ldots, i_\ell]$, $i_j \in \{1, 2, \ldots, n\}$, we define $\boldsymbol{w}[k] := [i_i, i_2, \ldots, i_\ell, k]$.

Definition 1.4. For each mutation sequence $\boldsymbol{w}$, define $r_i^{\boldsymbol{w}} \in \mathcal{W} \subset \mathcal{A}$ inductively with the initial elements $r_i = s_i$, $i = 1, \dots, n$, as follows:

$$r_i^{\boldsymbol{w}[k]} = \begin{cases} r_k^{\boldsymbol{w}} r_i^{\boldsymbol{w}} r_k^{\boldsymbol{w}} & \text{if } \ b_{ik}^{\boldsymbol{w}} c_k^{\boldsymbol{w}} > 0, \\ r_i^{\boldsymbol{w}} & \text{otherwise.} \end{cases} \tag{1.5}$$

Clearly, each $r_i^{\boldsymbol{w}}$ is written in the form

$$r_i^{\boldsymbol{w}} = g_i^{\boldsymbol{w}} s_i (g_i^{\boldsymbol{w}})^{-1}, \quad g_i^{\boldsymbol{w}} \in \mathcal{W}, \quad i = 1, \dots, n.$$

Definition 1.6. Fix a GIM A, and define

$$l_i^{\boldsymbol{w}} = g_i^{\boldsymbol{w}}(\alpha_i), \qquad i = 1, \dots, n.$$

Then the *L-matrix* $L^{\boldsymbol{w}}$ associated to A is defined to be the $n \times n$ matrix whose i^{th} row is $l_i^{\boldsymbol{w}}$ for $i = 1, \dots, n$, i.e.,

$$L^{\boldsymbol{w}} = \begin{bmatrix} l_1^{\boldsymbol{w}} \\ \vdots \\ l_n^{\boldsymbol{w}} \end{bmatrix},$$

and the vectors $l_i^{\boldsymbol{w}}$ are called the *l-vectors of A*.

Note that the L-matrix and l-vectors associated to a GIM A implicitly depend on the representation π which is suppressed from the notation. When multiple GIMs are being discussed we will use the notation $l_i^{A,\boldsymbol{w}}$ to distinguish between different sets of l-vectors.

When we fix a GIM, we will always choose a linear ordering $\prec$ on $\{1, 2, \dots, n\}$ and define the associated GIM $A = [a_{ij}]$ by

$$a_{ij} = \begin{cases} b_{ij} & \text{if } i \prec j, \\ 2 & \text{if } i = j, \\ -b_{ij} & \text{if } i \succ j. \end{cases} \tag{1.7}$$

An ordering $\prec$ provides a certain way for us to regard the skew-symmetrizable matrix B as acyclic even when it is not.

As our geometric model, we consider a Riemann surface and admissible curves (Definition 2.3), and define a map from the set of admissible curves to the set of monomials in s_i's in $\mathcal{W}$ (Definition 2.4). The first conjecture below extends our conjecture in [17] from acyclic quivers to skew-symmetrizable matrices. The second conjecture claims that we can choose a GIM A to obtain a set of reflections that only depend on the seed.

Conjecture 1.8. *Fix an ordering $\prec$ on $\{1, 2, \dots, n\}$ so that a GIM A is determined. Then for any mutation sequence $\boldsymbol{w}$, there exist non-self-intersecting admissible curves $\eta_i^{\boldsymbol{w}}$ such that $\pi(r_i^{\boldsymbol{w}}) = \pi\left(s(\eta_i^{\boldsymbol{w}})\right)$, where $s(\eta_i^{\boldsymbol{w}})$ are the monomials in $\mathcal{W}$ associated to $\eta_i^{\boldsymbol{w}}$ for $i = 1, 2, \dots, n$.*

Conjecture 1.9. *For any skew-symmetrizable matrix B, there exists a linear ordering $\prec$ and its associated GIM A such that if $\boldsymbol{w}$ and $\boldsymbol{v}$ are two mutation sequences with $C^{\boldsymbol{w}} = C^{\boldsymbol{v}}$ then $\pi(r_i^{\boldsymbol{w}}) = \pi(r_i^{\boldsymbol{v}})$, $i = 1, \dots, n$.*

As the main result of this paper, we show that the reflections $r_i^{\boldsymbol{w}}$ naturally arise in connection with the C-matrix. It also justifies potential importance of the matrix $L^{\boldsymbol{w}}$. The key idea is to maintain that we should have a "root system" for each mutation sequence $\boldsymbol{w}$ as in the acyclic case. More precisely, we choose a linear ordering $\prec$ and its associated GIM, and inductively define an n-tuple of elements $s_i^{\boldsymbol{w}} \in \mathcal{A}$ and an n-tuple of vectors $\lambda_i^{\boldsymbol{w}} \in \mathbb{Z}^n$ $(\cong \Gamma)$, $i = 1, 2, \dots, n$, so that the following formulae hold:

$$s_i^{\boldsymbol{w}}(\lambda_j^{\boldsymbol{w}}) = \begin{cases} \lambda_j^{\boldsymbol{w}} + b_{ji}^{\boldsymbol{w}} \lambda_i^{\boldsymbol{w}} & \text{if } i \prec j, \\ -\lambda_j^{\boldsymbol{w}} & \text{if } i = j, \\ \lambda_j^{\boldsymbol{w}} - b_{ji}^{\boldsymbol{w}} \lambda_i^{\boldsymbol{w}} & \text{if } i \succ j, \end{cases} \tag{1.10}$$

where $B^{\boldsymbol{w}} = [b_{ij}^{\boldsymbol{w}}]$. We denote by $\Lambda^{\boldsymbol{w}}$ the matrix whose rows are $\lambda_i^{\boldsymbol{w}}$.

Theorem 1.11. *Fix a linear ordering $\prec$ on $\{1, 2, \dots, n\}$ to obtain its associated GIM A. Then, for each mutation sequence $\boldsymbol{w}$, we have*

$$\Lambda^{\boldsymbol{w}} = C^{\boldsymbol{w}}.$$

Moreover,

$$s_i^{\boldsymbol{w}} \equiv r_i^{\boldsymbol{w}} \pmod{2\mathcal{A}}, \quad i = 1, 2, \dots, n.$$

As one can see from the flow chart in Table 1, the definitions of $s_i^{\boldsymbol{w}}$ and $\lambda_i^{\boldsymbol{w}}$ are somewhat convoluted and heavily depend on $\prec$. Nevertheless, in the end, we obtain $C^{\boldsymbol{w}}$ and $r_i^{\boldsymbol{w}}$ which do not depend on $\prec$. Moreover, this process reveals that $r_i^{\boldsymbol{w}}$ are certain leading terms in $s_i^{\boldsymbol{w}}$. Since $s_i^{\boldsymbol{w}}$ are related to $\lambda_i^{\boldsymbol{w}}$ and $r_i^{\boldsymbol{w}}$ to $l_i^{\boldsymbol{w}}$, the l-vectors $l_i^{\boldsymbol{w}}$ can be considered as "leading terms" of the c-vectors $c_i^{\boldsymbol{w}} (= \lambda_i^{\boldsymbol{w}})$. What Conjectures 1.8 and 1.9 claim is that these leading terms carry essential information.

To illustrate Theorem 1.11 and Conjecture 1.8, we present Example 1.12. Conjecture 1.9 is trivially satisfied for this matrix as each mutation sequence of a 3×3 skew-symmetrizable matrix produces a unique C-matrix [23]. A non-trivial example of Conjecture 1.9 is given in Example 2.11.

Example 1.12. Consider the symmetrizable matrix $B = \begin{bmatrix} 0 & 3 & -3 \\ -2 & 0 & 2 \\ 2 & -2 & 0 \end{bmatrix}$ with the symmetrizer $D = \mathrm{diag}(3, 2, 2)$, and the sequence of consecutive mutations at indices $2, 3, 2, 1, 2$:

$$\begin{bmatrix} B & I \end{bmatrix} \xrightarrow{[2,3,2,1,2]} \begin{bmatrix} 0 & -3 & 9 & 5 & 18 & 15 \\ 2 & 0 & -4 & -2 & -7 & -6 \\ -6 & 4 & 0 & 0 & -2 & -1 \end{bmatrix}$$

Thus we have obtained three c-vectors $(5, 18, 15)$, $(-2, -7, -6)$ and $(0, -2, -1)$.

We take the linear ordering $1 \succ 2 \succ 3$. Then its GIM A and the symmetrized matrix AD are as follows:

$$A = \begin{bmatrix} 2 & -3 & 3 \\ -2 & 2 & -2 \\ 2 & -2 & 2 \end{bmatrix}, \qquad AD = \begin{bmatrix} 6 & -6 & 6 \\ -6 & 4 & -4 \\ 6 & -4 & 4 \end{bmatrix}.$$

In accordance with (1.2), define a quadratic form by

$$q(x, y, z) = 6x^2 + 4y^2 + 4z^2 - 12xy - 8yz + 12zx.$$

Then we have

$$q(5, 18, 15) = 6, \quad q(-2, -7, -6) = 4, \quad q(0, -2, -1) = 4.$$

Thus all three c-vectors are Lösungen for A.

From Definition 1.4, we obtain

$$r_1^{\boldsymbol{v}} = s_3s_2s_1s_2s_3s_2s_3s_2s_1s_2s_3s_2s_3s_2s_1s_2s_3, \quad r_2^{\boldsymbol{v}} = s_3s_2s_1s_2s_3s_2s_3s_2s_1s_2s_3, \quad r_3^{\boldsymbol{v}} = s_2s_3s_2,$$

where $\boldsymbol{v}$ is the mutation sequence $[2, 3, 2, 1, 2]$. For the GIM A, Definition 1.6 gives rise to the l-vectors

$$l_1^{\boldsymbol{v}} = s_3s_2s_1s_2s_3s_2s_3s_2(\alpha_1) = (5, 18, 15),$$
$$l_2^{\boldsymbol{v}} = s_3s_2s_1s_2s_3(\alpha_2) = (2, 7, 6), \quad l_3^{\boldsymbol{v}} = s_2(\alpha_3) = (0, 2, 1).$$

On the other hand, following the definitions in Section 2, we obtain similar results for the $\lambda_i^{\boldsymbol{w}}$. In particular,

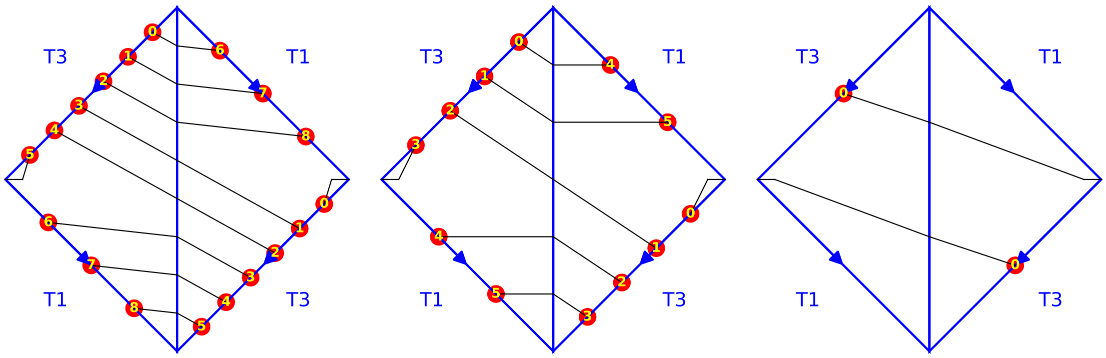
$$\lambda_1^{\boldsymbol{v}} = s_3s_2s_1s_2s_3s_2s_3s_2(\alpha_1) = (5, 18, 15),$$
$$\lambda_2^{\boldsymbol{v}} = -s_3s_2s_1s_2s_3(\alpha_2) = (-2, -7, -6), \quad \lambda_3^{\boldsymbol{v}} = -s_2(\alpha_3) = (0, -2, -1).$$

Thus the matrix $\Lambda^{\boldsymbol{v}} = \begin{bmatrix} 5 & 18 & 15 \\ -2 & -7 & -6 \\ 0 & -2 & -1 \end{bmatrix}$ equals the C-matrix.

However, l-vectors will not always be equal to positive c-vectors. Indeed, they need not even be sign-coherent. For the choice of GIM $A' = \begin{bmatrix} 2 & 3 & -3 \\ 2 & 2 & 2 \\ -2 & 2 & 2 \end{bmatrix}$ we see that

$$l_1^{A',\boldsymbol{v}} = (149, -462, 1341), \quad l_2^{A',\boldsymbol{v}} = (-10, 31, -90), \quad l_2^{A',\boldsymbol{v}} = (0, -2, 1).$$

We choose admissible curves $\eta_i^{\boldsymbol{v}}$ on a triangulated torus Σ_σ such that $r_i^{\boldsymbol{v}} = s(\eta_i^{\boldsymbol{v}})$ and draw the curves in Figure 1 to illustrate that they are non-self-intersecting. This verifies Conjecture 1.8



(A) The curve $\eta_1^{\boldsymbol{v}}$. (B) The curve $\eta_2^{\boldsymbol{v}}$. (C) The curve $\eta_3^{\boldsymbol{v}}$.

FIGURE 1. The curves $\eta_i^{\boldsymbol{v}}$ corresponding to Example 1.12 displayed on Σ_σ where $\sigma = (3, 1, 2) \in S_3$ written in one-line notation.

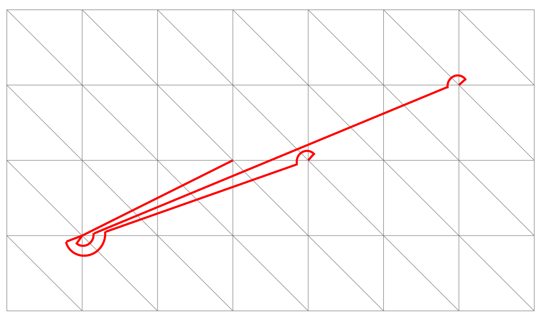


FIGURE 2. The curves from Example 1.12. The shortest curve corresponds to $\eta_3^{\boldsymbol{v}}$, and the longest one to $\eta_1^{\boldsymbol{v}}$.

for this example. (In this example, it is not necessary to go through π. The torus Σ_σ is precisely defined in the next section.) We also draw the curves on the universal cover of Σ_σ in Figure 2 to see that they have no pairwise intersections.

1.1. **Organization of the paper.** In Section 2, precise definitions will be made for the objects appeared in this introduction, and Conjectures 1.8 and 1.9 will be presented in a more refined way, and other examples will be given. In Section 3 the elements $s_i^{\boldsymbol{w}} \in \mathcal{A}$ and the vectors $\lambda_i^{\boldsymbol{w}}$ will be defined with a running example, and Theorem 1.11 will be stated more precisely. In Section 4, Theorem 1.11 will be proven through induction. The main induction step consists of six different cases, each of which has a few subcases.

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2. Conjectures

In this section, we present our conjectures in a more precise way after making necessary definitions.

For a nonzero vector $c = (c_1, \dots, c_n) \in \mathbb{Z}^n$, we define $c > 0$ if all c_i are non-negative, and $c < 0$ if all c_i are non-positive. This induces a partial ordering $<$ on $\mathbb{Z}^n$. Define $|c| = (|c_1|, \dots, |c_n|)$.

Assume that $M = [m_{ij}]$ be an $n \times 2n$ matrix of integers. Let $\mathcal{I} := \{1, 2, \dots, n\}$ be the set of indices. For $\boldsymbol{w} = [i_i, i_2, \dots, i_\ell]$, $i_j \in \mathcal{I}$, we define the matrix $M^{\boldsymbol{w}} = [m_{ij}^{\boldsymbol{w}}]$ inductively: the initial matrix is M for $\boldsymbol{w} = [\,]$, and assuming we have $M^{\boldsymbol{w}}$, define the matrix $M^{\boldsymbol{w}[k]} = [m_{ij}^{\boldsymbol{w}[k]}]$ for $k \in \mathcal{I}$ with $\boldsymbol{w}[k] := [i_i, i_2, \dots, i_\ell, k]$ by

$$m_{ij}^{\boldsymbol{w}[k]} = \begin{cases} -m_{ij}^{\boldsymbol{w}} & \text{if } i = k \text{ or } j = k, \\ m_{ij}^{\boldsymbol{w}} + \operatorname{sgn}(m_{ik}^{\boldsymbol{w}}) \max(m_{ik}^{\boldsymbol{w}} m_{kj}^{\boldsymbol{w}}, 0) & \text{otherwise,} \end{cases} \tag{2.1}$$

where $\operatorname{sgn}(a) \in \{1, 0, -1\}$ is the signature of a. The matrix $M^{\boldsymbol{w}[k]}$ is called the *mutation of* $M^{\boldsymbol{w}}$ at the index k.

Let $B = [b_{ij}]$ be an $n \times n$ skew-symmetrizable matrix and $D = \operatorname{diag}(d_1, \dots, d_n)$ be its symmetrizer such that BD is symmetric, $d_i \in \mathbb{Z}_{>0}$ and $\gcd(d_1, \dots, d_n) = 1$. Consider the $n \times 2n$ matrix $\begin{bmatrix} B & I \end{bmatrix}$ and a mutation sequence $\boldsymbol{w} = [i_1, \dots, i_k]$. After the mutations at the indices $i_1, \dots, i_k$ consecutively, we obtain $\begin{bmatrix} B^{\boldsymbol{w}} & C^{\boldsymbol{w}} \end{bmatrix}$. Write their entries as in (1.3). It is well-known that the c-vector $c_i^{\boldsymbol{w}}$ is non-zero for each i, and either $c_i^{\boldsymbol{w}} > 0$ or $c_i^{\boldsymbol{w}} < 0$ due to sign coherence of c-vectors ([8, 12]).

Choose a linear ordering $\prec$ on the set $\mathcal{I}$, and define a GIM $A = [a_{ij}]$ by (1.7). From Definition 1.1, we have Lösungen associated with A. Set $\lambda_1 = (1, 0, \dots, 0)$, $\lambda_2 = (0, 1, 0, \dots, 0), \dots, \lambda_n = (0, \dots, 0, 1)$ to be a basis of $\mathbb{Z}^n$. Recall that we have defined the algebra $\mathcal{A}$ in the introduction. Define a representation $\pi : \mathcal{A} \to \operatorname{End}(\mathbb{Z}^n)$ by

$$\pi(s_i)(\lambda_j) = \lambda_j - a_{ji}\lambda_i \quad \text{and} \quad \pi(e_i)(\lambda_j) = \delta_{ij}\lambda_i \qquad \text{for } i, j \in \mathcal{I}, \tag{2.2}$$

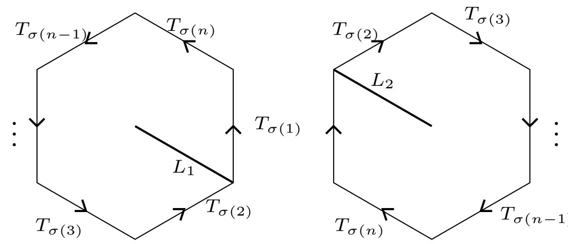
and by extending it through linearity, where δ_{ij} is the Kronecker delta. We will suppress π when we write the action of an element of $\mathcal{A}$ on $\mathbb{Z}^n$. As before, denote by $\mathcal{W}$ the subgroup of the units of $\mathcal{A}$ generated by s_i, $i = 1, \dots, n$.

To introduce our geometric model[2] for c-vectors, we need a Riemann surface equipped with n labeled curves as below. Let P_1 and P_2 be two identical copies of a regular n-gon. For $\sigma \in S_n$, label the edges of each of the two n-gons by $T_{\sigma(1)}, T_{\sigma(2)}, \dots, T_{\sigma(n)}$ counter-clockwise.

On P_i ($i = 1, 2$), let L_i be the line segment from the center of P_i to the common endpoint of $T_{\sigma(n)}$ and $T_{\sigma(1)}$. Later, these line segments will only be used to designate the end points of

[2] An alternative geometric model can be found in [10].

admissible curves and will not be used elsewhere. Fix the orientation of every edge of P_1 (resp. P_2) to be counter-clockwise (resp. clockwise) as in the following picture.



Let Σ_σ be the Riemann surface of genus $\lfloor \frac{n-1}{2} \rfloor$ obtained by gluing together the two n-gons with all the edges of the same label identified according to their orientations. The edges of the n-gons become n different curves in Σ_σ. If n is odd, all the vertices of the two n-gons are identified to become one point in Σ_σ and the curves obtained from the edges become loops. If n is even, two distinct vertices are shared by all curves. Let $T = T_1 \cup \cdots T_n \subset \Sigma_\sigma$, and V be the set of the vertex (or vertices) on T.

Let $\mathfrak{W}$ be the universal Coxeter group of rank n, which is by definition isomorphic to the free product of n-copies of $\mathbb{Z}/2\mathbb{Z}$, and let $\mathfrak{R}$ be the set of reflections in $\mathfrak{W}$. We will denote an element of $\mathfrak{W}$ as a word from the alphabet $\mathcal{I} = \{1, 2, ..., n\}$. In particular, an element $\mathfrak{v}$ of $\mathfrak{R}$ can be written as $\mathfrak{v} = i_1 i_2 \cdots i_k$ such that k is an odd integer and $i_j = i_{k+1-j}$ for all $j = 1, 2, \ldots, k$.

Definition 2.3. An *admissible curve* is a continuous function $\eta : [0, 1] \longrightarrow \Sigma_\sigma$ such that
1) $\eta(x) \in V$ if and only if $x \in \{0, 1\}$;
2) there exists $\epsilon > 0$ such that $\eta([0, \epsilon]) \subset L_1$ and $\eta([1 - \epsilon, 1]) \subset L_2$;
3) if $\eta(x) \in T \setminus V$ then $\eta([x - \epsilon, x + \epsilon])$ meets T transversally for sufficiently small $\epsilon > 0$;
4) $\upsilon(\eta) \in \mathfrak{R}$, where $\upsilon(\eta) := i_1 \cdots i_k \in \mathfrak{W}$ is given by

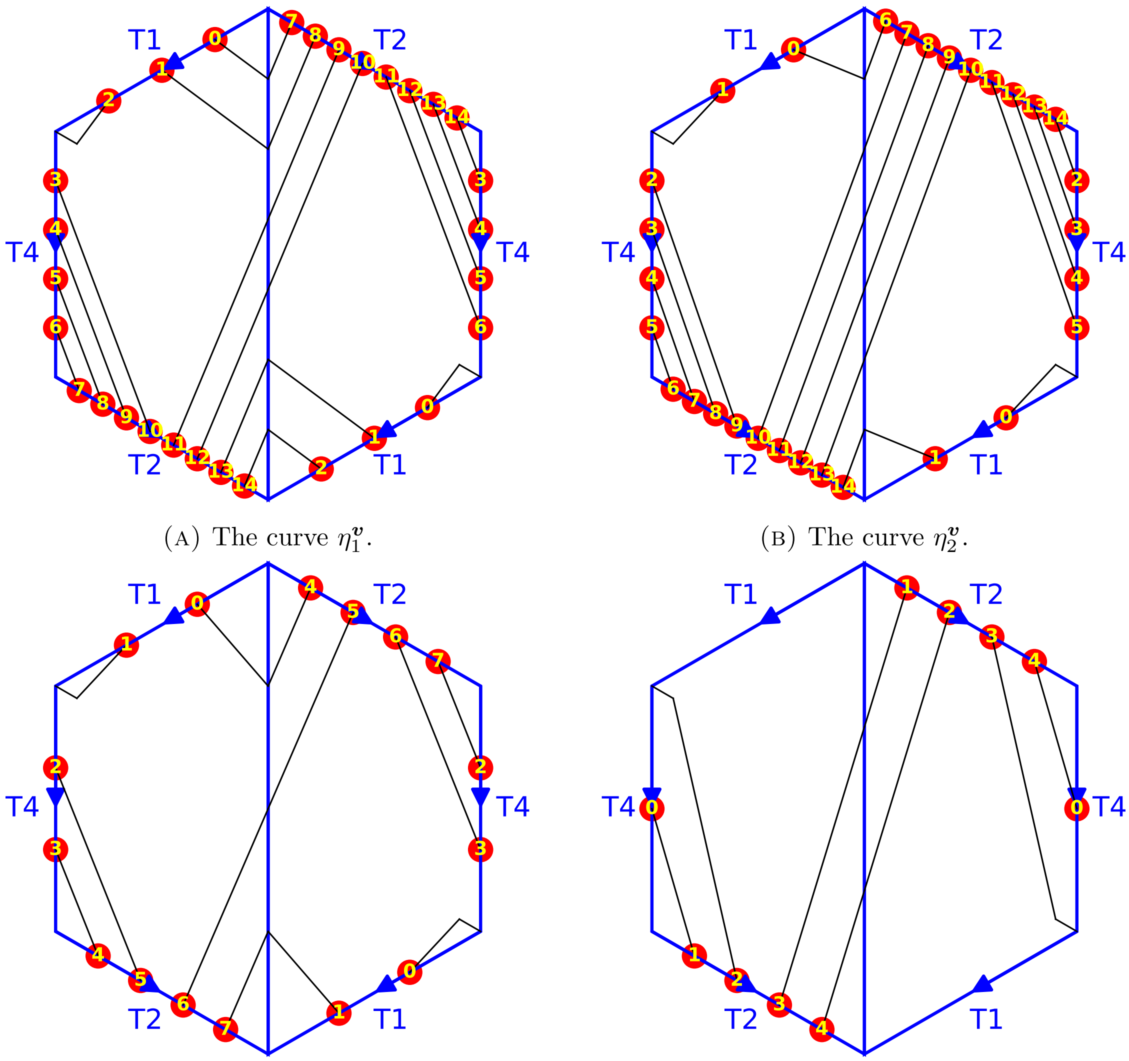
$$\{x \in (0, 1) \ : \ \eta(x) \in T\} = \{x_1 < \cdots < x_k\} \quad \text{and} \quad \eta(x_\ell) \in T_{i_\ell} \text{ for } \ell \in \{1, ..., k\}.$$

We consider curves up to isotopy. When $i_p = i_{p+1}$, $1 \le p \le k - 1$, for $\upsilon(\eta) = i_1 \cdots i_k$, the curve η is isotopic to a curve η_1 with $\upsilon(\eta_1) = i_1 \cdots i_{p-1} i_{p+2} \cdots i_k$. If η_1 and η_2 are curves with $\upsilon(\eta_1) = i_1 \cdots i_k$ and $\upsilon(\eta_2) = j_1 \cdots j_\ell$, define their *concatenation* $\eta_1\eta_2$ to be a curve such that $\upsilon(\eta_1\eta_2) = i_1 \cdots i_k j_1 \cdots j_\ell$.

Definition 2.4. For $\mathfrak{v} = i_1 i_2 \cdots i_k \in \mathfrak{W}$, define $s(\mathfrak{v}) = s_{i_1} ... s_{i_k} \in \mathcal{W} \subset \mathcal{A}$. We write $s(\eta) = s(\upsilon(\eta))$ for an admissible curve η.

Now we state Conjecture 1.8 in a more refined way.

Conjecture 2.5 (Conjecture 1.8)**.** *Fix an ordering on $\mathcal{I}$ so that a GIM A is determined. Then, for each mutation sequence $\boldsymbol{w}$, there exists a family of non-self-crossing admissible curves $\eta_i^{\boldsymbol{w}}$, $i = 1, \ldots, n$, on the Riemann surface Σ_σ for some $\sigma \in S_n$ such that $\pi(r_i^{\boldsymbol{w}}) = \pi\left(s(\eta_i^{\boldsymbol{w}})\right)$.*



(A) The curve $\eta_1^{\boldsymbol{v}}$.

(B) The curve $\eta_2^{\boldsymbol{v}}$.

(C) The curve $\eta_3^{\boldsymbol{v}}$.

(D) The curve $\eta_4^{\boldsymbol{v}}$.

FIGURE 3. The curves for Example 2.6 drawn on Σ_σ with $\sigma = (1, 4, 2, 3)$.

Example 2.6. Consider the matrix $B = \begin{bmatrix} 0 & -1 & -1 & 2 \\ 1 & 0 & 1 & -1 \\ 1 & -1 & 0 & -1 \\ -2 & 1 & 1 & 0 \end{bmatrix}$. It arises from a triangulation of the torus with one boundary component with one marked point. It is commonly referred to as the *dreaded torus*. With the mutation sequence $\boldsymbol{w} = [2,3,4,2,1,3]$, we have

$$\begin{bmatrix} B & I \end{bmatrix} \quad \xrightarrow{\boldsymbol{w}} \quad \begin{bmatrix} 0 & 1 & -1 & -1 & 0 & 2 & 3 & 2 \\ -1 & 0 & -1 & 2 & 2 & 3 & 3 & 2 \\ 1 & 1 & 0 & -1 & -1 & -2 & -3 & -2 \\ 1 & -2 & 1 & 0 & 0 & -2 & -2 & -1 \end{bmatrix}.$$

Choose the linear ordering $1 \prec 3 \prec 2 \prec 4$. From Definition 1.4, we obtain

$$\begin{aligned} r_1^{\boldsymbol{w}} &= s_1 s_3 (s_2 s_4 s_2 s_3)^2 s_1 (s_3 s_2 s_4 s_2)^2 s_3 s_1, \\ r_2^{\boldsymbol{w}} &= s_1 s_3 (s_2 s_4 s_2 s_3)^2 s_2 (s_3 s_2 s_4 s_2)^2 s_3 s_1, \\ r_3^{\boldsymbol{w}} &= s_1 s_3 s_2 s_4 s_2 s_3 s_2 s_4 s_2 s_3 s_1, \\ r_4^{\boldsymbol{w}} &= s_2 s_3 s_2 s_4 s_2 s_3 s_2. \end{aligned}$$

In Figure 3 we provide curves $\eta_i^{\boldsymbol{w}}$ such that $s(\eta_i^{\boldsymbol{w}}) = r_i^{\boldsymbol{w}}$ for all $i \in \mathcal{I}$. It is clear that they are non-self-intersecting on the surface Σ_σ with $\sigma = (1,4,2,3) \in S_4$ written in one-line notation. By inspection these curves can be seen to be pairwise non-crossing.

In Example 2.7 we show π is necessary in Conjecture 2.5 to avoid self-intersections.

Example 2.7. Consider the matrix $B = \begin{bmatrix} 0 & -2 & -2 & 3 \\ 2 & 0 & 4 & 2 \\ 2 & -4 & 0 & -1 \\ -3 & -2 & 1 & 0 \end{bmatrix}$. Applying to the mutation sequence $\boldsymbol{w} = [4,3,1,4,1]$ we have

$$r_4^{\boldsymbol{w}} = s_3 s_4 s_1 (s_4 s_3)^2 s_4 s_2 (s_4 s_3)^3 s_4 s_2 s_4 (s_3 s_4)^2 s_1 s_4 s_3.$$

Let η be the curve defined by $s(\eta) = r_4^{\boldsymbol{w}}$. Upon inspection, for any $\sigma \in S_4$ the curve η has a self-intersection in Σ_σ. However, for any choice of GIM we have $\pi((s_3 s_4)^3) = 1$ so the curve η' given by $\upsilon(\eta') = 34132423143 \in \mathfrak{W}$ satisfies $\pi(r_4^{\boldsymbol{w}}) = \pi(s(\eta'))$ and can be drawn with no self-intersections.

In order to refine Conjecture 1.9, we need a new definition. A sequence of indices $(i_1, \ldots, i_d)$ is said to be a *chordless cycle* in a skew-symmetrizable matrix B if

(1) $i_j = i_k$ if and only if $\{j,k\} = \{1,d\}$,
(2) for any distinct $j,k \in \{1,\ldots,d\}$ we have $b_{i_j,i_k} \neq 0$ if and only if $|j-k| = 1$,

Additionally, a chordless cycle is said to be *oriented* if and only if all entries $b_{i_j,i_{j+1}}$ for $j = 1,\ldots,d-1$ have the same sign. Two chordless cycles are considered equivalent if they have the same underlying set of indices.

Conjecture 2.8 (Conjecture 1.9)**.** *Let B be a skew-symmetrizable matrix.*

(1) *There exists a linear ordering $\prec$ on $\mathcal{I}$ such that every oriented chordless cycle $(i_1, \dots, i_d)$ in B has an odd number of positive $a_{i_j,i_{j+1}}$, $j = 1, \dots, d-1$, where $A = [a_{ij}]$ is the GIM determined by $\prec$.*

(2) *Fix an ordering $\prec$ and its GIM A satisfying the condition in (1). If $\boldsymbol{w}$ and $\boldsymbol{v}$ are two mutation sequences such that $C^{\boldsymbol{w}} = C^{\boldsymbol{v}}$ then $\pi(r_i^{\boldsymbol{w}}) = \pi(r_i^{\boldsymbol{v}})$, $i = 1, \dots, n$.*

The elements $\pi(r_i^{\boldsymbol{w}})$ can be viewed as elements of $\pi(\mathcal{W})$, and Conjecture 2.8 can be interpreted as a statement about relations in $\pi(\mathcal{W})$. Relations for these groups have been explored for particular skew-symmetrizable matrices and a restricted class of GIMs in [1, 9, 25]. A thorough investigation of relations in $\pi(\mathcal{W})$ and their application to Conjecture 2.8 will take place in a subsequent article. It is expected that all of the discovered relations will hold for any GIM satisfying the condition in Conjecture 2.8 (1) which is a weaker than Seven's notion of admissibility [25].

The lemma below provides a sufficient condition for existence of a linear ordering $\prec$ and its GIM A satisfying the condition in Conjecture 2.8 (1). If we do not require that a GIM is determined by a linear ordering, it can be proven that a GIM satisfying the condition of Conjecture 2.8 (1) always exists for any skew-symmetrizable matrix. But in order to define the elements $s_i^{\boldsymbol{w}} \in \mathcal{A}$ as in the next section, it is necessary that A arises from a linear ordering.

Lemma 2.9. *Let B be a skew-symmetrizable matrix and $\mathcal{G}$ be the directed graph with vertices in $\mathcal{I}$ and arrows $i \to j$ for $b_{ij} < 0$. If there exists a spanning tree of $\mathcal{G}$ which contains an arrow of every oriented chordless cycle of $\mathcal{G}$ then we can take a linear ordering $\prec$ on $\mathcal{I}$ to obtain a GIM $A = [a_{ij}]$ such that every oriented chordless cycle $(i_1, \dots, i_d)$ in B has an odd number of positive $a_{i_j,i_{j+1}}$, $j = 1, \dots, d-1$.*

Proof. For any collection of arrows $\mathcal{E} = \{e_1, \dots, e_p\}$ in $\mathcal{G}$, we can define a new directed graph $\mathcal{H}$ by reversing the direction of the arrows of $\mathcal{E}$. If $\mathcal{H}$ is acyclic we may define a linear order by setting $i \prec j$ if $i \to j$ is an arrow of $\mathcal{H}$ and extending it to a linear ordering on $\mathcal{I}$. We will show that there exists a sequence of arrows that contains an odd number of arrows from every chordless cycle of $\mathcal{G}$ such that $\mathcal{H}$ is acyclic. Therefore it follows from (1.7) that the associated GIM satisfies the condition in the statement of the lemma.

Let $\mathcal{T}$ be a spanning tree of $\mathcal{G}$ that contains an arrow of every oriented chordless cycle of $\mathcal{G}$. We will take $\mathcal{E}'$ to be the set of the arrows of $\mathcal{G} \setminus \mathcal{T}$ ordered so that every chordless cycle of the digraph $\mathcal{T} \cup \{e_1, \dots, e_i\}$ is a chordless cycle of $\mathcal{G}$ for all i. Let $\mathcal{C}_i$ be the chordless cycle uniquely defined by the arrow $e_i \in \mathcal{E}'$, and let $\bar{e}_i$ be the opposite arrow of e_i. We will now construct the desired sequence of arrows $\mathcal{E}$ as a subset of $\mathcal{E}'$ by iteratively taking e_i to be in $\mathcal{E}$ if and only if either

(1) $\mathcal{C}_i$ is oriented in $\mathcal{T} \cup \{\bar{e}_k | e_k \in \mathcal{E}, k < i\} \cup \{e_i\}$, or

(2) $\mathcal{C}_i$ is oriented in $\mathcal{G}$, and the number of arrows of $\mathcal{C}_i$ that are already in $\mathcal{E}$ is even, i.e., the cardinality of the set $|\{e_k \in \mathcal{E} | k < i\} \cap \mathcal{C}_i|$ is even.

$$\begin{bmatrix} 0 & -1 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & -1 \\ 0 & -1 & 0 & 1 & 0 \end{bmatrix}$$

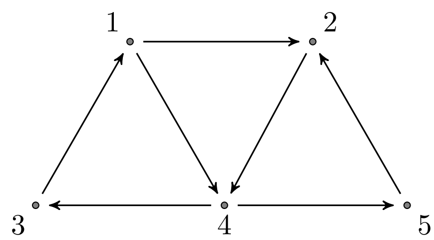


FIGURE 4. A skew-symmetric matrix B and the digraph associated to it in Lemma 2.9. The proof of the lemma is illustrated in Example 2.10.

Now define $\mathcal{H}$ from $\mathcal{G}$ by reversing the direction of the arrows of $\mathcal{E}$. Then for any oriented cycle of $\mathcal{G}$ we have not reversed all of its arrows in $\mathcal{H}$ as the arrow in $\mathcal{T}$ has been fixed, but we have reversed at least one arrow of the cycle by (2) so any oriented chordless of $\mathcal{G}$ is no longer oriented in $\mathcal{H}$. By (2) again, we chose $\mathcal{E}$ to contain an odd number of arrows of each chordless cycle. Furthermore every non-oriented cycle of $\mathcal{G}$ remains non-oriented in $\mathcal{H}$ by (1). Therefore all of the chordless cycles of $\mathcal{H}$ are non-oriented and it must be that $\mathcal{H}$ is acyclic. □

We now give an example illustrating the proof of Lemma 2.9.

Example 2.10. Let B be the skew-symmetric matrix given in Figure 4. The directed graph $\mathcal{G}$ associated to B in Lemma 2.9 is also shown in the figure. The matrix B has two oriented chordless cycles $(1, 3, 4, 1)$ and $(2, 4, 5, 2)$. Consider the spanning tree $\mathcal{T} = 4 \to 3 \to 1 \to 2 \leftarrow 5$. The tree $\mathcal{T}$ contains an edge of both of the oriented chordless cycles of $\mathcal{G}$. The edges of $\mathcal{G} \setminus \mathcal{T}$ are $e_1 = 1 \to 4$, $e_2 = 2 \to 4$, and $e_3 = 4 \to 5$. The cycles $\mathcal{C}_1, \mathcal{C}_2$, and $\mathcal{C}_3$ are $\{1, 3, 4\}$, $\{1, 2, 4\}$, and $\{2, 4, 5\}$, respectively. Note that $\mathcal{C}_2$ and $\mathcal{C}_3$ are not fundamental cycles of the graph $\mathcal{G}$. Now to construct $\mathcal{E}$ we see that $e_1 \in \mathcal{E}$ by condition (2), $e_2 \in \mathcal{E}$ by condition (1), and e_3 is not in $\mathcal{E}$. The covering relations dictated by the acyclic graph $\mathcal{H}$ obtained from $\mathcal{G}$ by reversing the arrows of $\mathcal{G}$ are $4 \prec 3 \prec 1$, $4 \prec 1 \prec 2$, and $4 \prec 5 \prec 2$. One extension of these relations to a linear ordering is $4 \prec 3 \prec 1 \prec 5 \prec 2$. It is straightforward to check that the associated GIM has exactly one positive entry for each oriented chordless cycle of B.

Recall the definition of an L-matrix from Definition 1.6. We now provide an example illustrating Conjecture 2.8 and l-vectors.

Example 2.11. Let B be the matrix from Example 2.6. For the two mutation sequences $\boldsymbol{w} = [3,4,1,3,4,3]$ and $\boldsymbol{v} = [4,1,3,4,1,3]$ we have $C^{\boldsymbol{w}} = C^{\boldsymbol{v}}$. On the other hand,

$$\begin{aligned} r_1^{\boldsymbol{w}} =& s_3s_4s_3s_1s_3s_4s_3, \\ r_2^{\boldsymbol{w}} =& s_3s_4s_3s_1s_3s_4s_2s_4s_3s_1s_3s_4s_3, \\ r_3^{\boldsymbol{w}} =& s_3s_4s_1s_3s_4s_3s_1s_3s_1s_3s_4s_3s_1s_4s_3, \\ r_4^{\boldsymbol{w}} =& s_3s_4s_1s_3s_4(s_3s_1)^2s_3s_4s_3(s_1s_3)^2s_4s_3s_1s_4s_3, \end{aligned}$$

and

$$\begin{aligned} r_1^{\boldsymbol{v}} =& s_3(s_4s_1)^2s_4s_3s_4s_1s_4s_3s_4(s_1s_4)^2s_3, \\ r_2^{\boldsymbol{v}} =& s_3(s_4s_1)^2s_4s_3s_4s_1s_4s_3(s_4s_1)^2s_4s_2s_4(s_1s_4)^2s_3s_4s_1s_4s_3s_4(s_1s_4)^2s_3, \\ r_3^{\boldsymbol{v}} =& s_3(s_4s_1)^2s_4s_3s_4(s_1s_4)^2s_3, \\ r_4^{\boldsymbol{v}} =& (s_3s_4s_1)^2s_4(s_1s_4s_3)^2. \end{aligned}$$

There are two oriented cycles on vertices $\{1,4,2\}$ and $\{1,4,3\}$ in B. Take the GIM arising from the linear ordering $1 \prec 2 \prec 3 \prec 4$. Then only the entry a_{14} is positive for the cycles, and the condition in Corollary 1.9 is satisfied. Direct computation shows that $\pi(r_i^{\boldsymbol{w}}) = \pi(r_i^{\boldsymbol{v}})$, and Conjecture 1.9 is verified.

We identify α_i with λ_i in Definition 1.6 and compute the l-vectors

$$\begin{aligned} l_1^{\boldsymbol{w}} =& s_3s_4s_3(\lambda_1) = (1,0,-1,-1), & l_2^{\boldsymbol{w}} =& s_3s_4s_3s_1s_3s_4(\lambda_2) = (-1,1,0,1), \\ l_3^{\boldsymbol{w}} =& s_3s_4s_1s_3s_4s_3s_1(\lambda_3) = (2,0,0,-3), & l_4^{\boldsymbol{w}} =& s_3s_4s_1s_3s_4(s_3s_1)^2s_3(\lambda_4) = (-3,0,0,4), \end{aligned}$$

and obtain the L-matrix

$$L^{\boldsymbol{w}} = \begin{bmatrix} 1 & 0 & -1 & -1 \\ -1 & 1 & 0 & 1 \\ 2 & 0 & 0 & -3 \\ -3 & 0 & 0 & 4 \end{bmatrix}.$$

On the other hand,

$$\begin{aligned} l_1^{\boldsymbol{v}} =& (-1,0,1,1) = -l_1^{\boldsymbol{w}}, & l_2^{\boldsymbol{v}} =& (-1,1,0,1) = l_2^{\boldsymbol{w}}, \\ l_3^{\boldsymbol{v}} =& (-2,0,0,3) = -l_3^{\boldsymbol{w}}, & l_4^{\boldsymbol{v}} =& (-3,0,0,4) = l_4^{\boldsymbol{w}}. \end{aligned}$$

One may hope that the reflections $r_i^{\boldsymbol{w}}$ would give a direct generalization of [28, Theorem 1.4] with the expectation that a product of $r_i^{\boldsymbol{w}}$'s might equal $s_{\tilde\sigma(1)}s_{\tilde\sigma(2)}\cdots s_{\tilde\sigma(n)}$ in $\mathcal{W}$ for some $\tilde\sigma \in S_n$. However Example 2.12 provides a counterexample.

Example 2.12. Let B be the matrix from Example 2.7. After the mutation sequence $\boldsymbol{w} = [2,3,2,1]$ we have

$$r_1^{\boldsymbol{w}} = s_1, \quad r_2^{\boldsymbol{w}} = s_1s_2s_1, \quad r_3^{\boldsymbol{w}} = s_2s_3s_2, \quad r_4^{\boldsymbol{w}} = s_3s_4s_3.$$

It is straightforward to check that $\prod_{i\in\mathcal{I}} r^{\boldsymbol{w}}_{\sigma(i)} \neq s_{\tilde\sigma(1)}s_{\tilde\sigma(2)}s_{\tilde\sigma(3)}s_{\tilde\sigma(4)}$ for any pair of $\sigma, \tilde\sigma \in S_4$. The same is true when considering the matrix representation of the s_i for any choice of GIM associated to B.

This collection $\{r_i^{\boldsymbol{w}}\}$ also provides an example where for any $\sigma \in S_4$ there will always be some pair of curves in $\eta_i^{\boldsymbol{w}}$ and $\eta_j^{\boldsymbol{w}}$ satisfying Conjecture 2.5 that intersect.

3. Main Theorem

In this section, we define the elements $s_i^{\boldsymbol{w}} \in \mathcal{A}$ and the vectors $\lambda_i^{\boldsymbol{w}}$ to present the main theorem of this paper precisely. This will manifest how the reflections $r_i^{\boldsymbol{w}}$ arise in relation to the c-vectors. The key idea is that we make the formulae (1.10) inductively hold for each mutation sequence $\boldsymbol{w}$. The process shows that there is a unique term in $s_i^{\boldsymbol{w}}$ that survives mod $2\mathcal{A}$ without regard to the choice of an ordering $\prec$.

Throughout this section, assume that $B = [b_{ij}]$ is a skew-symmetrizable matrix. Fix a linear ordering $\prec$ on $\mathcal{I}$ to obtain its associated GIM $A = [a_{ij}]$ from (1.7).

Example Λ-1. *As a running example in this section, we consider the skew-symmetrizable matrix*

$$B = \begin{bmatrix} 0 & 1 & -3 \\ -2 & 0 & -2 \\ 3 & 1 & 0 \end{bmatrix}$$

with symmetrizer $D = \mathrm{diag}(1, 2, 1)$ *and linear ordering* $1 \prec 2 \prec 3$. *Following the convention in* (1.7), *we produce the GIM*

$$A = \begin{bmatrix} 2 & 1 & -3 \\ 2 & 2 & -2 \\ -3 & -1 & 2 \end{bmatrix}.$$

Assume that a mutation sequence $\boldsymbol{w}$ is given. We will inductively define the elements $s_i^{\boldsymbol{w}} \in \mathcal{A}$ and the vectors $\lambda_i^{\boldsymbol{w}}$, $i \in \mathcal{I}$, in what follows. The procedure is summarized in Table 1.

For convenience, we recall the definition of $\mathcal{A}$ and its representation on $\mathbb{Z}^n$. As before, set $\lambda_1 = (1, 0, \dots, 0)$, $\lambda_2 = (0, 1, 0, \dots, 0), \dots,$ $\lambda_n = (0, \dots, 0, 1)$ to be a basis of $\mathbb{Z}^n$.

Definition Λ-1. Let $\mathcal{A}$ be the (unital) $\mathbb{Z}$-algebra generated by s_i, e_i, $i \in \mathcal{I}$, subject to the following relations:

$$s_i^2 = 1, \quad \sum_{i=1}^n e_i = 1, \quad s_i e_i = -e_i, \quad e_i s_j = \begin{cases} s_i + e_i - 1 & \text{if } i = j, \\ e_i & \text{if } i \neq j, \end{cases} \quad e_i e_j = \begin{cases} e_i & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Define a representation $\pi : \mathcal{A} \to \mathrm{End}(\mathbb{Z}^n)$ by

$$\pi(s_i)(\lambda_j) = \lambda_j - a_{ji}\lambda_i \quad \text{and} \quad \pi(e_i)(\lambda_j) = \delta_{ij}\lambda_i \qquad \text{for } i, j \in \mathcal{I}, \tag{3.1}$$

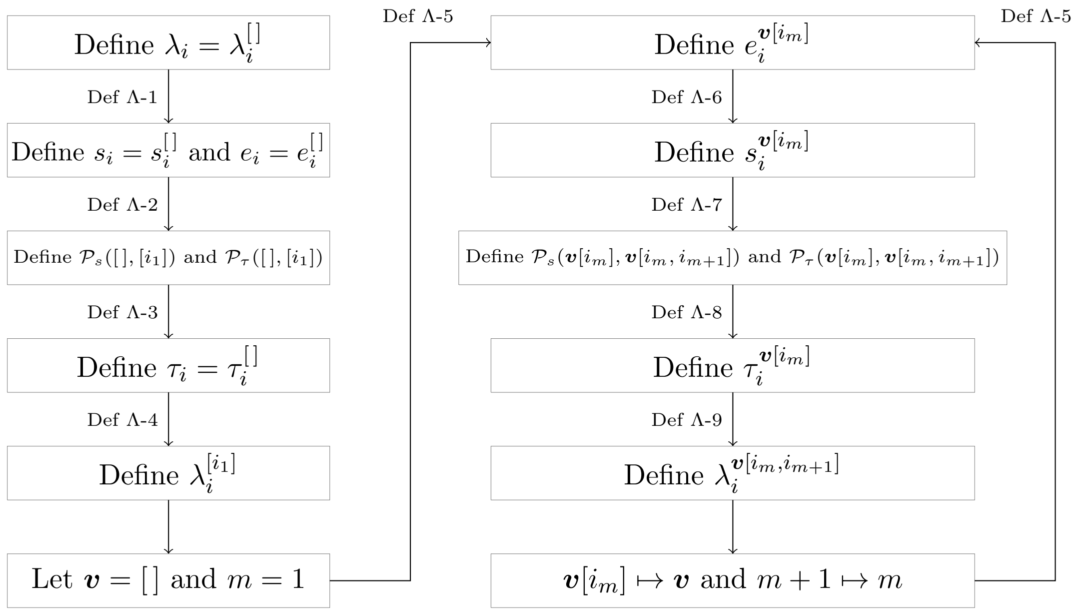


TABLE 1. Flow chart for defining $s_i^{\boldsymbol{w}}$ and $\lambda_i^{\boldsymbol{w}}$

and by extending it through linearity, where δ_{ij} is the Kronecker delta. We will suppress π when we write the action of an element of $\mathcal{A}$ on $\mathbb{Z}^n$.

Example Λ-2. *Continuing from Example Λ-1, the action of s_i, $i = 1, 2, 3$, are respectively given by the following matrices:*

$$\begin{bmatrix} -1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix}.$$

Here the action of s_i on the vector λ_j is to be understood by multiplication of the matrix on the right.

Definition Λ-2. Suppose that $\boldsymbol{w}$ starts with k. Let $\mathcal{P}_s([\,],[k])$ be the set of (i,j), $i,j \in \mathcal{I}$, such that

$$\lambda_i > s_k(\lambda_i) \text{ and } \lambda_j < s_k(\lambda_j) \text{ and } (k \prec i \prec j \text{ or } i \prec j \prec k), \text{ or}$$
$$\lambda_j < s_k(\lambda_j) \text{ and } k = i \prec j.$$

Let $\mathcal{P}_\tau([\,],[k])$ be the set of (i,j), $i,j \in \mathcal{I}$, such that

$$\lambda_i > s_k(\lambda_i) \text{ and } \lambda_j < s_k(\lambda_j) \text{ and } (k \prec i \prec j \text{ or } i \prec j \prec k), \text{ or}$$
$$\lambda_j > s_k(\lambda_j) \text{ and } k = i \succ j.$$

Definition Λ-3. Define

$$e_{\tau,i} = \sum e_j \in \mathcal{A},$$

where the sum is over j such that $(i,j) \in \mathcal{P}_\tau([\,],[k])$ or $(j,i) \in \mathcal{P}_\tau([\,],[k])$, and define

$$\tau_i = s_i + 2(1-s_i)e_{\tau,i} \qquad \text{for } i \in \mathcal{I}. \tag{3.2}$$

Definition Λ-4. Define

$$\lambda_i^{[k]} = \begin{cases} \tau_k(\lambda_i) & \text{if } \lambda_i < s_k(\lambda_i) \text{ and } k \prec i, \text{ or if } \lambda_i > s_k(\lambda_i) \text{ and } k \succ i, \text{ or if } i = k, \\ \lambda_i & \text{otherwise.} \end{cases} \tag{3.3}$$

Example Λ-3. *Continuing from Example Λ-2, take* $\boldsymbol{w} = [2,3]$ *so* $k = 2$. *We have* $\mathcal{P}_s([],[2]) = \{(2,3)\}$ *and* $\mathcal{P}_\tau([],[2]) = \{(2,1)\}$. *It follows that* $e_{\tau,1} = e_2$, $e_{\tau,2} = e_1$, *and* $e_{\tau,3} = 0$. *Putting everything together we see that*

$$\tau_1 = s_1 + 2(1-s_1)e_2, \quad \tau_2 = s_2 + 2(1-s_2)e_1, \quad \tau_3 = s_3.$$

We then have

$$\tau_2(\lambda_1) = (2-s_2)(\lambda_1) = (1,1,0), \quad \tau_2(\lambda_2) = s_2(\lambda_2) = (0,-1,0), \quad \tau_2(\lambda_3) = s_2(\lambda_3) = (0,1,1).$$

By (3.3) *we define* $\lambda_i^{[2]} := \tau_2(\lambda_i)$ *for all* $i \in \mathcal{I}$.

Definition Λ-5. Inductively, assume $\boldsymbol{w} = \boldsymbol{v}[k,\ell,\ldots,m]$, including the case $\boldsymbol{v} = [\,]$. For $i \neq k$, define

$$e_i^{\boldsymbol{v}[k]} = \begin{cases} \tau_k^{\boldsymbol{v}} e_i^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}} & \text{if } \lambda_i^{\boldsymbol{v}} < s_k^{\boldsymbol{v}}(\lambda_i^{\boldsymbol{v}}) \text{ and } k \prec i, \text{ or if } \lambda_i^{\boldsymbol{v}} > s_k^{\boldsymbol{v}}(\lambda_i^{\boldsymbol{v}}) \text{ and } k \succ i, \\ e_i^{\boldsymbol{v}} & \text{otherwise,} \end{cases} \tag{3.4}$$

and

$$e_k^{\boldsymbol{v}[k]} = e_k^{\boldsymbol{v}} - e_k^{\boldsymbol{v}} e_+^{\boldsymbol{v}[k]},$$

where we set

$$e_+^{\boldsymbol{v}[k]} = \sum_{j \neq k,\ \lambda_j^{\boldsymbol{v}[k]} \neq \lambda_j^{\boldsymbol{v}}} e_j^{\boldsymbol{v}[k]}.$$

Example Λ-4. *Continuing from Example Λ-3 we have* $k = 2, \ell = 3$, *and* $\boldsymbol{v} = []$. *For* $i = 1,3$ *we have* $e_i^{[2]} = \tau_2 e_i \tau_2$. *More explicitly,*

$$e_1^{[2]} = \tau_2 e_1 \tau_2 = (2-s_2)e_1, \qquad e_3^{[2]} = \tau_2 e_3 \tau_2 = s_2 e_3.$$

For $i = 2$,

$$e_+^{[2]} = e_1^{[2]} + e_3^{[2]} = 2e_1 - s_2(e_1 - e_3)$$

and finally

$$e_2^{[2]} = e_2(1 - e_1^{[2]} - e_3^{[2]}) = s_2(e_1 - e_3) - e_1 + e_2 + e_3.$$

Definition Λ-6. Define

$$e^{\boldsymbol{v}[k]}_{s,i} = \sum e^{\boldsymbol{v}[k]}_j,$$

where the sum is over j such that $(i,j) \in \mathcal{P}_s(\boldsymbol{v}, \boldsymbol{v}[k])$ or $(j,i) \in \mathcal{P}_s(\boldsymbol{v}, \boldsymbol{v}[k])$, and define

(3.5)

$$s^{\boldsymbol{v}[k]}_i = \begin{cases} \tau^{\boldsymbol{v}}_k \tau^{\boldsymbol{v}}_i \tau^{\boldsymbol{v}}_k + 2(1 - \tau^{\boldsymbol{v}}_k \tau^{\boldsymbol{v}}_i \tau^{\boldsymbol{v}}_k) e^{\boldsymbol{v}[k]}_{s,i} & \text{if } \lambda^{\boldsymbol{v}}_i < s^{\boldsymbol{v}}_k(\lambda^{\boldsymbol{v}}_i) \text{ and } k \prec i, \text{ or if } \lambda^{\boldsymbol{v}}_i > s^{\boldsymbol{v}}_k(\lambda^{\boldsymbol{v}}_i) \text{ and } k \succ i, \\ \tau^{\boldsymbol{v}}_i + 2(1 - \tau^{\boldsymbol{v}}_i) e^{\boldsymbol{v}[k]}_{s,i} & \text{otherwise.} \end{cases}$$

Example Λ-5. *In Example Λ-3 we computed $\mathcal{P}_s([], [2]) = \{(2,3)\}$ so*

$$e^{[2]}_{s,1} = 0, \qquad e^{[2]}_{s,2} = e^{[2]}_3, \qquad e^{[2]}_{s,3} = e^{[2]}_2.$$

Now by comparing $s_i(\lambda_i)$ given in Example Λ-2 to λ_i, we have

$$\begin{aligned} s^{[2]}_1 &= \tau_2\tau_1\tau_2 + 2(1 - \tau_2\tau_1\tau_2) e^{[2]}_{s,1} = \tau_2\tau_1\tau_2 \\ &= (2 - 2s_1 + s_2 s_1)s_2 + 2(1 - 2s_2 + 2s_1 s_2 - s_2 s_1 s_2)e_1 + 2(-2 + 2s_1 + 2s_2 - s_2 s_1)e_3, \\ s^{[2]}_2 &= \tau_2 + 2(1 - \tau_2)e^{[2]}_{s,2} = 2(e_1 - e_3) + s_2(1 - 2(e_1 - e_3)) = s_2 + 2(1 - s_2)(e_1 - e_3), \\ s^{[2]}_3 &= \tau_3 + 2(1 - \tau_3)e^{[2]}_{s,3} = s_2 s_3 s_2 + 2(1 + s_2 s_3)e_2 + 2(1 - 2s_2 - s_2 s_3 s_2)e_3. \end{aligned}$$

Definition Λ-7. Let $\mathcal{P}_s(\boldsymbol{v}[k], \boldsymbol{v}[k,\ell])$ be the collection of (i,j) such that

$(\ell \prec i \prec j$ or $i \prec j \prec \ell)$ and $\lambda^{\boldsymbol{v}[k]}_i > s^{\boldsymbol{v}[k]}_\ell(\lambda^{\boldsymbol{v}[k]}_i)$ and $\lambda^{\boldsymbol{v}[k]}_j < s^{\boldsymbol{v}[k]}_\ell(\lambda^{\boldsymbol{v}[k]}_j)$, or
$\ell = i \succ j$ and $\lambda^{\boldsymbol{v}[k]}_\ell < 0$ and $\lambda^{\boldsymbol{v}[k]}_j > s^{\boldsymbol{v}[k]}_\ell(\lambda^{\boldsymbol{v}[k]}_j)$, or
$\ell = i \prec j$ and $\lambda^{\boldsymbol{v}[k]}_\ell > 0$ and $\lambda^{\boldsymbol{v}[k]}_j < s^{\boldsymbol{v}[k]}_\ell(\lambda^{\boldsymbol{v}[k]}_j)$.

Similarly, let $\mathcal{P}_\tau(\boldsymbol{v}[k], \boldsymbol{v}[k,\ell])$ be the collection of (i,j) such that

$(\ell \prec i \prec j$ or $i \prec j \prec \ell)$ and $\lambda^{\boldsymbol{v}[k]}_i > s^{\boldsymbol{v}[k]}_\ell(\lambda^{\boldsymbol{v}[k]}_i)$ and $\lambda^{\boldsymbol{v}[k]}_j < s^{\boldsymbol{v}[k]}_\ell(\lambda^{\boldsymbol{v}[k]}_j)$, or
$\ell = i \succ j$ and $\lambda^{\boldsymbol{v}[k]}_\ell > 0$ and $\lambda^{\boldsymbol{v}[k]}_j > s^{\boldsymbol{v}[k]}_\ell(\lambda^{\boldsymbol{v}[k]}_j)$, or
$\ell = i \prec j$ and $\lambda^{\boldsymbol{v}[k]}_\ell < 0$ and $\lambda^{\boldsymbol{v}[k]}_j < s^{\boldsymbol{v}[k]}_\ell(\lambda^{\boldsymbol{v}[k]}_j)$.

Example Λ-6. *Continuing from Example Λ-5 we have*

$$s^{[2]}_3(\lambda^{[2]}_1) = (1,1,1), \quad s^{[2]}_3(\lambda^{[2]}_2) = (0,3,2), \quad s^{[2]}_3(\lambda^{[2]}_3) = (0,-4,-3)$$

so $\mathcal{P}_s([2],[2,3]) = \emptyset$ and $\mathcal{P}_\tau([2],[2,3]) = \{(3,2)\}$.

Definition Λ-8. Define

$$e^{\boldsymbol{v}[k]}_{\tau,i} = \sum e^{\boldsymbol{v}[k]}_j \in \mathcal{A},$$

where the sum is over j such that $(i,j) \in \mathcal{P}_\tau(\boldsymbol{v}[k], \boldsymbol{v}[k,\ell])$ or $(j,i) \in \mathcal{P}_\tau(\boldsymbol{v}[k], \boldsymbol{v}[k,\ell])$, and define

$$\tau^{\boldsymbol{v}[k]}_i = s^{\boldsymbol{v}[k]}_i + 2(1 - s^{\boldsymbol{v}[k]}_i) e^{\boldsymbol{v}[k]}_{\tau,i} \qquad \text{for } i \in \mathcal{I}. \tag{3.6}$$

Definition Λ-9. Finally, define

$$\lambda_j^{\boldsymbol{v}[k,\ell]} = \begin{cases} \tau_\ell^{\boldsymbol{v}[k]}(\lambda_j^{\boldsymbol{v}[k]}) & \text{if } \lambda_j^{\boldsymbol{v}[k]} < s_\ell^{\boldsymbol{v}[k]}(\lambda_j^{\boldsymbol{v}[k]}) \text{ and } \ell \prec j, \\ & \quad \text{or if } \lambda_j^{\boldsymbol{v}[k]} > s_\ell^{\boldsymbol{v}[k]}(\lambda_j^{\boldsymbol{v}[k]}) \text{ and } \ell \succ j, \text{ or if } \ell = j, \\ \lambda_j^{\boldsymbol{v}[k]} & \text{otherwise.} \end{cases} \tag{3.7}$$

Example Λ-7. *Continuing from Example Λ-6 we have*

$$e_{\tau,1}^{[2,3]} = 0, \quad e_{\tau,2}^{[2,3]} = e_3^{[2]}, \quad e_{\tau,3}^{[2,3]} = e_2^{[2]}.$$

Furthermore,

$$\tau_1^{[2]} = s_1^{[2]}, \quad \tau_2^{[2]} = s_2 - 2(1-s_2)e_1, \quad \tau_3^{[2]} = s_2s_3s_2 + 2(1 - s_2s_3s_2 + s_2s_3 - s_2)e_1.$$

In Example Λ-5 we computed $s_3^{[2]}(\lambda_i^{[2]})$. Finishing our running example we conclude that

$$\begin{aligned} \lambda_1^{[2,3]} &= \lambda_1^{[2]} = (2-s_2)(\lambda_1) = (1,1,0), \\ \lambda_2^{[2,3]} &= \tau_3^{[2]}(\lambda_2^{[2]}) = s_2s_3(\lambda_2) = (0,1,2), \\ \lambda_3^{[2,3]} &= \tau_3^{[2]}(\lambda_3^{[2]}) = -s_2(\lambda_3) = (0,-1,-1). \end{aligned}$$

For any mutation sequence $\boldsymbol{w}$, set

$$\Lambda^{\boldsymbol{w}} = \begin{bmatrix} \lambda_1^{\boldsymbol{w}} \\ \vdots \\ \lambda_n^{\boldsymbol{w}} \end{bmatrix}.$$

Now we restate the main theorem of this paper.

Theorem 3.8 (Theorem 1.11)**.** *Let B be a skew-symmetrizable matrix. Fix a linear ordering $\prec$ on $\mathcal{I}$ to obtain a GIM A. Then, for any mutation sequence $\boldsymbol{w}$, we have*

$$\lambda_i^{\boldsymbol{w}} = c_i^{\boldsymbol{w}} \qquad \text{for all } i \in \mathcal{I}, \tag{C1}$$

or equivalently,

$$\Lambda^{\boldsymbol{w}} = C^{\boldsymbol{w}};$$

for $i, j \in \mathcal{I}$,

$$s_i^{\boldsymbol{w}}(\lambda_j^{\boldsymbol{w}}) = \begin{cases} \lambda_j^{\boldsymbol{w}} + b_{ji}^{\boldsymbol{w}}\lambda_i^{\boldsymbol{w}} & \text{if } i \prec j, \\ -\lambda_j^{\boldsymbol{w}} & \text{if } i = j, \\ \lambda_j^{\boldsymbol{w}} - b_{ji}^{\boldsymbol{w}}\lambda_i^{\boldsymbol{w}} & \text{if } i \succ j, \end{cases} \qquad e_i^{\boldsymbol{w}}(\lambda_j^{\boldsymbol{w}}) = \delta_{ij}\lambda_j^{\boldsymbol{w}}; \tag{C2}$$

moreover, for all $i \in I$,

$$s_i^{\boldsymbol{w}} \equiv r_i^{\boldsymbol{w}} \pmod{2\mathcal{A}}. \tag{C3}$$

In what follows, we prove (C3). A proof of (C1) and (C2) will be given in Section 4.

Proof of (C3). Notice from (3.6) that $s_i^{\boldsymbol{w}} \equiv \tau_i^{\boldsymbol{w}}$ modulo $2\mathcal{A}$. Then the equation (3.5) becomes modulo $2\mathcal{A}$

$$s_i^{\boldsymbol{v}[k]} \equiv \begin{cases} s_k^{\boldsymbol{v}} s_i^{\boldsymbol{v}} s_k^{\boldsymbol{v}} & \text{if } \lambda_i^{\boldsymbol{v}} < s_k^{\boldsymbol{v}}(\lambda_i^{\boldsymbol{v}}) \text{ and } k \prec i, \text{ or if } \lambda_i^{\boldsymbol{v}} > s_k^{\boldsymbol{v}}(\lambda_i^{\boldsymbol{v}}) \text{ and } k \succ i, \\ s_i^{\boldsymbol{v}} & \text{otherwise.} \end{cases} \tag{3.9}$$

Using (C1) and (C2), both of the conditions $\lambda_i^{\boldsymbol{v}} < s_k^{\boldsymbol{v}}(\lambda_i^{\boldsymbol{v}}), k \prec i$ and $\lambda_i^{\boldsymbol{v}} > s_k^{\boldsymbol{v}}(\lambda_i^{\boldsymbol{v}}), k \succ i$ can be rewritten as

$$b_{ik}^{\boldsymbol{v}} \lambda_k^{\boldsymbol{v}} = b_{ik}^{\boldsymbol{v}} c_k^{\boldsymbol{v}} > 0,$$

which does not depend on the choice of a GIM. Now (C3) follows from the definitions (3.2), (3.5) and (3.6) and from induction. □

3.1. **Some observations.** We close this section with examples which show some relationship between c-vectors and Lösungen.

Example 3.10. Consider the matrix $B = \begin{bmatrix} 0 & -1 & -1 & -1 \\ 1 & 0 & 1 & -1 \\ 1 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 \end{bmatrix}$. The mutation sequence $[1, 2, 3, 4, 2]$ produces the c-vector $(5, 2, 2, 2)$ which is not a Lösung for any choice of GIM associated to B.

Example 3.11 below shows that even if a c-vector is a real Lösung our formula may not always express it as such.

Example 3.11. Consider the matrix $B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$. This is a finite-type matrix that corresponds to an orientation of the Dynkin diagram A_4. After the mutation sequence $\boldsymbol{w} = [2, 4, 2]$ with the GIM associated to the linear order $4 \prec 2 \prec 3 \prec 1$ our formula produces

$$\lambda_3^{\boldsymbol{w}} = -s_2 s_4 s_2 \lambda_3 - 2 s_2 \lambda_3 + 2\lambda_3 + 2 s_4 s_2 \lambda_3 = (0, 0, 1, 1).$$

However, we also have $s_2 s_4 s_2 \lambda_3 = (0, 0, 1, 1)$ so we see that $\lambda_3^{\boldsymbol{w}}$ could just be expressed as the real Lösung $s_2 s_4 s_2 \lambda_3$ as opposed to the linear combination of real Lösungen given above. For completeness, we have $s_2 \lambda_3 = (0, 1, 1, 0)$ and $s_4 s_2 \lambda_3 = (0, 1, 1, 1)$.

It is also worth noting that the matrix representation of $-s_2 s_4 s_2 - 2 s_2 + 2 + 2 s_4 s_2$ is not equal to the matrix representation of $s_2 s_4 s_2$. Furthermore, for any choice of linear ordering the expression for $\lambda_3^{\boldsymbol{w}}$ that our formula produces will always have three or four terms even though the vector is a real Lösung.

4. Proof of (C1) and (C2) in Theorem 3.8

In this section we prove Theorem 3.8. We start with the following proposition which shows that $s_i^{\boldsymbol{w}}, e_i^{\boldsymbol{w}}$ satisfy natural relations for each $\boldsymbol{w}$.

Proposition 4.1. *For $i, j \in \mathcal{I}$ and for any mutation sequence $\boldsymbol{w}$, the following relations hold:*

$$\sum_{i=1}^{n} e_i^{\boldsymbol{w}} = 1, \tag{4.2}$$

$$e_i^{\boldsymbol{w}} e_j^{\boldsymbol{w}} = \delta_{ij} e_i^{\boldsymbol{w}}, \tag{4.3}$$

$$e_i^{\boldsymbol{w}} s_j^{\boldsymbol{w}} = \begin{cases} s_i^{\boldsymbol{w}} + e_i^{\boldsymbol{w}} - 1 & \text{if } i = j, \\ e_i^{\boldsymbol{w}} & \text{if } i \neq j, \end{cases} \tag{4.4}$$

$$e_i^{\boldsymbol{w}} \tau_j^{\boldsymbol{w}} = \begin{cases} \tau_i^{\boldsymbol{w}} + e_i^{\boldsymbol{w}} - 1 & \text{if } i = j, \\ e_i^{\boldsymbol{w}} & \text{if } i \neq j, \end{cases} \tag{4.5}$$

$$s_i^{\boldsymbol{w}} s_i^{\boldsymbol{w}} = 1, \qquad \tau_i^{\boldsymbol{w}} \tau_i^{\boldsymbol{w}} = 1, \tag{4.6}$$

$$s_i^{\boldsymbol{w}} e_i^{\boldsymbol{w}} = -e_i^{\boldsymbol{w}}, \qquad \tau_i^{\boldsymbol{w}} e_i^{\boldsymbol{w}} = -e_i^{\boldsymbol{w}}. \tag{4.7}$$

Proof. We use induction. If $\boldsymbol{w} = []$, all the relations follow from the definitions. Assume the relations hold for $\boldsymbol{v}$. In what follows, we show that they hold for $\boldsymbol{v}[k]$, $k \in \mathcal{I}$.

Relation (4.2): Since $e_i^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}} = e_i^{\boldsymbol{v}}$ for $i \neq k$ by induction, we have $(1 - e_k^{\boldsymbol{v}})\tau_k^{\boldsymbol{v}} = (1 - e_k^{\boldsymbol{v}})$, and obtain

$$\begin{aligned}
\sum_{i=1}^{n} e_i^{\boldsymbol{v}[k]} &= e_k^{\boldsymbol{v}[k]} + \sum_{i \neq k} e_i^{\boldsymbol{v}[k]} \\
&= e_k^{\boldsymbol{v}} - e_k^{\boldsymbol{v}} \sum_{\substack{i \neq k \\ \lambda_i^{\boldsymbol{v}[k]} \neq \lambda_i^{\boldsymbol{v}}}} e_i^{\boldsymbol{v}[k]} + \sum_{\substack{i \neq k \\ \lambda_i^{\boldsymbol{v}[k]} \neq \lambda_i^{\boldsymbol{v}}}} e_i^{\boldsymbol{v}[k]} + \sum_{\substack{i \neq k \\ \lambda_i^{\boldsymbol{v}[k]} = \lambda_i^{\boldsymbol{v}}}} e_i^{\boldsymbol{v}[k]} \\
&= e_k^{\boldsymbol{v}} + (1 - e_k^{\boldsymbol{v}}) \sum_{\substack{i \neq k \\ \lambda_i^{\boldsymbol{v}[k]} \neq \lambda_i^{\boldsymbol{v}}}} \tau_k^{\boldsymbol{v}} e_i^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}} + \sum_{\substack{i \neq k \\ \lambda_i^{\boldsymbol{v}[k]} = \lambda_i^{\boldsymbol{v}}}} e_i^{\boldsymbol{v}} \\
&= e_k^{\boldsymbol{v}} + \sum_{\substack{i \neq k \\ \lambda_i^{\boldsymbol{v}[k]} \neq \lambda_i^{\boldsymbol{v}}}} (1 - e_k^{\boldsymbol{v}}) e_i^{\boldsymbol{v}} + \sum_{\substack{i \neq k \\ \lambda_i^{\boldsymbol{v}[k]} = \lambda_i^{\boldsymbol{v}}}} e_i^{\boldsymbol{v}} = \sum_{i=1}^{n} e_i^{\boldsymbol{v}} = 1.
\end{aligned}$$

Relations (4.3): Suppose that $i \neq k$ and $j \neq k$. Note that $e_i^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}} = e_i^{\boldsymbol{v}}$ and $e_j^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}} = e_j^{\boldsymbol{v}}$. Assume $e_i^{\boldsymbol{v}[k]} = e_i^{\boldsymbol{v}}$ and $e_j^{\boldsymbol{v}[k]} = e_j^{\boldsymbol{v}}$. Then

$$e_i^{\boldsymbol{v}[k]} e_j^{\boldsymbol{v}[k]} = e_i^{\boldsymbol{v}} e_j^{\boldsymbol{v}} = \delta_{ij} e_i^{\boldsymbol{v}} = \delta_{ij} e_i^{\boldsymbol{v}[k]}.$$

Assume $e_i^{\boldsymbol{v}[k]} = \tau_k^{\boldsymbol{v}} e_i^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}}$ and $e_j^{\boldsymbol{v}[k]} = e_j^{\boldsymbol{v}}$. Then

$$e_i^{\boldsymbol{v}[k]} e_j^{\boldsymbol{v}[k]} = \tau_k^{\boldsymbol{v}} e_i^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}} e_j^{\boldsymbol{v}} = \tau_k^{\boldsymbol{v}} e_i^{\boldsymbol{v}} e_j^{\boldsymbol{v}} = \delta_{ij} \tau_k^{\boldsymbol{v}} e_i^{\boldsymbol{v}} = \delta_{ij} e_i^{\boldsymbol{v}[k]}.$$

Assume $e_i^{\boldsymbol{v}[k]} = e_i^{\boldsymbol{v}}$ and $e_j^{\boldsymbol{v}[k]} = \tau_k^{\boldsymbol{v}} e_j^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}}$. Then

$$e_i^{\boldsymbol{v}[k]} e_j^{\boldsymbol{v}[k]} = e_i^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}} e_j^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}} = e_i^{\boldsymbol{v}} e_j^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}} = \delta_{ij} e_i^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}} = \delta_{ij} e_i^{\boldsymbol{v}[k]}.$$

Assume $e_i^{\boldsymbol{v}[k]} = \tau_k^{\boldsymbol{v}} e_i^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}}$ and $e_j^{\boldsymbol{v}[k]} = \tau_k^{\boldsymbol{v}} e_j^{\boldsymbol{v}} \tau_i^{\boldsymbol{v}}$. Then

$$e_i^{\boldsymbol{v}[k]} e_j^{\boldsymbol{v}[k]} = \tau_k^{\boldsymbol{v}} e_i^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}} e_j^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}} = \tau_k^{\boldsymbol{v}} e_i^{\boldsymbol{v}} e_j^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}} = \delta_{ij} \tau_k^{\boldsymbol{v}} e_i^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}} = \delta_{ij} e_i^{\boldsymbol{v}[k]}.$$

For $i \neq k$ and $j \neq k$, write $A = \left(1 - \sum_{i \neq k,\ \lambda_i^{\boldsymbol{v}[k]} \neq \lambda_i^{\boldsymbol{v}}} e_i^{\boldsymbol{v}[k]} \right)$ for the time being, and we get

$$e_k^{\boldsymbol{v}[k]} e_j^{\boldsymbol{v}[k]} = e_k^{\boldsymbol{v}} A e_j^{\boldsymbol{v}[k]} = \begin{cases} e_k^{\boldsymbol{v}} (e_j^{\boldsymbol{v}[k]} - e_j^{\boldsymbol{v}[k]}) = 0 & \text{if } \lambda_i^{\boldsymbol{v}[k]} \neq \lambda_i^{\boldsymbol{v}}, \\ e_k^{\boldsymbol{v}} e_j^{\boldsymbol{v}[k]} = e_k^{\boldsymbol{v}} e_j^{\boldsymbol{v}} = 0 & \text{if } \lambda_i^{\boldsymbol{v}[k]} = \lambda_i^{\boldsymbol{v}}, \end{cases}$$

$$e_i^{\boldsymbol{v}[k]} e_k^{\boldsymbol{v}[k]} = e_i^{\boldsymbol{v}[k]} e_k^{\boldsymbol{v}} A = \begin{cases} \tau_k^{\boldsymbol{v}} e_i^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}} e_k^{\boldsymbol{v}} A = \tau_k^{\boldsymbol{v}} e_i^{\boldsymbol{v}} e_k^{\boldsymbol{v}} A = 0 & \text{if } \lambda_i^{\boldsymbol{v}[k]} \neq \lambda_i^{\boldsymbol{v}}, \\ e_i^{\boldsymbol{v}} e_k^{\boldsymbol{v}} A = 0 & \text{if } \lambda_i^{\boldsymbol{v}[k]} = \lambda_i^{\boldsymbol{v}}, \end{cases}$$

$$e_k^{\boldsymbol{v}[k]} e_k^{\boldsymbol{v}[k]} = e_k^{\boldsymbol{v}} A e_k^{\boldsymbol{v}} A = (e_k^{\boldsymbol{v}} - \sum_{i \neq k,\ \lambda_i^{\boldsymbol{v}[k]} \neq \lambda_i^{\boldsymbol{v}}} e_k^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}} e_i^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}} e_k^{\boldsymbol{v}}) A = e_k^{\boldsymbol{v}} A = e_k^{\boldsymbol{v}[k]}.$$

We have proven

$$e_i^{\boldsymbol{v}[k]} e_j^{\boldsymbol{v}[k]} = \delta_{ij} e_i^{\boldsymbol{v}[k]}$$

for all $i, j \in \mathcal{I}$.

Relations (4.4): Assume that $i \neq j$ and $i \neq k$. Suppose that $e_i^{\boldsymbol{v}[k]} = e_i^{\boldsymbol{v}}$ and $e_j^{\boldsymbol{v}[k]} = e_j^{\boldsymbol{v}}$. Then we have

$$e_i^{\boldsymbol{v}[k]} s_j^{\boldsymbol{v}[k]} = e_i^{\boldsymbol{v}} (\tau_j^{\boldsymbol{v}} + 2(1 - \tau_j^{\boldsymbol{v}}) e_{s,j}^{\boldsymbol{v}[k]}) = e_i^{\boldsymbol{v}} + 2 e_i^{\boldsymbol{v}} (1 - \tau_j^{\boldsymbol{v}}) e_{s,j}^{\boldsymbol{v}[k]} = e_i^{\boldsymbol{v}} = e_i^{\boldsymbol{v}[k]}.$$

Suppose that $e_i^{\boldsymbol{v}[k]} = \tau_k^{\boldsymbol{v}} e_i^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}}$ and $e_j^{\boldsymbol{v}[k]} = e_j^{\boldsymbol{v}}$.

$$e_i^{\boldsymbol{v}[k]} s_j^{\boldsymbol{v}[k]} = \tau_k^{\boldsymbol{v}} e_i^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}} (\tau_j^{\boldsymbol{v}} + 2(1 - \tau_j^{\boldsymbol{v}}) e_{s,j}^{\boldsymbol{v}[k]}) = \tau_k^{\boldsymbol{v}} e_i^{\boldsymbol{v}} + 2(\tau_k^{\boldsymbol{v}} e_i^{\boldsymbol{v}} - \tau_k^{\boldsymbol{v}} e_i^{\boldsymbol{v}} \tau_j^{\boldsymbol{v}}) e_{s,j}^{\boldsymbol{v}[k]} = \tau_k^{\boldsymbol{v}} e_i^{\boldsymbol{v}} = e_i^{\boldsymbol{v}[k]}.$$

Suppose that $e_i^{\boldsymbol{v}[k]} = e_i^{\boldsymbol{v}}$ and $e_j^{\boldsymbol{v}[k]} = \tau_k^{\boldsymbol{v}} e_j^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}}$.

$$e_i^{\boldsymbol{v}[k]} s_j^{\boldsymbol{v}[k]} = e_i^{\boldsymbol{v}} \left[\tau_k^{\boldsymbol{v}} \tau_j^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}} + 2(1 - \tau_k^{\boldsymbol{v}} \tau_j^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}}) e_{s,j}^{\boldsymbol{v}[k]} \right] = e_i^{\boldsymbol{v}} + 2 e_i^{\boldsymbol{v}} (1 - \tau_k^{\boldsymbol{v}} \tau_j^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}}) e_{s,j}^{\boldsymbol{v}[k]} = e_i^{\boldsymbol{v}} = e_i^{\boldsymbol{v}[k]}.$$

Suppose that $e_i^{\boldsymbol{v}[k]} = \tau_k^{\boldsymbol{v}} e_i^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}}$ and $e_j^{\boldsymbol{v}[k]} = \tau_k^{\boldsymbol{v}} e_j^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}}$. Note that

$$\tau_k^{\boldsymbol{v}} e_i^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}} \tau_j^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}} = \tau_k^{\boldsymbol{v}} e_i^{\boldsymbol{v}} \tau_j^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}} = \tau_k^{\boldsymbol{v}} e_i^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}}.$$

Then we have

$$\begin{aligned} e_i^{\boldsymbol{v}[k]} s_j^{\boldsymbol{v}[k]} &= \tau_k^{\boldsymbol{v}} e_i^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}} \left[\tau_k^{\boldsymbol{v}} \tau_j^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}} + 2(1 - \tau_k^{\boldsymbol{v}} \tau_j^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}}) e_{s,j}^{\boldsymbol{v}[k]} \right] \\ &= \tau_k^{\boldsymbol{v}} e_i^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}} + 2(\tau_k^{\boldsymbol{v}} e_i^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}} - \tau_k^{\boldsymbol{v}} e_i^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}}) e_{s,j}^{\boldsymbol{v}[k]} = e_i^{\boldsymbol{v}[k]}. \end{aligned}$$

Assume that $i = k \neq j$. Suppose that $e_j^{\boldsymbol{v}[k]} = e_j^{\boldsymbol{v}}$. Note that

$$e_k^{\boldsymbol{v}[k]} \tau_j^{\boldsymbol{v}} = \left(e_k^{\boldsymbol{v}} - \sum_{\ell \neq k,\ \lambda_\ell^{\boldsymbol{v}[k]} \neq \lambda_\ell^{\boldsymbol{v}}} e_k^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}} e_\ell^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}} \right) \tau_j^{\boldsymbol{v}} = \left(e_k^{\boldsymbol{v}} - \sum_{\ell \neq k,\ \lambda_\ell^{\boldsymbol{v}[k]} \neq \lambda_\ell^{\boldsymbol{v}}} e_k^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}} e_\ell^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}} \right) = e_k^{\boldsymbol{v}[k]}.$$

Then we have

$$\begin{aligned} e_k^{\boldsymbol{v}[k]} s_j^{\boldsymbol{v}[k]} &= e_k^{\boldsymbol{v}[k]} (\tau_j^{\boldsymbol{v}} + 2(1 - \tau_j^{\boldsymbol{v}}) e_{s,j}^{\boldsymbol{v}[k]}) \\ &= e_k^{\boldsymbol{v}[k]} \tau_j^{\boldsymbol{v}} + 2 e_k^{\boldsymbol{v}[k]} (1 - \tau_j^{\boldsymbol{v}}) e_{s,j}^{\boldsymbol{v}[k]} = e_k^{\boldsymbol{v}[k]}. \end{aligned}$$

Suppose that $e_j^{\boldsymbol{v}[k]} = \tau_k^{\boldsymbol{v}} e_j^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}}$. Note that

$$\begin{aligned} e_k^{\boldsymbol{v}[k]} \tau_k^{\boldsymbol{v}} \tau_j^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}} &= (1 - \sum_{\ell \neq k} e_\ell^{\boldsymbol{v}[k]}) \tau_k^{\boldsymbol{v}} \tau_j^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}} \\ &= \tau_k^{\boldsymbol{v}} \tau_j^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}} - \sum_{\ell \neq k,\ \lambda_\ell^{\boldsymbol{v}[k]} = \lambda_\ell^{\boldsymbol{v}}} e_\ell^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}} \tau_j^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}} - \sum_{\ell \neq k,\ \lambda_\ell^{\boldsymbol{v}[k]} \neq \lambda_\ell^{\boldsymbol{v}}} \tau_k^{\boldsymbol{v}} e_\ell^{\boldsymbol{v}} \tau_j^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}} \\ &= \tau_k^{\boldsymbol{v}} \tau_j^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}} - \tau_k^{\boldsymbol{v}} e_j^{\boldsymbol{v}} \tau_j^b v \tau_k^{\boldsymbol{v}} - \sum_{\ell \neq k,\ \lambda_\ell^{\boldsymbol{v}[k]} = \lambda_\ell^{\boldsymbol{v}}} e_\ell^{\boldsymbol{v}} - \sum_{\ell \neq k,j,\ \lambda_\ell^{\boldsymbol{v}[k]} \neq \lambda_\ell^{\boldsymbol{v}}} \tau_k^{\boldsymbol{v}} e_\ell^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}} \\ &= 1 - \tau_k^{\boldsymbol{v}} e_j^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}} - \sum_{\ell \neq k,\ \lambda_\ell^{\boldsymbol{v}[k]} = \lambda_\ell^{\boldsymbol{v}}} e_\ell^{\boldsymbol{v}} - \sum_{\ell \neq k,j,\ \lambda_\ell^{\boldsymbol{v}[k]} \neq \lambda_\ell^{\boldsymbol{v}}} \tau_k^{\boldsymbol{v}} e_\ell^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}} \\ &= 1 - \sum_{\ell \neq k} e_\ell^{\boldsymbol{v}[k]} = e_k^{\boldsymbol{v}[k]}. \end{aligned}$$

Then we have

$$e_k^{\boldsymbol{v}[k]} s_j^{\boldsymbol{v}[k]} = e_k^{\boldsymbol{v}[k]} (\tau_k^{\boldsymbol{v}} \tau_j^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}} + 2(1 - \tau_k^{\boldsymbol{v}} \tau_j^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}}) e_{s,j}^{\boldsymbol{v}[k]}) = e_k^{\boldsymbol{v}[k]}.$$

Assume that $i = j \neq k$. Suppose that $\lambda_i^{\boldsymbol{v}[k]} = \lambda_i^{\boldsymbol{v}}$. Since $e_i^{\boldsymbol{v}} e_{s,i}^{\boldsymbol{v}[k]} = 0$, we get

$$\begin{aligned} e_i^{\boldsymbol{v}[k]} s_i^{\boldsymbol{v}[k]} &= e_i^{\boldsymbol{v}} (\tau_i^{\boldsymbol{v}} + 2(1 - \tau_i^{\boldsymbol{v}}) e_{s,i}^{\boldsymbol{v}[k]}) \\ &= e_i^{\boldsymbol{v}} \tau_i^{\boldsymbol{v}} - 2 e_i^{\boldsymbol{v}} \tau_i^{\boldsymbol{v}} e_{s,i}^{\boldsymbol{v}[k]} = \tau_i^{\boldsymbol{v}} + e_i^{\boldsymbol{v}} - 1 - 2(\tau_i^{\boldsymbol{v}} + e_i^{\boldsymbol{v}} - 1) e_{s,i}^{\boldsymbol{v}[k]} \\ &= \tau_i^{\boldsymbol{v}} + 2(1 - \tau_i^{\boldsymbol{v}}) e_{s,i}^{\boldsymbol{v}[k]} + e_i^{\boldsymbol{v}} - 1 = s_i^{\boldsymbol{v}[k]} + e_i^{\boldsymbol{v}[k]} - 1. \end{aligned}$$

The case $\lambda_i^{\boldsymbol{v}[k]} \neq \lambda_i^{\boldsymbol{v}}$ is similar to the case $\lambda_i^{\boldsymbol{v}[k]} = \lambda_i^{\boldsymbol{v}}$. We omit the computations for this case.

Assume that $i = j = k$. Then

$$e_k^{\boldsymbol{v}[k]} s_k^{\boldsymbol{v}[k]} = (1 - \sum_{\ell \neq k} e_k^{\boldsymbol{v}[k]}) s_k^{\boldsymbol{v}[k]} = s_k^{\boldsymbol{v}[k]} - \sum_{\ell \neq k} e_\ell^{\boldsymbol{v}[k]} s_k^{\boldsymbol{v}[k]} = s_k^{\boldsymbol{v}[k]} - \sum_{\ell \neq k} e_\ell^{\boldsymbol{v}[k]} = s_k^{\boldsymbol{v}[k]} + e_k^{\boldsymbol{v}[k]} - 1.$$

Relations (4.5): For $i \neq j$, we have $e_i^{\boldsymbol{v}[k]}(1 - s_i^{\boldsymbol{v}[k]}) = 0$ and

$$e_i^{\boldsymbol{v}[k]} \tau_j^{\boldsymbol{v}[k]} = e_i^{\boldsymbol{v}[k]} (s_j^{\boldsymbol{v}[k]} + 2(1 - s_j^{\boldsymbol{v}[k]}) e_{\tau,j}^{\boldsymbol{v}[k]}) = e_i^{\boldsymbol{v}[k]}.$$

For $i = j$, we get

$$\begin{aligned} e_i^{\boldsymbol{v}[k]} \tau_i^{\boldsymbol{v}[k]} &= e_i^{\boldsymbol{v}[k]} (s_j^{\boldsymbol{v}[k]} + 2(1 - s_i^{\boldsymbol{v}[k]}) e_{\tau,i}^{\boldsymbol{v}[k]}) = e_i^{\boldsymbol{v}[k]} s_j^{\boldsymbol{v}[k]} + 2 e_i^{\boldsymbol{v}[k]} (1 - s_i^{\boldsymbol{v}[k]}) e_{\tau,i}^{\boldsymbol{v}[k]} \\ &= s_i^{\boldsymbol{v}[k]} + e_i^{\boldsymbol{v}[k]} - 1 - 2 e_i^{\boldsymbol{v}[k]} s_i^{\boldsymbol{v}[k]} e_{\tau,i}^{\boldsymbol{v}[k]} = s_i^{\boldsymbol{v}[k]} + 2(1 - s_i^{\boldsymbol{v}[k]}) e_{\tau,i}^{\boldsymbol{v}[k]} + e_i^{\boldsymbol{v}[k]} - 1 \\ &= \tau_i^{\boldsymbol{v}[k]} + e_i^{\boldsymbol{v}[k]} - 1. \end{aligned}$$

Relations (4.6): Suppose that $i = k$ or $i \neq k$ and $\lambda_i^{\boldsymbol{v}[k]} = \lambda_i^{\boldsymbol{v}}$. Since $e_j^{\boldsymbol{v}} \tau_i^{\boldsymbol{v}} = e_j^{\boldsymbol{v}}$ and $\tau_k^{\boldsymbol{v}} e_j^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}} \tau_i^{\boldsymbol{v}} = \tau_k^{\boldsymbol{v}} e_j^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}}$ for $j \neq i, k$, we have

$$e_j^{\boldsymbol{v}[k]} \tau_i^{\boldsymbol{v}} = e_j^{\boldsymbol{v}[k]} \qquad \text{for } j \neq i.$$

Thus $e_{s,i}^{\boldsymbol{v}[k]} \tau_i^{\boldsymbol{v}} = e_{s,i}^{\boldsymbol{v}}$ or $e_{s,i}^{\boldsymbol{v}[k]}(1 - \tau_i^{\boldsymbol{v}}) = 0$, and we have

$$\begin{aligned} s_i^{\boldsymbol{v}[k]} s_i^{\boldsymbol{v}[k]} &= (\tau_i^{\boldsymbol{v}} + 2(1 - \tau_i^{\boldsymbol{v}}) e_{s,i}^{\boldsymbol{v}[k]})(\tau_i^{\boldsymbol{v}} + 2(1 - \tau_i^{\boldsymbol{v}}) e_{s,i}^{\boldsymbol{v}[k]}) \\ &= 1 + 2\tau_i^{\boldsymbol{v}}(1 - \tau_i^{\boldsymbol{v}}) e_{s,i}^{\boldsymbol{v}[k]} + 2(1 - \tau_i^{\boldsymbol{v}}) e_{s,i}^{\boldsymbol{v}[k]} \tau_i^{\boldsymbol{v}} + 4(1 - \tau_i^{\boldsymbol{v}}) e_{s,i}^{\boldsymbol{v}[k]} (1 - \tau_i^{\boldsymbol{v}}) e_{s,i}^{\boldsymbol{v}[k]} \\ &= 1 + 2(\tau_i^{\boldsymbol{v}} - 1) e_{s,i}^{\boldsymbol{v}[k]} + 2(1 - \tau_i^{\boldsymbol{v}}) e_{s,i}^{\boldsymbol{v}[k]} = 1. \end{aligned}$$

Suppose that $i \neq k$ and $\lambda_i^{\boldsymbol{v}[k]} \neq \lambda_i^{\boldsymbol{v}}$. Since $e_j^{\boldsymbol{v}[k]} \tau_k^{\boldsymbol{v}} \tau_i^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}} = e_j^{\boldsymbol{v}[k]}$ for $j \neq i$, the computation is similar to the previous case to obtain $s_i^{\boldsymbol{v}[k]} s_i^{\boldsymbol{v}[k]} = 1$ in this case as well. Furthermore, since $e_{\tau,i}^{\boldsymbol{v}[k]} s_i^{\boldsymbol{v}[k]} = e_{\tau,i}^{\boldsymbol{v}[k]}$, we get

$$\tau_i^{\boldsymbol{v}[k]} \tau_i^{\boldsymbol{v}[k]} = (s_i^{\boldsymbol{v}[k]} + 2(1 - s_i^{\boldsymbol{v}[k]}) e_{s,i}^{\boldsymbol{v}[k]})(s_i^{\boldsymbol{v}[k]} + 2(1 - s_i^{\boldsymbol{v}[k]}) e_{\tau,i}^{\boldsymbol{v}[k]}) = 1.$$

Relations (4.7): Assume $i \neq k$, and suppose that $\lambda_i^{\boldsymbol{v}[k]} \neq \lambda_i^{\boldsymbol{v}}$. Then

$$\begin{aligned} s_i^{\boldsymbol{v}[k]} e_i^{\boldsymbol{v}[k]} &= (\tau_k^{\boldsymbol{v}} \tau_i^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}} + 2(1 - \tau_k^{\boldsymbol{v}} \tau_i^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}}) e_{s,i}^{\boldsymbol{v}[k]}) e_i^{\boldsymbol{v}[k]} \\ &= \tau_k^{\boldsymbol{v}} \tau_i^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}} e_i^{\boldsymbol{v}[k]} = \tau_k^{\boldsymbol{v}} \tau_i^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}} e_i^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}} = -\tau_k^{\boldsymbol{v}} e_i^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}} = -e_i^{\boldsymbol{v}[k]}. \end{aligned}$$

The case $\lambda_i^{\boldsymbol{v}[k]} = \lambda_i^{\boldsymbol{v}}$ is similar. For $i = k$, we obtain

$$\begin{aligned} s_k^{\boldsymbol{v}[k]} e_k^{\boldsymbol{v}[k]} &= (\tau_k^{\boldsymbol{v}} + 2(1 - \tau_k^{\boldsymbol{v}}) e_{s,k}^{\boldsymbol{v}[k]}) e_k^{\boldsymbol{v}[k]} = \tau_k^{\boldsymbol{v}} e_k^{\boldsymbol{v}[k]} \\ &= \tau_k^{\boldsymbol{v}} e_k^{\boldsymbol{v}} (1 - \sum_{\ell \neq k,\ \lambda_\ell^{\boldsymbol{v}[k]} \neq \lambda_\ell^{\boldsymbol{v}}} e_j^{\boldsymbol{v}[k]}) = -e_k^{\boldsymbol{v}} (1 - \sum_{\ell \neq k,\ \lambda_\ell^{\boldsymbol{v}[k]} \neq \lambda_\ell^{\boldsymbol{v}}} e_j^{\boldsymbol{v}[k]}) = -e_k^{\boldsymbol{v}[k]}. \end{aligned}$$

For $i \in \mathcal{I}$, we have

$$\tau_i^{\boldsymbol{v}[k]} e_i^{\boldsymbol{v}[k]} = (s_i^{\boldsymbol{v}[k]} + 2(1 - s_i^{\boldsymbol{v}[k]}) e_{\tau,i}^{\boldsymbol{v}[k]}) e_i^{\boldsymbol{v}[k]} = s_i^{\boldsymbol{v}[k]} e_i^{\boldsymbol{v}[k]} = -e_i^{\boldsymbol{v}[k]}.$$

□

Proof of Theorem 3.8. The statements (C1) and (C2) are true for $\boldsymbol{w} = []$ from the definitions. Assume that (C1) and (C2) hold for $\boldsymbol{v}$. We will show that they also hold for $\boldsymbol{v}[k]$, $k \in \mathcal{I}$. There are cases (1)-(6) according to the order of i, j, k, and each case has several subcases. Since arguments are all similar, we will show details for the cases (1), (3), (4) and (6) and skip some details for the other cases.

To begin with, let us recall some definitions for ease of reference. From the definition of mutation in (2.1), we have

$$b_{ij}^{\boldsymbol{v}[k]} = \begin{cases} -b_{ij}^{\boldsymbol{v}} & \text{if } i = k \text{ or } j = k, \\ b_{ij}^{\boldsymbol{v}} + \operatorname{sgn}(b_{ik}^{\boldsymbol{v}})\, b_{ik}^{\boldsymbol{v}} b_{kj}^{\boldsymbol{v}} & \text{if } b_{ik}^{\boldsymbol{v}} b_{kj}^{\boldsymbol{v}} > 0, \\ b_{ij}^{\boldsymbol{v}} & \text{otherwise,} \end{cases} \tag{4.8}$$

and rewrite the definition of c-vectors as

$$c_i^{\boldsymbol{v}[k]} = \begin{cases} -c_i^{\boldsymbol{v}} & \text{if } i = k, \\ c_i^{\boldsymbol{v}} + \operatorname{sgn}(b_{ik}^{\boldsymbol{v}}) b_{ik}^{\boldsymbol{v}} c_k^{\boldsymbol{v}} & \text{if } b_{ik}^{\boldsymbol{v}} c_k^{\boldsymbol{v}} > 0, \\ c_i^{\boldsymbol{v}} & \text{otherwise.} \end{cases} \tag{4.9}$$

For $i \neq k$, consider the condition

$$\lambda_i^{\boldsymbol{v}} < s_k^{\boldsymbol{v}}(\lambda_i^{\boldsymbol{v}}) \text{ and } k \prec i, \text{ or } \lambda_i^{\boldsymbol{v}} > s_k^{\boldsymbol{v}}(\lambda_i^{\boldsymbol{v}}) \text{ and } k \succ i, \tag{$*$}$$

and rewrite (3.7), (3.4) and (3.5):

$$\lambda_i^{\boldsymbol{v}[k]} = \begin{cases} \tau_k^{\boldsymbol{v}}(\lambda_i^{\boldsymbol{v}}) & \text{if } (*) \text{ is true,} \\ \lambda_i^{\boldsymbol{v}} & \text{otherwise;} \end{cases} \tag{4.10}$$

$$e_i^{\boldsymbol{v}[k]} = \begin{cases} \tau_k^{\boldsymbol{v}} e_i^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}} & \text{if } (*) \text{ is true,} \\ e_i^{\boldsymbol{v}} & \text{otherwise;} \end{cases} \tag{4.11}$$

$$s_i^{\boldsymbol{v}[k]} = \begin{cases} \tau_k^{\boldsymbol{v}} \tau_i^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}} + 2(1 - \tau_k^{\boldsymbol{v}} \tau_i^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}}) e_{s,i}^{\boldsymbol{v}[k]} & \text{if } (*) \text{ is true,} \\ \tau_i^{\boldsymbol{v}} + 2(1 - \tau_i^{\boldsymbol{v}}) e_{s,i}^{\boldsymbol{v}[k]} & \text{otherwise.} \end{cases} \tag{4.12}$$

In each of the following cases (1)-(6), we will show the statements (C1) and (C2):

$$\lambda_i^{\boldsymbol{w}} = c_i^{\boldsymbol{w}} \qquad \text{for all } i \in \mathcal{I}; \tag{\textit{C1}}$$

for $i, j \in \mathcal{I}$,

$$(C2) \qquad e_i^{\boldsymbol{w}}(\lambda_j^{\boldsymbol{w}}) = \delta_{ij}\lambda_j^{\boldsymbol{w}}, \qquad s_i^{\boldsymbol{w}}(\lambda_j^{\boldsymbol{w}}) = \begin{cases} \lambda_j^{\boldsymbol{w}} + b_{ji}^{\boldsymbol{w}}\lambda_i^{\boldsymbol{w}} & \text{if } i \prec j, \\ -\lambda_j^{\boldsymbol{w}} & \text{if } i = j, \\ \lambda_j^{\boldsymbol{w}} - b_{ji}^{\boldsymbol{w}}\lambda_i^{\boldsymbol{w}} & \text{if } i \succ j. \end{cases}$$

1) Assume that $k \prec i \preceq j$. By induction we have

$$s_k^{\boldsymbol{v}}(\lambda_i^{\boldsymbol{v}}) = \lambda_i^{\boldsymbol{v}} + b_{ik}^{\boldsymbol{v}}\lambda_k^{\boldsymbol{v}}, \qquad s_k^{\boldsymbol{v}}(\lambda_j^{\boldsymbol{v}}) = \lambda_j^{\boldsymbol{v}} + b_{jk}^{\boldsymbol{v}}\lambda_k^{\boldsymbol{v}}.$$

a) Suppose $b_{ik}^{\boldsymbol{v}}\lambda_k^{\boldsymbol{v}} = -\lambda_i^{\boldsymbol{v}} + s_k^{\boldsymbol{v}}(\lambda_i^{\boldsymbol{v}}) < 0$ and $b_{jk}^{\boldsymbol{v}}\lambda_k^{\boldsymbol{v}} = -\lambda_j^{\boldsymbol{v}} + s_k^{\boldsymbol{v}}(\lambda_j^{\boldsymbol{v}}) < 0$. Then from (4.9), we have

$$c_i^{\boldsymbol{v}[k]} = c_i^{\boldsymbol{v}}, \qquad c_j^{\boldsymbol{v}[k]} = c_j^{\boldsymbol{v}},$$

and obtain from (4.10)

$$\lambda_i^{\boldsymbol{v}[k]} = \lambda_i^{\boldsymbol{v}}, \qquad \lambda_j^{\boldsymbol{v}[k]} = \lambda_j^{\boldsymbol{v}}.$$

By induction,

$$\lambda_i^{\boldsymbol{v}[k]} = c_i^{\boldsymbol{v}[k]}, \qquad \lambda_i^{\boldsymbol{v}[k]} = c_i^{\boldsymbol{v}[k]},$$

which proves (C1) in this case.

From (4.11),

$$e_i^{\boldsymbol{v}[k]} = e_i^{\boldsymbol{v}}, \qquad e_j^{\boldsymbol{v}[k]} = e_j^{\boldsymbol{v}},$$

and by induction,

$$\begin{aligned} e_i^{\boldsymbol{v}[k]}(\lambda_j^{\boldsymbol{v}[k]}) &= e_i^{\boldsymbol{v}}(\lambda_j^{\boldsymbol{v}}) = 0, & e_i^{\boldsymbol{v}[k]}(\lambda_i^{\boldsymbol{v}[k]}) &= e_i^{\boldsymbol{v}}(\lambda_i^{\boldsymbol{v}}) = \lambda_i^{\boldsymbol{v}} = \lambda_i^{\boldsymbol{v}[k]}, \\ e_j^{\boldsymbol{v}[k]}(\lambda_i^{\boldsymbol{v}[k]}) &= e_j^{\boldsymbol{v}}(\lambda_i^{\boldsymbol{v}}) = 0, & e_j^{\boldsymbol{v}[k]}(\lambda_j^{\boldsymbol{v}[k]}) &= e_j^{\boldsymbol{v}}(\lambda_j^{\boldsymbol{v}}) = \lambda_j^{\boldsymbol{v}} = \lambda_j^{\boldsymbol{v}[k]}. \end{aligned}$$

We also have

$$s_i^{\boldsymbol{v}[k]} = \tau_i^{\boldsymbol{v}} + 2(1-\tau_i^{\boldsymbol{v}})e_{s,i}^{\boldsymbol{v}[k]}, \qquad s_j^{\boldsymbol{v}[k]} = \tau_j^{\boldsymbol{v}} + 2(1-\tau_j^{\boldsymbol{v}})e_{s,j}^{\boldsymbol{v}[k]}.$$

From the definitions, $(i,j),(j,i) \notin \mathcal{P}_s(\boldsymbol{v},\boldsymbol{v}[k]) \cup \mathcal{P}_\tau(\boldsymbol{v},\boldsymbol{v}[k])$, and thus

$$\begin{aligned} s_i^{\boldsymbol{v}[k]}\lambda_j^{\boldsymbol{v}[k]} &= (\tau_i^{\boldsymbol{v}} + 2(1-\tau_i^{\boldsymbol{v}})e_{s,i}^{\boldsymbol{v}[k]})\lambda_j^{\boldsymbol{v}[k]} = \tau_i^{\boldsymbol{v}}\lambda_j^{\boldsymbol{v}[k]} = \tau_i^{\boldsymbol{v}}\lambda_j^{\boldsymbol{v}} \\ &= (s_i^{\boldsymbol{v}} + 2(1-s_i^{\boldsymbol{v}})e_{\tau,i}^{\boldsymbol{v}})\lambda_j^{\boldsymbol{v}} = s_i^{\boldsymbol{v}}\lambda_j^{\boldsymbol{v}} \\ &= \begin{cases} \lambda_i^{\boldsymbol{v}} + b_{ji}^{\boldsymbol{v}}\lambda_k^{\boldsymbol{v}} = \lambda_i^{\boldsymbol{v}[k]} + b_{ji}^{\boldsymbol{v}[k]}\lambda_k^{\boldsymbol{v}[k]} & \text{if } i \neq j, \\ -\lambda_i^{\boldsymbol{v}} = -\lambda_i^{\boldsymbol{v}[k]} & \text{if } i = j. \end{cases} \end{aligned}$$

Similarly, we get

$$s_j^{\boldsymbol{v}[k]}\lambda_i^{\boldsymbol{v}[k]} = \lambda_i^{\boldsymbol{v}[k]} - b_{ij}^{\boldsymbol{v}[k]}\lambda_k^{\boldsymbol{v}[k]} \qquad \text{for } i \neq j.$$

This proves (C2) in this case.

b) Suppose $b_{ik}^{\boldsymbol{v}}\lambda_k^{\boldsymbol{v}} = -\lambda_i^{\boldsymbol{v}} + s_k^{\boldsymbol{v}}(\lambda_i^{\boldsymbol{v}}) > 0$ and $b_{jk}^{\boldsymbol{v}}\lambda_k^{\boldsymbol{v}} = -\lambda_j^{\boldsymbol{v}} + s_k^{\boldsymbol{v}}(\lambda_j^{\boldsymbol{v}}) > 0$. From (4.9), we have

$$c_i^{\boldsymbol{v}[k]} = c_i^{\boldsymbol{v}} + \operatorname{sgn}(\lambda_k^{\boldsymbol{v}})b_{ik}^{\boldsymbol{v}}c_k^{\boldsymbol{v}}, \qquad c_j^{\boldsymbol{v}[k]} = c_j^{\boldsymbol{v}} + \operatorname{sgn}(\lambda_k^{\boldsymbol{v}})b_{jk}^{\boldsymbol{v}}c_k^{\boldsymbol{v}}.$$

On the other hand, we obtain from (4.10)

$$\lambda_i^{\boldsymbol{v}[k]} = \tau_k^{\boldsymbol{v}}(\lambda_i^{\boldsymbol{v}}) = (s_k^{\boldsymbol{v}} + 2(1 - s_k^{\boldsymbol{v}})e_{\tau,k}^{\boldsymbol{v}})(\lambda_i^{\boldsymbol{v}}).$$

If $\lambda_k^{\boldsymbol{v}} < 0$ then $(k, i) \in \mathcal{P}_\tau(\boldsymbol{v}, \boldsymbol{v}[k])$ and

$$\lambda_i^{\boldsymbol{v}[k]} = (s_k^{\boldsymbol{v}} + 2(1 - s_k^{\boldsymbol{v}}))(\lambda_i^{\boldsymbol{v}}) = 2\lambda_i^{\boldsymbol{v}} - s_k^{\boldsymbol{v}}(\lambda_i^{\boldsymbol{v}}) = \lambda_i^{\boldsymbol{v}} - b_{ik}^{\boldsymbol{v}}\lambda_k^{\boldsymbol{v}} = c_i^{\boldsymbol{v}[k]} \tag{4.13}$$

by induction. If $\lambda_k^{\boldsymbol{v}} > 0$ then $(k, i) \notin \mathcal{P}_\tau(\boldsymbol{v}, \boldsymbol{v}[k])$ and

$$\lambda_i^{\boldsymbol{v}[k]} = s_k^{\boldsymbol{v}}\lambda_i^{\boldsymbol{v}} = \lambda_i^{\boldsymbol{v}} + b_{ik}^{\boldsymbol{v}}\lambda_k^{\boldsymbol{v}} = c_i^{\boldsymbol{v}[k]}. \tag{4.14}$$

Similarly, $\lambda_j^{\boldsymbol{v}[k]} = c_j^{\boldsymbol{v}[k]}$. This proves (C1) in this case.

From (4.11),

$$e_i^{\boldsymbol{v}[k]} = \tau_k^{\boldsymbol{v}} e_i^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}}, \qquad e_j^{\boldsymbol{v}[k]} = \tau_k^{\boldsymbol{v}} e_j^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}},$$

and by induction,

$$\begin{aligned} e_i^{\boldsymbol{v}[k]}(\lambda_j^{\boldsymbol{v}[k]}) &= \tau_k^{\boldsymbol{v}} e_i^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}}(\tau_k^{\boldsymbol{v}}\lambda_j^{\boldsymbol{v}}) = \tau_k^{\boldsymbol{v}} e_i^{\boldsymbol{v}}(\lambda_j^{\boldsymbol{v}}) = 0, \\ e_i^{\boldsymbol{v}[k]}(\lambda_i^{\boldsymbol{v}[k]}) &= \tau_k^{\boldsymbol{v}} e_i^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}}(\tau_k^{\boldsymbol{v}}\lambda_i^{\boldsymbol{v}}) = \tau_k^{\boldsymbol{v}} e_i^{\boldsymbol{v}}(\lambda_i^{\boldsymbol{v}}) = \tau_k^{\boldsymbol{v}}\lambda_i^{\boldsymbol{v}} = \lambda_i^{\boldsymbol{v}[k]}. \end{aligned}$$

Similarly, $e_j^{\boldsymbol{v}[k]}(\lambda_i^{\boldsymbol{v}[k]}) = 0$ and $e_j^{\boldsymbol{v}[k]}(\lambda_j^{\boldsymbol{v}[k]}) = e_j^{\boldsymbol{v}[k]}$.

We have

$$s_i^{\boldsymbol{v}[k]} = \tau_k^{\boldsymbol{v}}\tau_i^{\boldsymbol{v}}\tau_k^{\boldsymbol{v}} + 2(1 - \tau_k^{\boldsymbol{v}}\tau_i^{\boldsymbol{v}}\tau_k^{\boldsymbol{v}})e_{s,i}^{\boldsymbol{v}[k]}, \qquad s_j^{\boldsymbol{v}[k]} = \tau_k^{\boldsymbol{v}}\tau_j^{\boldsymbol{v}}\tau_k^{\boldsymbol{v}} + 2(1 - \tau_k^{\boldsymbol{v}}\tau_j^{\boldsymbol{v}}\tau_k^{\boldsymbol{v}})e_{s,j}^{\boldsymbol{v}[k]}.$$

From the definitions, $(i, j), (j, i) \notin \mathcal{P}_s(\boldsymbol{v}, \boldsymbol{v}[k]) \cup \mathcal{P}_\tau(\boldsymbol{v}, \boldsymbol{v}[k])$, and thus

$$\begin{aligned} s_i^{\boldsymbol{v}[k]}\lambda_j^{\boldsymbol{v}[k]} &= (\tau_k^{\boldsymbol{v}}\tau_i^{\boldsymbol{v}}\tau_k^{\boldsymbol{v}} + 2(1 - \tau_k^{\boldsymbol{v}}\tau_i^{\boldsymbol{v}}\tau_k^{\boldsymbol{v}})e_{s,i}^{\boldsymbol{v}[k]})\lambda_j^{\boldsymbol{v}[k]} = \tau_k^{\boldsymbol{v}}\tau_i^{\boldsymbol{v}}\tau_k^{\boldsymbol{v}}\lambda_j^{\boldsymbol{v}[k]} = \tau_k^{\boldsymbol{v}}\tau_i^{\boldsymbol{v}}\lambda_j^{\boldsymbol{v}} \\ &= \tau_k^{\boldsymbol{v}}(s_i^{\boldsymbol{v}} + 2(1 - s_i^{\boldsymbol{v}})e_{\tau,i}^{\boldsymbol{v}})\lambda_j^{\boldsymbol{v}} = \tau_k^{\boldsymbol{v}} s_i^{\boldsymbol{v}}\lambda_j^{\boldsymbol{v}}. \end{aligned}$$

If $i \neq j$ and $\lambda_k^{\boldsymbol{v}} < 0$, then we obtain from (4.13)

$$\begin{aligned} s_i^{\boldsymbol{v}[k]}\lambda_j^{\boldsymbol{v}[k]} &= \tau_k^{\boldsymbol{v}} s_i^{\boldsymbol{v}}\lambda_j^{\boldsymbol{v}} = \tau_k^{\boldsymbol{v}}(\lambda_j^{\boldsymbol{v}} + b_{ji}^{\boldsymbol{v}}\lambda_i^{\boldsymbol{v}}) = \tau_k^{\boldsymbol{v}}\lambda_j^{\boldsymbol{v}} + b_{ji}^{\boldsymbol{v}}(s_k^{\boldsymbol{v}} + 2(1 - s_k^{\boldsymbol{v}})e_{\tau,k}^{\boldsymbol{v}})\lambda_i^{\boldsymbol{v}} \\ &= \lambda_j^{\boldsymbol{v}[k]} + b_{ji}^{\boldsymbol{v}[k]}(2 - s_k^{\boldsymbol{v}})\lambda_i^{\boldsymbol{v}} = \lambda_j^{\boldsymbol{v}[k]} + b_{ji}^{\boldsymbol{v}[k]}\lambda_i^{\boldsymbol{v}[k]}. \end{aligned}$$

If $i \neq j$ and $\lambda_k^{\boldsymbol{v}} > 0$, then it follows from (4.14) that

$$\begin{aligned} s_i^{\boldsymbol{v}[k]}\lambda_j^{\boldsymbol{v}[k]} &= \tau_k^{\boldsymbol{v}}\lambda_j^{\boldsymbol{v}} + b_{ji}^{\boldsymbol{v}}(s_k^{\boldsymbol{v}} + 2(1 - s_k^{\boldsymbol{v}})e_{\tau,k}^{\boldsymbol{v}})\lambda_i^{\boldsymbol{v}} \\ &= \lambda_j^{\boldsymbol{v}[k]} + b_{ji}^{\boldsymbol{v}[k]} s_k^{\boldsymbol{v}}\lambda_i^{\boldsymbol{v}} = \lambda_j^{\boldsymbol{v}[k]} + b_{ji}^{\boldsymbol{v}[k]}\lambda_i^{\boldsymbol{v}[k]}. \end{aligned}$$

Similarly, we get

$$s_j^{\boldsymbol{v}[k]}\lambda_i^{\boldsymbol{v}[k]} = \lambda_i^{\boldsymbol{v}[k]} - b_{ij}^{\boldsymbol{v}[k]}\lambda_k^{\boldsymbol{v}[k]} \qquad \text{for } i \neq j.$$

If $i = j$ then

$$\begin{aligned} s_i^{\boldsymbol{v}[k]}\lambda_i^{\boldsymbol{v}[k]} &= (\tau_k^{\boldsymbol{v}}\tau_i^{\boldsymbol{v}}\tau_k^{\boldsymbol{v}} + 2(1 - \tau_k^{\boldsymbol{v}}\tau_i^{\boldsymbol{v}}\tau_k^{\boldsymbol{v}})e_{s,i}^{\boldsymbol{v}[k]})\lambda_i^{\boldsymbol{v}[k]} = \tau_k^{\boldsymbol{v}}\tau_i^{\boldsymbol{v}}\tau_k^{\boldsymbol{v}}\lambda_i^{\boldsymbol{v}[k]} = \tau_k^{\boldsymbol{v}}\tau_i^{\boldsymbol{v}}\lambda_i^{\boldsymbol{v}} \\ &= \tau_k^{\boldsymbol{v}} s_i^{\boldsymbol{v}}\lambda_i^{\boldsymbol{v}} = -\tau_k^{\boldsymbol{v}}\lambda_i^{\boldsymbol{v}} = -\lambda_i^{\boldsymbol{v}[k]}. \end{aligned}$$

This proves (C2) in this case.

c) Suppose $b^{\boldsymbol{v}}_{ik}\lambda^{\boldsymbol{v}}_k = -\lambda^{\boldsymbol{v}}_i + s^{\boldsymbol{v}}_k(\lambda^{\boldsymbol{v}}_i) < 0$ and $b^{\boldsymbol{v}}_{jk}\lambda^{\boldsymbol{v}}_k = -\lambda^{\boldsymbol{v}}_j + s^{\boldsymbol{v}}_k(\lambda^{\boldsymbol{v}}_j) > 0$. From (4.9), we have

$$c^{\boldsymbol{v}[k]}_i = c^{\boldsymbol{v}}_i, \qquad c^{\boldsymbol{v}[k]}_j = c^{\boldsymbol{v}}_j + \operatorname{sgn}(\lambda^{\boldsymbol{v}}_k) b^{\boldsymbol{v}}_{jk} c^{\boldsymbol{v}}_k.$$

On the other hand, we obtain from (4.10)

$$\lambda^{\boldsymbol{v}[k]}_i = \lambda^{\boldsymbol{v}}_i, \qquad \lambda^{\boldsymbol{v}[k]}_j = \tau^{\boldsymbol{v}}_k(\lambda^{\boldsymbol{v}}_j) = (s^{\boldsymbol{v}}_k + 2(1-s^{\boldsymbol{v}}_k)e^{\boldsymbol{v}}_{\tau,k})(\lambda^{\boldsymbol{v}}_j).$$

Thus $\lambda^{\boldsymbol{v}[k]}_i = c^{\boldsymbol{v}[k]}_i$ by induction, and using the same argument as in (b), we also see that $\lambda^{\boldsymbol{v}[k]}_j = c^{\boldsymbol{v}[k]}_j$. Therefore (C1) is true in this case.

From (4.11),

$$e^{\boldsymbol{v}[k]}_i = e^{\boldsymbol{v}}_i, \qquad e^{\boldsymbol{v}[k]}_j = \tau^{\boldsymbol{v}}_k e^{\boldsymbol{v}}_j \tau^{\boldsymbol{v}}_k,$$

and it follows from similar computations to those in (a) and (b) that

$$\begin{aligned} e^{\boldsymbol{v}[k]}_i(\lambda^{\boldsymbol{v}[k]}_j) &= 0, & e^{\boldsymbol{v}[k]}_i(\lambda^{\boldsymbol{v}[k]}_i) &= \lambda^{\boldsymbol{v}[k]}_i, \\ e^{\boldsymbol{v}[k]}_j(\lambda^{\boldsymbol{v}[k]}_i) &= 0, & e^{\boldsymbol{v}[k]}_j(\lambda^{\boldsymbol{v}[k]}_j) &= \lambda^{\boldsymbol{v}[k]}_j. \end{aligned}$$

We have

$$s^{\boldsymbol{v}[k]}_i = \tau^{\boldsymbol{v}}_i + 2(1-\tau^{\boldsymbol{v}}_i)e^{\boldsymbol{v}[k]}_{s,i}, \qquad s^{\boldsymbol{v}[k]}_j = \tau^{\boldsymbol{v}}_k\tau^{\boldsymbol{v}}_j\tau^{\boldsymbol{v}}_k + 2(1-\tau^{\boldsymbol{v}}_k\tau^{\boldsymbol{v}}_j\tau^{\boldsymbol{v}}_k)e^{\boldsymbol{v}[k]}_{s,j}.$$

From the definitions, $(i,j) \in \mathcal{P}_s(\boldsymbol{v}, \boldsymbol{v}[k]) \cap \mathcal{P}_\tau(\boldsymbol{v}, \boldsymbol{v}[k])$, and thus

$$\begin{aligned} s^{\boldsymbol{v}[k]}_i\lambda^{\boldsymbol{v}[k]}_j &= (\tau^{\boldsymbol{v}}_i + 2(1-\tau^{\boldsymbol{v}}_i)e^{\boldsymbol{v}[k]}_{s,i})\lambda^{\boldsymbol{v}[k]}_j = \tau^{\boldsymbol{v}}_i\lambda^{\boldsymbol{v}[k]}_j + 2(1-\tau^{\boldsymbol{v}}_i)\lambda^{\boldsymbol{v}[k]}_j = 2\lambda^{\boldsymbol{v}[k]}_j - \tau^{\boldsymbol{v}}_i\lambda^{\boldsymbol{v}[k]}_j \\ &= 2\lambda^{\boldsymbol{v}[k]}_j - (s^{\boldsymbol{v}}_i + 2(1-s^{\boldsymbol{v}}_i)e^{\boldsymbol{v}}_{\tau,i})\lambda^{\boldsymbol{v}[k]}_j. \end{aligned}$$

If $i \neq j$ and $\lambda^{\boldsymbol{v}}_k < 0$, then $(k,i) \notin \mathcal{P}_\tau(\boldsymbol{v}, \boldsymbol{v}[k])$, $(k,j) \in \mathcal{P}_\tau(\boldsymbol{v}, \boldsymbol{v}[k])$, and thus $\lambda^{\boldsymbol{v}[k]}_j = \tau^{\boldsymbol{v}}_k(\lambda^{\boldsymbol{v}}_j) = \lambda^{\boldsymbol{v}}_j - b^{\boldsymbol{v}}_{jk}\lambda^{\boldsymbol{v}}_k$ and by (4.8)

$$\begin{aligned} s^{\boldsymbol{v}[k]}_i\lambda^{\boldsymbol{v}[k]}_j &= 2\lambda^{\boldsymbol{v}[k]}_j - (s^{\boldsymbol{v}}_i + 2(1-s^{\boldsymbol{v}}_i)e^{\boldsymbol{v}}_{\tau,i})(\lambda^{\boldsymbol{v}}_j - b^{\boldsymbol{v}}_{jk}\lambda^{\boldsymbol{v}}_k) \\ &= 2\lambda^{\boldsymbol{v}[k]}_j - (s^{\boldsymbol{v}}_i\lambda^{\boldsymbol{v}}_j - b^{\boldsymbol{v}}_{jk}(\lambda^{\boldsymbol{v}}_k - b^{\boldsymbol{v}}_{ki}\lambda^{\boldsymbol{v}}_i) + 2(1-s^{\boldsymbol{v}}_i)\lambda^{\boldsymbol{v}}_j) \\ &= 2\lambda^{\boldsymbol{v}[k]}_j - (2\lambda^{\boldsymbol{v}}_j - s^{\boldsymbol{v}}_i\lambda^{\boldsymbol{v}}_j - b^{\boldsymbol{v}}_{jk}\lambda^{\boldsymbol{v}}_k + b^{\boldsymbol{v}}_{jk}b^{\boldsymbol{v}}_{ki}\lambda^{\boldsymbol{v}}_i) \\ &= 2\lambda^{\boldsymbol{v}[k]}_j - (\lambda^{\boldsymbol{v}}_j - b^{\boldsymbol{v}}_{ji}\lambda^{\boldsymbol{v}}_i - b^{\boldsymbol{v}}_{jk}\lambda^{\boldsymbol{v}}_k + b^{\boldsymbol{v}}_{jk}b^{\boldsymbol{v}}_{ki}\lambda^{\boldsymbol{v}}_i) \\ &= \lambda^{\boldsymbol{v}[k]}_j + (b^{\boldsymbol{v}}_{ji} - b^{\boldsymbol{v}}_{jk}b^{\boldsymbol{v}}_{ki})\lambda^{\boldsymbol{v}[k]}_i = \lambda^{\boldsymbol{v}[k]}_j + b^{\boldsymbol{v}[k]}_{ji}\lambda^{\boldsymbol{v}[k]}_i. \end{aligned}$$

If $i \neq j$ and $\lambda^{\boldsymbol{v}}_k > 0$, then $(k,i),(k,j) \notin \mathcal{P}_\tau(\boldsymbol{v}, \boldsymbol{v}[k])$, and thus $\lambda^{\boldsymbol{v}[k]}_j = \tau^{\boldsymbol{v}}_k(\lambda^{\boldsymbol{v}}_j) = \lambda^{\boldsymbol{v}}_j + b^{\boldsymbol{v}}_{jk}\lambda^{\boldsymbol{v}}_k$ and by (4.8)

$$\begin{aligned} s^{\boldsymbol{v}[k]}_i\lambda^{\boldsymbol{v}[k]}_j &= 2\lambda^{\boldsymbol{v}[k]}_j - (s^{\boldsymbol{v}}_i + 2(1-s^{\boldsymbol{v}}_i)e^{\boldsymbol{v}}_{\tau,i})(\lambda^{\boldsymbol{v}}_j + b^{\boldsymbol{v}}_{jk}\lambda^{\boldsymbol{v}}_k) \\ &= \lambda^{\boldsymbol{v}[k]}_j + (b^{\boldsymbol{v}}_{ji} + b^{\boldsymbol{v}}_{jk}b^{\boldsymbol{v}}_{ki})\lambda^{\boldsymbol{v}[k]}_i = \lambda^{\boldsymbol{v}[k]}_j + b^{\boldsymbol{v}[k]}_{ji}\lambda^{\boldsymbol{v}[k]}_i. \end{aligned}$$

Similarly, we get

$$s_j^{\boldsymbol{v}[k]}\lambda_i^{\boldsymbol{v}[k]} = \lambda_i^{\boldsymbol{v}[k]} - b_{ij}^{\boldsymbol{v}[k]}\lambda_k^{\boldsymbol{v}[k]} \quad \text{for } i \neq j \quad \text{and} \quad s_i^{\boldsymbol{v}[k]}\lambda_i^{\boldsymbol{v}[k]} = -\lambda_i^{\boldsymbol{v}[k]}.$$

This proves (C2) in this case.

d) Suppose $b_{ik}^{\boldsymbol{v}}\lambda_k^{\boldsymbol{v}} = -\lambda_i^{\boldsymbol{v}} + s_k^{\boldsymbol{v}}(\lambda_i^{\boldsymbol{v}}) > 0$ and $b_{jk}^{\boldsymbol{v}}\lambda_k^{\boldsymbol{v}} = -\lambda_j^{\boldsymbol{v}} + s_k^{\boldsymbol{v}}(\lambda_j^{\boldsymbol{v}}) < 0$. This case is similar to case (c) right above.

<u>2) Assume that $i \preceq j \prec k$.</u> Since this case is similar to case (1), we omit the details.

<u>3) Assume that $i \prec k \prec j$.</u> By induction we have

$$s_k^{\boldsymbol{v}}(\lambda_i^{\boldsymbol{v}}) = \lambda_i^{\boldsymbol{v}} - b_{ik}^{\boldsymbol{v}}\lambda_k^{\boldsymbol{v}}, \qquad s_k^{\boldsymbol{v}}(\lambda_j^{\boldsymbol{v}}) = \lambda_j^{\boldsymbol{v}} + b_{jk}^{\boldsymbol{v}}\lambda_k^{\boldsymbol{v}}.$$

a) Suppose $b_{ik}^{\boldsymbol{v}}\lambda_k^{\boldsymbol{v}} = \lambda_i^{\boldsymbol{v}} - s_k^{\boldsymbol{v}}(\lambda_i^{\boldsymbol{v}}) < 0$ and $b_{jk}^{\boldsymbol{v}}\lambda_k^{\boldsymbol{v}} = -\lambda_j^{\boldsymbol{v}} + s_k^{\boldsymbol{v}}(\lambda_j^{\boldsymbol{v}}) < 0$. From (4.9), we have

$$c_i^{\boldsymbol{v}[k]} = c_i^{\boldsymbol{v}}, \qquad c_j^{\boldsymbol{v}[k]} = c_j^{\boldsymbol{v}}.$$

It follows from (4.10) that

$$\lambda_i^{\boldsymbol{v}[k]} = \lambda_i^{\boldsymbol{v}}, \qquad \lambda_j^{\boldsymbol{v}[k]} = \lambda_j^{\boldsymbol{v}}.$$

Thus $\lambda_i^{\boldsymbol{v}[k]} = c_i^{\boldsymbol{v}[k]}$ and $\lambda_j^{\boldsymbol{v}[k]} = c_j^{\boldsymbol{v}[k]}$ by induction. Thus (C1) is true in this case.

From (4.11),

$$e_i^{\boldsymbol{v}[k]} = e_i^{\boldsymbol{v}}, \qquad e_j^{\boldsymbol{v}[k]} = e_j^{\boldsymbol{v}},$$

and it follows from induction that

$$\begin{aligned} e_i^{\boldsymbol{v}[k]}(\lambda_j^{\boldsymbol{v}[k]}) &= 0, & e_i^{\boldsymbol{v}[k]}(\lambda_i^{\boldsymbol{v}[k]}) &= \lambda_i^{\boldsymbol{v}[k]}, \\ e_j^{\boldsymbol{v}[k]}(\lambda_i^{\boldsymbol{v}[k]}) &= 0, & e_j^{\boldsymbol{v}[k]}(\lambda_j^{\boldsymbol{v}[k]}) &= \lambda_j^{\boldsymbol{v}[k]}. \end{aligned}$$

We have

$$s_i^{\boldsymbol{v}[k]} = \tau_i^{\boldsymbol{v}} + 2(1-\tau_i^{\boldsymbol{v}})e_{s,i}^{\boldsymbol{v}[k]}, \qquad s_j^{\boldsymbol{v}[k]} = \tau_j^{\boldsymbol{v}} + 2(1-\tau_j^{\boldsymbol{v}})e_{s,j}^{\boldsymbol{v}[k]}.$$

Clearly, $(i,j),(j,i) \notin \mathcal{P}_s(\boldsymbol{v},\boldsymbol{v}[k]) \cup \mathcal{P}_\tau(\boldsymbol{v},\boldsymbol{v}[k])$, and thus

$$\begin{aligned} s_i^{\boldsymbol{v}[k]}\lambda_j^{\boldsymbol{v}[k]} &= (\tau_i^{\boldsymbol{v}} + 2(1-\tau_i^{\boldsymbol{v}})e_{s,i}^{\boldsymbol{v}[k]})\lambda_j^{\boldsymbol{v}[k]} = \tau_i^{\boldsymbol{v}}\lambda_j^{\boldsymbol{v}[k]} = (s_i^{\boldsymbol{v}} + 2(1-s_i^{\boldsymbol{v}})e_{\tau,i}^{\boldsymbol{v}})\lambda_j^{\boldsymbol{v}} \\ &= s_i^{\boldsymbol{v}}\lambda_j^{\boldsymbol{v}} = \lambda_j^{\boldsymbol{v}} + b_{ji}^{\boldsymbol{v}}\lambda_i^{\boldsymbol{v}} \\ &= \begin{cases} \lambda_i^{\boldsymbol{v}} + b_{ji}^{\boldsymbol{v}}\lambda_k^{\boldsymbol{v}} = \lambda_i^{\boldsymbol{v}[k]} + b_{ji}^{\boldsymbol{v}[k]}\lambda_k^{\boldsymbol{v}[k]} & \text{if } i \neq j, \\ -\lambda_i^{\boldsymbol{v}} = -\lambda_i^{\boldsymbol{v}[k]} & \text{if } i = j. \end{cases} \end{aligned}$$

Similarly, we get

$$s_j^{\boldsymbol{v}[k]}\lambda_i^{\boldsymbol{v}[k]} = \lambda_i^{\boldsymbol{v}[k]} - b_{ij}^{\boldsymbol{v}[k]}\lambda_k^{\boldsymbol{v}[k]} \quad \text{for } i \neq j \quad \text{and} \quad s_i^{\boldsymbol{v}[k]}\lambda_i^{\boldsymbol{v}[k]} = -\lambda_i^{\boldsymbol{v}[k]}.$$

This proves (C2) in this case.

b) Suppose $b_{ik}^{\boldsymbol{v}}\lambda_k^{\boldsymbol{v}} = \lambda_i^{\boldsymbol{v}} - s_k^{\boldsymbol{v}}(\lambda_i^{\boldsymbol{v}}) > 0$ and $b_{jk}^{\boldsymbol{v}}\lambda_k^{\boldsymbol{v}} = -\lambda_j^{\boldsymbol{v}} + s_k^{\boldsymbol{v}}(\lambda_j^{\boldsymbol{v}}) > 0$. From (4.9), we have

$$c_i^{\boldsymbol{v}[k]} = c_i^{\boldsymbol{v}} + \operatorname{sgn}(\lambda_k^{\boldsymbol{v}})b_{ik}^{\boldsymbol{v}}c_k^{\boldsymbol{v}}, \qquad c_j^{\boldsymbol{v}[k]} = c_j^{\boldsymbol{v}} + \operatorname{sgn}(\lambda_k^{\boldsymbol{v}})b_{jk}^{\boldsymbol{v}}c_k^{\boldsymbol{v}}.$$

We obtain from (4.10)

$$\lambda_i^{\boldsymbol{v}[k]} = \tau_k^{\boldsymbol{v}}(\lambda_i^{\boldsymbol{v}}) = (s_k^{\boldsymbol{v}} + 2(1 - s_k^{\boldsymbol{v}})e_{\tau,k}^{\boldsymbol{v}})(\lambda_i^{\boldsymbol{v}}).$$

If $\lambda_k^{\boldsymbol{v}} > 0$ then $(k,i) \in \mathcal{P}_\tau(\boldsymbol{v}, \boldsymbol{v}[k])$ and

$$\lambda_i^{\boldsymbol{v}[k]} = (s_k^{\boldsymbol{v}} + 2(1 - s_k^{\boldsymbol{v}}))(\lambda_i^{\boldsymbol{v}}) = 2\lambda_i^{\boldsymbol{v}} - s_k^{\boldsymbol{v}}(\lambda_i^{\boldsymbol{v}}) = \lambda_i^{\boldsymbol{v}} + b_{ik}^{\boldsymbol{v}}\lambda_k^{\boldsymbol{v}} = c_i^{\boldsymbol{v}[k]} \tag{4.15}$$

by induction. If $\lambda_k^{\boldsymbol{v}} < 0$ then $(k,i) \notin \mathcal{P}_\tau(\boldsymbol{v}, \boldsymbol{v}[k])$ and

$$\lambda_i^{\boldsymbol{v}[k]} = s_k^{\boldsymbol{v}}\lambda_i^{\boldsymbol{v}} = \lambda_i^{\boldsymbol{v}} - b_{ik}^{\boldsymbol{v}}\lambda_k^{\boldsymbol{v}} = c_i^{\boldsymbol{v}[k]}. \tag{4.16}$$

Similarly, $\lambda_j^{\boldsymbol{v}[k]} = c_j^{\boldsymbol{v}[k]}$. This proves (C1) in this case.

From (4.11),

$$e_i^{\boldsymbol{v}[k]} = \tau_k^{\boldsymbol{v}} e_i^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}}, \qquad e_j^{\boldsymbol{v}[k]} = \tau_k^{\boldsymbol{v}} e_j^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}},$$

and it follows from induction that

$$\begin{aligned} e_i^{\boldsymbol{v}[k]}(\lambda_j^{\boldsymbol{v}[k]}) &= 0, & e_i^{\boldsymbol{v}[k]}(\lambda_i^{\boldsymbol{v}[k]}) &= \lambda_i^{\boldsymbol{v}[k]}, \\ e_j^{\boldsymbol{v}[k]}(\lambda_i^{\boldsymbol{v}[k]}) &= 0, & e_j^{\boldsymbol{v}[k]}(\lambda_j^{\boldsymbol{v}[k]}) &= \lambda_j^{\boldsymbol{v}[k]}. \end{aligned}$$

We have

$$s_i^{\boldsymbol{v}[k]} = \tau_k^{\boldsymbol{v}}\tau_i^{\boldsymbol{v}}\tau_k^{\boldsymbol{v}} + 2(1 - \tau_k^{\boldsymbol{v}}\tau_i^{\boldsymbol{v}}\tau_k^{\boldsymbol{v}})e_{s,i}^{\boldsymbol{v}[k]}, \qquad s_j^{\boldsymbol{v}[k]} = \tau_k^{\boldsymbol{v}}\tau_j^{\boldsymbol{v}}\tau_k^{\boldsymbol{v}} + 2(1 - \tau_k^{\boldsymbol{v}}\tau_j^{\boldsymbol{v}}\tau_k^{\boldsymbol{v}})e_{s,j}^{\boldsymbol{v}[k]}.$$

Clearly, $(i,j), (j,i) \notin \mathcal{P}_s(\boldsymbol{v}, \boldsymbol{v}[k]) \cup \mathcal{P}_\tau(\boldsymbol{v}, \boldsymbol{v}[k])$, and as in (1)-(b),

$$s_i^{\boldsymbol{v}[k]}\lambda_j^{\boldsymbol{v}[k]} = \tau_k^{\boldsymbol{v}} s_i^{\boldsymbol{v}} \lambda_j^{\boldsymbol{v}}.$$

If $\lambda_k^{\boldsymbol{v}} > 0$, then we obtain from (4.15)

$$\begin{aligned} s_i^{\boldsymbol{v}[k]}\lambda_j^{\boldsymbol{v}[k]} &= \tau_k^{\boldsymbol{v}} s_i^{\boldsymbol{v}} \lambda_j^{\boldsymbol{v}} = \tau_k^{\boldsymbol{v}}(\lambda_j^{\boldsymbol{v}} + b_{ji}^{\boldsymbol{v}}\lambda_i^{\boldsymbol{v}}) = \tau_k^{\boldsymbol{v}}\lambda_j^{\boldsymbol{v}} + b_{ji}^{\boldsymbol{v}}(s_k^{\boldsymbol{v}} + 2(1 - s_k^{\boldsymbol{v}})e_{\tau,k}^{\boldsymbol{v}})\lambda_i^{\boldsymbol{v}} \\ &= \lambda_j^{\boldsymbol{v}[k]} + b_{ji}^{\boldsymbol{v}[k]}(2 - s_k^{\boldsymbol{v}})\lambda_i^{\boldsymbol{v}} = \lambda_j^{\boldsymbol{v}[k]} + b_{ji}^{\boldsymbol{v}[k]}\lambda_i^{\boldsymbol{v}[k]}. \end{aligned}$$

If $\lambda_k^{\boldsymbol{v}} < 0$, then it follows from (4.16) that

$$\begin{aligned} s_i^{\boldsymbol{v}[k]}\lambda_j^{\boldsymbol{v}[k]} &= \tau_k^{\boldsymbol{v}}\lambda_j^{\boldsymbol{v}} + b_{ji}^{\boldsymbol{v}}(s_k^{\boldsymbol{v}} + 2(1 - s_k^{\boldsymbol{v}})e_{\tau,k}^{\boldsymbol{v}})\lambda_i^{\boldsymbol{v}} \\ &= \lambda_j^{\boldsymbol{v}[k]} + b_{ji}^{\boldsymbol{v}[k]} s_k^{\boldsymbol{v}}\lambda_i^{\boldsymbol{v}} = \lambda_j^{\boldsymbol{v}[k]} + b_{ji}^{\boldsymbol{v}[k]}\lambda_i^{\boldsymbol{v}[k]}. \end{aligned}$$

Similarly, we get

$$s_j^{\boldsymbol{v}[k]}\lambda_i^{\boldsymbol{v}[k]} = \lambda_i^{\boldsymbol{v}[k]} - b_{ij}^{\boldsymbol{v}[k]}\lambda_k^{\boldsymbol{v}[k]}.$$

This proves (C2) in this case.

c) Suppose $b_{ik}^{\boldsymbol{v}}\lambda_k^{\boldsymbol{v}} = \lambda_i^{\boldsymbol{v}} - s_k^{\boldsymbol{v}}(\lambda_i^{\boldsymbol{v}}) < 0$ and $b_{jk}^{\boldsymbol{v}}\lambda_k^{\boldsymbol{v}} = -\lambda_j^{\boldsymbol{v}} + s_k^{\boldsymbol{v}}(\lambda_j^{\boldsymbol{v}}) > 0$. From (4.9), we have

$$c_i^{\boldsymbol{v}[k]} = c_i^{\boldsymbol{v}}, \qquad c_j^{\boldsymbol{v}[k]} = c_j^{\boldsymbol{v}} + \mathrm{sgn}(\lambda_k^{\boldsymbol{v}}) b_{jk}^{\boldsymbol{v}} c_k^{\boldsymbol{v}}.$$

On the other hand, we obtain from (4.10)

$$\lambda_i^{\boldsymbol{v}[k]} = \lambda_i^{\boldsymbol{v}}, \qquad \lambda_j^{\boldsymbol{v}[k]} = \tau_k^{\boldsymbol{v}}(\lambda_j^{\boldsymbol{v}}) = (s_k^{\boldsymbol{v}} + 2(1 - s_k^{\boldsymbol{v}})e_{\tau,k}^{\boldsymbol{v}})(\lambda_j^{\boldsymbol{v}}).$$

Thus $\lambda_i^{\boldsymbol{v}[k]} = c_i^{\boldsymbol{v}[k]}$ by induction, and using the same argument as in (b), we also see that $\lambda_j^{\boldsymbol{v}[k]} = c_j^{\boldsymbol{v}[k]}$. Therefore (C1) is true in this case.

From (4.11),

$$e_i^{\boldsymbol{v}[k]} = e_i^{\boldsymbol{v}}, \qquad e_j^{\boldsymbol{v}[k]} = \tau_k^{\boldsymbol{v}} e_j^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}},$$

and it follows from induction that

$$e_i^{\boldsymbol{v}[k]}(\lambda_j^{\boldsymbol{v}[k]}) = 0, \qquad e_i^{\boldsymbol{v}[k]}(\lambda_i^{\boldsymbol{v}[k]}) = \lambda_i^{\boldsymbol{v}[k]},$$
$$e_j^{\boldsymbol{v}[k]}(\lambda_i^{\boldsymbol{v}[k]}) = 0, \qquad e_j^{\boldsymbol{v}[k]}(\lambda_j^{\boldsymbol{v}[k]}) = \lambda_j^{\boldsymbol{v}[k]}.$$

We have

$$s_i^{\boldsymbol{v}[k]} = \tau_i^{\boldsymbol{v}} + 2(1-\tau_i^{\boldsymbol{v}}) e_{s,i}^{\boldsymbol{v}[k]}, \qquad s_j^{\boldsymbol{v}[k]} = \tau_k^{\boldsymbol{v}} \tau_j^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}} + 2(1-\tau_k^{\boldsymbol{v}} \tau_j^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}}) e_{s,j}^{\boldsymbol{v}[k]}.$$

From the definitions, $(i,j),(j,i) \notin \mathcal{P}_s(\boldsymbol{v}, \boldsymbol{v}[k])$, and thus

$$s_i^{\boldsymbol{v}[k]} \lambda_j^{\boldsymbol{v}[k]} = (\tau_i^{\boldsymbol{v}} + 2(1-\tau_i^{\boldsymbol{v}}) e_{s,i}^{\boldsymbol{v}[k]}) \lambda_j^{\boldsymbol{v}[k]} = \tau_i^{\boldsymbol{v}} \lambda_j^{\boldsymbol{v}[k]}.$$

If $\lambda_k^{\boldsymbol{v}} < 0$, then $(k,i) \notin \mathcal{P}_\tau(\boldsymbol{v}, \boldsymbol{v}[k])$, $(k,j) \in \mathcal{P}_\tau(\boldsymbol{v}, \boldsymbol{v}[k])$, and thus $\lambda_j^{\boldsymbol{v}[k]} = \tau_k^{\boldsymbol{v}}(\lambda_j^{\boldsymbol{v}}) = \lambda_j^{\boldsymbol{v}} - b_{jk}^{\boldsymbol{v}} \lambda_k^{\boldsymbol{v}}$ and by (4.8)

$$\begin{aligned} s_i^{\boldsymbol{v}[k]} \lambda_j^{\boldsymbol{v}[k]} &= \tau_i^{\boldsymbol{v}} \lambda_j^{\boldsymbol{v}[k]} = \tau_i^{\boldsymbol{v}}(\lambda_j^{\boldsymbol{v}} - b_{jk}^{\boldsymbol{v}} \lambda_k^{\boldsymbol{v}}) = (s_i^{\boldsymbol{v}} + 2(1-s_i^{\boldsymbol{v}}) e_{\tau,i}^{\boldsymbol{v}})(\lambda_j^{\boldsymbol{v}} - b_{jk}^{\boldsymbol{v}} \lambda_k^{\boldsymbol{v}}) \\ &= s_i^{\boldsymbol{v}} \lambda_j^{\boldsymbol{v}} - b_{jk}^{\boldsymbol{v}} s_i^{\boldsymbol{v}} \lambda_k^{\boldsymbol{v}} = \lambda_j^{\boldsymbol{v}} + b_{ji}^{\boldsymbol{v}} \lambda_i^{\boldsymbol{v}} - b_{jk}^{\boldsymbol{v}}(\lambda_k^{\boldsymbol{v}} + b_{ki}^{\boldsymbol{v}} \lambda_i^{\boldsymbol{v}}) \\ &= \lambda_j^{\boldsymbol{v}} - b_{jk}^{\boldsymbol{v}} + (b_{ji}^{\boldsymbol{v}} - b_{jk}^{\boldsymbol{v}} b_{ki}^{\boldsymbol{v}}) \lambda_i^{\boldsymbol{v}} = \lambda_j^{\boldsymbol{v}[k]} + b_{ji}^{\boldsymbol{v}[k]} \lambda_i^{\boldsymbol{v}[k]}. \end{aligned}$$

If $\lambda_k^{\boldsymbol{v}} > 0$, then $(k,i),(k,j) \notin \mathcal{P}_\tau(\boldsymbol{v}, \boldsymbol{v}[k])$ and thus $\lambda_j^{\boldsymbol{v}[k]} = \tau_k^{\boldsymbol{v}}(\lambda_j^{\boldsymbol{v}}) = \lambda_j^{\boldsymbol{v}} + b_{jk}^{\boldsymbol{v}} \lambda_k^{\boldsymbol{v}}$ and by (4.8)

$$\begin{aligned} s_i^{\boldsymbol{v}[k]} \lambda_j^{\boldsymbol{v}[k]} &= \tau_i^{\boldsymbol{v}} \lambda_j^{\boldsymbol{v}[k]} = (s_i^{\boldsymbol{v}} + 2(1-s_i^{\boldsymbol{v}}) e_{\tau,i}^{\boldsymbol{v}})(\lambda_j^{\boldsymbol{v}} + b_{jk}^{\boldsymbol{v}} \lambda_k^{\boldsymbol{v}}) \\ &= s_i^{\boldsymbol{v}} \lambda_j^{\boldsymbol{v}} + b_{jk}^{\boldsymbol{v}} s_i^{\boldsymbol{v}} \lambda_k^{\boldsymbol{v}} = \lambda_j^{\boldsymbol{v}} + b_{ji}^{\boldsymbol{v}} \lambda_i^{\boldsymbol{v}} + b_{jk}^{\boldsymbol{v}}(\lambda_k^{\boldsymbol{v}} + b_{ki}^{\boldsymbol{v}} \lambda_i^{\boldsymbol{v}}) \\ &= \lambda_j^{\boldsymbol{v}[k]} + (b_{ji}^{\boldsymbol{v}} + b_{jk}^{\boldsymbol{v}} b_{ki}^{\boldsymbol{v}}) \lambda_i^{\boldsymbol{v}[k]} = \lambda_j^{\boldsymbol{v}[k]} + b_{ji}^{\boldsymbol{v}[k]} \lambda_i^{\boldsymbol{v}[k]}. \end{aligned}$$

Similarly, we get

$$s_j^{\boldsymbol{v}[k]} \lambda_i^{\boldsymbol{v}[k]} = \lambda_i^{\boldsymbol{v}[k]} - b_{ij}^{\boldsymbol{v}[k]} \lambda_k^{\boldsymbol{v}[k]} \quad \text{for } i \neq j \quad \text{and} \quad s_i^{\boldsymbol{v}[k]} \lambda_i^{\boldsymbol{v}[k]} = -\lambda_i^{\boldsymbol{v}[k]}.$$

This proves (C2) in this case.

d) Suppose $b_{ik}^{\boldsymbol{v}} \lambda_k^{\boldsymbol{v}} = \lambda_i^{\boldsymbol{v}} - s_k^{\boldsymbol{v}}(\lambda_i^{\boldsymbol{v}}) > 0$ and $b_{jk}^{\boldsymbol{v}} \lambda_k^{\boldsymbol{v}} = -\lambda_j^{\boldsymbol{v}} + s_k^{\boldsymbol{v}}(\lambda_j^{\boldsymbol{v}}) < 0$. This case is similar to (c) and we omit the details.

4) Assume that $i \prec k = j$. By induction we have

$$s_k^{\boldsymbol{v}}(\lambda_i^{\boldsymbol{v}}) = \lambda_i^{\boldsymbol{v}} - b_{ik}^{\boldsymbol{v}} \lambda_k^{\boldsymbol{v}}, \qquad s_k^{\boldsymbol{v}}(\lambda_k^{\boldsymbol{v}}) = -\lambda_k^{\boldsymbol{v}}.$$

a) Suppose $b_{ik}^{\boldsymbol{v}} \lambda_k^{\boldsymbol{v}} = \lambda_i^{\boldsymbol{v}} - s_k^{\boldsymbol{v}}(\lambda_i^{\boldsymbol{v}}) < 0$. From (4.9), we have

$$c_i^{\boldsymbol{v}[k]} = c_i^{\boldsymbol{v}}, \qquad c_k^{\boldsymbol{v}[k]} = -c_k^{\boldsymbol{v}}.$$

Since $(k,k) \notin \mathcal{P}_\tau(\boldsymbol{v}, \boldsymbol{v}[k])$, we obtain from (3.7) and induction

$$\lambda_i^{\boldsymbol{v}[k]} = \lambda_i^{\boldsymbol{v}},$$
$$\lambda_k^{\boldsymbol{v}[k]} = \tau_k^{\boldsymbol{v}}(\lambda_k^{\boldsymbol{v}}) = (s_k^{\boldsymbol{v}} + 2(1-s_k^{\boldsymbol{v}})e_{\tau,k}^{\boldsymbol{v}})\lambda_k^{\boldsymbol{v}} = s_k^{\boldsymbol{v}}\lambda_k^{\boldsymbol{v}} = -\lambda_k^{\boldsymbol{v}}. \tag{4.17}$$

Thus $\lambda_i^{\boldsymbol{v}[k]} = c_i^{\boldsymbol{v}[k]}$ and $\lambda_k^{\boldsymbol{v}[k]} = c_k^{\boldsymbol{v}[k]}$ by induction, and (C1) is true in this case.

From (4.11) and (4.2),

$$e_i^{\boldsymbol{v}[k]} = e_i^{\boldsymbol{v}}, \qquad e_k^{\boldsymbol{v}[k]} = 1 - \sum_{\ell \neq k} e_\ell^{\boldsymbol{v}[k]},$$

and it follows from induction that

$$e_i^{\boldsymbol{v}[k]}(\lambda_k^{\boldsymbol{v}[k]}) = e_i^{\boldsymbol{v}}(-\lambda_k^{\boldsymbol{v}}) = 0, \qquad e_i^{\boldsymbol{v}[k]}(\lambda_i^{\boldsymbol{v}[k]}) = e_i^{\boldsymbol{v}}\lambda_i^{\boldsymbol{v}} = \lambda_i^{\boldsymbol{v}[k]},$$
$$e_k^{\boldsymbol{v}[k]}(\lambda_i^{\boldsymbol{v}[k]}) = (1 - \sum_{\ell \neq k} e_\ell^{\boldsymbol{v}[k]})\lambda_i^{\boldsymbol{v}[k]} = \lambda_i^{\boldsymbol{v}[k]} - \lambda_i^{\boldsymbol{v}[k]} = 0.$$

We have

$$s_i^{\boldsymbol{v}[k]} = \tau_i^{\boldsymbol{v}} + 2(1-\tau_i^{\boldsymbol{v}})e_{s,i}^{\boldsymbol{v}[k]}, \qquad s_k^{\boldsymbol{v}[k]} = \tau_k^{\boldsymbol{v}} + 2(1-\tau_k^{\boldsymbol{v}})e_{s,k}^{\boldsymbol{v}[k]}.$$

We see that $(k,i) \notin \mathcal{P}_s(\boldsymbol{v}, \boldsymbol{v}[k]) \cup \mathcal{P}_\tau(\boldsymbol{v}, \boldsymbol{v}[k])$, and thus

$$s_i^{\boldsymbol{v}[k]}\lambda_k^{\boldsymbol{v}[k]} = (\tau_i^{\boldsymbol{v}} + 2(1-\tau_i^{\boldsymbol{v}})e_{s,i}^{\boldsymbol{v}[k]})\lambda_k^{\boldsymbol{v}[k]} = \tau_i^{\boldsymbol{v}}\lambda_k^{\boldsymbol{v}[k]} = -(s_i^{\boldsymbol{v}} + 2(1-s_i^{\boldsymbol{v}})e_{\tau,i}^{\boldsymbol{v}})\lambda_k^{\boldsymbol{v}}$$
$$= -s_i^{\boldsymbol{v}}\lambda_k^{\boldsymbol{v}} = -\lambda_k^{\boldsymbol{v}} - b_{ki}^{\boldsymbol{v}}\lambda_i^{\boldsymbol{v}} = \lambda_k^{\boldsymbol{v}[k]} + b_{ki}^{\boldsymbol{v}[k]}\lambda_i^{\boldsymbol{v}[k]}.$$

Similarly, we get

$$s_k^{\boldsymbol{v}[k]}\lambda_i^{\boldsymbol{v}[k]} = \lambda_i^{\boldsymbol{v}[k]} - b_{ik}^{\boldsymbol{v}[k]}\lambda_k^{\boldsymbol{v}[k]}.$$

This proves (C2) in this case.

b) Suppose $b_{ik}^{\boldsymbol{v}}\lambda_k^{\boldsymbol{v}} = \lambda_i^{\boldsymbol{v}} - s_k^{\boldsymbol{v}}(\lambda_i^{\boldsymbol{v}}) > 0$. From (4.9), we have

$$c_i^{\boldsymbol{v}[k]} = c_i^{\boldsymbol{v}} + \operatorname{sgn}(\lambda_k^{\boldsymbol{v}})b_{ik}^{\boldsymbol{v}}c_k^{\boldsymbol{v}}, \qquad c_k^{\boldsymbol{v}[k]} = -c_k^{\boldsymbol{v}}.$$

On the other hand, we obtain from (3.7)

$$\lambda_i^{\boldsymbol{v}[k]} = \tau_k^{\boldsymbol{v}}(\lambda_i^{\boldsymbol{v}}) = (s_k^{\boldsymbol{v}} + 2(1-s_k^{\boldsymbol{v}})e_{\tau,k}^{\boldsymbol{v}})(\lambda_i^{\boldsymbol{v}}), \qquad \lambda_k^{\boldsymbol{v}[k]} = -\lambda_k^{\boldsymbol{v}}.$$

If $\lambda_k^{\boldsymbol{v}} < 0$ then $(k,i) \notin \mathcal{P}_\tau(\boldsymbol{v}, \boldsymbol{v}[k])$ and $\lambda_i^{\boldsymbol{v}[k]} = s_k^{\boldsymbol{v}}\lambda_i^{\boldsymbol{v}} = \lambda_i^{\boldsymbol{v}} - b_{ik}^{\boldsymbol{v}}\lambda_k^{\boldsymbol{v}}$; if $\lambda_k^{\boldsymbol{v}} > 0$ then $(k,i) \in \mathcal{P}_\tau(\boldsymbol{v}, \boldsymbol{v}[k])$ and $\lambda_i^{\boldsymbol{v}[k]} = (2 - s_k^{\boldsymbol{v}})\lambda_i^{\boldsymbol{v}} = \lambda_i^{\boldsymbol{v}} + b_{ik}^{\boldsymbol{v}}\lambda_k^{\boldsymbol{v}}$. Thus $\lambda_i^{\boldsymbol{v}[k]} = c_i^{\boldsymbol{v}[k]}$ and $\lambda_k^{\boldsymbol{v}[k]} = c_k^{\boldsymbol{v}[k]}$ by induction, and (C1) is true in this case.

From (4.11), (4.2) and (4.5),

$$e_i^{\boldsymbol{v}[k]} = \tau_k^{\boldsymbol{v}}e_i^{\boldsymbol{v}}\tau_k^{\boldsymbol{v}} = \tau_k^{\boldsymbol{v}}e_i^{\boldsymbol{v}}, \qquad e_k^{\boldsymbol{v}[k]} = 1 - \sum_{\ell \neq k} e_\ell^{\boldsymbol{v}[k]},$$

and it follows from induction that

$$e_i^{\boldsymbol{v}[k]}(\lambda_k^{\boldsymbol{v}[k]}) = \tau_k^{\boldsymbol{v}} e_i^{\boldsymbol{v}}(-\lambda_k^{\boldsymbol{v}}) = 0, \qquad e_i^{\boldsymbol{v}[k]}(\lambda_i^{\boldsymbol{v}[k]}) = \tau_k^{\boldsymbol{v}} e_i^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}} \lambda_i^{\boldsymbol{v}} = \tau_k^{\boldsymbol{v}} \lambda_i^{\boldsymbol{v}} = \lambda_i^{\boldsymbol{v}[k]},$$
$$e_k^{\boldsymbol{v}[k]}(\lambda_i^{\boldsymbol{v}[k]}) = (1 - \sum_{\ell \neq k} e_\ell^{\boldsymbol{v}[k]})\lambda_i^{\boldsymbol{v}[k]} = \lambda_i^{\boldsymbol{v}[k]} - \lambda_i^{\boldsymbol{v}[k]} = 0.$$

We have

$$s_i^{\boldsymbol{v}[k]} = \tau_k^{\boldsymbol{v}} \tau_i^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}} + 2(1 - \tau_k^{\boldsymbol{v}} \tau_i^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}}) e_{s,i}^{\boldsymbol{v}[k]}, \qquad s_k^{\boldsymbol{v}[k]} = \tau_k^{\boldsymbol{v}} + 2(1 - \tau_k^{\boldsymbol{v}}) e_{s,k}^{\boldsymbol{v}[k]}.$$

If $\lambda_k^{\boldsymbol{v}} < 0$, then $(k,i) \notin \mathcal{P}_\tau(\boldsymbol{v}, \boldsymbol{v}[k])$ and $(k,i) \in \mathcal{P}_s(\boldsymbol{v}, \boldsymbol{v}[k])$, and thus

$$\begin{aligned} s_i^{\boldsymbol{v}[k]} \lambda_k^{\boldsymbol{v}[k]} &= (\tau_k^{\boldsymbol{v}} \tau_i^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}} + 2(1 - \tau_k^{\boldsymbol{v}} \tau_i^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}}) e_{s,i}^{\boldsymbol{v}[k]}) \lambda_k^{\boldsymbol{v}[k]} = (2 - \tau_k^{\boldsymbol{v}} \tau_i^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}})(-\lambda_k^{\boldsymbol{v}}) \\ &= -2\lambda_k^{\boldsymbol{v}} - \tau_k^{\boldsymbol{v}} \tau_i^{\boldsymbol{v}} \lambda_k^{\boldsymbol{v}} = -2\lambda_k^{\boldsymbol{v}} - \tau_k (s_i^{\boldsymbol{v}} + 2(1 - s_i^{\boldsymbol{v}}) e_{\tau,i}^{\boldsymbol{v}}) \lambda_k^{\boldsymbol{v}} \\ &= -2\lambda_k^{\boldsymbol{v}} - \tau_k^{\boldsymbol{v}} s_i^{\boldsymbol{v}} \lambda_k^{\boldsymbol{v}} = -2\lambda_k^{\boldsymbol{v}} - \tau_k^{\boldsymbol{v}} (\lambda_k^{\boldsymbol{v}} + b_{ki}^{\boldsymbol{v}} \lambda_i^{\boldsymbol{v}}) \\ &= -2\lambda_k^{\boldsymbol{v}} + \lambda_k^{\boldsymbol{v}} - b_{ki}^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}} \lambda_i^{\boldsymbol{v}} = \lambda_k^{\boldsymbol{v}[k]} + b_{ki}^{\boldsymbol{v}[k]} \lambda_i^{\boldsymbol{v}[k]}, \end{aligned}$$

and since $\lambda_i^{\boldsymbol{v}[k]} = \tau_k^{\boldsymbol{v}} \lambda_i^{\boldsymbol{v}} = s_k^{\boldsymbol{v}} \lambda_i^{\boldsymbol{v}} = \lambda_i^{\boldsymbol{v}} - b_{ik}^{\boldsymbol{v}} \lambda_k^{\boldsymbol{v}}$, we have

$$\begin{aligned} s_k^{\boldsymbol{v}[k]} \lambda_i^{\boldsymbol{v}[k]} &= (\tau_k^{\boldsymbol{v}} + 2(1 - \tau_k^{\boldsymbol{v}}) e_{s,i}^{\boldsymbol{v}}) \lambda_i^{\boldsymbol{v}[k]} = (2 - \tau_k^{\boldsymbol{v}}) \tau_k^{\boldsymbol{v}} \lambda_i^{\boldsymbol{v}} = 2\tau_k^{\boldsymbol{v}} \lambda_i^{\boldsymbol{v}} - \lambda_i^{\boldsymbol{v}} \\ &= 2(\lambda_i^{\boldsymbol{v}} - b_{ik}^{\boldsymbol{v}} \lambda_k^{\boldsymbol{v}}) - \lambda_i^{\boldsymbol{v}} = (\lambda_i^{\boldsymbol{v}} - b_{ik}^{\boldsymbol{v}} \lambda_k^{\boldsymbol{v}}) - b_{ik}^{\boldsymbol{v}} \lambda_k^{\boldsymbol{v}} = \lambda_i^{\boldsymbol{v}[k]} - b_{ik}^{\boldsymbol{v}[k]} \lambda_k^{\boldsymbol{v}[k]}. \end{aligned}$$

If $\lambda_k^{\boldsymbol{v}} > 0$, then $(k,i) \in \mathcal{P}_\tau(\boldsymbol{v}, \boldsymbol{v}[k])$ and $(k,i) \notin \mathcal{P}_s(\boldsymbol{v}, \boldsymbol{v}[k])$, and the computations are similar to the case right above. This proves (C2) in this case.

5) Assume that $i = k \prec j$. Since this case is similar to case (4), we omit the details.

6) Assume that $i = j = k$. From (4.9), we have $c_k^{\boldsymbol{v}[k]} = -c_k^{\boldsymbol{v}}$. As seen in (4.17), we have $\lambda_k^{\boldsymbol{v}[k]} = -\lambda_k^{\boldsymbol{v}}$. Thus by induction $c_k^{\boldsymbol{v}[k]} = \lambda_k^{\boldsymbol{v}[k]}$, and (C1) holds. In cases (4) and (5), it is proven that $e_\ell^{\boldsymbol{v}[k]} \lambda_k^{\boldsymbol{v}[k]} = 0$ for $\ell \neq k$. Thus using (4.2), we have

$$e_k^{\boldsymbol{v}[k]} \lambda_k^{\boldsymbol{v}[k]} = (1 - \sum_{\ell \neq k} e_\ell^{\boldsymbol{v}[k]}) \lambda_k^{\boldsymbol{v}[k]} = \lambda_k^{\boldsymbol{v}[k]}.$$

Finally, since $(k,k) \notin \mathcal{P}_s(\boldsymbol{v}, \boldsymbol{v}[k])$, we see that

$$s_k^{\boldsymbol{v}[k]} \lambda_k^{\boldsymbol{v}[k]} = (\tau_k^{\boldsymbol{v}} + 2(1 - \tau_k^{\boldsymbol{v}}) e_{s,k}^{\boldsymbol{v}[k]}) \lambda_k^{\boldsymbol{v}[k]} = \tau_k^{\boldsymbol{v}} \lambda_k^{\boldsymbol{v}[k]} = \tau_k^{\boldsymbol{v}} \tau_k^{\boldsymbol{v}} \lambda_k^{\boldsymbol{v}} = \lambda_k^{\boldsymbol{v}} = -\lambda_k^{\boldsymbol{v}[k]},$$

where we use (4.6). This proves (C2) in this case, and a proof of Thoerem 3.8 has been completed. □

References

[1] M. Barot, R. J. Marsh, *Reflection group presentations arising from cluster algebras*, Trans. Amer. Math. Soc. **367** (2015), 1945–1967.

[2] M. Barot and D. Rivera, *Generalized Serre relations for Lie algebras associated with positive unit forms*, J. Pure Appl. Algebra **211** (2007), no. 2, 360–373.

[3] M. Barot, D. Kussin and H. Lenzing, *The Lie algebra associated to a unit form*, J. Algebra **296** (2006), no. 1, 1–17.

[4] B. Baumeister, M. Dyer, C. Stump and P. Wegener, *A note on the transitive Hurwitz action on decompositions of parabolic Coxeter elements*, Proc. Amer. Math. Soc. Ser. B **1** (2014), 149–154.

[5] G. Benkart and E. Zelmanov, *Lie algebras graded by finite root systems and intersection matrix algebras*, Invent. Math. **126** (1996), no. 1, 1–45.

[6] S. Berman and R. V. Moody, *Lie algebras graded by finite root systems and the intersection matrix algebras of Slodowy*, Invent. Math. **108** (1992), no. 2, 323–347.

[7] H. Derksen, J. Weyman and A. Zelevinsky, *Quivers with potentials and their representations I: mutations*, Selecta Math. **14** (2008), no. 1, 59–119.

[8] ———, *Quivers with potentials and their representations II: applications to cluster algebras*, J. Amer. Math. Soc. **23** (2010), no. 3, 749–790.

[9] A. Felikson and P. Tumarkin, *Coxeter groups and their quotients arising from cluster algebras*, Int. Math. Res. Not. IMRN 2016, **17** (2016), 5135–5186.

[10] ———, *Acyclic cluster algebras, reflections groups, and curves on a punctured disc*, Adv. Math. **340** (2018), 855–882.

[11] S. Fomin and A. Zelevinsky, *Cluster algebras IV: Coefficients*, Compos. Math. **143** (2007), 112–164.

[12] M. Gross, P. Hacking, S. Keel, and M. Kontsevich, *Canonical bases for cluster algebras*, J. Amer. Math. Soc. **31** (2018), no. 2, 497–608.

[13] M. Gupta, *A formula for F-polynomials in terms of C-cectors and stabilization of F-polynomials*, preprint, arXiv:1812.01910.

[14] A. Hubery and H. Krause, *A categorification of non-crossing partitions*, J. Eur. Math. Soc. **18** (2016), no. 10, 2273–2313.

[15] K. Igusa and R. Schiffler, *Exceptional sequences and clusters*, J. Algebra **323** (2010), no. 8, 2183–2202.

[16] V. G. Kac, *Infinite root systems, representations of graphs and invariant theory*, Invent. Math. **56** (1980), 57–92.

[17] K.-H. Lee and K. Lee, *A correspondence between rigid modules over path algebras and simple curves on Riemann surfaces*, to appear in Exp. Math., arXiv:1703.09113.

[18] T. Nakanishi and A. Zelevinsky, *On tropical dualities in cluster algebras*, Contemporary Math. **565** (2012), 217–226.

[19] A. Nájera Chávez, *On the c-vectors of an acyclic cluster algebra*, Int. Math. Res. Not. IMRN 2015, no. 6, 1590–1600.

[20] P.-G. Plamondon, *Cluster algebras via cluster categories with infinite-dimensional morphism spaces*, Compos. Math. **147** (2011), 1921–1954.

[21] K. Saito and D. Yoshii, *Extended affine root system. IV. Simply-laced elliptic Lie algebras*, Publ. Res. Inst. Math. Sci. **36** (2000), no. 3, 385–421.

[22] A. Schofield, *General representations of quivers*, Proc. London Math. Soc. (3) **65** (1992), no. 1, 46–64.

[23] A. Seven, *Mutation classes of skew-symmetrizable* 3×3 *matrices*, Proc. Amer. Math. Soc. **141** (2013), 1493–1504.

[24] ______, *Cluster algebras and symmetric matrices*, Proc. Amer. Math. Soc. **143** (2015), 469–478.

[25] ______, *Reflection group relations arising from cluster algebras*, Proc. Amer. Math. Soc. **144** (2016), 4641–4650.

[26] P. Slodowy, *Singularitäten, Kac–Moody Lie-Algebren, assoziierte Gruppen und Verallgemeinerungen*, Habilitationsschrift, Universität Bonn, March 1984.

[27] ______, *Beyond Kac-Moody algebras and inside*, Can. Math. Soc. Conf. Proc. **5** (1986), 361–371.

[28] D. Speyer and H. Thomas, *Acyclic cluster algebras revisited*, Algebras, quivers and representations, 275–298, Abel Symp. **8**, Springer, Heidelberg, 2013.

[29] L.-m. Xia and N. Hu, *A class of Lie algebras arising from intersection matrices*, Front. Math. China **10** (2015), no. 1, 185–198.

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