POINCARÉ-BIRKHOFF-WITT BASES FOR TWO-PARAMETER QUANTUM GROUPS

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Abstract. We construct Poincaré-Birkhoff-Witt (PBW) bases for the two-parameter quantum groups corresponding to $\mathfrak{gl}_{n+1}$ and $\mathfrak{sl}_{n+1}$. For this purpose, we derive some useful commutation relations, which hold in the positive part of the algebra, and show that the relations actually determine a Gröbner-Shirshov basis.

0. Introduction

In this paper, we construct Poincaré-Birkhoff-Witt (PBW) type bases for the two-parameter quantum groups $\widetilde{U} = U_{r,s}(\mathfrak{gl}_{n+1})$ and $U = U_{r,s}(\mathfrak{sl}_{n+1})$ introduced by Takeuchi (see [30, 31]). As shown in [3, 4], these quantum groups are Drinfeld doubles and have an R-matrix. They are related to the down-up algebras in [1, 2] and to the multi-parameter quantum groups of Chin and Musson [10] and Dobrev and Parashar [12]. In the analogous quantum function algebra setting, allowing two parameters unifies the Drinfeld-Jimbo quantum groups ($r = q, s = q^{-1}$) in [13] with the Dipper-Donkin quantum groups ($r = 1, s = q^{-1}$) in [11].

For the one-parameter quantum group $U_q(\mathfrak{g})$ of a finite-dimensional simple Lie algebra $\mathfrak{g}$, there is a sizeable literature ([9, 15, 19–29, 33, 34]) dealing with PBW bases. The approach taken in many of these papers is to combine braid group actions and direct calculations, starting from the defining relations, to build a PBW basis. An alternate approach has been developed by Ringel ([26–29]) and Green [15] using the Hall algebra associated to the Cartan matrix of $\mathfrak{g}$. In this setting, the basis elements of the positive part of $U_q(\mathfrak{g})$ have an interpretation as indecomposable modules for a certain finite-dimensional hereditary algebra.

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In the special case that $r = q$ and $s = q^{-1}$, our PBW basis for $\widetilde{U}$ (or U) exactly coincides with that in Ringel's paper [29]. When both r and s are roots of unity, the commutation relations developed here play an essential role in [5], where finite-dimensional restricted two-parameter quantum groups $\mathfrak{u}_{r,s}(\mathfrak{gl}_{n+1})$ and $\mathfrak{u}_{r,s}(\mathfrak{sl}_{n+1})$ are constructed. As shown in [5], these restricted two-parameter quantum groups are quasitriangular Hopf algebras and often are ribbon Hopf algebras.

1. Gröbner-Shirshov bases

In this section, we briefly recall the Gröbner-Shirshov basis theory for associative algebras. We refer the reader to ([6, 7, 17, 18]) for further details.

Let X be a set and let X^* be the free monoid of associative monomials on X. We denote by 1 the empty monomial and by $l(u)$ the *length* of a monomial u . Thus, $l(1) = 0$.

Definition 1.1. A total ordering $\prec$ on X^* is called a *monomial order* if $x \prec y$ implies $axb \prec ayb$ for all $a, b \in X^*$.

Fix a monomial order $\prec$ on X^*, and let $\mathcal{A}_X$ be the free associative algebra over a field $\mathbb{K}$ generated by X. Given a nonzero element $p \in \mathcal{A}_X$, we denote by $\overline{p}$ the maximal monomial appearing in p under the ordering $\prec$. Thus $p = \alpha\overline{p} + \sum \beta_i w_i$ where $\alpha, \beta_i \in \mathbb{K}$, $w_i \in X^*$, $\alpha \neq 0$, and $w_i \prec \overline{p}$. If $\alpha = 1$, p is said to be *monic*.

Let S be a subset of monic elements of $\mathcal{A}_X$, and let I be the ideal of $\mathcal{A}_X$ generated by S. Then we say that the algebra $A = \mathcal{A}_X/I$ is *defined by* S and denote the image of $p \in \mathcal{A}_X$ in A under the canonical quotient map also by p.

Definition 1.2. Assume S is a subset of monic elements of $\mathcal{A}_X$. A monomial $u \in X^*$ is *S-standard* if $u \neq a\overline{s}b$ for any $s \in S$ and $a, b \in X^*$. Otherwise, the monomial u is said to be *S-reducible.*

Proposition 1.3. [7, 18] *Every $p \in \mathcal{A}_X$ can be expressed as*

$$p = \sum_i \alpha_i a_i s_i b_i + \sum_j \beta_j u_j, \tag{1.4}$$

where $\alpha_i, \beta_j \in \mathbb{K}$, $a_i, b_i, u_j \in X^*$, $s_i \in S$, $a_i\overline{s_i}b_i \preceq \overline{p}$, $u_j \preceq \overline{p}$, and each u_j is S-standard.

The term $\sum_j \beta_j u_j$ in expression (1.4) is called a *normal form* (or a *remainder*) of p with respect to the set S (and also with respect to the monomial order $\prec$). As an immediate corollary of Proposition 1.3, we obtain

Corollary 1.5. *The set of S-standard monomials spans the algebra $A = \mathcal{A}_X/I$ defined by the set S.*

Definition 1.6. A subset S of monic elements of $\mathcal{A}_X$ is a *Gröbner-Shirshov basis* if the set of S-standard monomials forms a linear basis of the algebra $A = \mathcal{A}_X/I$ defined by the set S. In this case, we say that S is a *Gröbner-Shirshov basis* for the algebra $A = \mathcal{A}_X/I$.

Let p and q be monic elements of $\mathcal{A}_X$ with leading terms $\overline{p}$ and $\overline{q}$ respectively. We define the *composition* of p and q as follows.

Definition 1.7. (a) If there exist a and b in X^* such that $\overline{p}a = b\overline{q} = w$ with $l(\overline{p}) > l(b)$, then the *composition of intersection* is defined to be $(p, q)_w = pa - bq$.

(b) If there exist a and b in X^* such that $b \neq 1$, $\overline{p} = a\overline{q}b = w$, then the *composition of inclusion* is defined to be $(p, q)_w = p - aqb$.

(c) A *composition* $(p, q)_w$ is a composition of intersection or a composition of inclusion.

Corresponding to a subset S of monic elements and a word $w \in X^*$, there is a congruence relation on $\mathcal{A}_X$ defined as follows: For $p, q \in \mathcal{A}_X$, $p \equiv q \mod (S; w)$ if and only if $p - q = \sum_i \alpha_i a_i s_i b_i$, where $\alpha_i \in \mathbb{K}$, $a_i, b_i \in X^*$, $s_i \in S$, and $a_i \overline{s_i} b_i \prec w$.

Definition 1.8. A subset S of monic elements in $\mathcal{A}_X$ is *closed under composition* if $(p, q)_w \equiv 0 \mod (S; w)$ for all $p, q \in S$, $w \in X^*$, whenever the composition $(p, q)_w$ is defined.

Lemma 1.9. [6, 7, 17] *Assume S is a subset of monic elements in the free associative algebra $\mathcal{A}_X$ generated by X, and let $A = \mathcal{A}_X/I$ be the associative algebra defined by S. If S is closed under composition, and the image of $p \in \mathcal{A}_X$ is trivial in A, then the word $\overline{p}$ is S-reducible.*

As a consequence, we obtain

Theorem 1.10. [6, 7, 18] *Let S be a subset of monic elements in $\mathcal{A}_X$. Then the following conditions are equivalent*:

(a) *S is a Gröbner-Shirshov basis.*

(b) *S is closed under composition.*

(c) *For each $p \in \mathcal{A}_X$, the normal form of p is unique.*

2. Two-parameter quantum groups

Assume that Φ is a finite root system of type A_n with a base Π of simple roots. We regard Φ as a subset of a Euclidean space $\mathbb{R}^{n+1}$ with an inner product $\langle\, ,\, \rangle$. Let $\epsilon_1, \dots, \epsilon_{n+1}$ denote an orthonormal basis of $\mathbb{R}^{n+1}$, and suppose that $\Pi = \{\alpha_j = \epsilon_j - \epsilon_{j+1} \mid j = 1, \dots, n\}$ and that $\Phi = \{\epsilon_i - \epsilon_j \mid 1 \le i \ne j \le n+1\}$.

Fix nonzero elements r, s in a field $\mathbb{K}$ such that $r \ne s$. Let $\widetilde{U} = U_{r,s}(\mathfrak{gl}_{n+1})$ be the unital associative algebra over $\mathbb{K}$ generated by the elements e_j, f_j $(1 \le j \le n)$, and $a_i^{\pm 1}, b_i^{\pm 1}$ $(1 \le i \le n+1)$, which satisfy the following relations.

(R1) The $a_i^{\pm 1}$, $b_j^{\pm 1}$ all commute with one another, and $a_i a_i^{-1} = b_j b_j^{-1} = 1$,

(R2) $a_i\, e_j = r^{\langle \epsilon_i, \alpha_j\rangle} e_j\, a_i$ and $a_i\, f_j = r^{-\langle \epsilon_i, \alpha_j\rangle} f_j\, a_i$,

(R3) $b_i\, e_j = s^{\langle \epsilon_i, \alpha_j\rangle} e_j\, b_i$ and $b_i\, f_j = s^{-\langle \epsilon_i, \alpha_j\rangle} f_j\, b_i$,

(R4) $[e_i, f_j] = \dfrac{\delta_{i,j}}{r-s}\Big(a_i b_{i+1} - a_{i+1} b_i\Big)$,

(R5) $[e_i, e_j] = [f_i, f_j] = 0$ if $|i-j| > 1$,

(R6) $e_{i+1}^2 e_i - (r^{-1} + s^{-1}) e_{i+1} e_i e_{i+1} + r^{-1} s^{-1}\, e_i e_{i+1}^2 = 0$,
$e_{i+1} e_i^2 - (r^{-1} + s^{-1}) e_i e_{i+1} e_i + r^{-1} s^{-1}\, e_i^2 e_{i+1} = 0$,

(R7) $f_{i+1}^2 f_i - (r+s) f_{i+1} f_i f_{i+1} + rs f_i f_{i+1}^2 = 0$,
$f_{i+1} f_i^2 - (r+s) f_i f_{i+1} f_i + rs f_i^2 f_{i+1} = 0$.

Let $U = U_{r,s}(\mathfrak{sl}_{n+1})$ be the subalgebra of $\widetilde{U} = U_{r,s}(\mathfrak{gl}_{n+1})$ generated by the elements e_j, f_j, ω_j, and ω_j' $(1 \le j \le n)$, where

$$\omega_j = a_j b_{j+1} \quad \text{and} \quad \omega_j' = a_{j+1} b_j.$$

These elements satisfy (R5)-(R7) along with the following relations:

(R1') The $\omega_i^{\pm 1}$, $\omega_j^{\pm 1}$ all commute with one another, and $\omega_i \omega_i^{-1} = \omega_j' (\omega_j')^{-1} = 1$,

(R2') $\omega_i e_j = r^{\langle \epsilon_i, \alpha_j\rangle} s^{\langle \epsilon_{i+1}, \alpha_j\rangle} e_j \omega_i$ and $\omega_i f_j = r^{-\langle \epsilon_i, \alpha_j\rangle} s^{-\langle \epsilon_{i+1}, \alpha_j\rangle} f_j \omega_i$, (R3') $\omega_i' e_j = r^{\langle \epsilon_{i+1}, \alpha_j \rangle} s^{\langle \epsilon_i, \alpha_j \rangle} e_j \omega_i'$ and $\omega_i' f_j = r^{-\langle \epsilon_{i+1}, \alpha_j \rangle} s^{-\langle \epsilon_i, \alpha_j \rangle} f_j \omega_i'$,

(R4') $[e_i, f_j] = \dfrac{\delta_{i,j}}{r-s}(\omega_i - \omega_i')$.

Now it follows from the defining relations that $\widetilde{U} = U^- \widetilde{U^0} U^+$, where U^+ (resp. U^-) is the subalgebra generated by the elements e_i (resp. f_i). In the standard way, (see [16] for example), one can prove that $\widetilde{U}$ has a triangular decomposition, that is, there is an isomorphism of vector spaces $U^- \otimes \widetilde{U^0} \otimes U^+ \simeq \widetilde{U}$.

3. Commutation relations in U^+

The commutation relations for U^+, which we derive in this section, will determine a Gröbner-Shirshov basis S for U^+. Our relations are similar to those in Yamane's paper [34], which treats the special case $r = q^2, s = q^{-2}$. However, it should be noted that the definition of the commutator in [34] differs from the one given below.

Fix $r, s \in \mathbb{K}^\times$ and assume that $r + s \neq 0$ (or equivalently, $r^{-1} + s^{-1} \neq 0$). We define inductively

$$\mathcal{E}_{j,j} = e_j \quad \text{and} \quad \mathcal{E}_{i,j} = e_i \mathcal{E}_{i-1,j} - r^{-1} \mathcal{E}_{i-1,j} e_i \quad (i > j). \tag{3.1}$$

The defining relations for U^+ in (R6) can be reformulated as saying

$$e_{i+1}\,\mathcal{E}_{i+1,i} = s^{-1}\mathcal{E}_{i+1,i}\,e_{i+1}, \tag{3.2}$$

$$\mathcal{E}_{i+1,i}\,e_i = s^{-1}e_i\,\mathcal{E}_{i+1,i}. \tag{3.3}$$

Next we state the main result of this section.

Theorem 3.4. *Assume $(i,j) > (k,l)$ in the lexicographic order. Then the following relations hold in the algebra U^+ :*

(1) $\mathcal{E}_{i,j}\mathcal{E}_{k,l} - r^{-1}\mathcal{E}_{k,l}\mathcal{E}_{i,j} - \mathcal{E}_{i,l} = 0$ if $j = k+1$,

(2) $\mathcal{E}_{i,j}\mathcal{E}_{k,l} - \mathcal{E}_{k,l}\mathcal{E}_{i,j} = 0$ if $i > k \geq l > j$ or $j > k+1$,

(3) $\mathcal{E}_{i,j}\mathcal{E}_{k,l} - s^{-1}\mathcal{E}_{k,l}\mathcal{E}_{i,j} = 0$ if $i = k \geq j > l$ or $i > k \geq j = l$,

(4) $\mathcal{E}_{i,j}\mathcal{E}_{k,l} - r^{-1}s^{-1}\mathcal{E}_{k,l}\mathcal{E}_{i,j} + (r^{-1} - s^{-1})\mathcal{E}_{k,j}\mathcal{E}_{i,l} = 0$ if $i > k \geq j > l$.

The proof of Theorem 3.4 will be achieved through a sequence of lemmas.

Lemma 3.5. *The relations*

$$\begin{array}{lll} \text{(i)} & \mathcal{E}_{i,j}\mathcal{E}_{k,l} - \mathcal{E}_{k,l}\mathcal{E}_{i,j} = 0 & (i \geq j > k+1 \geq l+1), \\ \text{(ii)} & \mathcal{E}_{i,j}\mathcal{E}_{k,l} - r^{-1}\mathcal{E}_{k,l}\mathcal{E}_{i,j} - \mathcal{E}_{i,l} = 0 & (i \geq j = k+1 \geq l+1), \\ \text{(iii)} & \mathcal{E}_{i,j}e_j - s^{-1}\, e_j\mathcal{E}_{i,j} = 0 & (i > j) \end{array}$$

hold in U^+.

Proof. The relations in (i) are obvious.

For (ii), we fix j and l with $j > l$ and use induction on i. If $i = j$, this is just the definition of $\mathcal{E}_{i,l}$ from (3.1). Assume that $i > j$. Then we have

$$\begin{aligned} \mathcal{E}_{i,j}\mathcal{E}_{j-1,l} &= e_i\mathcal{E}_{i-1,j}\mathcal{E}_{j-1,l} - r^{-1}\mathcal{E}_{i-1,j}e_i\mathcal{E}_{j-1,l} \\ &= r^{-1}e_i\mathcal{E}_{j-1,l}\mathcal{E}_{i-1,j} + e_i\mathcal{E}_{i-1,l} - r^{-2}\mathcal{E}_{j-1,l}\mathcal{E}_{i-1,j}e_i - r^{-1}\mathcal{E}_{i-1,l}e_i \\ &= r^{-1}\mathcal{E}_{j-1,l}\mathcal{E}_{i,j} + \mathcal{E}_{i,l} \end{aligned}$$

by part (i) and the induction hypothesis.

To establish (iii), we fix j and use induction on i. When $i = j+1$, the relation is simply (3.3) with j instead of i. Assume that $i > j+1$. Then we have

$$\begin{aligned} \mathcal{E}_{i,j}e_j &= e_i\mathcal{E}_{i-1,j}e_j - r^{-1}\mathcal{E}_{i-1,j}e_je_i \\ &= s^{-1}\, e_je_i\mathcal{E}_{i-1,j} - r^{-1}s^{-1}\, e_j\mathcal{E}_{i-1,j}e_i \\ &= s^{-1}\, e_j\mathcal{E}_{i,j} \end{aligned}$$

by (i) and induction. □

Lemma 3.6. *In* U^+,

$$\begin{array}{lll} \text{(i)} & \mathcal{E}_{i,j}\mathcal{E}_{j,l} - r^{-1}s^{-1}\mathcal{E}_{j,l}\mathcal{E}_{i,j} + (r^{-1} - s^{-1})e_j\mathcal{E}_{i,l} = 0 & (i > j > l), \\ \text{(ii)} & \mathcal{E}_{i,j}\mathcal{E}_{k,l} - \mathcal{E}_{k,l}\mathcal{E}_{i,j} = 0 & (i > k \geq l > j). \end{array}$$

Proof. The following expression can be easily verified by induction on l:

$$\mathcal{E}_{i,j}\mathcal{E}_{j,l} - r^{-1}s^{-1}\mathcal{E}_{j,l}\mathcal{E}_{i,j} + r^{-1}\mathcal{E}_{i,l}e_j - s^{-1}\, e_j\mathcal{E}_{i,l} = 0 \quad (i > j > l). \tag{3.7}$$

We claim that

$$\mathcal{E}_{j+1,j-1}e_j - e_j\mathcal{E}_{j+1,j-1} = 0. \tag{3.8}$$

Indeed, we have $e_j\mathcal{E}_{j,j-1} = s^{-1}\mathcal{E}_{j,j-1}e_j$ as in (3.2), and using this we get

$$\begin{aligned}
&\mathcal{E}_{j+1,j}\mathcal{E}_{j,j-1} - r^{-1}s^{-1}\mathcal{E}_{j,j-1}\mathcal{E}_{j+1,j}\\
&\quad = e_{j+1}e_j\mathcal{E}_{j,j-1} - r^{-1}e_je_{j+1}\mathcal{E}_{j,j-1} - r^{-1}s^{-1}\mathcal{E}_{j,j-1}e_{j+1}e_j + r^{-2}s^{-1}\mathcal{E}_{j,j-1}e_je_{j+1}\\
&\quad = s^{-1}\, e_{j+1}\mathcal{E}_{j,j-1}e_j - r^{-1}e_je_{j+1}\mathcal{E}_{j,j-1} - r^{-1}s^{-1}\mathcal{E}_{j,j-1}e_{j+1}e_j + r^{-2}e_j\mathcal{E}_{j,j-1}e_{j+1}\\
&\quad = s^{-1}\mathcal{E}_{j+1,j-1}e_j - r^{-1}\, e_j\mathcal{E}_{j+1,j-1}.
\end{aligned}$$

On the other hand, we also have from (3.7)

$$\mathcal{E}_{j+1,j}\mathcal{E}_{j,j-1} - r^{-1}s^{-1}\mathcal{E}_{j,j-1}\mathcal{E}_{j+1,j} = s^{-1}\, e_j[\mathcal{E}_{j+1,j-1}] - r^{-1}\mathcal{E}_{j+1,j-1}e_j,$$

so that

$$(r^{-1}+s^{-1})\mathcal{E}_{j+1,j-1}e_j - (r^{-1}+s^{-1})e_j\mathcal{E}_{j+1,j-1} = 0.$$

Since we have assumed that $r^{-1}+s^{-1} \neq 0$, this implies (3.8).

Now to demonstate that

$$\mathcal{E}_{i,j}e_k - e_k\mathcal{E}_{i,j} = 0 \quad (i > k > j), \tag{3.9}$$

we fix k, and assume first that $j = k-1$. The argument proceeds by induction on i. If $i = k+1$, then the expression in (3.9) becomes (3.8) (with k instead of j there). When $i > k+1$, then

$$\begin{aligned}
\mathcal{E}_{i,k-1}e_k &= e_i\mathcal{E}_{i-1,k-1}e_k - r^{-1}\mathcal{E}_{i-1,k-1}e_ke_i\\
&= e_ke_i\mathcal{E}_{i-1,k-1} - r^{-1}e_k\mathcal{E}_{i-1,k-1}e_i = e_k\mathcal{E}_{i,k-1}.
\end{aligned}$$

For the case $j < k-1$, we have by induction on j,

$$\begin{aligned}
\mathcal{E}_{i,j}e_k &= \mathcal{E}_{i,j+1}e_je_k - r^{-1}\, e_j\mathcal{E}_{i,j+1}e_k\\
&= e_k\mathcal{E}_{i,j+1}e_j - r^{-1}\, e_ke_j\mathcal{E}_{i,j+1} = e_k\mathcal{E}_{i,j},
\end{aligned}$$

so that (3.9) is verified.

As a consequence, the relations in part (i) follow from (3.7) and (3.9); while the ones in (ii) can be derived easily from (3.9) by fixing i, j, k and using induction on l. □

Lemma 3.10. *The relations*

(i) $\mathcal{E}_{i,j}\mathcal{E}_{k,j} - s^{-1}\mathcal{E}_{k,j}\mathcal{E}_{i,j} = 0 \qquad (i > k > j)$

(ii) $\mathcal{E}_{i,j}\mathcal{E}_{k,l} - r^{-1}s^{-1}\mathcal{E}_{k,l}\mathcal{E}_{i,j} + (r^{-1}-s^{-1})\mathcal{E}_{k,j}\mathcal{E}_{i,l} = 0 \qquad (i > k > j > l)$

hold in U^+.

Proof. Part (i) follows from Lemma 3.5 (iii) and Lemma 3.6 (ii). For (ii), we apply induction on l. When $l = j-1$, part (i), Lemma 3.5 (ii), and Lemma 3.6 (ii) imply that

$$\begin{aligned}
\mathcal{E}_{i,j}\mathcal{E}_{k,j-1} &= \mathcal{E}_{i,j}\mathcal{E}_{k,j}e_{j-1} - r^{-1}\mathcal{E}_{i,j}e_{j-1}\mathcal{E}_{k,j} \\
&= s^{-1}\mathcal{E}_{k,j}\mathcal{E}_{i,j}e_{j-1} - r^{-1}\mathcal{E}_{i,j}e_{j-1}\mathcal{E}_{k,j} \\
&= r^{-1}s^{-1}\mathcal{E}_{k,j}e_{j-1}\mathcal{E}_{i,j} + s^{-1}\mathcal{E}_{k,j}\mathcal{E}_{i,j-1} - r^{-2}\, e_{j-1}\mathcal{E}_{i,j}\mathcal{E}_{k,j} - r^{-1}\mathcal{E}_{i,j-1}\mathcal{E}_{k,j} \\
&= r^{-1}s^{-1}\mathcal{E}_{k,j}e_{j-1}\mathcal{E}_{i,j} + s^{-1}\mathcal{E}_{k,j}\mathcal{E}_{i,j-1} - r^{-2}s^{-1}\, e_{j-1}\mathcal{E}_{k,j}\mathcal{E}_{i,j} - r^{-1}\mathcal{E}_{k,j}\mathcal{E}_{i,j-1} \\
&= r^{-1}s^{-1}\mathcal{E}_{k,j-1}\mathcal{E}_{i,j} + (s^{-1} - r^{-1})\mathcal{E}_{k,j}\mathcal{E}_{i,j-1}.
\end{aligned}$$

Now assume that $l < j-1$. Then $\mathcal{E}_{i,j}e_l = e_l\mathcal{E}_{i,j}$ and $\mathcal{E}_{k,j}e_l = e_l\mathcal{E}_{k,j}$ by Lemma 3.5 (i), and so by Lemma 3.5 (ii), we obtain

$$\begin{aligned}
\mathcal{E}_{i,j}\mathcal{E}_{k,l} &= \mathcal{E}_{i,j}\mathcal{E}_{k,l+1}e_l - r^{-1}\mathcal{E}_{i,j}e_l\mathcal{E}_{k,l+1} \\
&= r^{-1}s^{-1}\mathcal{E}_{k,l+1}e_l\mathcal{E}_{i,j} + (s^{-1} - r^{-1})\mathcal{E}_{k,j}\mathcal{E}_{i,l+1}e_l \\
&\qquad - r^{-2}s^{-1}\, e_l\mathcal{E}_{k,l+1}\mathcal{E}_{i,j} - r^{-1}(s^{-1} - r^{-1})e_l\mathcal{E}_{k,j}\mathcal{E}_{i,l+1} \\
&= r^{-1}s^{-1}\mathcal{E}_{k,l}\mathcal{E}_{i,j} + (s^{-1} - r^{-1})\mathcal{E}_{k,j}\mathcal{E}_{i,l}
\end{aligned}$$

by the induction assumption. □

Lemma 3.11. *In* U^+,

$$\mathcal{E}_{i,j}\mathcal{E}_{i,l} - s^{-1}\mathcal{E}_{i,l}\mathcal{E}_{i,j} = 0 \quad (i \geq j > l). \tag{3.12}$$

Proof. First consider the case $i = j$. If $l = i-1$, the above relation is just the defining relation in (3.2). Assume that $l < i-1$. By induction on l, we have

$$\begin{aligned}
e_i\mathcal{E}_{i,l} &= e_i\mathcal{E}_{i,l+1}e_l - r^{-1}\, e_ie_l\mathcal{E}_{i,l+1} \\
&= s^{-1}\mathcal{E}_{i,l+1}e_le_i - r^{-1}s^{-1}\, e_l\mathcal{E}_{i,l+1}e_i = s^{-1}\mathcal{E}_{i,l}e_i.
\end{aligned}$$

When $i > j$, then by induction on j and Lemma 3.6 (ii), we get

$$\begin{aligned}
\mathcal{E}_{i,j}\mathcal{E}_{i,l} &= \mathcal{E}_{i,j+1}e_j\mathcal{E}_{i,l} - r^{-1}e_j\mathcal{E}_{i,j+1}\mathcal{E}_{i,l} \\
&= \mathcal{E}_{i,j+1}\mathcal{E}_{i,l}e_j - r^{-1}s^{-1}\, e_j\mathcal{E}_{i,l}\mathcal{E}_{i,j+1}, \\
&= s^{-1}\mathcal{E}_{i,l}\mathcal{E}_{i,j+1}e_j - r^{-1}s^{-1}\mathcal{E}_{i,l}e_j\mathcal{E}_{i,j+1} \\
&= s^{-1}\mathcal{E}_{i,l}\mathcal{E}_{i,j}.
\end{aligned}$$

□

The proof of Theorem 3.4 is now complete, because we have

(1) $\Longleftrightarrow$ Lemma 3.5 (ii);

(2) $\Longleftrightarrow$ Lemma 3.5 (i) and Lemma 3.6 (ii);

(3) $\Longleftrightarrow$ Lemma 3.5 (iii), Lemma 3.10 (i), and Lemma 3.11;

(4) $\Longleftrightarrow$ Lemma 3.6 (i) and Lemma 3.10 (ii).

4. Gröbner-Shirshov bases and PBW-type bases

In this section we determine a Gröbner-Shirshov basis and a PBW basis for the algebra U^+. This will be achieved by showing that the set of relations obtained in the previous section is closed under composition. To simplify compositions of relations, we consider an algebra with sufficiently many generators and (essentially) the same defining relations as the ones in Theorem 3.4, which will turn out to be isomorphic to U^+.

Let $E = \{e_1, e_2, \ldots, e_n\}$ be the set of generators of the algebra U^+. We introduce a linear ordering $\prec$ on E by saying $e_i \prec e_j$ if and only if $i < j$. We extend this ordering to the set E^* of monomials in E so that it becomes the *degree-lexicographic order*; that is, for $u = u_1u_2\cdots u_p$ and $v = v_1v_2\cdots v_q$, then $u \prec v$ if and only if $p < q$ or $p = q$ and $u_i \prec v_i$ for the first i such that $u_i \neq v_i$. Let $\mathcal{S} \subset \mathcal{A}_E$ be the set consisting of the following elements:

$$\begin{array}{ll}
\mathcal{E}_{i,j}\mathcal{E}_{k,l} - \mathcal{E}_{k,l}\mathcal{E}_{i,j} & \text{if } \ i > k \geq l > j \ \text{ or } \ j > k+1, \\
\mathcal{E}_{i,j}\mathcal{E}_{k,l} - s^{-1}\mathcal{E}_{k,l}\mathcal{E}_{i,j} & \text{if } \ i = k \geq j > l \ \text{ or } \ i > k \geq j = l, \\
\mathcal{E}_{i,j}\mathcal{E}_{k,l} - r^{-1}s^{-1}\mathcal{E}_{k,l}\mathcal{E}_{i,j} + (r^{-1} - s^{-1})\mathcal{E}_{k,j}\mathcal{E}_{i,l} & \text{if } \ i > k \geq j > l.
\end{array}$$

The elements of $\mathcal{S}$ just correspond to relations (2), (3), and (4) of Theorem 3.4. Note that we may take $\mathcal{S}$ to be the set of defining relations for the algebra U^+, since $\mathcal{S}$ contains all the (original) defining relations (R5) and (R6) of U^+ and the other relations in $\mathcal{S}$ are all consequences of (R5) and (R6).

Now we introduce the algebra $\widehat{U}^+$. Let $\widehat{E} = \{E_{i,j} \mid 1 \leq j \leq i \leq n\}$ with the linear ordering $\ll$ defined by

$$E_{i,j} \ll E_{k,l} \Longleftrightarrow (i,j) < (k,l) \text{ lexicographically} .$$

The ordering $\ll$ may be extended to the set $\widehat{E}^*$ of monomials in $\widehat{E}$ to give the degree-lexicographic order. Let $\mathcal{A}_{\widehat{E}}$ denote the free associative algebra generated by $\widehat{E}$ over $\mathbb{K}$. Fix $r, s \in \mathbb{K}^\times$ with $r + s \neq 0$ as before. Let $\widehat{\mathcal{S}} \subset \mathcal{A}_{\widehat{E}}$ consist of the following elements for all $(i,j) > (k,l)$ in the lexicographic order:

$$E_{i,j}E_{k,l} - r^{-1}E_{k,l}E_{i,j} - E_{i,l} \qquad \text{if } \ j = k+1, \tag{4.1}$$

$$E_{i,j}E_{k,l} - E_{k,l}E_{i,j} \qquad \text{if } \ i > k \geq l > j \text{ or } \ j > k+1, \tag{4.2}$$

$$E_{i,j}E_{k,l} - s^{-1}E_{k,l}E_{i,j} \qquad \text{if } \ i = k \geq j > l \text{ or } \ i > k \geq j = l, \tag{4.3}$$

$$E_{i,j}E_{k,l} - r^{-1}s^{-1}E_{k,l}E_{i,j} + (r^{-1} - s^{-1})E_{k,j}E_{i,l} \qquad \text{if } \ i > k \geq j > l. \tag{4.4}$$

The indices here are precisely the same ones as in Theorem 3.4. Then we define the algebra $\widehat{U}^+$ to be the associative algebra generated by $\widehat{E}$ with defining relations $\widehat{S}$.

Proposition 4.5. *The algebra $\widehat{U}^+$ is isomorphic to the algebra U^+.*

Proof. It is easy to verify that the map $\phi : U^+ \to \widehat{U}^+, \ \ e_i \mapsto E_{i,i}$, gives a well-defined algebra homomorphism. For example, we have by (4.1) and (4.3),

$$\begin{aligned}
&E_{i+1,i+1}^2 E_{i,i} - (r^{-1} + s^{-1})E_{i+1,i+1}E_{i,i}E_{i+1,i+1} + r^{-1}s^{-1}E_{i,i}E_{i+1,i+1}^2 \\
&\quad = E_{i+1,i+1}E_{i+1,i} - s^{-1}E_{i+1,i+1}E_{i,i}E_{i+1,i+1} + r^{-1}s^{-1}E_{i,i}E_{i+1,i+1}^2 \\
&\quad = s^{-1}E_{i+1,i}E_{i+1,i+1} - r^{-1}s^{-1}E_{i,i}E_{i+1,i+1}^2 - s^{-1}E_{i+1,i}E_{i+1,i+1} + r^{-1}s^{-1}E_{i,i}E_{i+1,i+1}^2 = 0.
\end{aligned}$$

Conversely, it is a consequence of Theorem 3.4 that the map $\psi : \widehat{U}^+ \to U^+, \ \ E_{i,j} \mapsto \mathcal{E}_{i,j}$, is a well-defined algebra homomorphism.

Now the definition of the commutator (3.1) implies that $(\psi \circ \phi)(e_i) = \mathcal{E}_{i,i} = e_i$ for $1 \leq i \leq n$. We claim that $\phi(\mathcal{E}_{i,l}) = E_{i,l}$ for $i \geq l$. To see this, we fix l and use induction on i. If $i = l$, then

$\phi(\mathcal{E}_{l,l}) = \phi(e_l) = E_{l,l}$. If $i > l$, then it follows from (3.1), (4.1) and the induction hypothesis that

$$\phi(\mathcal{E}_{i,l}) = \phi(e_i\mathcal{E}_{i-1,l} - r^{-1}\mathcal{E}_{i-1,l}e_i) = E_{i,i}E_{i-1,l} - r^{-1}E_{i-1,l}E_{i,i} = E_{i,l}.$$

Therefore, $(\phi \circ \psi)(E_{i,l}) = E_{i,l}$ for $1 \le l \le i \le n$, so that ϕ and ϕ are inverses of each other. □

Now we consider compositions of elements in $\widehat{\mathcal{S}}$. To begin, we define

$$\begin{aligned}
C_1 &= \{(i,j,k,l) \in \mathbb{N}^4 \mid n \ge i \ge j = k+1 \ge l+1 \ge 2\},\\
C_2 &= \{(i,j,k,l) \in \mathbb{N}^4 \mid n \ge i = k \ge j > l \ge 1\},\\
C_3 &= \{(i,j,k,l) \in \mathbb{N}^4 \mid n \ge i > k \ge j = l \ge 1\},\\
C_4 &= \{(i,j,k,l) \in \mathbb{N}^4 \mid n \ge i > k \ge j > l \ge 1\},\\
C_5 &= \{(i,j,k,l) \in \mathbb{N}^4 \mid n \ge i > k \ge l > j \ge 1\},\\
C_6 &= \{(i,j,k,l) \in \mathbb{N}^4 \mid n \ge i \ge j > k+1 \ge l+1 \ge 2\}.
\end{aligned}$$

Note that all the elements of $\widehat{\mathcal{S}}$ can be written in the form

$$E_{i,j}E_{k,l} - \varepsilon_{ij}^{kl}E_{k,l}E_{i,j} + X_{ij}^{kl},$$

where

$$\begin{array}{lll}
\varepsilon_{ij}^{kl} = r^{-1}, & X_{ij}^{kl} = -E_{i,l} & \text{if } (i,j,k,l) \in C_1,\\
\varepsilon_{ij}^{kl} = s^{-1}, & X_{ij}^{kl} = 0 & \text{if } (i,j,k,l) \in C_2 \cup C_3,\\
\varepsilon_{ij}^{kl} = r^{-1}s^{-1}, & X_{ij}^{kl} = (r^{-1} - s^{-1})E_{k,j}E_{i,l} & \text{if } (i,j,k,l) \in C_4,\\
\varepsilon_{ij}^{kl} = 1, & X_{ij}^{kl} = 0 & \text{if } (i,j,k,l) \in C_5 \cup C_6.
\end{array}$$

For $f, g \in \widehat{\mathcal{S}}$, the composition $(f,g)_w$ can occur only if

$$f = E_{i,j}E_{k,l} - \varepsilon_{ij}^{kl}E_{k,l}E_{i,j} + X_{ij}^{kl}, \qquad g = E_{k,l}E_{p,q} - \varepsilon_{kl}^{pq}E_{p,q}E_{k,l} + X_{kl}^{pq}$$

and $w = E_{i,j}E_{k,l}E_{p,q}$ where $(i,j) > (k,l) > (p,q)$ lexicographically. We see that

$$\begin{aligned} (f,g)_w &= -\varepsilon_{ij}^{kl}E_{k,l}E_{i,j}E_{p,q} + X_{ij}^{kl}E_{p,q} + \varepsilon_{kl}^{pq}E_{i,j}E_{p,q}E_{k,l} - E_{i,j}X_{kl}^{pq} \\ &\equiv -\varepsilon_{ij}^{kl}\varepsilon_{ij}^{pq}E_{k,l}E_{p,q}E_{i,j} + \varepsilon_{ij}^{kl}E_{k,l}X_{ij}^{pq} + X_{ij}^{kl}E_{p,q} \\ &\quad +\varepsilon_{kl}^{pq}\varepsilon_{ij}^{pq}E_{p,q}E_{i,j}E_{k,l} - \varepsilon_{kl}^{pq}X_{ij}^{pq}E_{k,l} - E_{i,j}X_{kl}^{pq} \\ &\equiv \varepsilon_{ij}^{kl}\varepsilon_{ij}^{pq}X_{kl}^{pq}E_{i,j} + \varepsilon_{ij}^{kl}E_{k,l}X_{ij}^{pq} + X_{ij}^{kl}E_{p,q} \\ &\quad -\varepsilon_{kl}^{pq}\varepsilon_{ij}^{pq}E_{p,q}X_{ij}^{kl} - \varepsilon_{kl}^{pq}X_{ij}^{pq}E_{k,l} - E_{i,j}X_{kl}^{pq} \qquad \mathrm{mod}(\widehat{\mathcal{S}}; w). \end{aligned}$$

With a careful analysis of the conditions on $\{(i,j),(k,l),(p,q)\}$, we find that there are 62 cases to be considered. (See the table at the end of this section and also [9, 34].) In all the cases, it is straightforward to check that $(f,g)_w \equiv 0 \ \mathrm{mod}(\widehat{\mathcal{S}}; w)$ is satisfied. For example, if $(i,j,k,l) \in C_4$, $(k,l,p,q) \in C_3$, and $(i,j,p,q) \in C_1$, then

$$i > k \geq j = p+1 \geq l+1 = q+1,$$

and

$$\begin{aligned} (f,g)_w &\equiv -r^{-1}s^{-1}\,E_{k,l}E_{i,l} + (r^{-1}-s^{-1})E_{k,j}E_{i,l}E_{p,l} \\ &\qquad - r^{-1}s^{-1}(r^{-1}-s^{-1})\,E_{p,l}E_{k,j}E_{i,l} + s^{-1}\,E_{i,l}E_{k,l} \\ &\equiv -r^{-1}s^{-1}\,E_{k,l}E_{i,l} + s^{-1}(r^{-1}-s^{-1})\,E_{k,j}E_{p,l}E_{i,l} \\ &\qquad - r^{-1}s^{-1}(r^{-1}-s^{-1})E_{p,l}E_{k,j}E_{i,l} + s^{-2}\,E_{k,l}E_{i,l} \\ &\equiv -r^{-1}s^{-1}E_{k,l}E_{i,l} + s^{-1}(r^{-1}-s^{-1})\,E_{k,l}E_{i,l} + s^{-2}\,E_{k,l}E_{i,l} = 0 \qquad \mathrm{mod}(\widehat{\mathcal{S}}; w). \end{aligned}$$

The remaining cases are of the same level of difficulty to verify. Hence, we have the first part of the next lemma, and the second part follows from the lexicographic ordering of the indices in Theorem 3.4.

Lemma 4.6. *Assume that $r, s \in \mathbb{K}^\times$ and $r + s \neq 0$.*

(1) *The set $\widehat{\mathcal{S}}$ is a Gröbner-Shirshov basis for the algebra $\widehat{U}^+$.*
(2) *The set of $\widehat{\mathcal{S}}$-standard monomials is given by*

$$\widehat{\mathcal{B}} = \{E_{i_1,j_1}E_{i_2,j_2}\cdots E_{p_1,q_1} \,|\, (i_1,j_1) \leq (i_2,j_2) \leq \cdots \leq (i_p,j_p) \text{ lexicographically}\},$$

and $\widehat{\mathcal{B}}$ is a linear basis of the algebra $\widehat{U}^+$.

This brings us to the main result of the paper.

Theorem 4.7. *Assume that $r, s \in \mathbb{K}^\times$ and $r+s \neq 0$. Then*

(1) $\mathcal{B}_0 = \{\mathcal{E}_{i_1,j_1}\mathcal{E}_{i_2,j_2}\cdots\mathcal{E}_{i_p,j_p} \,|\, (i_1,j_1) \leq (i_2,j_2) \leq \cdots \leq (i_p,j_p) \text{ lexicographically}\}$ *is a linear basis of the algebra U^+.*

(2) $\mathcal{B}_1 = \{e_{i_1,j_1}e_{i_2,j_2}\cdots e_{i_p,j_p} \,|\, (i_1,j_1) \leq (i_2,j_2) \leq \cdots \leq (i_p,j_p) \text{ lexicographically}\}$ *is a linear basis of the algebra U^+, where $e_{i,j} = e_i e_{i-1}\cdots e_j$ for $i \geq j$.*

(3) *The set $\mathcal{S}$ is a Gröbner-Shirshov basis for the algebra U^+.*

Proof. The assertion in (1) follows from Proposition 4.5 and Lemma 4.6 (2). For (2), note that $\mathcal{B}_1$ is exactly the set of $\mathcal{S}$-standard monomials. By Corollary 1.5, the set $\mathcal{B}_1$ spans the algebra U^+. If we consider the root space decomposition $U^+ = \bigoplus_{\alpha\in Q^+} U_\alpha$, with $Q^+ = \sum_{i=1}^n \mathbb{Z}_{\geq 0}\alpha_i$, the homogeneous space U_α is finite-dimensional, and the number of elements in $\mathcal{B}_0 \cap U_\alpha$ is clearly equal to $\mathcal{B}_1 \cap U_\alpha$ for each $\alpha \in Q^+$. Since $\mathcal{B}_0$ is a linear basis of U^+, the set $\mathcal{B}_1$ is also a linear basis of U^+. The last statement follows from the definition of a Gröbner-Shirshov basis. □

Remark 4.8. If we define inductively $\mathcal{F}_{i,j}$ to be

$$\mathcal{F}_{j,j} = f_j \quad \text{and} \quad \mathcal{F}_{i,j} = f_i\,\mathcal{F}_{i-1,j} - r\mathcal{F}_{i-1,j}\,f_i \quad (i > j),$$

and denote by $f_{i,j}$ the monomial $f_{i,j} = f_i f_{i-1}\cdots f_j$ $(i \geq j)$, then we have linear bases for the algebra U^- as in Theorem 4.7. Note that $\widetilde{U}^0$ and U^0 have obvious linear bases. All together and from the triangular decomposition $\widetilde{U} = U^-\widetilde{U}^0U^+$ (resp. $U = U^-U^0U^+$) we have "PBW-bases" for the algebra $\widetilde{U}$ (resp. U).

5. Iterated skew polynomial ring structure

As applications of the previous results of the paper, we will show that the algebra U^+ is an iterated skew polynomial ring over $\mathbb{K}$, and that any prime ideal P of U^+ is completely prime (that is, U^+/P is a domain) when r and s are "generic" (see Proposition 5.5 for the precise statement). Our approach is similar to that of [29], which treats the one-parameter quantum group case.

In this section we assume that $r+s \neq 0$ as before and that $(k,l) < (i,j)$ always means "relative to the lexicographic ordering". Let $U^+_{i,j}$ be the subalgebra of U^+ generated by $\mathcal{E}_{k,l}$, $(k,l) < (i,j)$.

For each (i,j), $1 \le j \le i \le n$, we define an automorphism $\iota_{i,j}$ of $U^+_{i,j}$ by

$$\iota_{i,j}(\mathcal{E}_{k,l}) = \begin{cases} r^{-1}\mathcal{E}_{k,l} & \text{if } j = k+1, \\ \mathcal{E}_{k,l} & \text{if } i > k \ge l > j \text{ or } j > k+1, \\ s^{-1}\mathcal{E}_{k,l} & \text{if } i = k \ge j > l \text{ or } i > k \ge j = l, \\ r^{-1}s^{-1}\mathcal{E}_{k,l} & \text{if } i > k \ge j > l \end{cases}$$

for $(k,l) < (i,j)$. In order to see that $\iota_{i,j}$ is well-defined, we may check the relations in Theorem 3.4 owing to Proposition 4.5. Note that there is nothing to prove concerning the relations in (2) and (3) of Theorem 3.4. A case-by-case investigation shows that the above definition of $\iota_{i,j}$ is also compatible with relations (1) and (4) of Theorem 3.4. For example, consider relation (4),

$$\mathcal{E}_{i,j}\mathcal{E}_{k,l} - r^{-1}s^{-1}\mathcal{E}_{k,l}\mathcal{E}_{i,j} + (r^{-1} - s^{-1})\mathcal{E}_{k,j}\mathcal{E}_{i,l} = 0$$

with $i > k \ge j > l$. Applying $\iota_{p,q}$ such that $p > i \ge q > j$ and $q = k+1$, gives

$$\iota_{p,q}(\mathcal{E}_{i,j}\mathcal{E}_{k,l}) = \iota_{p,q}(\mathcal{E}_{i,j})\iota_{p,q}(\mathcal{E}_{k,l}) = (r^{-1}s^{-1}\mathcal{E}_{i,j})(r^{-1}\mathcal{E}_{k,l}) = r^{-2}s^{-1}\mathcal{E}_{i,j}\mathcal{E}_{k,l}.$$

Similarly, we have $\iota_{p,q}(\mathcal{E}_{k,l}\mathcal{E}_{i,j}) = r^{-2}s^{-1}\mathcal{E}_{k,l}\mathcal{E}_{i,j}$ and $\iota_{p,q}(\mathcal{E}_{k,j}\mathcal{E}_{i,l}) = r^{-2}s^{-1}\mathcal{E}_{k,j}\mathcal{E}_{i,l}$. Thus the relation is preserved.

Now we define $\iota_{i,j}$-derivation $\vartheta_{i,j}$ on $U^+_{i,j}$ by

$$\vartheta_{i,j}(\mathcal{E}_{k,l}) = \mathcal{E}_{i,j}\mathcal{E}_{k,l} - \iota_{i,j}(\mathcal{E}_{k,l})\mathcal{E}_{i,j} = \begin{cases} \mathcal{E}_{i,l} & \text{if } j = k+1, \\ (r^{-1} - s^{-1})\mathcal{E}_{k,j}\mathcal{E}_{i,l} & \text{if } i > k \ge j > l, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that $\vartheta_{i,j}$ is indeed an $\iota_{i,j}$-derivation; that is, $\vartheta_{i,j}(uv) = \vartheta_{i,j}(u)\iota_{i,j}(v) + u\vartheta_{i,j}(v)$ for all $u, v \in U^+_{i,j}$ (cf. Lemma 3, p. 62 of [29]). With $\iota_{i,j}$ and $\vartheta_{i,j}$ at hand, the next proposition will follow immediately.

Proposition 5.1. *The algebra U^+ is an iterated skew polynomial ring whose structure is given by*

$$U^+ = \mathbb{K}[\mathcal{E}_{1,1}][\mathcal{E}_{2,1}, \iota_{2,1}, \vartheta_{2,1}] \cdots [\mathcal{E}_{n,n}, \iota_{n,n}, \vartheta_{n,n}]. \tag{5.2}$$

Proof. Note that all the relations in Theorem 3.4 can be condensed into a single expression:

$$\mathcal{E}_{i,j}\mathcal{E}_{k,l} = \iota_{i,j}(\mathcal{E}_{k,l})\mathcal{E}_{i,j} + \vartheta_{i,j}(\mathcal{E}_{k,l}), \quad (i,j) > (k,l). \tag{5.3}$$

Furthermore, Proposition 4.5 asserts that relations in (5.3) are all the relations needed to define the algebra U^+, which means U^+ is an iterated skew polynomial ring with the structure given by (5.2). □

The other result of this section requires an additional lemma.

Lemma 5.4. *The automorphism $\iota_{i,j}$ and the $\iota_{i,j}$-derivation $\vartheta_{i,j}$ of $U^+_{i,j}$ satisfy*

$$\iota_{i,j}\vartheta_{i,j} = (r^{-1}s)\vartheta_{i,j}\iota_{i,j}.$$

Proof. For $(k,l) < (i,j)$, the definitions imply that

$$(\iota_{i,j}\vartheta_{i,j})(\mathcal{E}_{k,l}) = \begin{cases} s^{-1}\mathcal{E}_{i,l} & \text{if } j = k+1, \\ (r^{-1}-s^{-1})s^{-2}\mathcal{E}_{k,j}\mathcal{E}_{i,l} & \text{if } i > k \geq j > l, \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, for $(k,l) < (i,j)$,

$$(\vartheta_{i,j}\iota_{i,j})(\mathcal{E}_{k,l}) = \begin{cases} r^{-1}\mathcal{E}_{i,l} & \text{if } j = k+1, \\ (r^{-1}-s^{-1})r^{-1}s^{-1}\mathcal{E}_{k,j}\mathcal{E}_{i,l} & \text{if } i > k \geq j > l, \\ 0 & \text{otherwise.} \end{cases}$$

Comparing these two calculations, we arrive at the result. □

Now we obtain:

Proposition 5.5. *Assume that the subgroup of $\mathbb{K}^\times$ generated by r and s is torsion-free. Then all prime ideals of U^+ are completely prime.*

Proof. This follows directly from Proposition 5.1, Lemma 5.4 and Theorem 2.3 of [14]. □

6. Appendix

In this appendix, we display the table of conditions on $\{(i,j),(k,l),(p,q)\}$ required for the calculation of $(f,g)_w$ in Lemma 4.6. A row in this table with an entry a in column $(ijkl)$, b in column $(klpq)$, and $t_1, t_2, \cdots, t_m$ in column $(ijpq)$ signifies that if $(i,j,k,l) \in C_a$ and $(k,l,p,q) \in C_b$, then the sets that can contain (i,j,p,q) are $C_{t_1}, C_{t_2}, \cdots, C_{t_m}$.

$(ijkl)$	$(klpq)$	$(ijpq)$	$(ijkl)$	$(klpq)$	$(ijpq)$	$(ijkl)$	$(klpq)$	$(ijpq)$
1	1	6	3	1	1	5	1	$3, 4, 5$
	2	1		2	4		2	$3, 4, 5$
	3	6		3	3		3	5
	4	6		4	4		4	$3, 4, 5$
	5	6		5	5		5	5
	6	6		6	6		6	$1, 3, 4, 5, 6$
2	1	6	4	1	6	6	1	6
	2	2		2	4		2	6
	3	$1, 4, 6$		3	$1, 4, 6$		3	6
	4	$1, 4, 6$		4	$1, 4, 6$		4	6
	5	$1, 3, 4, 5, 6$		5	$1, 3, 4, 5, 6$		5	6
	6	6		6	6		6	6

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