EISENSTEIN SERIES AND THEIR APPLICATIONS

HENRY H. KIM*

Eisenstein series provide very concrete examples of modular forms. Langlands was led to his functoriality principle by studying Eisenstein series and their constant terms. We describe Eisenstein series, both classical and adelic, and their applications. See [1] for the details and references therein.

1. Lecture One: Classical Eisenstein series

In Lecture one, we look at Eisenstein series as functions on the upper half plane.

1.1. **classical holomorphic Eisenstein series.** We follow [7], Chapter VII. Let $\mathbb{H} = \{z = x + iy, y > 0\}$ be the upper half plane. Let, for $k > 1$ positive integer,

$$E_{2k}(z) = \frac{1}{2} \sum_{(c,d)=1} (cz+d)^{-2k}.$$

Here the sum $(c,d) = 1$ is the same as $\gamma \in \Gamma_\infty \backslash \Gamma$, where $\Gamma = SL_2(\mathbb{Z})$, and $\Gamma_\infty = \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} n \in \mathbb{Z} \right\}$.

The series converges absolutely and uniformly if $k > 1$, and E_{2k} is a modular form of weight $2k$, i.e., $E_{2k}(\gamma z) = (cz+d)^{2k} E_{2k}(z)$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We have the Fourier expansion

$$E_{2k}(z) = 1 + \gamma_{2k} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n,$$

where $q = e^{2\pi i z}$, $\sigma_s(n) = \sum_{d|n} d^s$, and $\gamma_{2k} = (-1)^k \frac{4k}{B_{2k}}$, and B_{2k} are the Bernoulli numbers

$$\frac{x}{e^x - 1} = 1 - \frac{x}{2} + \sum_{k=1}^{\infty} (-1)^{k+1} B_{2k} \frac{x^{2k}}{(2k)!}.$$

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Let $\Delta(z) = \frac{E_4^3 - E_6^2}{1728} = \sum_{n=1}^{\infty} \tau(n) q^n$, where $\tau(n)$ is the Ramanujan τ-function. Then Δ is a cusp form of weight 12, and we have an infinite product expansion:

$$\Delta(z) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}.$$

We have many remarkable congruences which have natural explanation from Galois representations:

$$\tau(n) \equiv n^2 \sigma_7(n) \bmod 3^3$$

$$\tau(n) \equiv n \sigma_3(n) \bmod 7$$

$$\tau(n) \equiv \sigma_{11}(n) \bmod 691$$

Here the prime 691 comes from

$$E_{12}(z) = 1 + \frac{65520}{691} \sum_{n=1}^{\infty} \sigma_{11}(n) q^n, \quad 65520 = 2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13.$$

Conjecture 1.1. *(Lehmer's Conjecture) $\tau(n) \neq 0$ for all $n \geq 1$.*

Since $\tau(n)$ is multiplicative, i.e., $\tau(mn) = \tau(m)\tau(n)$ for $(m, n) = 1$ and $\tau(p^{k+1}) = \tau(p)\tau(p^k) - p^{11}\tau(p^{k-1})$ for p prime and $k > 1$, it can be shown that it is enough to prove $\tau(p) \neq 0$ for all prime p.

Let $j(z) = \frac{E_4^3}{\Delta}$. Then j is a modular function of weight 0, and it plays an important role:

$$j(z) = \frac{1}{q} + 744 + \sum_{n=1}^{\infty} c(n) q^n = \frac{1}{q} + 744 + 196884q + 21493760q^2 + 864299970q^3 + ...$$

It is a remarkable fact that $c(n)$'s are related to dimensions of irreducible representations of the Monster $\mathbf{M}$, the largest of the sporadic simple groups, with group order

$$|\mathbf{M}| = 2^{45} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 \sim 8 \times 10^{53}.$$

John McKay observed first $196884 = 196883 + 1$, and Thompson found

$$\begin{aligned} 196884 &= 196883 + 1 \\ 21493760 &= 21296876 + 196883 \\ 864299970 &= 842609326 + 2 \cdot 21296876 + 2 \cdot 196883 + 2 \cdot 1, \end{aligned}$$

where the numbers on the right are dimensions of irreducible representations of the Monster.

1.2. **classical non-holomorphic Eisenstein series.** We follow [1], p. 215. Let

$$E(z,s)=\frac{1}{2}\sum_{(c,d)=1}\frac{y^s}{|cz+d|^{2s}}=\frac{1}{2}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}Im(\gamma z)^s.$$

The normalized Eisenstein series is

$$E^*(z,s)=\frac{1}{2}\pi^{-s}\Gamma(s)\zeta(2s)E(z,s)=\frac{1}{2}\pi^{-s}\Gamma(s)\sum_{(m,n)\neq(0,0)}\frac{y^s}{|mz+n|^{2s}}.$$

The series converges for $Re(s)>1$, and it becomes an eigenfunction for the Laplacian $\Delta=-y^2(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2})$, i.e., $\Delta E(z,s)=s(1-s)E(z,s)$. We have the Fourier expansion

$$E^*(z,s)=\pi^{-s}\Gamma(s)\zeta(2s)y^s+\pi^{s-1}\Gamma(1-s)\zeta(2-2s)y^{1-s}+2\sum_{r\neq 0}|r|^{s-\frac{1}{2}}\sigma_{1-2s}(|r|)\sqrt{y}K_{s-\frac{1}{2}}(2\pi|r|y)e^{2\pi irx},$$

where $K_s(y)=\frac{1}{2}\int_0^\infty e^{-y\frac{t+t^{-1}}{2}}t^s\frac{dt}{t}$ is the K-Bessel function. This show the functional equation $E^*(z,1-s)=E^*(z,s)$.

1.2.1. *Non-vanishing of* $\zeta(1+it)$. Eisenstein series can be used to show that $\zeta(1+it)\neq 0$ for all $t\in\mathbb{R}$, which is equivalent to the prime number theorem, $\pi(x)=\sum_{p<x}1\sim\frac{x}{\log x}$ as $x\to\infty$. Note that

$$E(z,s)=y^s+\frac{\pi^{2s-1}\Gamma(1-s)\zeta(2-2s)}{\Gamma(s)\zeta(2s)}y^{1-s}+\frac{2\pi^s}{\Gamma(s)\zeta(2s)}\sum_{r\neq 0}|r|^{s-\frac{1}{2}}\sigma_{1-2s}(|r|)\sqrt{y}K_{s-\frac{1}{2}}(2\pi|r|y)e^{2\pi irx}.$$

Then $\int_0^1 E(z,s)e^{2\pi irx}\,dx=\frac{2|r|^{s-\frac{1}{2}}\sigma_{1-2s}(|r|)\sqrt{y}K_{s-\frac{1}{2}}(2\pi|r|y)}{\pi^{-s}\Gamma(s)\zeta(2s)}$. The numerator is the Whittaker function which is entire. Now the holomorphy of $E(z,s)$ on $Re(s)=\frac{1}{2}$ implies the non-vanishing of $\zeta(s)$ on $Re(s)=1$. Namely, the information of Eisenstein series give rise to the information on L-function. This is the beginning observation of Langlands-Shahidi method.

Sarnak [6] refined this more to obtain

$$|\zeta(1+it)|\gg\frac{1}{(\log t)^3}.$$

1.2.2. *Rankin-Selberg method.* Let $f(z)=\sum_{n=1}^\infty a(n)q^n$ be a cusp form of weight k for $SL_2(\mathbb{Z})$. Let $L(s,f)=\sum_{n=1}^\infty a(n)n^{-\frac{k-1}{2}}n^{-s}$ be the L-function. We can show easily that $L(s,f)$ has an analytic continuation to all of $\mathbb{C}$, and satisfies the functional equation $\Lambda(s,f)=(2\pi)^s\Gamma(s+\frac{k-1}{2})L(s,f)=(-1)^k\Lambda(1-s,f)$. Now let $L(s)=\sum_{n=1}^\infty|a(n)|^2n^{-s}$. Here $L(s)$ is related to the symmetric square L-function $L(s, Sym^2 f) = \zeta(2s)\sum_{n=1}^{\infty} a(n^2)n^{-s-k+1}$. In order to study the analytic property of $L(s)$, we need Rankin-Selberg method:

$$(4\pi)^{-s-k+1}\Gamma(s+k-1)\sum_{n=1}^{\infty}|a(n)|^2 n^{-s-k+1} = \int_{\Gamma\backslash\mathbb{H}} y^k |f(z)|^2 E(z,s)\,\frac{dxdy}{y^2}.$$

Here note that $y^2|f(z)|^2$ is Γ-invariant, and $\frac{dxdy}{y^2}$ is Γ-invariant measure on $\mathbb{H}$. The analytic continuation and functional equation of $E(z,s)$ give rise to those of $L(s)$. Let $\psi(s,f) = \pi^{-2s-k+1}2^{-2s}\Gamma(s)\Gamma(s+k-1)\zeta(2s)\sum_{n=1}^{\infty}|a(n)|^2 n^{-s-k+1}$. Here

$$L(s, f\times f) = \zeta(2s)\sum_{n=1}^{\infty}|a(n)|^2 n^{-s-k+1}, \quad \zeta(s)L(s, Sym^2 f) = L(s, f\times f).$$

Then $\psi(s,f)$ has a meromorphic continuation to all of $\mathbb{C}$, with a simple pole at $s=1$, and the residue is $\frac{3}{\pi}(f,f)$, and satisfies the functional equation $\psi(1-s,f) = \psi(s,f)$.

1.2.3. *Roelcke-Selberg Spectral decomposition.* Let

$$L^2(\Gamma\backslash\mathbb{H}) = L^2_{dis}(\Gamma\backslash\mathbb{H}) \oplus L^2_{cont}(\Gamma\backslash\mathbb{H}),$$

be the spectral decomposition of the Laplacian Δ into a discrete part and continuous part. Let $L^2_{cusp}(\Gamma\backslash\mathbb{H})$ be the space of cusp forms, i.e., those f such that $\int_0^1 f(x+iy)\,dx = 0$ for almost all y. Then $L^2_{dis}(\Gamma\backslash\mathbb{H}) = L^2_{cusp}(\Gamma\backslash\mathbb{H}) \oplus \mathbb{C}\cdot 1$. Now the Eisenstein series $E(z,s)$ is not in $L^2(\Gamma\backslash\mathbb{H})$ for any $s\in\mathbb{C}$, but $E(z,\frac{1}{2}+it)$ spans a continuous spectrum, i.e., the map

$$L^2(\mathbb{R}_{\geq 0}) \longrightarrow L^2(\Gamma\backslash\mathbb{H}), \quad h \longmapsto Eh = \int h(it)E(z,\frac{1}{2}+it)\,dt,$$

is an isometry onto $L^2_{cont}(\Gamma\backslash\mathbb{H})$, and $\Delta(Eh) = E((\frac{1}{4}-t^2)h)$. This means that any $f\in L^2(\Gamma\backslash\mathbb{H})$ has a spectral decomposition

$$f(z) = \sum_{j=0}^{\infty}(f,u_j)u_j(z) + \frac{1}{4\pi}\int_{-\infty}^{\infty}(f, E(\cdot,\frac{1}{2}+it))E(z,\frac{1}{2}+it)\,dt,$$

where the first sum runs over an orthonormal basis of the discrete part.

1.2.4. *Volume of the fundamental domain.* Langlands used Eisenstein series to compute the volume of the fundamental domain. Let's see it in our special case: Now $E(z,s)\notin L^1(\Gamma\backslash\mathbb{H})$ for $Re(s)>1$, but $E(z,s)\in L^1(\Gamma\backslash\mathbb{H})$ if $0<Re(s)<1$, so that $I(s) = \int_{\Gamma\backslash\mathbb{H}} E(z,s)\,dz$ makes sense there. For any $h\in C_c^{\infty}(\mathbb{R}_{>0})$, consider $\theta_h(z) = \frac{1}{2}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} f(Im\,\gamma z)$. This is a finite sum, and

θ_h is compactly supported. Let $\hat{h}(s) = \int_{\mathbb{R}_{>0}} h(y)y^{-s}\frac{dy}{y}$ be the Mellin transform. By Mellin inversion, $h(y) = \frac{1}{2\pi i}\int_{Re(s)=s_0} \hat{h}(s)y^s\,ds$. Then for $s_0 > 1$,

$$\theta_h(z) = \frac{1}{2}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \frac{1}{2\pi i}\int_{Re(s)=s_0} \hat{h}(s)(Im\,\gamma z)^s\,ds = \frac{1}{2\pi i}\int_{Re(s)=s_0} \hat{h}(s)E(z,s)\,ds.$$

We compute $I = \int_{\Gamma\backslash\mathbb{H}} \theta_h(z)\,dz$ in two different ways: On the one hand, move the contour to $Re(s) = \frac{1}{2}$:

$$I = vol(\Gamma\backslash\mathbb{H})\frac{3}{\pi}\hat{h}(\frac{1}{2}) + \frac{1}{2\pi i}\int_{Re(s)=\frac{1}{2}} \hat{h}(s)I(s)\,ds.$$

On the other hand, we can compute I directly using the definition of θ_h by the unfolding method:

$$I = \int_{\Gamma_\infty\backslash\mathbb{H}} h(Im\,z)\,dz = \int_{\mathbb{R}_{>0}} h(y)\,\frac{dy}{y^2} = \hat{h}(\frac{1}{2}).$$

Hence $vol(\Gamma\backslash\mathbb{H}) = \frac{\pi}{3}$, and $I(s) \equiv 0$. We have used the following lemma:

Lemma 1.2. *Suppose that $I(t)$ is bounded, and $\int_{\mathbb{R}_{>0}} \hat{h}(it)I(t)\,dt = a\hat{h}(1)$ for all $h \in C_c^\infty(\mathbb{R}_{>0})$. Then $I \equiv 0$ and $a = 0$.*

Proof. By taking $h_1 = yh' - h$, we have $\hat{h}_1 = (s-1)\hat{h}$, and hence $\int_{\mathbb{R}_{>0}} \hat{h}(it)I_1(t)\,dt = 0$ for $I_1(t) = (it-1)I(t)$. Since this is true for all h, $I_1 \equiv 0$, and hence $I \equiv 0$. ☐

1.3. **Siegel's Eisenstein series.** Siegel generalized the classical holomorphic Eisenstein series to Siegel domain. $\mathbb{H}_g = \{Z = X+iY : Y > 0\}$, where X, Y are $g\times g$ symmetric matrices, and Y is positive definite. Let $\Gamma = Sp_{2g}(\mathbb{Z}) = \{\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : AB^t = BA^t, CD^t = DC^t, AD^t - BC^t = I\}$.

Then we have the action $\gamma Z = (AZ+B)(CZ+D)^{-1}$. Let $\Gamma_\infty = \{\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in \Gamma\}$, and define the Siegel Eisenstein series

$$E_k(Z) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} det(CZ+D)^{-k}.$$

Then $E_k(Z)$ has the Fourier expansion

$$E_k(Z) = \sum_{T\geq 0} a(T)e^{2\pi i Tr(TZ)},$$

where $a(T)$ satisfies, when $g=2$,

$$a\begin{pmatrix} n & \frac{r}{2} \\ \frac{r}{2} & m \end{pmatrix} = \sum_{d|(n,r,m),\, d>0} d^{k-1} a\begin{pmatrix} 1 & \frac{r}{2d} \\ \frac{r}{2d} & \frac{mn}{d^2} \end{pmatrix}.$$

This means that E_k satisfies the Maass relations. The Maass subspace $S_k^*(\Gamma_2)$ is a subspace of cusp forms generated by Saito-Kurokawa lift. They satisfy the Maass relations. We have an isomorphism $S^+_{k-\frac{1}{2}}$ (Kohnen plus space) $\longrightarrow S_k^*(\Gamma_2)$, given by

$$g = \sum_{\substack{n\geq 1 \\ n\equiv 3,4 \bmod 4}} c(n)q^n \longmapsto \sum_{T>0} a(T)e^{2\pi i tr(TZ)},$$

where

$$a(T) = \sum_{d|(n,r,m),\, d>0} d^{k-1} c\left(\frac{4mn-r^2}{d^2}\right), \quad T = \begin{pmatrix} n & \frac{r}{2} \\ \frac{r}{2} & m \end{pmatrix}.$$

1.3.1. *Ikeda lift.* By using the Eisenstein series, Ikeda constructed cusp form F of weight $k+g$ on $\mathbb{H}_{2g}$ attached to an elliptic cusp form f of weight $2k$ such that $k\equiv g \pmod 2$. When $g=1$, it is Saito-Kurokawa lift.

Let $E_{k+g}(Z) = \sum_{T\geq 0} a(T)e^{2\pi i Tr(TZ)}$. When $T>0$, $a(T)$ is given by

$$a(T) = (*)L(1-k,\chi_T)\mathfrak{f}_T^{k-\frac{1}{2}} \prod_{p|D_T} \tilde{F}_p(T;p^{k-\frac{1}{2}}),$$

where $D_T = det(2T) = \mathfrak{d}_T\mathfrak{f}_T^2$, i.e., $\mathfrak{d}_T$ is the discriminant of $\mathbb{Q}[\sqrt{(-1)^k D_T}]$, and χ_T is the quadratic character. Here $\tilde{F}_p(T;X)$ is a Laurent polynomial which satisfies $\tilde{F}_p(T;X^{-1}) = \tilde{F}_p(T;X)$.

Now let $f(z) = \sum_{n=1}^\infty a(n)q^n$. Then $a(p) = p^{k-\frac{1}{2}}(\alpha_p + \alpha_p^{-1})$, where $|\alpha_p|=1$.

Let $h(z) = \sum_{(-1)^k n\equiv 0,1 (\text{mod } 4)} c(n)q^n$ be the weight $k+\frac{1}{2}$ Hecke eigenform which corresponds to f by the Shimura correspondence. Note that the Cohen Eisenstein series $H_{k+\frac{1}{2}}(z)$ corresponds to the ordinary Eisenstein series $E_{2k}(z)$ by the Shimura correspondence, and the $\mathfrak{d}_T$th Fourier coefficient of $H_{k+\frac{1}{2}}$ is $L(1-k,\chi_T)$.

Then

$$F(Z) = \sum_{T>0} c(\mathfrak{d}_T)\mathfrak{f}_T^{k-\frac{1}{2}} \prod_p \tilde{F}_p(T;\alpha_p)e^{2\pi i Tr(TZ)},$$

is the Ikeda lift.

2. Lecture Two: Adelic Eisenstein series

We follow [1], p. 129. In Lecture Two, we define adelic Eisenstein series and describe their role in representation theory.

2.1. **Eisenstein series in adelic setting.**

First, we reformulate Siegel's Eisenstein series in terms of adelic language and induced representation. We can take Γ to be any congruence subgroup such as $\Gamma(N) = \{\gamma \in Sp_{2n}(\mathbb{Z}) : \gamma \equiv I \pmod{N}\}$. We can also consider vector-valued Eisenstein series.

Let $G = Sp_{2n}$, and $\mathbb{A} = \mathbb{R} \otimes \mathbb{A}_f$ be the ring of adeles and $\mathbb{A}_f = \otimes'_p \mathbb{Q}_p$ be its finite part. Then we have the strong approximation theorem: $G(\mathbb{A}) = G(\mathbb{Q})G(\mathbb{R})K'$, where K' is any compact open subgroup of $G(\mathbb{A}_f)$. Usually we take K' to be a subgroup of finite index in $K = \prod_p K_p$, where $K_p = Sp_{2n}(\mathbb{Z}_p)$. Then $\Gamma = G(\mathbb{Q}) \cap G(\mathbb{R})K = Sp_{2n}(\mathbb{Z})$, and $\Gamma' = G(\mathbb{Q}) \cap G(\mathbb{R})K'$.

Let P be the Siegel parabolic subgroup, i.e., $P = MN$, $M = \{\begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix}\} \simeq GL_n$, and $N = \{\begin{pmatrix} I & S \\ 0 & I \end{pmatrix} : S = {}^t S\}$. Note that $P(\mathbb{Q}) \cap \Gamma = \Gamma_\infty$. Then $P(\mathbb{Q})\backslash G(\mathbb{Q}) \simeq \Gamma_\infty \backslash \Gamma$.

Hence the sum over $\Gamma_\infty\backslash\Gamma$ can be translated over the sum over $P(\mathbb{Q})\backslash G(\mathbb{Q})$. This gives rise to the concept of induced representation. A special case is called the degenerate principal series: Let χ be a grössencharacter. Let $I(s, \chi)$ be the space of all smooth K-finite functions $\Phi(g, s)$ on $G(\mathbb{A})$ such that $\Phi(nm(a)g, s) = \chi(det(a))|det(a)|^{s+\rho_n}\Phi(g, s)$, where $n \in N$, $m(a) = \begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix}$, and $\rho_n = \frac{n+1}{2}$, which is for normalization. Then $I(s, \chi) = \otimes_v I_v(s, \chi_v)$, where $I_v(s, \chi_v)$ is the local induced representation. We can take $\Phi(g, s) = \otimes_v \Phi_v(g, s)$, and the restriction of $\Phi(g, s)$ to K is independent of s. Note that $\Phi(g, s)$ is determined by its restriction to K.

For any $\Phi(g, s) \in I(s, \chi)$, the Siegel Eisenstein series is defined by

$$E(g, s, \Phi) = \sum_{\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})} \Phi(\gamma g, s).$$

Then $E(g, s, \Phi)$ has a meromorphic continuation to the whole $\mathbb{C}$, and satisfies the functional equation

$$E(g, s, \Phi) = E(g, -s, M(s)\Phi),$$

where $M(s)$ is the global intertwining operator $M(s) : I(s, \chi) \longrightarrow I(-s, \chi^{-1})$, defined by

$$M(s)\Phi(g, s) = \int_{N(\mathbb{A})} \Phi(ng, s)\, dn.$$

Notice that we have shifted the line of symmetry to $Re(s) = 0$.

Special case: Let $\Phi(g,s) = \Phi_\infty(s)\prod_p \Phi_p(s)$. Take Φ_p to be the normalized spherical section for all p. ($\Phi_p(k,s) = 1$ for all $k \in K$.) For a positive integer l, let $\Phi_\infty(k,s) = (det\,\mathbf{k})^l$, where $\mathbf{k} \in U(n)$ corresponds to $k \in K_\infty$ by $K_\infty = \{\begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a + bi \in U(n)\}$. Here note that det^l is the simplest representation of $U(n)$. Any representation of $U(n)$ is indexed by $(l_1, ..., l_n)$, $l_1 \geq \cdots \geq l_n$. One can take an arbitrary representation ϕ of $U(n)$.

Let $z = x + iy = x + i^t vv = g(i \cdot I_n) \in \mathbb{H}_n$, where $g = n(x)m(v) \in P(\mathbb{R})$. Then we can show that

$$E(g,s,\Phi) = det(y)^{\frac{1}{2}(s+\rho_n+l)} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} det(cz+d)^{-l} |det(cz+d)|^{-s-\rho_n+l}.$$

So when $s = l - \rho_n$, we obtain the classical Siegel Eisenstein series of weight l. The advantage of this construction is that we can generalize for any compact subgroup K', and for any representation of ϕ of $U(n)$. Also we can choose Φ_p, different from spherical section to allow ramification. We can also generalize it to an arbitrary number field.

2.2. **Eisenstein series attached to induced representation.** The above construction can be generalized to any reductive group and any parabolic subgroup.

Let (π, W) be an irreducible admissible representation of $M(F)$, where $P = MN \subset G$ is a maximal parabolic subgroup. Here F is a local or global field. Let $I(s\tilde{\alpha}, \pi)$ be the induced representation of $G(F)$;

$$I(s\tilde{\alpha}, \pi) = Ind_P^G\, \pi \otimes e^{\langle s\tilde{\alpha}, H_P(\,)\rangle} \otimes 1,$$

where $\tilde{\alpha}$ is the fundamental weight corresponding to α, namely, $\tilde{\alpha} = \langle \rho_P, \alpha \rangle^{-1} \rho_P$.

The representation space $V = V(s\tilde{\alpha}, \pi)$ is the vector space of all smooth functions $f : \mathbf{G}(F) \longrightarrow W$ such that

$$f(mng) = \pi(m) e^{\langle s\tilde{\alpha} + \rho_P, H_P(m)\rangle} f(g),$$

for all $m \in \mathbf{M}(F), n \in \mathbf{N}(F), g \in \mathbf{G}(F)$. The action is

$$(I(s\tilde{\alpha}, \pi)(g)f)(h) = f(hg),$$

for all $f \in V$ and $g, h \in \mathbf{G}(F)$. The reason we add ρ_P is to normalize $I(s\tilde{\alpha}, \pi)$ so that if π is unitary, then $I(s\tilde{\alpha}, \pi)$ is unitary for $s \in i\mathbb{R}$. The Eisenstein series is defined by, for $f_s \in I(s\tilde{\alpha}, \pi)$, $E(s, \pi, f_s, g) = \sum_{\gamma \in P(\mathbb{Q}) \backslash G(\mathbb{Q})} f_s(\gamma g)$.

More generally, let $P = MN \subset G$ be an arbitrary parabolic subgroup. Given an irreducible representation of $M(F)$ and a certain parameter Λ, one can define the induced representation $I(\Lambda, \pi)$, and Eisenstein series $E(g, \Lambda, \pi)$.

2.3. **Spectral decomposition.** As in GL_2 case, Eisenstein series provide the continuous spectrum of $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$.

It has the orthogonal decomposition;

$$L^2(G(\mathbb{Q})\backslash G(\mathbb{A})) = \bigoplus_{(M,\pi)} L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))_{(M,\pi)},$$

where M is a Levi subgroup of G, and π is a cuspidal representation of $M(\mathbb{A})$ taken modulo conjugacy (Here we normalize π so that the action of the maximal split torus in the center of G at the archimedean place is trivial.). Here $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))_{(M,\pi)}$ is generated by the pseudo-Eisenstein series

$$\hat{\phi}(g) = \frac{1}{(2\pi i)^{dim A}} \int_{Re(\Lambda)=\Lambda_0} E(g, \phi, \Lambda)\, d\Lambda,$$

where A is the maximal split torus of M. Here the subspace

$$\oplus_{(G,\pi)} L^2_{dis}(G(\mathbb{Q})\backslash G(\mathbb{A}))_{(G,\pi)},$$

is the space of cuspidal representations $L^2_{cusp}(G(\mathbb{Q})\backslash G(\mathbb{A}))$. It belongs to the discrete spectrum $L^2_{dis}(G(\mathbb{Q})\backslash G(\mathbb{A}))$. Moreover, the residues of the Eisenstein series span a part of the discrete spectrum $L^2_{dis}(G(\mathbb{Q})\backslash G(\mathbb{A}))_{(M,\pi)}$. Hence we have

$$\begin{aligned} &L^2(G(F)\backslash G(\mathbb{A})) = L^2_{dis}(G(F)\backslash G(\mathbb{A})) \oplus L^2_{cont}(G(\mathbb{Q})\backslash G(\mathbb{A})); \\ &L^2_{dis}(G(\mathbb{Q})\backslash G(\mathbb{A})) = \bigoplus_{(M,\pi)} L^2_{dis}(G(\mathbb{Q})\backslash G(\mathbb{A}))_{(M,\pi)} = L^2_{cusp}(G(\mathbb{Q})\backslash G(\mathbb{A})) \oplus L^2_{res}(G(\mathbb{Q})\backslash G(\mathbb{A})). \end{aligned}$$

Its orthogonal complement of the cuspidal spectrum in $L^2_{dis}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ is called the residual spectrum. Write

$$L^2_{cusp} = \oplus_{\Pi}\, m_{cusp}(\Pi)\, \Pi, \quad L^2_{res} = \oplus_{\Pi}\, m_{res}(\Pi)\, \Pi,$$

where $m_{cusp}(\Pi), m_{res}(\Pi)$ are called multiplicity of Π. Residual representations occur with multiplicity one (if the inducing cuspidal representation has multiplicity one) in L^2_{res}. But cuspidal representations can occur with arbitrarily large multiplicity in some groups such as G_2.

2.4. **Residual spectrum.** We give several examples of residual spectrum. The trace formula cannot distinguish cuspidal spectrum from residual spectrum. The trace formula gives a formula in terms of Arthur packets like

$$\text{sum over all representations of Arthur packets} = \text{sum of some orbital integrals.}$$

Here Arthur packet contains both cuspidal representations and residual representations.

We compute the residue of Eisenstein series attached to the trivial character of Borel subgroup and everywhere unramified for $G = Sp_8$ and G_2. Here $G = Sp_8$ is the smallest symplectic group which has non-trivial residues.

Let $G = Sp_8$. Let $E(g, \Lambda, \Phi) = \sum_{\gamma \in B(\mathbb{Q})\backslash G(\mathbb{Q})} f(\gamma g)$ be the Eisenstein series attached to $I(\Lambda)$, where $\Lambda = (\lambda_1, ..., \lambda_4) \in \mathbb{C}^4$. Let $E_0(g, \Lambda, f)$ be the constant term along $B = TU$, namely,

$$E_0(g, \Lambda, f) = \int_{U(\mathbb{Q})\backslash U(\mathbb{A})} E(ug, \Lambda, f)\, du = \sum_{w \in W} M(w, \Lambda) f(g),$$

where $M(w, \Lambda) f(g) = \int_{U \cap w\overline{U}w^{-1}} f(w^{-1}ug)\, du$. Here W is the Weyl group of Sp_8, namely, $W \simeq S_4 \rtimes \mathbb{Z}_2^4$.

Then the iterated residue is square integrable only when $\Lambda = (4, 3, 2, 1)$, in which case we get the constant, and $\Lambda = (2, 1, 0, -1), (1, 0, 2, 1)$, in which case we get the non-trivial residue.

Let $G = G_2$. This is done by Langlands [4]. We follow the notation from Moeglin-Waldspurger [5]: Simple roots are β_1 (long), β_6 (short). Let $\beta_2 = \beta_1 + \beta_6$, $\beta_3 = 2\beta_1 + 3\beta_6$, $\beta_4 = \beta_1 + 2\beta_6$, $\beta_5 = \beta_1 + 3\beta_6$. Then $\{\beta_1, ..., \beta_6\}$ are positive roots. The Weyl group $W \simeq D_{12}$, the dihedral group of order 12. Then the constant term of $E(g, \Lambda, f)$ is

$$E_0(g, \Lambda, f) = \sum_{w \in W} M(w, \Lambda) f(g).$$

If $f = \otimes f_v$, unramified everywhere, $M(w, \Lambda) f$ is a quotient of product of the completed Riemann zeta functions; Let $\xi(s) = \pi^{-\frac{s}{2}} \Gamma(\frac{s}{2}) \zeta(s)$. Then

$$M(w, \Lambda) f = \prod_{\alpha > 0,\, w\alpha < 0} \frac{\xi(\langle \Lambda, \alpha^\vee \rangle)}{\xi(\langle \Lambda, \alpha^\vee \rangle + 1)} f.$$

Let $\Lambda = x\beta_3 + y\beta_4$ with $x, y \in \mathbb{C}$. Then

$$E_0(e,\Lambda,f) = 1 + \frac{\xi(x)}{\xi(x+1)} + \frac{\xi(y)}{\xi(y+1)} + \frac{\xi(x)\xi(3x+y)}{\xi(x+1)\xi(3x+y+1)} + \frac{\xi(y)\xi(x+y)}{\xi(y+1)\xi(x+y+1)}$$
$$+\frac{\xi(x)\xi(3x+y)\xi(2x+y)}{\xi(x+1)\xi(3x+y+1)\xi(2x+y+1)} + \frac{\xi(y)\xi(x+y)\xi(3x+2y)}{\xi(y+1)\xi(x+y+1)\xi(3x+2y+1)}$$
$$+\frac{\xi(x)\xi(3x+y)\xi(2x+y)\xi(3x+2y)}{\xi(x+1)\xi(3x+y+1)\xi(2x+y+1)\xi(3x+2y+1)}$$
$$+\frac{\xi(y)\xi(x+y)\xi(2x+y)\xi(3x+2y)}{\xi(y+1)\xi(x+y+1)\xi(2x+y+1)\xi(3x+2y+1)}$$
$$+\frac{\xi(x)\xi(x+y)\xi(3x+y)\xi(2x+y)\xi(3x+2y)}{\xi(x+1)\xi(x+y+1)\xi(3x+y)\xi(2x+y+1)\xi(3x+2y+1)}$$
$$+\frac{\xi(y)\xi(x+y)\xi(2x+y)\xi(3x+y)\xi(3x+2y)}{\xi(y+1)\xi(x+y+1)\xi(2x+y+1)\xi(3x+y+1)\xi(3x+2y+1)}$$
$$+\frac{\xi(x)\xi(y)\xi(x+y)\xi(2x+y)\xi(3x+y)\xi(3x+2y)}{\xi(x+1)\xi(y+1)\xi(x+y+1)\xi(2x+y+1)\xi(3x+y+1)\xi(3x+2y+1)}.$$

Then the iterated residue is square integrable only when $\Lambda = \beta_3 + \beta_4$, in which case we get the constant, and when $\Lambda = \beta_2$, in which case we get non-trivial residue.

If we allow ramification, then it becomes a little more complicated: Namely, we allow $f = \otimes f_v$ to be arbitrary. Then f_p is spherical for almost all p. When f_p is not spherical, then the residue is the image $(1 + \frac{1}{2}E)R(\rho_2, \beta_2)I(\beta_2)$, where $R(\rho_2, \beta_2)$ is the normalized intertwining operator, and $R(\rho_2, \beta_2)I(\beta_2) = \pi_{1p} \oplus \pi_{2p}$, and $E(f_p) = \begin{cases} f_p, & \text{if } f_p \in \pi_{1p} \\ -2f_p, & \text{if } f_p \in \pi_{2p} \end{cases}.$

Let $\pi^S = \otimes_{v \notin S} \pi_{1v} \otimes \otimes_{v \in S} \pi_{2v}$. Then π^S occurs in $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$ if and only if $|S| \neq 1$. Here π^S belongs to an Arthur packet. Its multiplicity in the discrete spectrum is bigger than $\frac{1}{6}(2^{\#S} + (-1)^{\#S}2)$.

2.5. **Volume of the fundamental domain.** We follow [3]. For $T \gg 0$, let $\wedge^T$ be the Arthur truncation operator for ϕ on $G(\mathbb{Q})\backslash G(\mathbb{A})$. For the constant function 1, $\wedge^T 1$ is the characteristic function of a compact subset $\mathfrak{F}(T)$ of $G(\mathbb{Q})\backslash G(\mathbb{Q})^1$, where $G(\mathbb{A})^1 = \{g \in G(\mathbb{A}) : H_G(g) = 0\}$ and H_G is a certain homomorphism. For example, if $G = GL_n$, $G(\mathbb{A})^1 = \{g \in GL_n(\mathbb{A}) : |det(g)| = 1\}$. Then as $T \to \infty$, we obtain the fundamental domain $\mathfrak{F}$. We fix a Haar measure on $G(\mathbb{Q})\backslash G(\mathbb{A})^1$ so that K gets measure 1. We want to find the volume of the fundamental domain of $G(\mathbb{Q})\backslash G(\mathbb{A})^1$.

We use the formula by Jacquet-Lapid-Rogawski [2]: Let $E(g,\lambda,1)$ be the Eisenstein series associated to the trivial character of the Borel subgroup. Then

$$\int_{\mathfrak{F}(T)} E(g,\lambda,1)\,dg = v\sum_{w\in W}\frac{e^{\langle w\lambda-\rho,T\rangle}}{\prod_{\alpha\in\Delta}\langle w\lambda-\rho,\alpha^\vee\rangle}\prod_{\alpha>0,\,w\alpha<0}\frac{\xi(\langle\lambda,\alpha^\vee\rangle)}{\xi(\langle\lambda,\alpha^\vee\rangle+1)},$$

where $v = Vol(\{\sum_{\alpha\in\Delta} a_\alpha\alpha^\vee : 0\le a_\alpha<1\}$.

We look at the residue at $\lambda=\rho$, the half sum of positive roots. We know that the residue of $E(g,\lambda,1)$ at $\lambda=\rho$ is a constant, which is $\prod_{i>1}\xi(i)^{n_i}$, where $n_i = \#\{\alpha>0 : \langle\rho,\alpha^\vee\rangle = i\} - \#\{\alpha > 0 : \langle\rho,\alpha^\vee\rangle = i-1\}$. From this, we get

$$Vol(\mathfrak{F}(T)) = v\prod_{i>1}\xi(i)^{-n_i}\sum_{J\subset\Delta}(-1)^{\mathrm{rank}P_J}\frac{\prod_{i>1}\xi(i)^{n_{i,J}}}{\prod_{\alpha\in\Delta-w_JJ}(1-\langle w_J\rho,\alpha^\vee\rangle)}e^{\langle w_J\rho-\rho,T\rangle},$$

where P_J is the parabolic subgroup corresponding to J, and w_J is the element of the maximal length in D_J, distinguished coset representatives of W/W_J. Also $n_{i,J} = \#\{\alpha>0, w_J\alpha<0 : \langle\rho,\alpha^\vee\rangle = i\} - \#\{\alpha>0, w_J\alpha<0 : \langle\rho,\alpha^\vee\rangle = i-1\}$.

As $T\to\infty$, only the term corresponding to $w=1$ survives, and hence

$$Vol(\mathfrak{F}) = v\prod_{i>1}\xi(i)^{-n_i}.$$

3. Lecture Three: Langlands-Shahidi method

In Lecture Three, we describe Langlands-Shahidi method. We follow [1]. Langlands observed that the constant term of Eisenstein series associated to cuspidal representation of maximal parabolic subgroups contains many new automorphic L-functions. The meromorphic continuation and functional equation of Eisenstein series give rise to the same for the automorphic L-functions.

Let $\pi = \otimes_v\pi_v$ be a cuspidal representation of $M(\mathbb{A})$, and let $r : {}^LM \longrightarrow GL_N(\mathbb{C})$ be a finite-dimensional representation. For almost all v (say, $v\notin S$), π_v is spherical. So it is uniquely determined by a semi-simple conjugacy class $\{t_v\}\subset {}^LT$. We form a local Langlands L-function

$$L(s,\pi_v,r) = det(I - r(t_v)q_v^{-s})^{-1}.$$

Let $L_S(s,\pi,r) = \prod_{v\notin S} L(s,\pi_v,r)$ be a partial L-function. It converges absolutely for $Re(s)\gg 0$.

Conjecture 3.1. *(Langlands) $L_S(s,\pi,r)$ has a meromorphic continuation to all of $\mathbb{C}$. Fix an additive character $\psi=\otimes\psi_v$ of $\mathbb{A}/\mathbb{Q}$. For each $v\in S$, we can define a local L-function $L(s,\pi_v,r)$ and a local root number $\epsilon(s,\pi_v,r,\psi_v)$ such that the completed L-function $L(s,\pi,r)=\prod_{all\ v} L(s,\pi_v,r)$*

has a meromorphic continuation and satisfies a functional equation $L(s,\pi,r)=\epsilon(s,\pi,r)L(1-s,\pi,\tilde{r})$*, where* $\epsilon(s,\pi,r)=\prod_v \epsilon(s,\pi_v,r,\psi_v)$*, and* $\tilde{r}(g)={}^t r(g)^{-1}$*.*

Even the meromorphic continuation is not obvious: For example, it is known that the Euler product $\prod_{p\equiv 1(4)}(1-p^{-s})^{-1}$ has a natural boundry at $Re(s)=0$.

Let $\pi=\otimes\pi_v$ be a cuspidal representation of GL_2. Let $diag(\alpha_p,\beta_p)$ be a semi-simple conjugacy class of π_p. Let $Sym^m: GL_2(\mathbb{C})\longrightarrow GL_{m+1}(\mathbb{C})$ be the mth symmetric power representation. Then

$$L(s,\pi_p,Sym^m)=\prod_{i=0}^m (1-\alpha_p^{m-i}\beta_p^i p^{-s})^{-1}.$$

Holomorphic continuation and functional equations of m-th symmetric power L-functions are outstanding open problems.

They provide test cases of Langlands functoriality: Let $\psi_p: W_{\mathbb{Q}_p}\times SL_2(\mathbb{C})\longrightarrow GL_2(\mathbb{C})$ be the parametrization of π_p. Then we have $Sym^m\psi_p: W_{\mathbb{Q}_p}\times SL_2(\mathbb{C})\longrightarrow GL_{m+1}(\mathbb{C})$. By the local Langlands correspondence proved by Harris-Taylor and Henniart, we have an irreducible admissible representation $Sym^m\pi_p$ corresponding to $Sym^m\psi_p$. Let $Sym^m\pi=\otimes Sym^m\pi_p$. It is an irreducible admissible representation of $GL_{m+1}(\mathbb{A})$.

Conjecture 3.2. *(Langlands functoriality conjecture)* $Sym^m\pi=\otimes Sym^m\pi_p$ *is an automorphic representation of* $GL_{m+1}(\mathbb{A})$*.*

If we know that $Sym^m\pi$ is an automorphic representation of GL_{m+1}, then by Jacquet-Shalika, $|\alpha_p^m|<p^{\frac{1}{2}}$. So $|\alpha_p|<p^{\frac{1}{2m}}$. If it is true for all m, then $|\alpha_p|=1$. It is the Ramanujan conjecture. The current best bound is $|\alpha_p|\leq p^{\frac{7}{64}}$.

If λ_1 is the first positive eigenvalue of the Laplacian on the corresponding hyperbolic space, we would have $\lambda_1>\frac{1}{4}(1-(\frac{2}{m})^2)$ for all m. So $\lambda_1\geq\frac{1}{4}$. This is called the Selberg conjecture. The current best bound is $\lambda_1\geq\frac{1}{4}(1-(\frac{7}{32})^2)=\frac{975}{4096}=0.238...$

It also implies that the L-functions $L(s,\pi,Sym^m)$ is non-vanishing for all m. It implies the Sato-Tate conjecture; write $\alpha_p=e^{i\theta_p}$ with $0\leq\theta_p\leq\pi$. Sato-Tate conjecture says that $\{\theta_p\}$ is equidistributed with respect to the measure $\frac{2}{\pi}\sin^2\theta d\theta$. Let $(a,b)\subset[0,\pi]$. Then

$$\lim_{x\to\infty}\frac{1}{\pi(x)}\#\{p\leq x,\theta_p\in(a,b)\}=\int_a^b\frac{2}{\pi}\sin^2\theta\,d\theta.$$

Currently, we only know for $m=2,3,4$ for arbitrary cusp form. When π comes from a holomorphic cusp form, we have a recent potential modularity result due to Taylor and others.

It gives rise meromorphic continuation to all of $\mathbb{C}$, and non-vanishing for $Re(s) \geq 1$. It is enough to prove Sato-Tate conjecture.

Langlands proved meromorphic continuation for many L-functions, and Shahidi computed non-constant terms for globally generic cuspidal representations and proved the functional equations.

Let G be a Chevalley group, and Δ be the set of simple roots. Let $P = P_\theta = MN$ be a maximal parabolic subgroup of G, where $\theta = \Delta - \{\alpha\}$. Let w_0 be the unique element in W such that $w_0(\theta) \subset \Delta$ and $w_0(\alpha) < 0$. Then P is called self-conjugate if $P = P' = P_{w_0(\theta)}$.

Let π be a cuspidal representation of $M(\mathbb{A})$ and $I(s,\pi)$ be the induced representation. For $f_s \in I(s,\pi)$, let $E(s,\pi,f_s,g) = \sum_{\gamma\in P(\mathbb{Q})\backslash G(\mathbb{Q})} f_s(\gamma g)$ be the Eisenstein series. For a parabolic subgroup $Q = M_Q N_Q$, let $E_Q(s,\pi,f_s,g) = \int_{N_Q(\mathbb{Q})\backslash N_Q(\mathbb{A})} E(s,\pi,f_s,ng)\,dn$.

Theorem 3.3. *Unless $Q = P, P'$, $E_Q(s,\pi,f_s,g) = 0$. If P is self-conjugate, then*

$$E_P(s,\pi,f_s,g) = f_s + M(s,\pi)f_s.$$

If P is not self-conjugate, then

$$E_P(s,\pi,f_s,g) = f_s, \quad E_{P'}(s,\pi,f_s,g) = M(s,\pi)f_s.$$

Let $M(s,\pi) = \otimes A(s,\pi_v,w_0)$, and $f = \otimes f_v$, where f_v is spherical for almost all v. When f_v is spherical, we have Gindikin-Karpelevich formula: Let $\pi_v \hookrightarrow I(\chi_v)$, where χ_v is a unitary character.

$$A(s,\pi_v,w_0)f_v(e) = \prod_{\beta>0,\, w_0\beta<0} \frac{L(s\langle\tilde{\alpha},\beta^\vee\rangle, \chi_v\circ\beta^\vee)}{L(1+s\langle\tilde{\alpha},\beta^\vee\rangle, \chi_v\circ\beta^\vee)}.$$

Langlands observed that $\langle\tilde{\alpha},\beta^\vee\rangle = i$ for $i = 1,...,m$; Let V_i be the subspace of ${}^L\mathfrak{n}$, generated by $E_{\beta^\vee}$ such that $\langle\tilde{\alpha},\beta^\vee\rangle = i$. Here ${}^L\mathfrak{n}$ is the Lie algebra of ${}^LN$. For each i, the adjoint action of ${}^LM$ leaves V_i stable. Let r be the adjoint representation of ${}^LM$ on ${}^L\mathfrak{n}$, and $r_i = r|_{V_i}$. Then

Theorem 3.4. *(Langlands-Shahidi) r_i is irreducible for each i, and the weights of r_i are the roots $\beta^\vee$ in ${}^L\mathfrak{n}$ which restricts to $i\alpha^\vee$ in ${}^LA$.*

Therefore, we have

$$A(s,\pi_v,w_0)f_v(e) = \prod_{i=1}^m \frac{L(is,\pi_v,r_i)}{L(1+is,\pi_v,r_i)},$$

where $L(s,\pi_v,r_i) = \prod_{\beta>0,\,\langle\tilde{\alpha},\beta^\vee\rangle=i} L(s,\chi_v\circ\beta^\vee)$.

Example 1: Let $G = GL_{k+l}$ and $M \simeq GL_k \times GL_l$, and $N = \{\begin{pmatrix} I & X \\ 0 & I \end{pmatrix}\}$. Then ${}^LM \simeq GL_k(\mathbb{C}) \times GL_l(\mathbb{C})$, and $r(diag(g_1, g_2))X = g_1 X g_2^{-1}$. So r is irreducible. Hence $m = 1$. Let $\pi_1 = \pi(\mu_1, ..., \mu_k)$ and $\pi_2 = \pi(\nu_1, ..., \nu_l)$ be spherical representations of GL_k, GL_l, resp. Then

$$L(s, \pi_1 \otimes \pi_2, r) = L(s, \pi_1 \times \tilde{\pi}_2) = \prod_{i,j} L(s, \mu_i \nu_j^{-1}).$$

Example 2: Let $G = G_2$, and P be attached to $\Delta - \{\beta_1\}$. Then $P = MN$, and $M \simeq GL_2$. In this case, $m = 2$, and $r_1 = Sym^3(\rho_2) \otimes (\wedge^2 \rho_2)^{-1}$ and $r_2 = \wedge^2 \rho_2$, where $\rho_2 : GL_2(\mathbb{C}) \longrightarrow GL_2(\mathbb{C})$. We obtain the symmetric cube L-function.

Example 3: Let $G = E_8$, and $P = MN$, where the derived group of M is $SL_3 \times SL_2 \times SL_5$. Then $m = 6$ (maximum length), and r_1 gives rise to the triple product L-function of $GL_3 \times GL_2 \times GL_5$.

3.1. **List of L-functions via Langlands-Shahidi method (split reductive groups).**

3.1.1. A_n *case.* $GL_m \times GL_n \subset GL_{m+n}$ gives the Rankin-Selberg L-function $L(s, \pi_1 \times \pi_2)$, where $\pi_i's$ are cuspidal representations of GL_m, GL_n, resp.

3.1.2. B_n *case.* $B_n - 1$: $m = 2$; r_1 gives the Rankin-Selberg L-function of $GL_k \times GSpin(2l+1)$; r_2 gives the twisted symmetric square L-function of GL_k. If $G = SO(2n+1)$, r_1 gives the Rankin-Selberg L-function of $GL_k \times SO(2l+1)$; r_2 gives the symmetric square L-function of GL_k.

3.1.3. C_n *case.* $C_n - 1$: $m = 2$; r_1 gives the Rankin-Selberg L-function of $GL_k \times Sp(2l)$; r_2 gives the exterior square L-function of GL_k, if $k \neq 1$ (If $k = 1$, then $m = 1$).

3.1.4. D_n *case.* $D_n - 1$: $m = 2$; r_1 gives the Rankin-Selberg L-function of $GL_k \times GSpin(2l)$; r_2 gives the twisted exterior square L-function of GL_k. If $G = SO(2n)$, r_1 gives the Rankin-Selberg L-function of $GL_k \times SO(2l)$; r_2 gives the exterior square L-function of GL_k (If $k = 1$, then $m = 1$).

$D_n - 2$: $m = 2$; r_1 gives the triple L-function of $GL_{n-2} \times GL_2 \times GL_2$; r_2 gives the twisted exterior square L-function of GL_{n-2}.

$D_n - 3$: $m = 2$; r_1 gives $L(s, \sigma \otimes \tau, \rho_{n-3} \otimes \wedge^2 \rho_4)$; r_2 gives the twisted exterior square L-function of GL_{n-3}.

3.1.5. F_4 *case.* F_4-1: $m=4$; r_1 gives $L(s,\sigma\times\tau)$ which is entire; r_2 gives $L(s,\sigma\otimes\tau, Sym^2\rho_2\otimes\rho_3)$ which has a pole at $s=1$ when $\tilde{\tau}\simeq Sym^2\sigma$.

F_4-2: $m=3$; r_1 gives $L(s,\sigma\otimes\tau, Sym^2\rho_3\otimes\rho_2)$; r_2 gives $L(s,\sigma, Sym^2\rho_3\otimes\omega_\tau)$ which has a pole at $s=1$ always.

$(xviii)$: $m=2$; $M=GSpin(7)\subset F_4$; $dim\, r_2=1$; r_1 is the 14-dim'l irreducible representation of $Sp_6(\mathbb{C})$, called spherical harmonic.

$(xxii)$: $m=2$; $M=GSp_6\subset F_4$; r_1=8-dim'l spin representation of $Spin(7,\mathbb{C})$; r_2 gives the standard L-function of $SO_7(\mathbb{C})$ (7-dimensional).

3.1.6. E_6 *case.* E_6-1: $m=3$; r_1 gives the triple L-function of $GL_3\times GL_2\times GL_3$; r_2 gives the standard L-function of $GL_3\times GL_3$.

E_6-2: $m=2$; $r_1=\wedge^2\rho_5\otimes\rho_2$; r_2 gives the Rankin-Selberg L-function of $GL_5\times GL_2$ which is entire.

(x): $m=2$; $dim\, r_2=1$; r_1 gives the exterior cube L-function of $GL_6(\mathbb{C})$ (20 dim'l irreducible representation of $GL_6(\mathbb{C})$)

$(xxiv)$: $m=1$; r_1=16-dimensional half-spin representation of $Spin(10)$.

3.1.7. E_7 *case.* E_7-1: $m=4$; r_1 gives the triple L-function of $GL_3\times GL_2\times GL_4$; r_2 comes from D_6-3 case.

E_7-2: $m=3$; $r_1=\wedge^2\rho_5\otimes\rho_3$; r_2 gives the Rankin-Selberg L-function of $GL_5\times GL_3$ which is entire.

E_7-3: $m=2$; r_1 gives the L-function $L(s,\sigma\otimes\tau, Spin^{16}\otimes\rho_2)$, where σ,τ are cuspidal representations of $GSpin_{10}, GL_2$, resp. and $Spin^{16}$ is the 16-dimensional half-spin representation of $Spin_{16}(\mathbb{C})$.

E_7-4: $m=3$; r_1 gives the L-function $L(s,\sigma\otimes\tau,\wedge^2\rho_6\otimes\rho_2)$, where σ,τ are cuspidal representations of GL_6, GL_2, resp.

(xi): $m=2$; r_1 gives the exterior cube L-function of $GL_7(\mathbb{C})$ (35-dimensional representation of $GL_7(\mathbb{C})$); r_2 gives the standard L-function of $GL_7(\mathbb{C})$ which is entire.

$(xxvi)$: $m=2$; $dim\, r_2=1$ and r_1 gives the degree $32=2^5$ spin L-function of $Spin_{12}$

(xxx): $m=1$; r_1 gives the standard L-function of E_6.

3.1.8. E_8 *case.* E_8-1: $m=6$; r_1 gives the triple L-function of $GL_3\times GL_2\times GL_5$; r_2 comes from E_7-2 case.

E_8-2: $m=5$; r_1 gives the L-function $L(s,\sigma\otimes\tau,\wedge^2\rho_5\otimes\rho_4)$, where σ,τ are cuspidal representations of GL_5, GL_4, resp.

E_8-3: $m=4$; r_1 gives the L-function $L(s,\sigma\otimes\tau, Spin^{16}\otimes\rho_3)$, where σ,τ are cuspidal representations of $GSpin_{10}, GL_3$, resp. and $Spin^{16}$ is the16-dimensional half-spin representation of $Spin_{16}(\mathbb{C})$.

E_8-4: $m=3$; r_1 gives the standard L-function of $E_6\times GL_2$; r_2 gives the standard L-function of E_6 ((xxx) case).

E_8-5: $m=4$; r_1 gives the L-function $L(s,\sigma\otimes\tau,\wedge^2\rho_7\otimes\rho_2)$, where σ,τ are cuspidal representations of GL_7, GL_2, resp.

$(xiii)$: $m=3$; r_1 gives the degree 56 exterior cube L-function of GL_8

$(xxviii)$: $m=2$; r_1 gives the degree $64=2^6$ spin L-function of $Spin_{14}$

$(xxxii)$: $m=2$; $dim\, r_2=1$; r_1 gives the standard L-function of E_7.

3.1.9. G_2 *case.* (xv) (attached to the maximal parabolic subgroup generated whose unipotent radical contains the long simple root subgroup) : $m=2$; $dim\, r_2=1$, r_1 gives the third symmetric power L-function of GL_2

(xvi) (attached to the maximal parabolic subgroup generated whose unipotent radical contains the short simple root subgroup): $m=3$; $dim\, r_2=1$, r_1 gives the standard L-function of GL_2

We have

$$M(s,\pi)f=\prod_{i=1}^m \frac{L_S(is,\pi,r_i)}{L_S(1+is,\pi,r_i)}\otimes_{v\notin S}\tilde{f}_v^0\otimes\otimes_{v\in S}A(s,\pi_v,w_0)f_v.$$

The meromorphic continuation of the Eisenstein series gives rise to the same for $M(s,\pi)$. By induction on m, it gives rise to the meromorphic continuation of $L_S(s,\pi,r_i)$. For induction, we show that r_i, $i\geq 2$, appears at r_1' for some other group. However, it does not give the desired functional equation. We only get the functional equation of the quotient

$$\frac{L_S(s,\pi,r_i)}{L_S(1+s,\pi,r_i)}=\prod_{v\in S}\mu_i(s,\pi_v)\frac{L_S(1-s,\pi,\tilde{r}_i)}{L_S(-s,\pi,\tilde{r}_i)},$$

where μ_i is some meromorphic function. In order to obtain the functional equation, we need to isolate $L_S(s,\pi,r_i)$. Shahidi computed non-constant term of Eisenstein series for globally generic cuspidal representations.

Let $B=TU$. Then $U/[U,U]\simeq\prod_{\alpha\in\Delta}U_\alpha$. Hence if ψ is a character of $U(\mathbb{Q})\backslash U(\mathbb{A})$, $\psi=\prod_{\alpha\in\Delta}\psi_\alpha$. We say that ψ is generic or non-degenerate if each ψ_α is non-trivial. Let π be a cuspidal representation of $M(\mathbb{A})$, and let φ be a function in the space of π. Let

$$W_\varphi(g) = \int_{U(\mathbb{Q})\backslash U(\mathbb{A})} \varphi(ug)\overline{\psi(u)}\, du.$$

W_φ is called the Whittaker function. We say that π is globally generic if $W_\varphi \neq 0$ for some φ. Then $W_\varphi = \otimes W_{\varphi_v}$, i.e., π_v is locally generic for all v. Now let

$$E_\psi(s, \pi, f_s, g) = \int_{U(\mathbb{Q})\backslash U(\mathbb{A})} E(s, \pi, f_s, ug)\overline{\psi(u)}\, du.$$

It is called ψ-th Fourier coefficient (or non-constant term) of $E(s, \pi, f_s, g)$. Then by Casselman-Shalika formula,

Theorem 3.5. *Suppose $f_s = \otimes f_v$, where $f_v = f_v^0$, spherical for $v \notin S$. Then*

$$E_\psi(s, \pi, f_s, e) = \frac{\prod_{v\in S} W_{f_v}(e)}{\prod_{i=1}^m L_S(1 + is, \pi, r_i)}.$$

By induction, this gives rise to functional equation of the form

$$L_S(s, \pi, r_i) = \prod_{v\in S} \gamma(s, \pi_v, r_i, \psi_v) L_S(1 - s, \pi, \tilde{r}_i).$$

By the theory of local coefficients, Shahidi was able to refine this to get the desired functional equation.

Also from the fact that the Eisenstein series is holomorphic for $Re(s) = 0$, we obtain the fact that $\prod_{i=1}^m L_S(1+is, \pi, r_i)$ is nonvanishing for $Re(s) = 0$. For example, for cuspidal representations π_1, π_2 of GL_k, GL_l, resp., $L_S(s, \pi_1 \times \pi_2)$ has no zeros for $Re(s) \geq 1$. By Sarnak's method, Gelbart-Lapid was able to get zero-free region of $L_S(s, \pi, r_i)$.

3.2. **Application to functoriality.**

The key theorem is

Theorem 3.6. *Suppose P is a maximal parabolic subgroup. Unless P is self-conjugate and $w_0\pi \simeq \pi$, $L^2_{dis}(G(\mathbb{Q})\backslash G(\mathbb{A}))_{(M,\pi)} = 0$, i.e., the Eisenstein series has no poles for $Re(s) \geq 0$, and $M(s, \pi)$ is holomorphic for $Re(s) \geq 0$.*

We can choose a grössencharacter χ where χ_p is highly ramified for some p such that $w_0(\pi\otimes\chi) \not\simeq \pi \otimes \chi$. Then $M(s, \pi \otimes \chi)$ is holomorphic for $Re(s) \geq 0$. We normalize the intertwining operator $A(s, \pi_p, w_0)$ as

$$A(s, \pi_p, w_0) = \prod_{i=1}^m \frac{L(is, \pi_p, r_i)}{L(1 + is, \pi_p, r_i)\epsilon(is, \pi_p, r_i)} N(s, \pi_p, w_0).$$

Then

$$M(s,\pi\otimes\chi)=\prod_{i=1}^{m}\frac{L(is,\pi\otimes\chi,r_i)}{L(1+is,\pi\otimes\chi,r_i)\epsilon(is,\pi\otimes\chi,r_i)}\otimes_p N(s,\pi_p\otimes\chi_p,w_0).$$

We prove that $N(s,\pi_p,w_0)$ is holomorphic and non-vanishing for $Re(s)\geq\frac{1}{2}$. By induction, we prove that $L(s,\pi\otimes\chi,r_i)$ is holomorphic for $Re(s)\geq\frac{1}{2}$, and non-vanishing for $Re(s)\geq 1$. By the functional equation, we prove that $L(s,\pi\otimes\chi,r_i)$ is entire. We also have

Theorem 3.7. *(Gelbart-Shahidi) Suppose $L(s,\pi,r_i)$ is entire. Then it is bounded in vertical strips.*

We can now apply the converse theorem of Cogdell and Piatetski-Shapiro.

Theorem 3.8. *Suppose $\Pi=\otimes\Pi_p$ is an irreducible admission representation of $GL_N(\mathbb{A})$ such that $\omega_\Pi=\otimes\omega_{\Pi_p}$ is a grössencharacter. Let S be a finite set of primes and let $\mathcal{T}^S(m)$ be the set of cuspidal representations of $GL_m(\mathbb{A})$ that are unramified at all primes $p\in S$. Suppose $L(s,\sigma\times\Pi)$ is entire, bounded in vertical strips, and satisfies a functional equation for all $\sigma\in\mathcal{T}^S(m)$, $m<n-1$. Then there exists an automorphic representation Π' of $GL_N(\mathbb{A})$ such that $\Pi_p\simeq\Pi'_p$ for all $p\notin S$.*

3.2.1. *Functoriality of $Sym^3\pi$.* Functoriality of $Sym^3\pi$ follows from the functoriality of the tensor product $\pi\boxtimes Sym^2\pi$. Consider $\phi: GL_2(\mathbb{C})\times GL_3(\mathbb{C})\longrightarrow GL_6(\mathbb{C})$ be the tensor product. Then given cuspidal representations π_1,π_2 of $GL_2(\mathbb{A}),GL_3(\mathbb{A})$, resp, one can define an irreducible admission representation $\pi_1\boxtimes\pi_2=\otimes(\pi_{1p}\boxtimes\pi_{2p})$ by the local Langlands correspondence. Langlands functoriality conjecture says that $\pi_1\boxtimes\pi_2$ is an automorphic representation of $GL_6(\mathbb{A})$. In order to prove it by the converse theorem, we need the triple product L-functions $L(s,\sigma\times\pi_1\times\pi_2)$, where σ is a cuspidal representation of $GL_m(\mathbb{A})$, $m=1,2,3,4$. They are available from Langlands-Shahidi method:

(1) D_5-2 case: $G=Spin(10)$, the derived group of M is $SL_2\times SL_2\times SL_3$;
(2) E_6-1 case: $G=E_6$, the derived group of M is $SL_3\times SL_2\times SL_3$;
(3) E_7-2 case: $G=E_7$, the derived group of M is $SL_4\times SL_2\times SL_3$.

Here $Sym^2\pi$ is the symmetric square, or Gelbart-Jacquet lift. It is a cuspidal representation of $GL_3(\mathbb{A})$ if it is not of dihedral type. Then when π has the trivial central character, $\pi\boxtimes Sym^2\pi=\pi\boxplus Sym^3\pi$.

3.2.2. *Functoriality of $Sym^4\pi$.* Functoriality of $Sym^4\pi$ follows from the functoriality of the exterior square $\wedge^2\Pi$, where Π is a cuspidal representation of $GL_4(\mathbb{A})$; Consider $\wedge^2: GL_4(\mathbb{C})\longrightarrow$

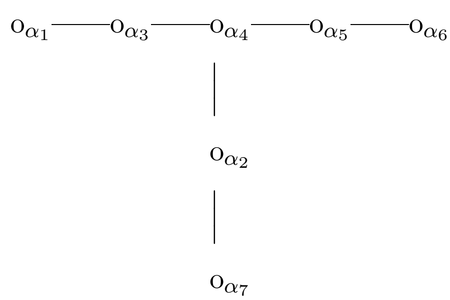
$GL_6(\mathbb{C})$ be the exterior square. Then given a cuspidal representation Π of $GL_4(\mathbb{A})$, one can define an irreducible admission representation $\wedge^2\Pi$ by the local Langlands correspondence. Langlands functoriality conjecture says that $\wedge^2\Pi$ is an automorphic representation of $GL_6(\mathbb{A})$. In order to prove it by the converse theorem, we need the L-functions $L(s, \sigma \times \Pi, \rho_m \otimes \wedge^2\rho_4)$, where σ is a cuspidal representation of $GL_m(\mathbb{A})$, $m = 1, 2, 3, 4$, and ρ_m is the standard representation of $GL_m(\mathbb{C})$. They are available from Langlands-Shahidi method:

$D_n - 3$ case, $n = 4, 5, 6, 7$: $G = Spin(2n)$, the derived group of M is $SL_{n-3} \times SL_4$.

Then when π has the trivial central character, $\wedge^2(Sym^3\pi) = 1 \boxplus Sym^4\pi$. In this way, we prove that $Sym^4\pi$ is an automorphic representation of $GL_5(\mathbb{A})$.

3.3. **Speculation on Kac-Moody groups.** It is evident that Langlands-Shahidi method gives rise to limited cases of L-functions due to the fact that there are only finitely many exceptional Lie groups. On the other hand, we can construct Kac-Moody groups attached to extended Dynkin diagrams. It is tempting to generalize the theory of Eisenstein series to Kac-Moody groups and Langlands-Shahidi method. However, the theory of Eisenstein series, in particular, the ones attached to maximal parabolic subgroups, is still in its infancy. Furthermore, if G is an infinite dimensional group, and P is a finite dimensional maximal parabolic subgroup, Shahidi [8] showed that P can never be self-conjugate. It means that the constant term of the Eisenstein series, even if it may be defined, may be trivial.

Let $G = E_6^{(1)}$, the affine Kac-Moody group, attached to the affine Dynkin diagram



Consider $P = P_\theta$, where $\theta = \Delta - \{\alpha_4\}$. Then $P = MN$, and the derived group of M is $SL_3 \times SL_3 \times SL_3$. Shahidi [8] calculated the adjoint action of $\mathfrak{m}_{\mathbb{C}}$ on $\mathfrak{n}_{\mathbb{C}}$. Then the following representations appear infinitely many times:

$$r_1 = \delta_1 + \delta_7 + \delta_6, \quad r_2 = \delta_3 + \delta_2 + \delta_5, \quad r_3 = (\delta_1 + \delta_3) \oplus (\delta_2 + \delta_7) \oplus (\delta_5 + \delta_6),$$

where δ_i's denote the fundamental weights attached to α_i's. So r_1 is the triple tensor product of standard representations of $sl_3(\mathbb{C})$, and $r_2 = \tilde{r}_1$, and r_3 is the direct sum of three 8-dimensional adjoint representations $\delta_1 + \delta_3, \delta_2 + \delta_7, \delta_5 + \delta_6$ of these $sl_3(\mathbb{C})$. So if π_1, π_2, π_3, are cuspidal representations of GL_3, we obtain the triple product L-function $L(s, \pi_1 \times \pi_2 \times \pi_3)$.

Similarly, if we take G is the Kac-Moody group of type $T_{p,q,r}$, and $P = MN$, maximal parabolic subgroup such that the derived group of M is $SL_p \times SL_q \times SL_r$, then we would get the triple product L-function $L(s, \pi_1 \times \pi_2 \times \pi_3)$, where π_i, $i = 1, 2, 3$, is a cuspidal representation of GL_p, GL_q, GL_r, resp. Then the converse theorem would imply the functoriality of $\pi_1 \boxtimes \pi_2$ as an automorphic representation of GL_{pq}.

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Department of mathematics, University of Toronto, Toronto, Ontario M5S 2E4, CANADA, and Korea Institute for Advanced Study, Seoul, KOREA

E-mail address: `henrykim@math.toronto.edu`