Fourier coefficients and nilpotent orbits for small representations

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Workshop on Eisenstein series on Kac–Moody groups
KIAS, Seoul, Nov 18, 2015

With Henrik Gustafsson and Daniel Persson

[GKP] arXiv:1412.5625 [math.NT]

Also review with Philipp Fleig: arXiv:1511.04265 [math.NT]

Overview

- Fourier coefficients and nilpotent orbits
- Whittaker coefficients and wavefront sets
- Small representations
- Kac–Moody case
- K-types
- Quantum cosmology
- Beyond Eisenstein series and automorphic forms?

Setting

Consider

- $G = G(\mathbb{R})$ a split real, reductive and f.d. algebraic group
- $\Gamma = G(\mathbb{Z}) \subset G(\mathbb{R})$ arithmetic Chevalley subgroup
- Global setting $G \to G(\mathbb{A})$ ($\mathbb{A} = \mathbb{A}_{\mathbb{Q}}$)

An automorphic form $\varphi \in \pi$ has Fourier expansion w.r.t. a unipotent subgroup $U \subset G$ (organised in derived series):

$$\varphi(g) = \sum_{\psi} F_\psi(\varphi, g) + \text{'non-abelian terms'}$$

unitary character $\psi : U(\mathbb{Q})\backslash U(\mathbb{A}) \to U(1)$

$$F_\psi(\varphi, g) = \int_{U(\mathbb{Q})\backslash U(\mathbb{A})} \varphi(ug)\overline{\psi(u)}du$$

Connection to orbits

Seek a rearrangement of the Fourier expansion in terms of an expansion in terms of orbits

$$\varphi(g) = \sum_{\mathcal{O} \in \mathrm{WF}(\pi)} \mathcal{F}_{\mathcal{O}}(\varphi, g)$$

- $\mathrm{WF}(\pi)$: (global) wave-front set ($\subset$ {nilpotent orbits})
- $\mathcal{F}_{\mathcal{O}}$: (linear) combination of orbit Fourier coefficients $F_{\psi_{\mathcal{O}}}$

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Remark: Cf. in local case Harish-Chandra–Howe expansion

$$\chi_{\pi_p} = \sum_{\mathcal{O} \in \mathrm{WF}(\pi_p)} c_{\mathcal{O}} \hat{\mu}_{\mathcal{O}}$$

and work by [Mœglin, Waldspurger; Matumoto; Ginzburg]. Global results by [Jiang, Liu, Savin; Gan, Savin]

Orbit Fourier coefficients

Let $\mathcal{O}$ be a nilpotent orbit through $X \in \mathfrak{g}$. Let (X, Y, H) be a Jacobson–Morozov $\mathfrak{sl}(2)$ triple and $\mathfrak{g} = \oplus_{i=-m}^{m} \mathfrak{g}_i$ a decomposition into H-eigenspaces. Define:

$$\mathfrak{p}_{\mathcal{O}} = \bigoplus_{i=0}^{m} \mathfrak{g}_i, \quad \mathfrak{l}_{\mathcal{O}} = \mathfrak{g}_0, \quad \mathfrak{u}_{\mathcal{O}} = \bigoplus_{i=1}^{m} \mathfrak{g}_i, \quad \mathfrak{v}_{\mathcal{O}} = \bigoplus_{i=2}^{m} \mathfrak{g}_i$$

Lie algebras of $P_{\mathcal{O}} = L_{\mathcal{O}} U_{\mathcal{O}}$, and $V_{\mathcal{O}} \subset U_{\mathcal{O}}$.

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Lie algebras of $P_\mathcal{O} = L_\mathcal{O} U_\mathcal{O}$, and $V_\mathcal{O} \subset U_\mathcal{O}$.

Let $\psi_\mathcal{O}$ be a unitary character on $V_\mathcal{O}$ (same stabilizer type as $\mathcal{O}$). The orbit Fourier coefficient is

$$F_{\psi_\mathcal{O}}(\varphi, g) = \int_{V_\mathcal{O}(\mathbb{Q})\backslash V_\mathcal{O}(\mathbb{A})} \varphi(vg)\overline{\psi_\mathcal{O}(v)}dv$$

If there is a $\psi_\mathcal{O}$ such that $F_{\psi_\mathcal{O}} \neq 0$ then $\mathcal{O} \in \mathrm{WF}(\pi)$.

Wave-front set and orbit coefficients

Some questions:

- How to determine $\mathrm{WF}(\pi)$?
- How to determine $F_{\psi_{\mathcal{O}}}$?
- How to determine $\mathcal{F}_{\mathcal{O}} = \sum F_{\psi_{\mathcal{O}}}$?
- How to determine F_{ψ} for ψ on some unipotent U?

(In particular last question of interest for string theory.)

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We will tackle these questions for Eisenstein series and using (degenerate) Whittaker vectors. (cf. [Mœglin, Waldspurger; Matumoto; Gourevitch, Sahi])

Very explicit results for $SL(3)$ and $SL(4)$ in [GKP].

(Degenerate) principal series

$$E(\chi, g) = \sum_{\gamma \in B(\mathbb{Q})\backslash G(\mathbb{Q})} \chi(\gamma g)$$

Eisenstein series induced from Borel $B = NA \subset G$ using a generic quasi-character $\chi : B \to \mathbb{C}^\times$, extended to G.

- Spherical function in the principal series
- $\mathrm{GKdim}\,\pi = \dim N = \frac{1}{2}\dim \mathcal{O}_{\mathrm{reg}}, \quad \mathrm{WF}(\pi) = \overline{\mathcal{O}_{\mathrm{reg}}}$

Induced from a parabolic subgroup $P = LU$ with $\chi_P : P \to \mathbb{C}^\times$. Generically degenerate principal series

- $\mathrm{GKdim}\,\pi = \dim U = \frac{1}{2}\dim \mathcal{O}_P, \quad \mathrm{WF}(\pi) = \overline{\mathcal{O}_P}$

For non-generic characters can have reductions of $\mathrm{WF}(\pi)$ and contributions to the residual spectrum.

Rewriting the $SL(3)$ Fourier expansion

Nilpotent orbits $\mathcal{O}$ for $SL(3)$

Bala–Carter	partition	dim	$V_{\mathcal{O}}$
0	(1^3)	0	$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$
A_1	(21)	4	$\begin{pmatrix} 1 & & * \\ & 1 & \\ & & 1 \end{pmatrix}$
A_2	(3)	6	$\begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix}$

For trivial orbit define the orbit Fourier coefficient to be the constant term (along N):

$$F_{\psi_0}(\chi, g) = \int\limits_{N(\mathbb{Q})\backslash N(\mathbb{A})} E(\chi, ng)dn = \mathcal{F}_0$$

$SL(3)$ regular orbit

Since $V_{\mathcal{O}_{\text{reg}}} = \left\{ \begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix} \right\} = N$, orbit Fourier coefficients equal non-degenerate Whittaker coefficients

$$F_{\psi_{A_2}}(\chi, g) = \int\limits_{N(\mathbb{Q})\backslash N(\mathbb{A})} E(\chi, ng)\overline{\psi_{A_2}(n)}dn = W_{[m_1,m_2]}(\chi, g)$$

where ($m_1 m_2 \neq 0$)

$$\psi_{A_2}\left(\begin{pmatrix} 1 & x_1 & * \\ & 1 & x_2 \\ & & 1 \end{pmatrix}\right) = \exp(2\pi i(m_1 x_1 + m_2 x_2))$$

(Could be calculated locally using Casselman–Shalika.)

$SL(3)$ minimal orbit

With $V_{\mathcal{O}_{\mathrm{reg}}} = \left\{ \begin{pmatrix} 1 & & * \\ & 1 & \\ & & 1 \end{pmatrix} \right\} = [N, N] = N^{(2)}$, orbit Fourier coefficients equal 'non-abelian' Whittaker coefficients

$$F_{\psi_{A_1}}(\chi, g) = \int\limits_{N^{(2)}(\mathbb{Q})\backslash N^{(2)}(\mathbb{A})} E(\chi, ng)\overline{\psi_{A_1}(n)}dn$$

These do not as such cover all terms in the Fourier expansion: Degenerate Whittaker vectors $W_{[m,0]}$ and $W_{[0,m]}$ (on N) are missing.

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Are also associated with the minimal orbit. E.g.

$$W_{[m,0]}(\chi, g) = \int_{(\mathbb{Q}\backslash\mathbb{A})^2} F_{\psi_{A_1}}\left(\chi, \begin{pmatrix} -1 & & \\ & & -1 \\ & -1 & \end{pmatrix} \begin{pmatrix} 1 & & u_2 \\ & 1 & u_3 \\ & & 1 \end{pmatrix} g\right) du_2 du_3$$

$SL(3)$ **orbit Fourier expansion**

Theorem [GKP] *Any Eisenstein series* $E(\chi, g)$ *on* $SL(3)$ *can be written as*

$$E(\chi, g) = \mathcal{F}_0(\chi, g) + \mathcal{F}_{A_1}(\chi, g) + \mathcal{F}_{A_2}(\chi, g)$$

with

$$\begin{aligned} \mathcal{F}_0(\chi, g) &= \text{const. term} = W_{[0,0]}(\chi, g) \\ \mathcal{F}_{A_1}(\chi, g) &= \sum_{m \neq 0} W_{[m,0]}(\chi, g) + \sum_{m \neq 0} W_{[0,m]}(\chi, g) + \sum_{m \neq 0} F_{\psi_{A_1}}(\chi, g) \\ \mathcal{F}_{A_2}(\chi, g) &= \sum_{m_1 m_2 \neq 0} W_{[m_1,m_2]}(\chi, g) \end{aligned}$$

‘non-abelian’ Whittaker

Proof: Follows from considerations above.

Checks / Consequences

For $SL(3)$ the principal series can be parametrised by two complex numbers s_1, s_2

$$\chi_{s_1,s_2}(a) = v_1^{s_1} v_2^{s_2}, \quad a = \begin{pmatrix} v_1 & & \\ & v_1^{-1} v_2 & \\ & & v_2^{-1} \end{pmatrix}$$

- Minimal series for $s_1 = 0$ (or $s_2 = 0$ or a Weyl image). From explicit formulas for Whittakers [Bump] $\mathcal{F}_{A_2}(\chi, g) = 0$ and only minimal orbit in Fourier expansion of minimal series:

$$E(\chi_{A_1}, g) = \mathcal{F}_0 + \mathcal{F}_{A_1}, \quad \mathrm{WF}(\pi_{A_1}) = \overline{\mathcal{O}_{A_1}}$$

  $\mathcal{F}_{A_1}$ functions same 'functional type' (single K-Bessel).

- For generic series all terms and $\mathrm{WF}(\pi_{A_2}) = \overline{\mathcal{O}_{A_2}}$.

$SL(3)$ unipotent coefficients

Consider the (mirabolic) unipotent $U = \left\{ \begin{pmatrix} 1 & u_1 & u_2 \\ & 1 & \\ & & 1 \end{pmatrix} \right\}$.

Not the unipotent associated with any orbit. What is $F_\psi(\chi, g)$ for unitary character $\psi = \exp(2\pi i(m_1 u_1 + m_2 u_2))$ on U?

Study the way Levi orbits on U intersect the nilpotent G-orbits and then relate to orbit coefficients:

$$F_\psi(\chi, g) = \int_{\mathbb{Q}\backslash\mathbb{A}} F_{A_1}\Big(\chi, \begin{pmatrix} 1 & u_1 & 0 \\ & 1 & \\ & & 1 \end{pmatrix} l_{m_1,m_2} g\Big) du_1$$

l_{m_1,m_2} belongs to Levi. In minrep translate of deg. Whittaker.

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Simplicity for $GL(n)$ [Piatetski-Shapiro; Shalika]. In general: orbit structure much more intricate [Miller, Sahi].

Generalization to $SL(4)$

- Same logic works for $SL(4)$

Orbit $\mathcal{O}$	Weighted Dynkin diagram	$\dim(\mathcal{O})$	C	V
$(4) = A_3$	2 — 2 — 2	12	$\mathbb{1}$	$\begin{pmatrix} 1 & * & * & * \\ & 1 & * & * \\ & & 1 & * \\ & & & 1 \end{pmatrix}$
$(31) = A_2$	2 — 0 — 2	10	T_1	$\begin{pmatrix} 1 & * & * & * \\ & 1 & & * \\ & & 1 & * \\ & & & 1 \end{pmatrix}$
$(2^2) = 2A_1$	0 — 2 — 0	8	A_1	$\begin{pmatrix} 1 & & * & * \\ & 1 & * & * \\ & & 1 & \\ & & & 1 \end{pmatrix}$
$(21^2) = A_1$	1 — 0 — 1	6	$A_1 \times T_1$	$\begin{pmatrix} 1 & & & * \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$
$(1^4) = 0$	0 — 0 — 0	0	A_3	$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

- Orbit of type $2A_1$ – next-to-minimal
- Our methods show also that the next-to-minimal representation is determined by degenerate Whittaker coefficients of type $2A_1$

Remarks on small representations

Let G be a simply-laced split group (ADE). Its nilpotent orbits of A-type can be labelled by a choice of simple roots determining an A-type subgroup $G' \subset G$. ‘Small’ orbits are of A-type.

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Conjecture: *The A-type orbits of the wave-front set of π with $E(\chi, g)$ as spherical vector can be determined by looking at the degenerate Whittaker coefficients of $E(\chi, g)$. If π is a small representation, $\mathrm{WF}(\pi)$ can be fully determined.*

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Theorem: [FKP; Hashizume] *Let ψ be a deg. character on N and $G' \subset G$ the semi-simple subgroup on whose maximal unipotent $N' \subset N$ ψ is generic. Then for $\chi(nak) = a^{\lambda+\rho}$*

$$W_\psi^G(\lambda, a) = \sum_{w_c w_0' \in \mathcal{W}/\mathcal{W}'} a^{(w_c w_0')^{-1}\lambda+\rho} M(w_c^{-1}, \lambda) W_\psi^{G'}(w_c^{-1}\lambda, 1).$$

Remarks on small representations (II)

Theorem works for any type of subgroup G' but for application to wave-front set of small representations A-type most useful.

- For minimal representation only A_1 type
- For next-to-minimal only $2A_1$ type (definition...)
- Other example: $E\left[\begin{smallmatrix} & & \frac{5}{2} & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 \end{smallmatrix}\right]$ for $E_7(\mathbb{R})$. $\mathrm{WF}(\pi) = \overline{\mathcal{O}_{A_2+A_1}}$

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Question: Does this simplify the calculation of arbitrary Fourier coefficients? (Min. rep.: yes [Gan, Savin])

Remarks on small representations (III)

Theorem (formally) applicable to Kac–Moody extension. E.g. hyperbolic E_{10} (see talk by Philipp Fleig yesterday). Inducing from constant on parabolic with semi-simple D_9

- $s = \frac{3}{2}$: only A_1-type Whittaker functions $\Rightarrow$ Eisenstein series should be attached to minimal representation.

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<u>Question:</u> Classification of Kac–Moody nilpotent orbits?

More Kac–Moody questions

- K-types

For discrete series often non-trivial K-types necessary.
Possibilities for Kac–Moody?

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At the level of Lie algebras $\mathfrak{k} \subset \mathfrak{g}$ over $\mathbb{R}$.

(1) ∞-dim'l fixed point Lie algebra of (Chevalley) involution.
(2) $\mathfrak{k}$ is <u>not</u> a Kac–Moody algebra.
(3) $\mathfrak{k}$ is not a simple algebra. It has ∞-dim'l ideals.

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For $\mathfrak{k}$ of hyperbolic $\mathfrak{g} = \mathfrak{e}_{10}$ one has irreducible (spinor) representations of dimensions [Damour, AK, Nicolai]

$$\mathbf{32}, \quad \mathbf{320}, \quad \mathbf{1728}, \quad \mathbf{7040}$$

with quotients $\mathfrak{so}(32), \quad \mathfrak{so}(288, 32), \quad ?, \quad ?$

K-types

(Some of) these representations can be lifted to the Weyl group W and (covers of) K [Ghatei, Horn, Köhl, Weiss].

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For other Kac–Moody groups, e.g. $\begin{pmatrix} 2 & -2 & \\ -2 & 2 & -1 \\ & -1 & 2 \end{pmatrix}$ other quotients possible, also with $U(1)$ factors
$\Rightarrow$ holomorphic discrete series?

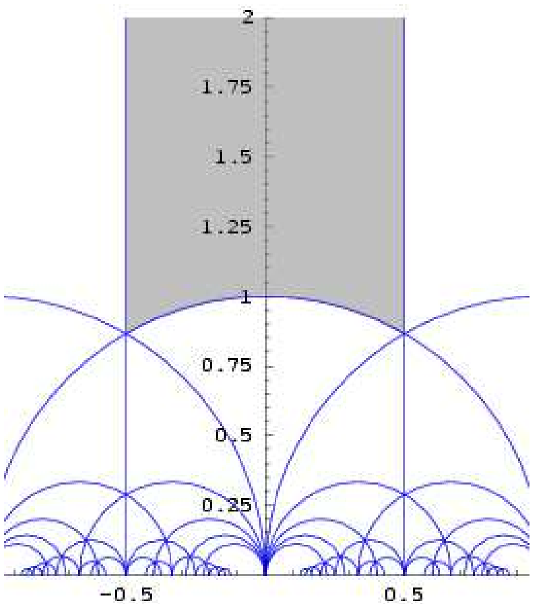
<u>Question:</u> Spherical vectors for Kac–Moody reps?

Yet more Kac–Moody connections

- Hyperbolic Weyl groups

Standard upper half plane can be seen as projection of (open) Tits cone $\mathcal{C} \subset \mathfrak{h}$ for the hyperbolic Kac–Moody algebra

with $A = \begin{pmatrix} 2 & -2 & \\ -2 & 2 & -1 \\ & -1 & 2 \end{pmatrix}$



(Real) upper half plane

$z = x + iy;\ x \in \mathbb{R},\ y > 0$

Action of $PSL(2,\mathbb{Z})$ on this space $=$ Action of $W^+ \subset W$

[Feingold, Frenkel]

Generalization

Generalizes to other hyperbolic Kac–Moody algebras by replacing $x \in \mathbb{R}$ by $x \in \mathbb{K}$, another division algebra.

E.g. take $\mathbb{K} = \mathbb{O}$ (octonions) and consider

$$z = x + iy, \quad x \in \mathbb{O}, \; y > 0 \qquad \text{‘octonionic upper half plane’}$$

Define transformations [Feingold, AK, Nicolai]

$$w_{-1}(z) = \frac{1}{\bar{z}}, \quad w_0(z) = -\bar{z} + 1, \quad w_j(z) = -\varepsilon_j \bar{z} \varepsilon_j$$

Here: ε_j ($j = 1, \ldots, 8$) are E_8 simple roots in the root lattice $Q(E_8) \cong \mathtt{O}$ (integer Cayley numbers) s.t. hst root $= 1$

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These generate the Weyl group $W = W(E_{10})$!

E_{10} Weyl group

Even part W^+ is generated by 'holomorphic'

$$s_{-1}(z) = -\frac{1}{z}, \quad s_0(z) = z + 1, \quad s_j(z) = \varepsilon_j z \varepsilon_j$$

They correspond to modular group $PSL(2, \mathtt{O})$ over integer octonions. Finite volume fundamental domain.

<u>Question:</u> What about modular forms in this set-up?

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Question: What about modular forms in this set-up?

Can define Poincaré series [AK, Nicolai, Palmkvist]

$$P_s(z) = \frac{1}{240} \sum_{(c,d)=1} \frac{y^s}{|cz + d|^{2s}} \qquad (\text{Re}(s) > 4)$$

Fourier expansion similar to the $SL(2)$ case; involves K_{s-4}.
Arithmetic properties not studied. Cusp forms?

Can also study other hyperbolic Kac–Moody algebras.

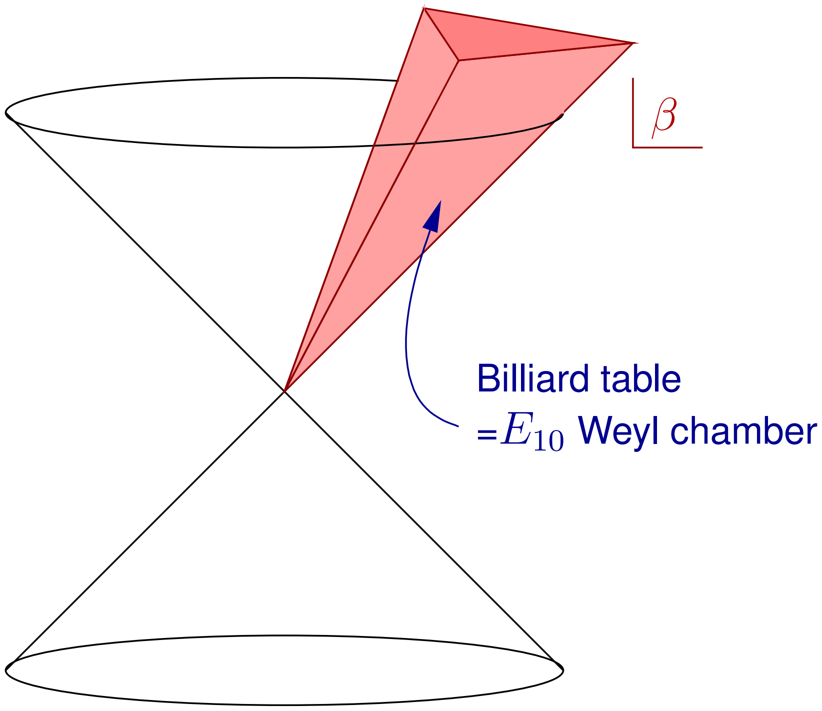
Hyperbolic Weyl groups

$\mathbb{K}$	$\mathfrak{g}$	Ring	$W(\mathfrak{g})$	$W^+(\mathfrak{g}^{++})$
$\mathbb{R}$	A_1	$\mathbb{Z}$	$2 \equiv \mathbb{Z}_2$	$PSL_2(\mathbb{Z})$
$\mathbb{C}$	A_2	Eisenstein $\mathtt{E}$	$\mathbb{Z}_3 \rtimes 2$	$PSL_2(\mathtt{E})$
$\mathbb{C}$	$B_2 \equiv C_2$	Gaussian $\mathtt{G}$	$\mathbb{Z}_4 \rtimes 2$	$PSL_2(\mathtt{G}) \rtimes 2$
$\mathbb{C}$	G_2	Eisenstein $\mathtt{E}$	$\mathbb{Z}_6 \rtimes 2$	$PSL_2(\mathtt{E}) \rtimes 2$
$\mathbb{H}$	A_4	Icosians $\mathtt{I}$	$\mathfrak{S}_5$	$PSL_2^{(0)}(\mathtt{I})$
$\mathbb{H}$	B_4	Octahedral $\mathtt{R}$	$2^4 \rtimes \mathfrak{S}_4$	$PSL_2^{(0)}(\mathtt{H}) \rtimes 2$
$\mathbb{H}$	C_4	Octahedral $\mathtt{R}$	$2^4 \rtimes \mathfrak{S}_4$	$\widetilde{PSL}_2^{(0)}(\mathtt{H}) \rtimes 2$
$\mathbb{H}$	D_4	Hurwitz $\mathtt{H}$	$2^3 \rtimes \mathfrak{S}_4$	$PSL_2^{(0)}(\mathtt{H})$
$\mathbb{H}$	F_4	Octahedral $\mathtt{R}$	$2^5 \rtimes (\mathfrak{S}_3 \times \mathfrak{S}_3)$	$PSL_2(\mathtt{H}) \rtimes 2$
$\mathbb{O}$	E_8	Octavians $\mathtt{O}$	$2 \,.\, O_8^+(2) \,.\, 2$	$PSL_2(\mathtt{O})$

Connection to physics

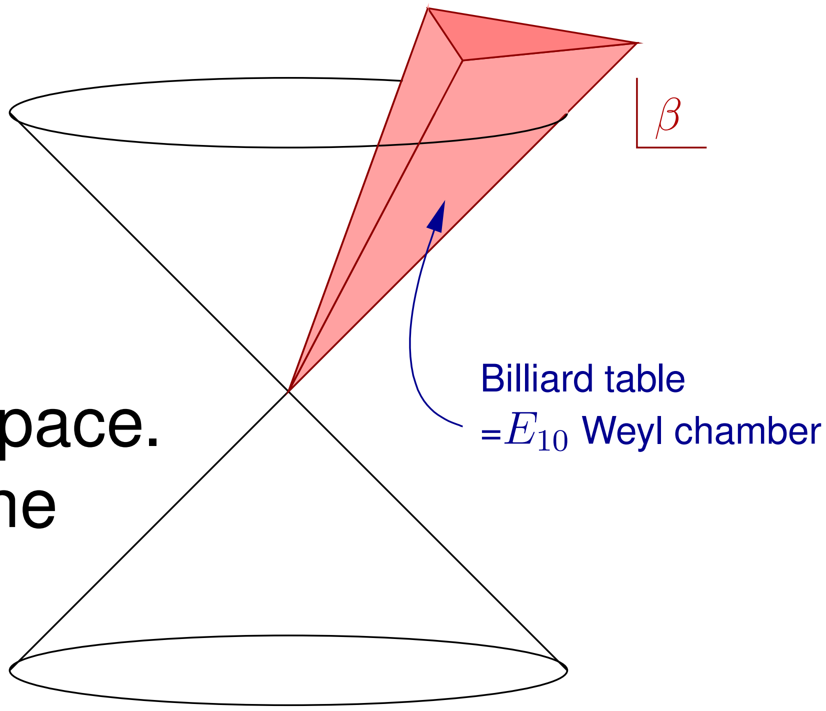
Max. supergravity near a space-like singularity (e.g. big bang) can be given effective description in terms of cosmological billiard. [Belinski, Khalatnikov, Lifschitz; Damour, Henneaux]

Classically: Free ball in hyperbolic space bouncing off the walls of Weyl chamber



Connection to physics

Max. supergravity near a space-like singularity (e.g. big bang) can be given effective description in terms of cosmological billiard. [Belinski, Khalatnikov, Lifschitz; Damour, Henneaux]



Classically: Free ball in hyperbolic space bouncing off the walls of Weyl chamber

Quantize: ‘Wave-function’ on this space. Unfold to octonionic upper half plane with odd boundary conditions
$\Rightarrow$ cups forms?

Could be related to ‘singularity resolution’ in quantum cosmology [AK, Koehn, Nicolai]

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Eisenstein series $E(s,z)$ etc.

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'Next order' requires $SL(2,\mathbb{Z})$-invariant function $f(z)$ with [Green, Vanhove]

$$\Delta f(z) = 12f(z) - 4\zeta(3)^2 E(3/2,z)^2$$

Recently solved [Green, Miller, Vanhove]

$$f(z) = \sum_{\gamma\in\Gamma_\infty\backslash SL(2,\mathbb{Z})} \Phi(\gamma z)$$

One more thing (II)

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with ($z = x + iy$)

$$\Phi(z) = 2\zeta(3)^2 y^3 + \frac{1}{9}\pi^2 y + \sum_{n \neq 0} c_n(y) e^{2\pi i n x}$$

$$c_n(y) = 8\zeta(3)\sigma_{-2}(n) y \Bigg[\left(1 + \frac{10}{\pi^2 n^2 y^2}\right) K_0(2\pi|n|y)$$

$$+ \left(\frac{6}{\pi|n|y} + \frac{10}{\pi^3 |n|^3 y^3}\right) K_1(2\pi|n|y) - \frac{16}{\pi(|n|y)^{1/2}} K_{7/2}(2\pi|n|y) \Bigg]$$

For higher rank dualities (in progress with Olof Ahlén).

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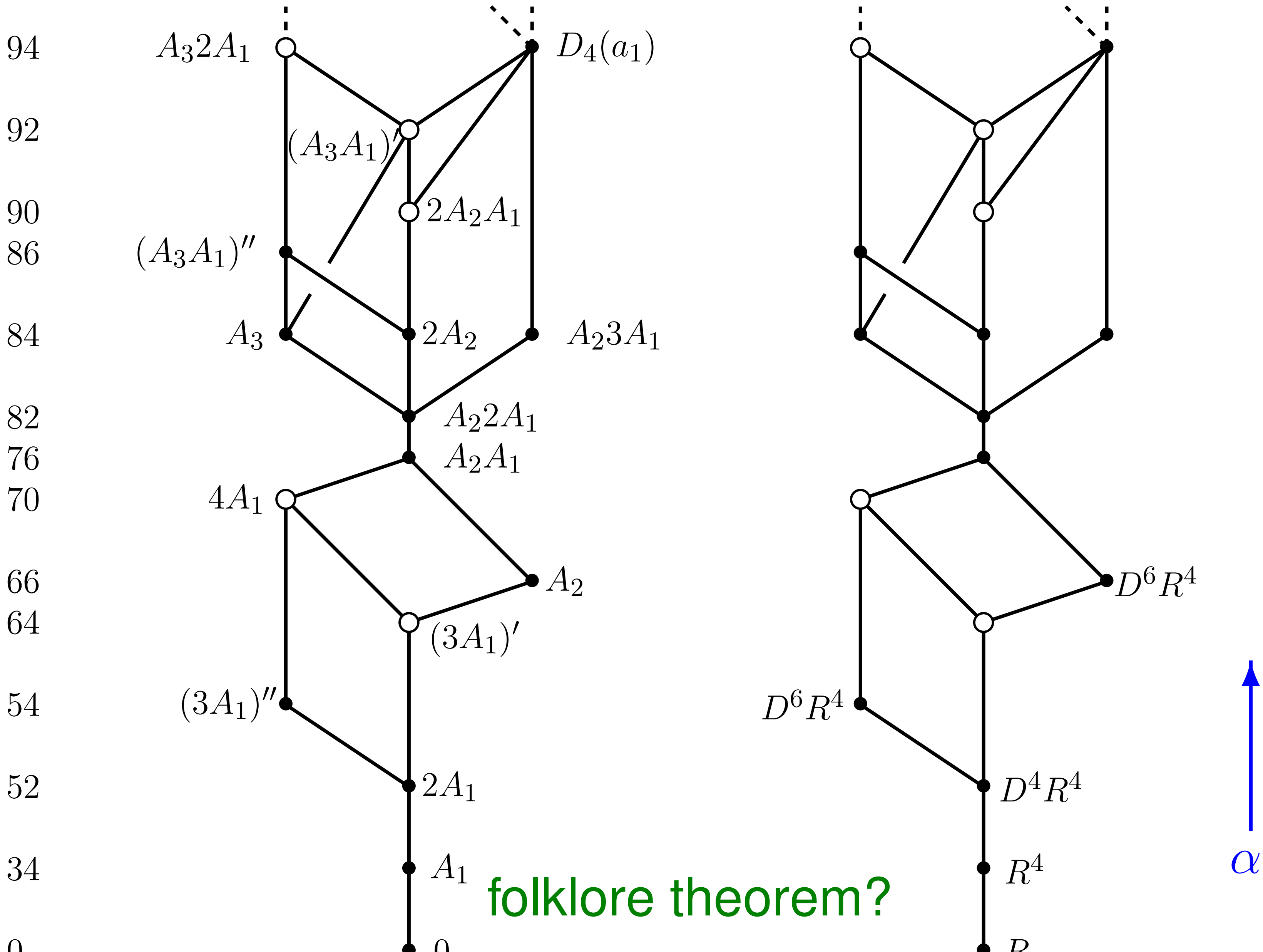
Question: Rep. theory? Global version? Aut. distributions?

One more thing (III)



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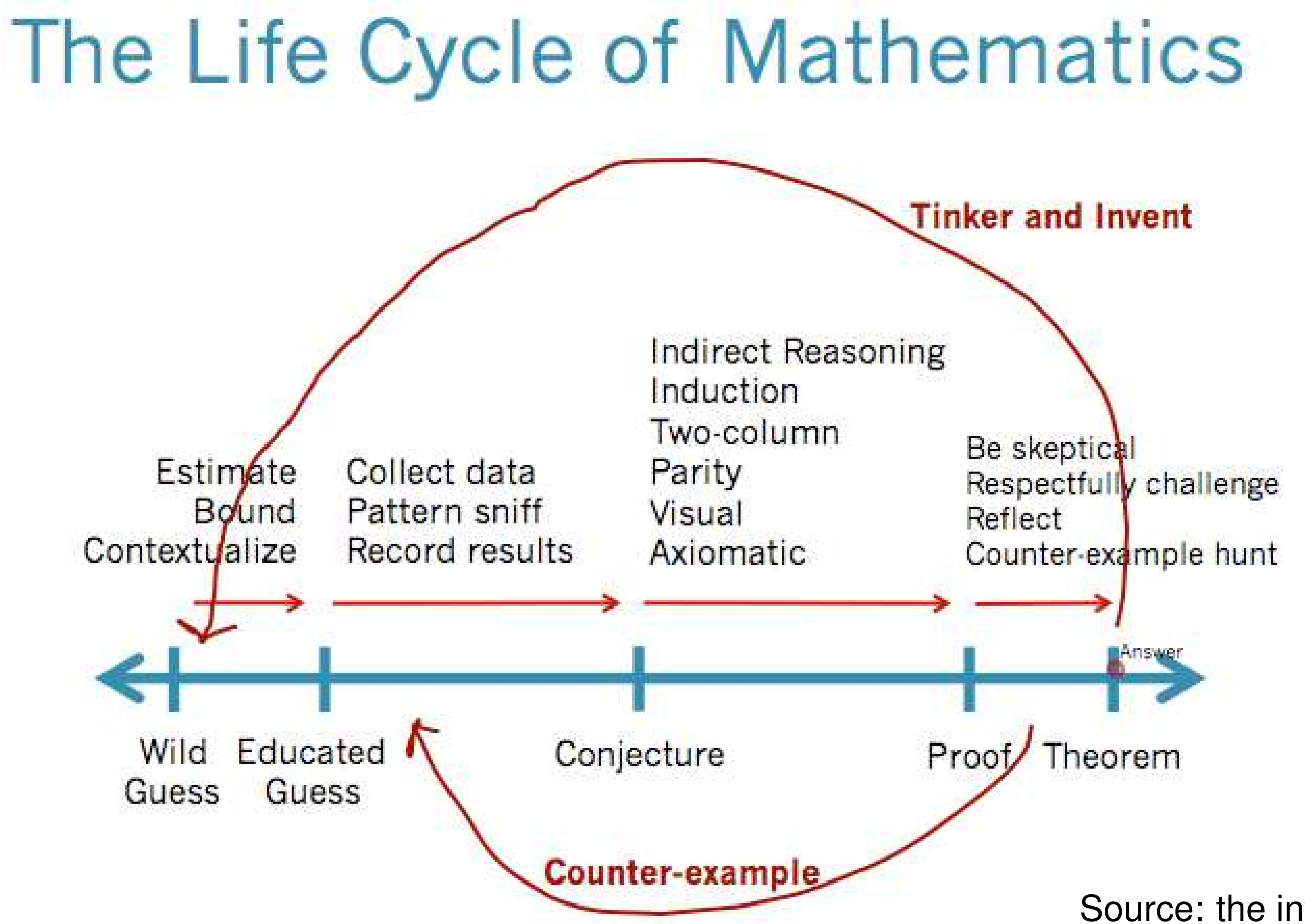

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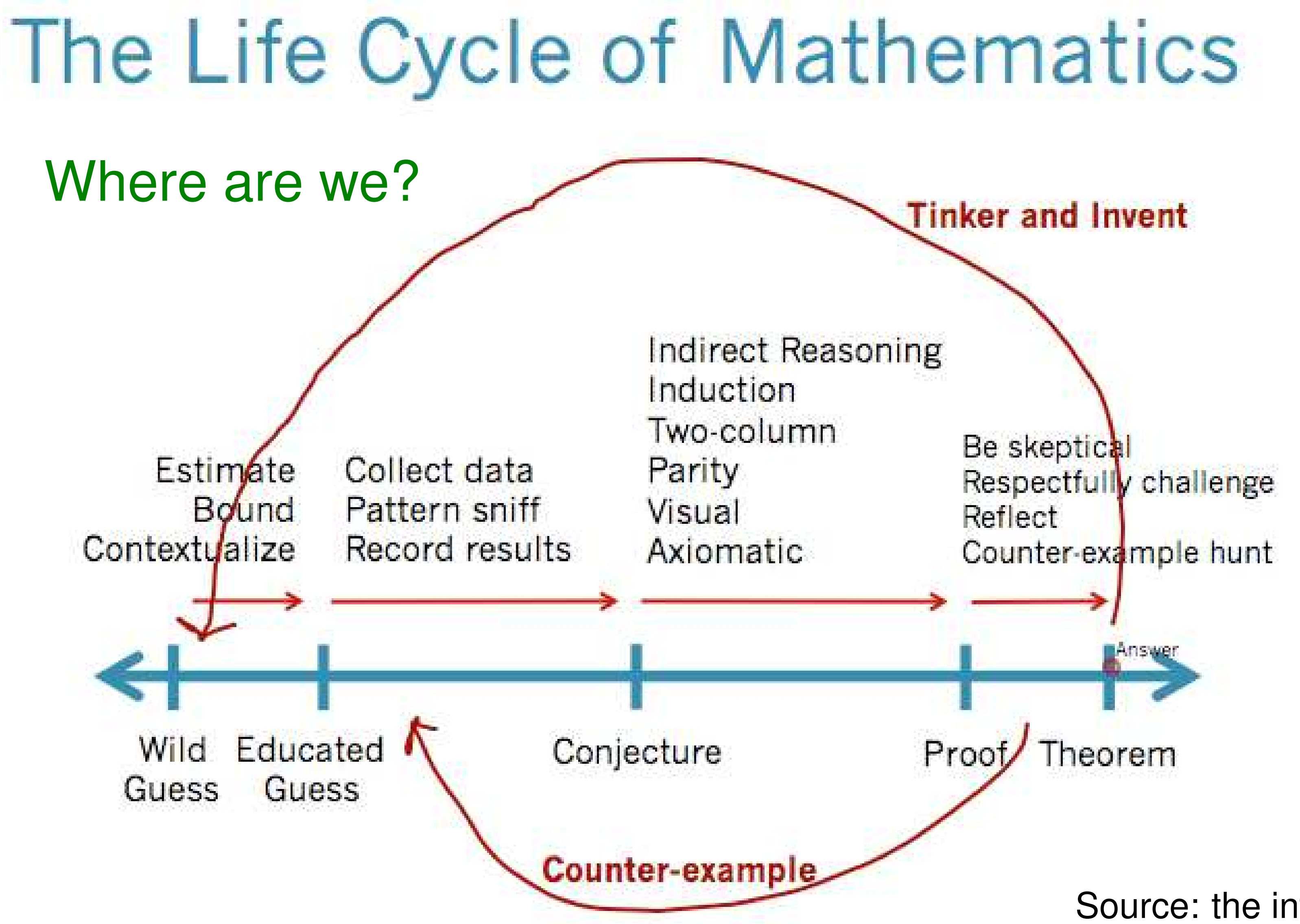

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