Section 03   Math 227 Practice Midterm Exam 2   December 5, 2001

Show all work clearly. For full credit, include all steps necessary for your answer to be a clear and logical consequence of the preceding work. To be eligible for partial credit, all row-reduction work must carry clear labels indicating which operations you perform.

Budget time by allotting no more than 7 minutes for each 10 points.

1. Let $B = \begin{bmatrix} 1 & 2 & 2 \\ -6 & -11 & -12 \\ 1 & -4 & 3 \end{bmatrix}$.

   (a) (10 points) Find the LU-decomposition of B.

   (b) (5 points) Without further work, calculate det B and explain why B is invertible.

   (c) (5 points) Calculate U^{-1}.

   (d) (5 points) Use your work to write L as a product of elementary matrices.

   (e) (5 points) Calculate L^{-1} from (d).

   (f) (5 points) Calculate B^{-1} from (c) and (e). (This is the procedure MATLAB actually follows to invert matrices.)

   (g) (5 points) Write B as a product of elementary matrices. (Do *not* check by multiplying out the product!)

2. (15 points) Consider the linear transformation T whose standard matrix is

$$C = \begin{bmatrix} 1 & 2 & 2 \\ 1 & -4 & 3 \\ -6 & -11 & -12 \end{bmatrix}.$$

   (a) What is the value of det C? (b) Is $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ in the range of T? Why or why not?

   (c) Describe the image under T of the unit cube determined by the vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ in $\mathbf{R}^3$.

3. (10 points) Suppose that I, O, X, Y, A, B, C, and D have appropriate sizes so that

$$\begin{bmatrix} I & X \\ O & Y \end{bmatrix} \begin{bmatrix} A & C \\ B & O \end{bmatrix} = \begin{bmatrix} D & C \\ I & O \end{bmatrix}.$$

   Express X and Y in terms of A, B, C, and D.

4. (15 points, 5 per part) Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ be the bases of $\mathbf{R}^2$ for which $\mathbf{b}_1 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, $\mathbf{c}_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$, and $\mathbf{c}_2 = \begin{bmatrix} -2 \\ 9 \end{bmatrix}$.

(a) Find the coordinate vector $[\mathbf{v}]_{\mathcal{B}}$ of $\mathbf{v} = \begin{bmatrix} -3 \\ 7 \end{bmatrix}$ relative to the basis $\mathcal{B}$.

(b) Find the change-of-basis matrix $P = P_{\mathcal{C} \leftarrow \mathcal{B}}$.

(b) Give the matrix-vector equation that connects $[\mathbf{v}]_{\mathcal{B}}$ and $[\mathbf{v}]_{\mathcal{C}}$. Use that to calculate $[\mathbf{v}]_{\mathcal{C}}$.

5. (20 points, 5 per part) Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & -1 \\ -1 & 3 & 5 \end{bmatrix}$.

   (a) Find the rank of A.

   (b) What is the dimension of the column space $R(A)$ of A? Give a basis for $R(A)$.

   (c) What is the dimension of the null space $N(A)$ of A? Give a basis for $N(A)$.

   (d) What is the dimension of the column space of A^T? What is the dimension of the null space of A^T?

6. **Bonus Question** (10 points maximum) **Attempt this only after having completed and checked the earlier problems.**

   Two square matrices A and B *anticommute* if $AB = -BA$.

   (a) Show that if two 3-by-3 matrices anticommute, then at least one of them is not invertible.

   (b) What if A and B are 2-by-2 matrices?