Exam 2 Answers **Math 220** **December 8, 1999**

1. Since $l(x, y, z) = \sqrt{x^2 + y^2 + z^2}$**, note that** $\nabla l = \frac{1}{\sqrt{x^2+y^2+z^2}}(x, y, z)$**.**

(a) At $(a, b, c) \neq \mathbf{0}$**, the function** l **thus increase most rapidly in the direction of the gradient:** $\frac{1}{\sqrt{a^2+b^2+c^2}}(a, b, c)$**. The maximum rate of increase of** l **at** (a, b, c) **is** $\|\nabla l(a, b, c)\| = 1$**.**

(b) The function l **decreases most rapidly at** (a, b, c) **in the direction of** $-\nabla(a, b, c)$**; the maximum rate of decrease is** -1**.**

2. For $w = x^2 - 2y^2$**, with** $x = r\cos\theta, y = r\sin\theta$**,**

$$\frac{\partial w}{\partial \theta} = \big(2x, -4y\big) \cdot \big(-r\sin\theta, r\cos\theta\big) = -6r^2 \sin\theta\cos\theta.$$

3. (a) $F(x, y, z) = x^2+y^2+z^2-9$ **does define** z **as a differentiable function of** x **and** y **near** $(-1, 2, 2)$**, because** $F_z(-1, 2, 2) = 4 \neq 0$ **and** F_x, F_y, F_z **are all continuous.**

(b) Implicit differentiation gives $\partial z/\partial x = -2x/2z = 1/2$ **at** $(-1, 2, 2)$ **and** $\partial z/\partial y = -2y/2z = -1$ **there. So the tangent plane is:** $z - 2 = \frac{1}{2}(x + 1) - (y - 2)$**. At** $(-1.02, 2.01)$**, that gives approximate value** $z \approx 1.98$**.**

4. (a) Since $f(x, y) = x^2 - xy - y^2 + 5y - 1$**,** $f_x = 2x - y$ **and** $f_y = -x - 2y + 5$**. Setting those to zero and solving gives the sole critical point** $(1, 2)$**. Since** $f_{xx} = 2 > 0, f_{xy} = f_{yx} = -1, f_{yy} = -2$**, we find that** $D = \det H_f = -5 < 0$**, so** $(1, 2)$ **is a saddle point.**

(b) The problem asks you to optimize $f(x, y, z) = x^2+y^2+z^2$ **subject to the two constraints** $g_1(x, y, z) = \sqrt{x^2 + y^2} - z = 0$ **and** $g_2(x, y, z) = x^2 + y^2 - 2 - z = 0$**. Solution by Lagrange multipliers requires solution of the system of five equations in the variables** x, y, z, λ, μ **that results from** $\nabla f(x, y, z) = \lambda\nabla g_1(x, y, z) + \mu\nabla g_2(x, y, z)$ **and the two constraints:**

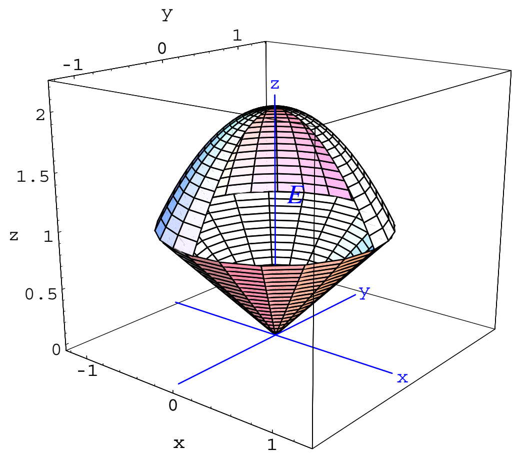
$$2x = \frac{\lambda x}{\sqrt{x^2 + y^2}} + 2\mu x, \qquad 2y = \frac{\lambda y}{\sqrt{x^2 + y^2}} + 2\mu y, \qquad 2z = -\lambda - \mu,$$

$$\sqrt{x^2 + y^2} - z = 0, \qquad x^2 + y^2 - 2 - z = 0.$$

5. (a) The region is bounded by the y**-axis, the horizontal line** $y = e$ **on its top, and by the graph of** $x = \ln y \iff y = e^x$**.**

(b) So $\iint_S f(x, y)\,dA = \int_1^e dy \int_0^{\ln y} f(x, y)\,dx = \int_0^1 \int_{e^x}^e f(x, y)\,dy\,dx$.

6. (a) The graphs of $z = \sqrt{x^2+y^2}$ and $z = 2 - x^2 - y^2$ intersect when $2 - x^2 - y^2 = \sqrt{x^2+y^2}$, that is when $2 - r^2 = r$. That gives $r^2 + r - 2 = (r+2)(r-1) = 0$, which means $r = 1$ since $r \geq 0$. The intersection thus occurs on the plane $z = 1$, which is almost clear from the plot. So $V = \int_0^{2\pi} \int_0^1 (2 - r^2 - r)\, r dr\, d\theta = \frac{5}{6}\pi$.

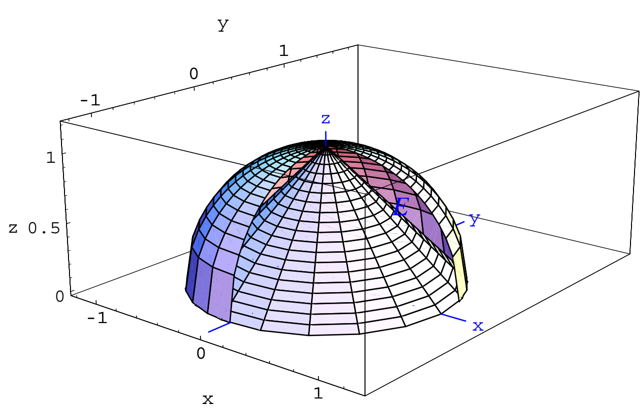


(b) This time the surfaces intersect when $z = \sqrt{1 - x^2 - y^2} = 1 - \sqrt{x^2+y^2}$, that is, when $\sqrt{1-r^2} = 1 - r$. Squaring gives

$$1 - r^2 = 1 - 2r + r^2 \Rightarrow 2r^2 - 2r = 0 \Rightarrow r = 0, 1.$$

The first solution is not helpful: just the point $(0, 0, 1)$. The second gives the actual curve of intersection in the xy-plane, since $z = 1 - r = \sqrt{1-r^2} = 0$ when $r = 1$. Again, this is almost clear from the plot. So

$$\iiint_E xyz\, dV = \int_0^{\pi/2} \int_0^1 \int_{z=1-r}^{z=\sqrt{1-r^2}} r\cos\theta\, r\sin\theta\, z dz\, r dr\, d\theta = \frac{1}{60}.$$



7. Solve the two equations $2 = \frac{3}{5} f_x + \frac{4}{5} f_y$ and $-1 = -\frac{4}{5} f_x + \frac{3}{5} f_y$ for $f_x = 2,\ f_y = 1$ to find that $\nabla f(a, b) = 2\mathbf{i} + \mathbf{j}$.