Exam 2 | **Math 220** | **December 8, 1999**

Name: ____

Social Security Number: ____________________________

Before starting to work, make sure that you have a *complete* exam: **5** numbered pages stapled to this page.

The point value of each question appears after its number. Budget your time accordingly: *about five minutes for each ten points.* That will leave time to check your work or to attempt the bonus question. The maximum score for averaging purposes is 100, although you can earn up to 110 points because of the bonus question.

Do not write anything on this cover page below the following solid line.

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1. ______

2. ______

3. ______

4. ______

5. ______

6. ______

7. ______ **(Bonus)**

TOTAL SCORE: ______

Show all work clearly in the space provided. For full credit, solution methods must be logical and understandable, and must involve *only* techniques and results developed thus far in *this course, Math 120–121, 115–116, or 112–114.* Clearly label your answers, be sure that they give *all* the information asked for, and that they follow logically from your preceding work. Read each question carefully! Provide *clear and explicit* directions to any work that is outside the workspace for its question.

1. Like many functions, the *length* function $l : \mathbf{R}^3 \to \mathbf{R}$ with formula $l(x, y, z) = \|x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}\| = \sqrt{x^2 + y^2 + z^2}$ changes at different rates as (x, y, z) varies in different directions.

   (a) (5 points) At any point $(a, b, c) \in \mathbf{R}^3$ other than the origin, in which direction is l increasing most rapidly? What is the maximum rate of increase of l at (a, b, c)?

   (b) (5 points) In which direction is l decreasing most rapidly at (a, b, c)? What is that maximum rate of decrease?

2. (10 points) Suppose that $w = x^2 - 2y^2$, where $x = r\ \cos\theta$, $y = r\ \sin\theta$. Without expressing w in terms of r and θ, find a formula for $\dfrac{\partial w}{\partial\theta}$ in terms of r and θ.

3. (15 points) Consider the function F with formula $F(x, y, z) = x^2 + y^2 + z^2 - 9$.

   (a) Does the equation $F(x, y, z) = 0$ define z as a differentiable function of x and y near the point $(-1, 2, 2)$? Why can you be confident of your answer?

   (b) If possible, use the tangent approximation to estimate the value of z on the graph of $F(x, y, z) = 0$ when $x = -1.02$ and $y = 2.01$. If that is not possible, explain why not.

4. (a) (10 points) Find and classify the critical points of the function f if $f(x, y) = x^2 - xy - y^2 + 5y - 1$.

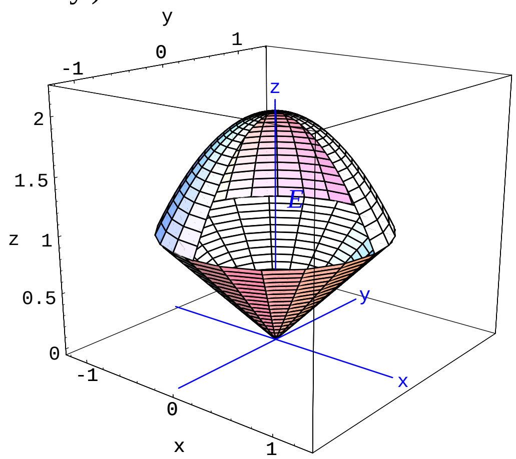
(b) (10 points) Suppose that $f(x, y, z) = x^2 + y^2 + z^2$. To find the maximum and minimum values of f on the curve of intersection of the cone $z = \sqrt{x^2 + y^2}$ and the paraboloid $z = x^2 + y^2 - 2$, what system of equations would you solve? (You do **not** need to carry out the solution!)

5. The double integral of a function f over a region S in the xy-plane is $\iint_S f(x, y)dA = \int_1^e dy \int_0^{\ln y} f(x, y)\, dx$.

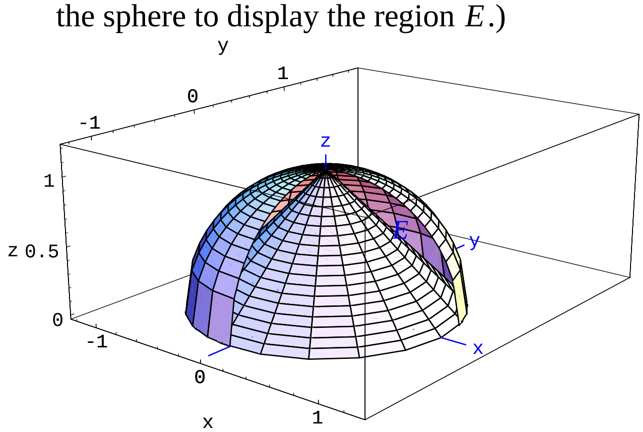
(a) (5 points) Sketch the region S.

(b) (5 points) Express the double integral as an iterated integral in which the first integration proceeds with respect to y rather than x.

6. (a) (15 points) Use multiple integration to calculate the volume of the region E in $\mathbf{R}^3$ that lies above the graph of $z = \sqrt{x^2 + y^2}$ and below the graph of $z = 2 - x^2 - y^2$. (In the accompanying plot, part of the second surface is omitted to show the region E more clearly.)



(b) (20 points) Evaluate $\iiint_E xyz\, dV$ if E is the first-octant region inside the sphere $x^2 + y^2 + z^2 = 1$ and above the graph of $z = 1 - \sqrt{x^2 + y^2}$. (As above, the plot omits part of the sphere to display the region E.)



7. **Bonus Question** (10 points maximum, partial credit possible). Attempt this problem *only* after having finished and checked over all earlier questions.

   For a certain differentiable function $f : \mathbf{R}^2 \to \mathbf{R}$, its directional derivatives in the directions $\mathbf{u}_1 = \frac{3}{5}\,\mathbf{i} + \frac{4}{5}\,\mathbf{j}$ and $\mathbf{u}_2 = -\frac{4}{5}\,\mathbf{i} + \frac{3}{5}\,\mathbf{j}$ at the point (a, b) are, respectively, 2 and -1. What is the total derivative $\mathbf{f}\,'(a, b)$? Explain fully and clearly how you arrive at your answer!