Curves and Tangents in *Mathematica*

Consider the parametric curve from **Exercise 6, Section 2.1**, of *Multivariable Calculus* by James Hurley: $\mathbf{x} = \mathbf{x}(t) = t \sin t\, \mathbf{i} + 3\, t\, \mathbf{j} + t \cos t\, \mathbf{k}$, for $t \geq 0$. To find the tangent line at the point corresponding to given value of t, such as $t = \pi/2$, the first step is to differentiate the function $\mathbf{x}$. *Mathematica* can do that, but in this case the formula is certainly simple enough for hand differentiation:

$$\mathbf{x}'(t) = (\sin t + t \cos t)\, \mathbf{i} + 3\, \mathbf{j} + (\cos t - t \sin t)\, \mathbf{k} \quad \Rightarrow \quad \mathbf{x}'(\tfrac{\pi}{2}) = \mathbf{i} + 3\, \mathbf{j} - \tfrac{\pi}{2}\mathbf{k}.$$

Since $\mathbf{x}(\pi/2) = (\pi/2, 3\pi/2, 0)$, the tangent line to the curve at the point $P(\pi/2, 3\pi/2, 0)$ has vector equation

$$\mathbf{x} = \mathbf{x}(t) = (\tfrac{\pi}{2}, \tfrac{3\pi}{2}, 0) + t\,(1, 3, -\tfrac{\pi}{2}) \quad \Rightarrow \quad x = \tfrac{\pi}{2} + t,\ y = \tfrac{3\pi}{2} + 3t,\ z = -\tfrac{\pi}{2}t$$

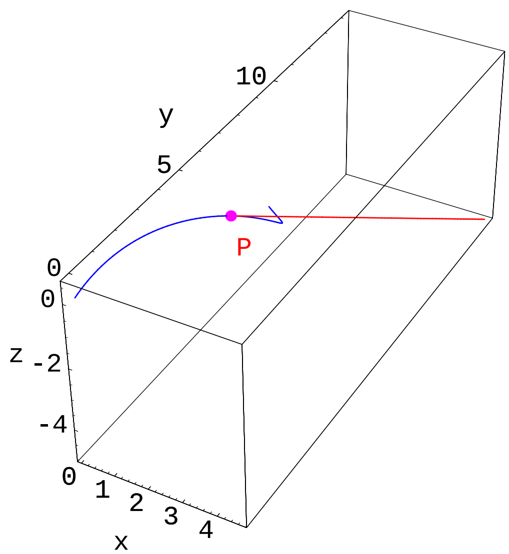
The following *Mathematica* routine plots the curve in blue and its tangent line at the point $(\pi/2, 3\pi/2, 0)$ in red inside a coordinate box. (The term (-1, 2) in the `Text` command prints the label P one unit to the right and two units below its actual position, so as not to overwrite the dot that marks$P$.) As usual, to generate the plot, move the cursor to the end of the last blue line of code and press the Enter key, or press the Shift and Return keys together.

```
In[1]:= Curve = ParametricPlot3D[ { { t*Sin[t], 3*t, t*Cos[t],
                                    RGBColor[0, 0, 1]},
                                   {Pi/2 + t,
            3*Pi/2 + 3*t, - Pi*t/2,
                                    RGBColor[1, 0, 0] } }, {t, 0, Pi},
                                  AxesLabel -> {x, y, z} ];

        Basepoint = Graphics3D[{ RGBColor[1, 0, 1] ,
                               {PointSize[0.025],
            Point[{Pi/2, 3*Pi/2, 0}]} }]

        Legend = Graphics3D[ {RGBColor[1, 0, 0],
                             Text[ "P", {Pi/2, 3*Pi/2, 0}, {-1, 2}]}]

        Show[Curve, Basepoint, Legend]
```



As a second example, consider the curve in **Exercise 3, Section 2.1**, of *Multivariable Calculus:*

$$\mathbf{x} = \mathbf{x}(t) = (t - 1)\,\mathbf{i} + (t^2 + 1)\,\mathbf{j} + (\,t^3\; - 1)\,\mathbf{k}.$$

The question asks for the tangent line at the point that corresponds to $t = 2$, and also for the speed there. The formula gives $\mathbf{x}(2) = (1, 5, 7)$ as the point of tangency. Again, although *Mathematica* can easily differentiate the vector function $\mathbf{x}$, hand differentiation is likely quicker:

$$\mathbf{x}'(t) = \mathbf{i} + 2t\,\mathbf{j} + 3t^2\,\mathbf{k} = \mathbf{i} + 4\,\mathbf{j} + 12\,\mathbf{k} \text{ at the point } P(1, 5, 7).$$

From this, the speed is $\|\mathbf{x}'(2)\| = \sqrt{1 + 16 + 144} = \sqrt{161}$. The tangent line to the curve at the point P has vector equation

$$\mathbf{x} = \boldsymbol{x_0} + t\,(1, 4, 12) = (1, 5, 7) + (t, 4t, 12t) = (1 + t, 5 + 4t, 7 + 12t).$$

The following routine graphs the curve and the tangent line at P.

```
In[37]:= Curve = ParametricPlot3D[ { t - 1, t^2 + 1, t^3 - 1, RGBColor[0, 0, 1]},
                                     {t, 0, 2.25} ]

         Tanln = ParametricPlot3D[{ 1 + t, 5 + 4 * t, 7 + 12 * t, RGBColor[1, 0, 0]},
                                     {t, -.15, .15} ]

         Basepoint = Graphics3D[{ RGBColor[1, 0, 1] ,
                                  {PointSize[0.05], Point[{1, 5, 7}]} }]

         Legend = Graphics3D[ {RGBColor[1, 0, 0],
                               Text[ "P", {1, 5, 7}, {-2, 0}]}]

         Show[Curve, Tanln, Basepoint, Legend, AxesLabel -> {x, y, z}]
```