Plotting Planes in *Mathematica*

This notebook discusses planes in 3-space, and the *Mathematica* code to generate plots of them.

1. Normal Form. The best algebraic representation of planes is the *normal-form equation*, whose geometric basis is simple: a plane is uniquely determined by one point P on it and a normal vector $\mathbf{n}$.

Definition. The *normal-form equation* of the plane through $P(x_0, y_0, z_0)$ with the normal vector $\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k} = (a, b, c)$ is

(1) $$\mathbf{n} \cdot (\mathbf{x} - \mathbf{x}_0) = 0,$$

where $\mathbf{x} = (x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = \mathbf{OQ}$ for any point $Q(x, y, z)$ on the plane and $\mathbf{x}_0 = \mathbf{OP} = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k} = (x_0, y_0, z_0)$ is the vector from the origin to the point P.

Equation (1) says that a point $Q(x, y, z)$ lies on the plane if and only if the plane's normal vector $\mathbf{n}$ is perpendicular to the vector $\mathbf{x} - \mathbf{x}_0$ from the given point P to Q. The vector equation leads easily to a scalar equation. The definition of dot product and (1) give

$$(a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}) \cdot [(x - x_0)\,\mathbf{i} + (y - y_0)\,\mathbf{j} + (z - z_0)\,\mathbf{k}] = 0,$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$ax + by + cz = ax_0 + by_0 + cz_0$$

(2) $$ax + by + cz = d, \quad \text{where } d = \mathbf{n} \cdot \mathbf{x}_0.$$

Conversely, any linear equation of the form (2) is the equation of a plane in 3-space, provided that not all three of a, b, and c are 0. To see why, let $P(x_0, y_0, z_0)$ be any point

that satisfies (2). Then

(3) $$ax_0 + by_0 + cz_0 = d.$$

Let $\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k} = (a, b, c)$ and $\mathbf{x_0} = \mathbf{OP} = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k} = (x_0, y_0, z_0)$. Then (2) and (3) give

$$\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{x_0} = d \quad \Rightarrow \quad \mathbf{n} \cdot (\mathbf{x} - \mathbf{x_0}) = 0,$$

which is the normal-form equation of a plane.

To plot the graph of a plane in *Mathematica*, the simplest approach is to plot its equation as that of a (simple flat) surface with equation $z = f(x, y)$. That is possible unless the variable z does not appear in the equation (1). To plot a plane with equation in the form (1), where the coefficient c of z is not 0, solve (1) for the variable z:

(4) $$z = (ax_0 + by_0 + cz_0 - ax - by)/c$$

The following routine illustrates how to plot the graph of the plane by plotting the equation $z = f(x, y)$, where $f(x, y)$ is the right side of (4). The values of a, b, c, and d come from the plane with normal vector $\mathbf{n} = 3\mathbf{i} + \mathbf{j} + 5\mathbf{k} = (3, 1, 5)$ through the point $P(-2, 1, 3)$. As usual, to generate the plot execute the routine by placing the cursor at the end of the last blue line and pressing the Enter key.

```
In[11]:= (* Mathematica Routine to plot graph of a plane
          ax + by + cz = d, where c is not 0 *)
        a := 3;
        b := 1;
        c := 5;
        x0 := -2;
        y0 := 1;
        z0 := 3;
        F[x_, y_] := (a x0 + b y0 + c z0 - a x - b y)/c;
        Plot3D[ F[x, y], {x, -2, 5}, {y, -2, 10},
                AxesLabel -> {x, y, z} ]
```
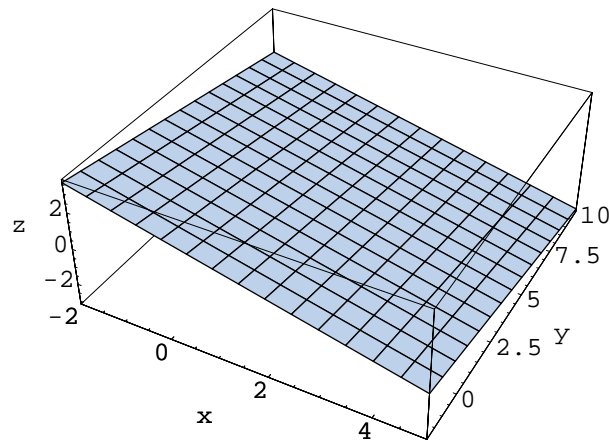
Out[19]=   ▪ SurfaceGraphics ▪

As before, it is possible to add coordinate axes, at the expense of complicating the code. The following routine outputs the original rendering first, and then repeats that with coordinate axes. Use whichever routine that gives your eye the better image.

```
In[21]:= (* Mathematica Routine to plot graph of a plane
         ax + by + cz = d, where c is not 0
         with coordinate axes drawn in *)
       a := 3;
       b := 1;
       c := 5;
       x0 := -2;
       y0 := 1;
       z0 := 3;
       F[x_, y_] := (a x0 + b y0 + c z0 - a x - b y)/c;
       planeplot = Graphics3D[
                   Plot3D[ F[x, y], {x, -2, 5}, {y, -2, 10},
                       AxesLabel -> {x, y, z} ]
                          ];
       coordaxes = Graphics3D[{
                       {RGBColor[1, 0, 0],
                        Line[{{-2, 0, 0}, {6, 0, 0}}],
                        Text["x", {7, 0, 0}]},
                       {RGBColor[1, 0, 0],
                        Line[{{0, -3, 0}, {0, 12, 0}}],
                        Text["y", {0, 13, 0}]},
                       {RGBColor[1, 0, 0],
                        Line[{{0, 0, -3}, {0, 0, 9}}],
                        Text["z", {0, 0, 9}]}
                          }];
       Show[planeplot, coordaxes]
```
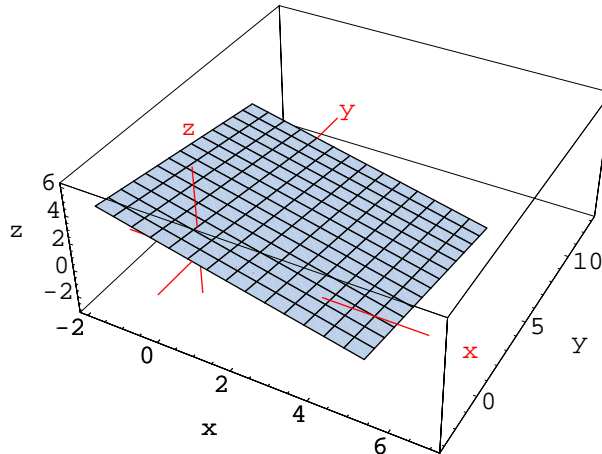
Out[31]=   - Graphics3D -

What if $c = 0$, that is, the plane is vertical? Then the equation of the plane involves only the two variables x and y, so it is not possible to plot it as the graph of an equation $z = F(x, y)$. The best approach is to treat the plane as a simple example of a *parametric surface*. If the normal vector $\mathbf{n} = a\mathbf{i} + b\mathbf{j} + 0\mathbf{k} = (a, b, 0)$, then the normal-form equation (1) becomes

$$(a\mathbf{i} + b\mathbf{j}) \cdot [(x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}] = 0 ,$$

$$a(x - x_0) + b(y - y_0) = 0$$

$$ax + by = ax_0 + by_0$$

(5) $ax + by = d ,$ where $d = \mathbf{n} \cdot \mathbf{x}_0 .$

In (5), not both a and b can be zero, because the normal vector $\mathbf{n}$ is nonzero. Suppose for example that b is nonzero. Then $y = d/b - ax/b = (d - ax)/b$. Since the equation of the plane does not involve z, it can vary over all real numbers. Any point on the plane then satisfies the *triple* of parametric equations

$$x = u, \; y = (d - au)/b, \; z = v, \quad \text{where } u \in (-\infty, +\infty) \text{ and } v \in (-\infty, +\infty).$$

Note that there are *two* parameters, reflecting the fact that a plane is two dimensional. The following *Mathematica* routine plots the plane through $P(1, 3, 2)$ with normal vector $\mathbf{n} = 3\,\mathbf{i} - \mathbf{j} = (3, -1, 0)$. Since $d = \mathbf{n} \cdot \mathbf{x}_0$, for this plane $d = (3, -1, 0) \cdot (1, 3, 2) = 3$

$-\ 3 = 0$. The plane is vertical, so there is no need to add the coordaxes routine to display the coordinate axes. The default plot's box displays the nature of the plane quite well.

```
In[33]:= (* Mathematica Routine to plot graph of a plane
            ax + by = d, where b is not 0 *)
         a := 3;
         b := -1;
         c := 0;
         x0 := 1;
         y0 := 3;
         z0 := 2;
         d := Dot[{a, b, c}, {x0, y0, z0}]
         ParametricPlot3D[ {u, (d - a u)/b, v}, {u, -4, 4}, {v, -2, 8},
                 AxesLabel -> {x, y, z} ]
```
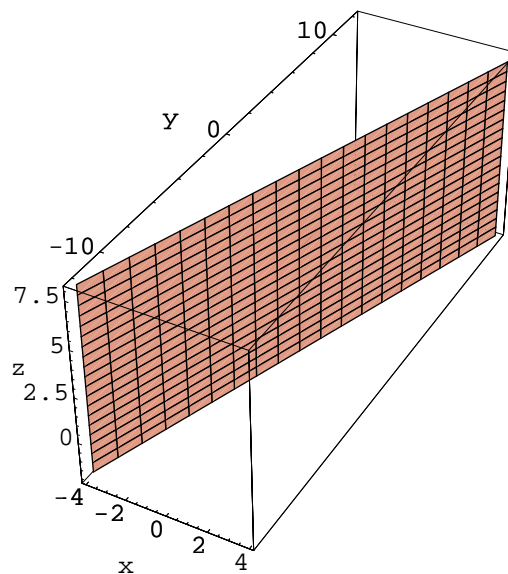


```
Out[41]= ▪ Graphics3D ▪
```

2. Vector cross product and planes. Euclidean geometry arose many centuries before vectors. The classical ideal of a plane in 3-space was the surface determined by three points P, Q, and R that are *non-collinear* (that is, do not lie in a common line) or by two non-skew lines (lines that are either parallel or intersect). One tool makes it easy to find the normal-form equation of a plane with either description. That tool is the *vector cross product* of 3-dimensional vectors. It produces a vector perpendicular to two given vectors, say **v** and **w**.

The definition of $\mathbf{v} \times \mathbf{w}$ is somewhat involved, so it is worth noting first that finding a vector perpendicular to each of two given vectors is indeed enough to determine the equation of a plane through three non-collinear points or two non-skew lines. Consider

first that two lines: l_1 with vector equation $\mathbf{x} = \mathbf{x}_0 + t\mathbf{v}$ and l_2 with equation $\mathbf{x} = \mathbf{x}_1 + s\mathbf{w}$. Then $\mathbf{v}$ and $\mathbf{w}$ are direction vectors in the directions of the respective lines. So a normal vector $\mathbf{n}$ to the plane determined by $\mathbf{v}$ and $\mathbf{w}$ is indeed perpendicular to both $\mathbf{v}$ and $\mathbf{w}$.

Similarly, given three points *P*, *Q*, and *R* not on a common line, the vectors $\mathbf{v} = \mathbf{PQ}$ and $\mathbf{w} = \mathbf{PR}$ are in the plane determined by *P*, *Q*, and *R*. So once again, a normal vector to that plane is perpendicular to both $\mathbf{v}$ and $\mathbf{w}$. The following definition thus provides the promised tool for finding the equation of a plane with either description.

Definition. The ***cross product*** of two vectors $\mathbf{v} = (v_1, v_2, v_3) = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ and $\mathbf{w} = (w_1, w_2, w_3) = w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}$ in Euclidean 3-space is

(6) $\qquad \mathbf{v} \times \mathbf{w} = (v_2 w_3 - v_3 w_2)\mathbf{i} + (v_3 w_1 - v_1 w_3)\mathbf{j} + (v_1 w_2 - v_2 w_1)\mathbf{k}.$

The complexity of the formulas for the coordinates results from solving the system of equations

$$\mathbf{v} \cdot \mathbf{n} = 0 \quad \text{and} \quad \mathbf{w} \cdot \mathbf{n} = 0$$

for $\mathbf{n} = (x, y, z)$. (See the text for details.) *Mathematica* has a command for calculating cross products. The following short routine illustrates not only that but also *Mathematica*'s capacity to carry out symbolic algebra.

```
In[43]:= (* Mathematica Routine to illustrate symbolic computation
            of the cross product of two vectors *)
         v = {v1, v2, v3};
         w = {w1, w2, w3};
         Cross[v, w]
```

```
Out[46]= {-v3 w2 + v2 w3, v3 w1 - v1 w3, -v2 w1 + v1 w2}
```

For a numerical example, simply input a couple of specific vectors $\mathbf{v}$ and $\mathbf{w}$.

```
In[47]:= (* Mathematica Routine to illustrate numerical computation
            of the cross product of two vectors *)
         v = {2, -1, 3};
         w = {1, 5, -3};
         Cross[v, w]
```

```
Out[50]= {-12, 9, 11}
```

From this it follows that the plane through the two lines $\mathbf{x} = (2, 0, 3) + s(2, -1, 3)$ and $\mathbf{x} = (-4, 3, -6) + t(1, 5, -3)$ has normal vector $\mathbf{n} = (-12, 9, 11)$. Its normal-form equation is therefore

$$(-12, 9, 11) \cdot (x - 2, y - 0, z - 3) = 0 \quad \Rightarrow \quad -12(x - 2) + 9y + 11(z - 3) = 0,$$

which simplifies to $-12x + 9y + 11z = 9$. (**Note**: both points (2, 0, 3) and (–4, 3, –6) from the two given equations satisfy this equation, which confirms that there has been no error in the calculation. It is always advisable to substitute a known point from each line into the final equation of the plane, to verify that the lines are not skew. If they are, one of the points will satisfy the final equation, but the other will not.)

Mathematica can plot two vectors and their cross product. The following modification of a routine from *VectPlot* illustrates that. Execute it to view the plot.

```
In[51]:=  (* Mathematica Routine to illustrate graphically
            the cross product of two vectors *)
          v = {2, -1, 3};
          w = {1, 5, -3};
          k = Cross[v, w];
          q = Sqrt[v.v];
          r = Sqrt[w.w];
          s = Sqrt[(v + w).(v + w)];
          coordaxes = Graphics3D[{
                             {RGBColor[0, 1, 0],
                              Line[{{-2, 0, 0}, {6, 0, 0}}],
                              Text["x", {8, 0, 0}]},
                             {RGBColor[0, 1, 0],
                              Line[{{0, -3, 0}, {0, 12, 0}}],
                              Text["y", {0, 14, 0}]},
                             {RGBColor[0, 1, 0],
                              Line[{{0, 0, -3}, {0, 0, 9}}],
                              Text["z", {0, 0, 10}]}
                                 }];
          vplot = Graphics3D[{
                  {Line[{{0, 0, 0}, v}],
                   Polygon[{v,{.8 v[[1]], .8 v[[2]], .8 v[[3]]
                             - 2/q},
                            {.8 v[[1]], .8 v[[2]], .8 v[[3]]
                             + 2/q}   }],
                  Text[FontForm["v", {"Times-Bold", 12}],
                      {1.5 v[[1]], 1.5 v[[2]], 1.5 v[[3]]}]
                   },
                  {Line[{{0, 0, 0}, w}],
                   Polygon[{w, {.8 w[[1]], .8 w[[2]], .8 w[[3]]
                             - 2/r},
                            {.8 w[[1]], .8 w[[2]], .8 w[[3]]
                             + 2/r}   }],
                  Text[FontForm["w", {"Times-Bold", 12}],
                      {1.5 w[[1]], 1.5 w[[2]], 1.5 w[[3]]}]
                   },
                  {RGBColor[1, 0, 1], Line[{{0, 0, 0},
                                    Cross[v, w]}],
                     Polygon[{k,{.9 (k)[[1]] - 1, .9 (k)[[2]]
                               + 1, .9 (k)[[3]] - 1},
                              {.9 (k)[[1]] + 1, .9 (k)[[2]]
                               + 1, .9 (k)[[3]] - 1}
                              }],
                  Text[FontForm["v x w", {"Times-Bold", 12}],
                                {.8 (k)[[1]], .8 (k)[[2]],
```

```
                         .7 (k)[[3]]}]
            }
                  }]
      Show[vplot, coordaxes, Axes -> True, Lighting -> False]
```

Out[59]= ▪ Graphics3D ▪
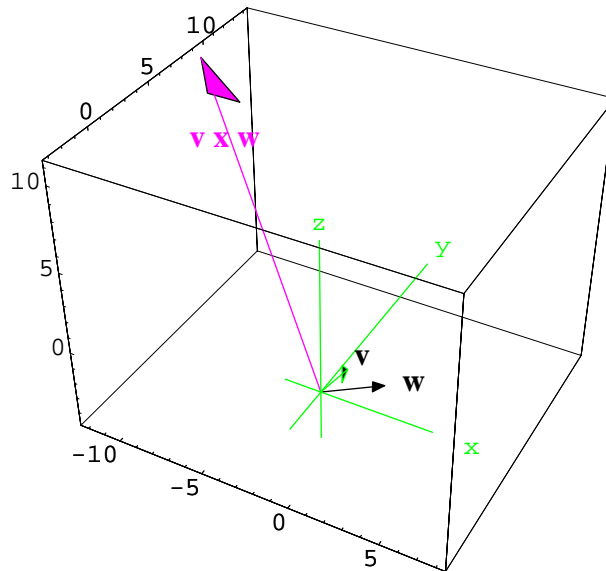
Out[60]= ▪ Graphics3D ▪

No new routine is needed to plot the plane: just modify the earlier routine for plotting the normal-form equation. There is no text figure with which to compare this plot, but the next edition will have one!

```
In[61]:=        (* Mathematica Routine to plot graph of a plane
                ax + by + cz = d, where c is not 0 *)
                a = -12;
                b = 9;
                c = 11;
                x0 = 2;
                y0 = 0;
                z0 = 3;
                F[x_, y_] := (a x0 + b y0 + c z0 - a x - b y)/c;
                planeplot = Graphics3D[
                        Plot3D[ F[x, y], {x, -2, 5}, {y, -2, 10},
                        AxesLabel -> {"x", "y", "z"} ]
                                        ];
                coordaxes = Graphics3D[{{RGBColor[1, 0, 0],
                                Line[{{-2, 0, 0}, {5, 0, 0}}],
                                Text["x", {6, 0, 0}]},
                               {RGBColor[1, 0, 0],
                                Line[{{0, -2, 0}, {0, 10, 0}}],
                                Text["y", {0, 11, 0}]},
```
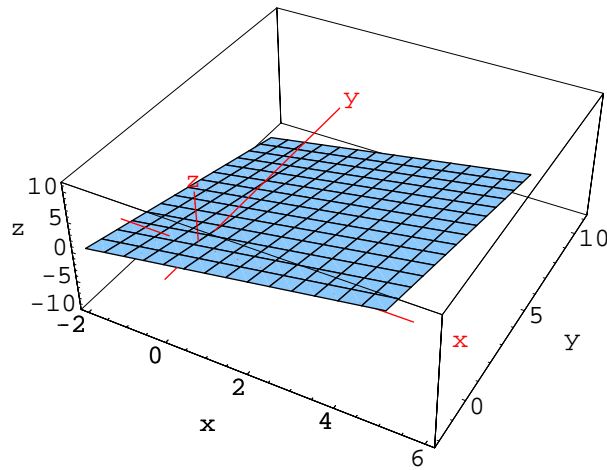
```
                    {RGBColor[1, 0, 0],
                     Line[{{0, 0, -2}, {0, 0, 9}}],
                     Text["z", {0, 0, 11}]}
                          }];
        Show[planeplot, coordaxes]
```


Out[71]=   ◾ Graphics3D ◾