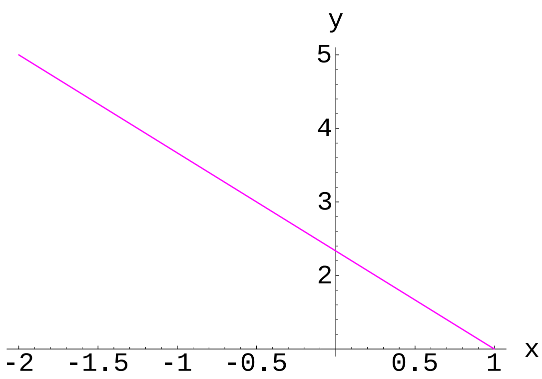
Plotting Lines in Mathematica

This notebook discusses lines in the plane and 3-space and illustrates how to plot them in *Mathematica*.

1. Lines. *Mathematica* has a built-in command to plot the line *segment* between two points $P(a, b)$ and $Q(c, d)$. The following 4-line routine illustrates its use. Execute it by placing your cursor after the last line and pressing the Enter key.

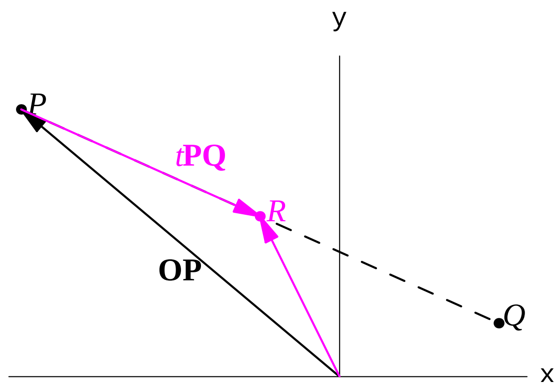
```
In[1]:= (* Mathematica Routine to plot line segment
           through two points in the plane*)
        P := {-2, 5};
        Q := {1, 1};
        lineseg = Graphics[ { RGBColor[1, 0, 1], Line[{P, Q}] }  ]
        Show[lineseg, Axes -> True, AxesLabel -> {x, y}]

Out[4]=  - Graphics -
```




```
Out[5]=  - Graphics -
```

A more powerful tool for plotting lines through two points uses *vector parametric representation*. Such a representation of the line L through two points $P(a, b)$ and $Q(c, d)$ comes from the parallelogram law of vector addition. Namely, to get to any point $R(x, y)$ on the line, start at the origin and go first to the point $P(a, b)$ on the line, by viewing P as the endpoint of the vector $\mathbf{x}_0 = \mathbf{OP} = (a, b)$. From P travel along the line by adding an appropriate multiple t of the vector $\mathbf{v} = \mathbf{PQ} = (c - a, d - b)$. See the figure.



This leads to the *parametric vector equation*

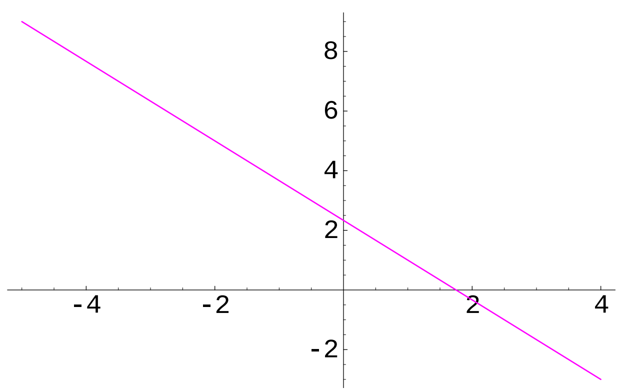
$$\mathbf{x} = \mathbf{x}_0 + t\,\mathbf{v} = (a, b) + t\,(c - a, d - b),\ t \in \mathbf{R}$$

for L. An easy way to plot such a vector equation in *Mathematica* is via the `ParametricPlot` command. For the line L through the two points (–2, 5) and (1, 1), a vector in the direction of L is $\mathbf{v} = 3\,\mathbf{i} - 4\,\mathbf{j} = (3, -4)$. A vector equation for L is then $\mathbf{x} = (-2, 5) + t\,(3, -4) = (-2 + 3t, 5 - 4t)$. That corresponds to the *scalar parametric equations*

$$x = -2 + 3t,\ y = 5 - 4t,\ t \in \mathbf{R}.$$

The following simple Mathematica routine plots this for the range $t \in [-1, 2]$.Try it!

```
In[7]:= ParametricPlot[ {-2 + 3 t, 5 - 4 t}, {t, -1, 2},
                        PlotStyle -> RGBColor[1, 0, 1] ]
```




```
Out[7]= - Graphics -
```

There are really no algebraic differences between lines in 2-space or 3-space. In the latter space, points have three coordinates instead of two, but the above reasoning still applies to yield a *parametric vector equation* for the line through two points $P(a, b, c)$ and $Q(p, q, r)$:

$$\mathbf{x} = \mathbf{x}_0 + t\,\mathbf{v} = (a, b, c\,) + t\,(p - a, q - b, r - c),\ t \in \mathbf{R}.$$

This in turn gives the *scalar parametric equations*

$$x = a + t\,(p - a),\ y = b + t(q - b),\ z = c + t(r - c),\ t \in \mathbf{R}.$$

Mathematica plots such a line by means of its 3-dimensional parametric plotting command, `ParametricPlot3D`. For example, the following routine plots the line L through the points $P(1, 2, 3)$ and $Q(-1, 1, 4)$. First, note that a vector in the direction of L is **PQ**= (–2, –1, 1). A *parametric vector equation* for the line is therefore

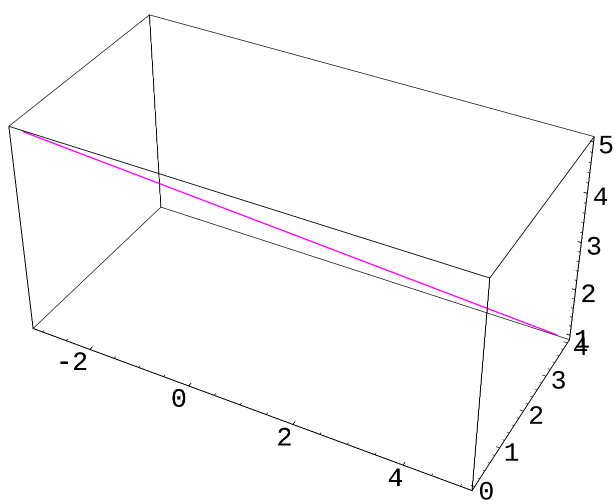
$$\mathbf{x} = (1, 2, 3) + t\,(-2, -1, 1) = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + t\,(-2\mathbf{i} - \mathbf{j} + \mathbf{k}).$$

A set of *parametric scalar equations* is then

$$x = 1 - 2t,\ y = 2 - t,\ z = 3 + t,\ t \in \mathbf{R}.$$

The following *Mathematica* routine provides a plot.Execute the routine to generate the graph.

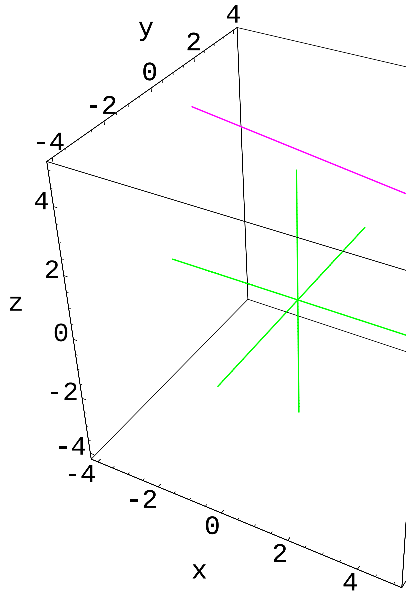
```
In[8]:= plotline = Graphics3D[ParametricPlot3D[ {1 - 2 t, 2 - t, 3  + t,
                              RGBColor[1, 0, 1]}, {t, -2, 2} ]]
```




```
Out[8]=  - Graphics3D -
```

To add coordinate axes inside the surrounding coordinate box, execute the following cell.

```
In[9]:=     ParametricPlot3D[ {{1 - 2 t, 2 - t, 3 + t, RGBColor[1, 0, 1]},
                               {2 t, 0, 0, RGBColor[0, 1, 0]},
                               {0, 2 t, 0, RGBColor[0, 1, 0]},
                               {0, 0, 2 t, RGBColor[0, 1, 0]}},
                               {t, -2, 2}, AxesLabel -> {x, y, z} ]
```



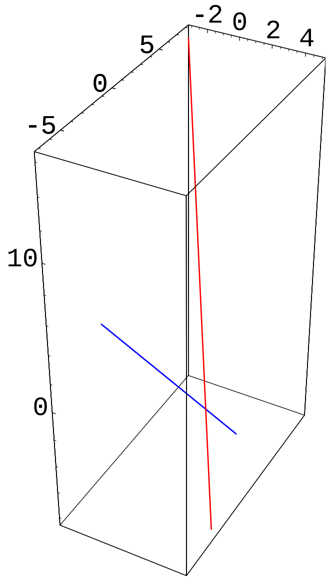

```
Out[9]= - Graphics3D -
```

Two lines in space can be parallel, perpendicular, or *skew*. The latter term applies to non-parallel lines that do not intersect (because they lie in parallel planes).

3.6. Definition. Two lines $\mathbf{x} = \mathbf{x}_1 + s\mathbf{v}$ and $\mathbf{x} = \mathbf{x}_2 + t\mathbf{w}$ are ***parallel*** if their direction vectors **v** and **w** are parallel vectors. The lines are ***perpendicular*** if **v** and **w** are perpendicular vectors (that is, $\mathbf{v} \cdot \mathbf{w} = 0$) *and* the two lines intersect.

The next routine plots a *pair* of lines with parametric equations, and can help you decide whether or not the two lines likely intersect. To confirm that, solve the two sets of scalar equations simultaneously to determine algebraically whether the two lines in fact do have a common point.

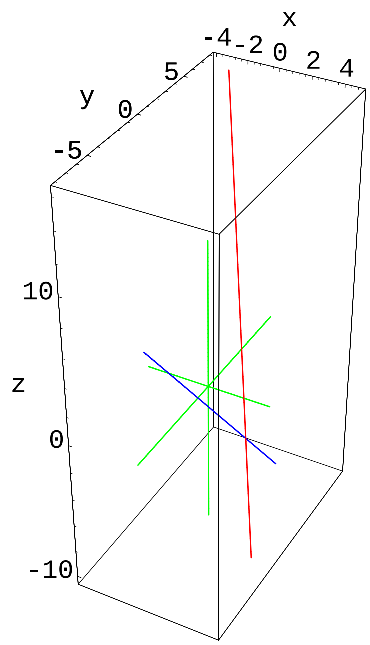
```
In[10]:= lines = Graphics3D[ParametricPlot3D[{
                          {1 - 2 t, 2 + 3 t, 4 + 6 t, RGBColor[1, 0, 0]},
                          {2 + t, -3 + 2 t, 1 - 3 t, RGBColor[0, 0, 1]}},
                          {t, -2, 2}    ] ]
```



Out[10]= - Graphics3D -

To add coordinate axes inside the surrounding coordinate box, execute the following cell.

```
ParametricPlot3D[ {
          {1 - 2 t, 2 + 3 t, 4 + 6 t, RGBColor[1, 0, 0]},
          {2 + t, -3 + 2 t, 1 - 3 t, RGBColor[0, 0, 1]},
          {2 t, 0, 0, RGBColor[0, 1, 0]},
          {0, 4 t, 0, RGBColor[0, 1, 0]},
          {0, 0, 5 t, RGBColor[0, 1, 0]} },
          {t, -2, 2}, AxesLabel -> {x, y, z} ]
```



The figure suggests that intersection is possible. To investigate further, equate the respective x-coordinate expressions $2 + t$ and $1 - 2s$ and the y-coordinate expressions $-3 + 2t$ and $2 + 3s$. From $2 + t = 1 - 2s$, it follows that $t = -1 - 2s$. Substitution of that into the equation $-3 + 2t = 2 + 3s$ gives

$$-3 + 2(-1 - 2s) = 2 + 3s \;\Rightarrow\; -5 - 4s = 2 + 3s \;\Rightarrow\; -7 = 7s \;\Rightarrow\; s = -1.$$

The corresponding value of $t = -1 - 2s = 1$. To see finally whether the lines intersect, substitute the values $s = -1$ and $t = 1$ into the two z-coordinate expressions to see whether a valid equation results. In this case that gives from the first line

$$z = 1 - 3t = 1 - 3(1) = -2$$

and from the second

$$z = 4 + 6s = 4 + 6(-1) = -2.$$

Hence, the two lines do intersect — at the point $(3, -1, -2)$.