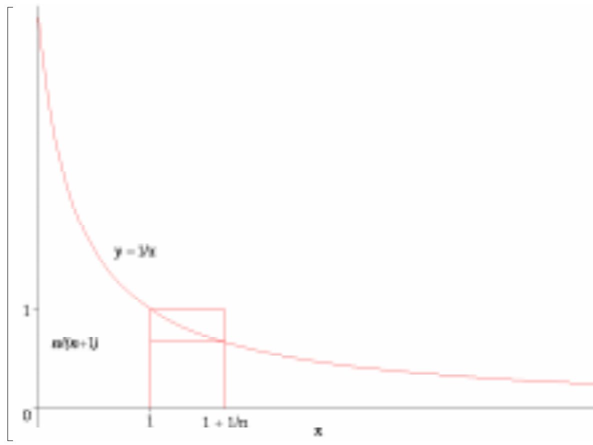
The Number *e* as a Limit

This document derives two descriptions of the number e, the base of the natural logarithm function, as limits:

$$\lim_{x\to 0}(1+x)^{1/x} = e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n . \tag{8, 9}$$

These equations appear with those numbers in Section 7.4 (p. 442) and in Section 7.4* (p. 467) of Stewart's text *Calculus*, *4th Ed.*, Brooks/Cole, 1999. Reversing the approach of the text, the following derivation first establishes (9) and then shows that (8) follows from that.



Suppose that $n > 0$. (Very similar reasoning applies if $n < 0$.) From the figure, the area under the graph of $y = 1/x$ between $x = 1$ and $x = 1 + 1/n$ lies between the areas of the two rectangles above the interval $[1, 1 + 1/n]$. The larger rectangle has height 1, while the height of the smaller is $\frac{n}{n+1}$. Since the base in each case is $1/n$, the area relations translate into the inequalities

$$\frac{1}{n}\cdot\frac{n}{n+1} \le \int_1^{1+1/n}\frac{1}{x}\,dx \le \frac{1}{n}\cdot 1 \quad\Longrightarrow\quad \frac{1}{n+1} \le \ln\left(1+\frac{1}{n}\right) \le \frac{1}{n}.$$

Mutliplying the last set of inequalities through by the positive number n gives

$$\frac{n}{n+1} \le n\,\ln\left(1+\frac{1}{n}\right) = \ln\left(1+\frac{1}{n}\right)^n \le 1.$$

Next, let $n \to +\infty$ and use the fact that taking limits preserves the relation $\le$:

$$1 \le \lim_{n\to+\infty}\ln\left(1+\frac{1}{n}\right)^n \le 1 \quad\Longrightarrow\quad \lim_{n\to+\infty}\ln\left(1+\frac{1}{n}\right)^n = 1.$$

Since exp is continuous at $x = 1$, for $u = \ln(1 + 1/n)^n$ we have $e^u \to e^1 = e$ as $u \to 1$). Hence,

$$\text{as } n \to +\infty,\quad e^{\ln(1+1/n)^n} = \left(1+\frac{1}{n}\right)^n \to e. \tag{9}$$

Next, let $x = 1/n$, so that $n \to +\infty$ is equivalent to $x \to 0^+$. Thus (9) is equivalent to

$$\lim_{x\to 0^+}(1+x)^{1/x} = e.$$

What about the limit as $x \to 0^-$? The similar reasoning referred to above shows that $\lim_{n\to-\infty}(1 + 1/n)^n = e$ also holds. So letting $x = 1/n$ leads to the left-hand limit also being e, which means that (8) holds.