Math 116 Final Examination Worksheet

1. (7.5 minutes) Find the area of the region in the first quadrant that lies above the x-axis and below both the graphs of the functions f and g, if $f(x) = x^2 + 1$ and $g(x) = 3 - x$. **Answer:** $\frac{10}{3}$.

2. Evaluate:

   (a) (5 minutes) $\lim_{h \to 0^+} \frac{e^{-1/h^2}}{h}$. *Hint:* Let $t = 1/h$. **Answer:** 0.

   (b) (5 minutes) $\int \frac{(1+\ln x)^2}{x}\, dx$ **Answer:** $(1 + \ln x)^3/3 + C$.

   (c) (5 minutes) $\int \frac{1+e^x}{e^x}\, dx$ **Answer:** $-e^{-x} + x + C$.

   (d) (5 minutes) $\int \frac{dx}{x^2 - 2x + 5}$ **Answer:** $\frac{1}{2} \arctan \frac{x-1}{2} + C$

   (e) (2.5 minutes) $\sin(\arctan\left(\frac{x}{2}\right))$ **Answer:** $\frac{x}{\sqrt{x^2+4}}$

3. (7.5 minutes) Evaluate $\lim_{x \to 0^+} (1 + 2x)^{1/x}$ **Answer:** e^2

4. Evaluate the following integrals.

   (a) (7.5 minutes) $\int x \ln x\, dx$ **Answer:** $\frac{x^2}{2} \ln x - \frac{1}{4}x^2 + C$

   (b) (7.5 points) $\int \frac{x}{(x+1)(x+2)}\, dx$ **Answer:** $-\ln|x+1| + 2\ln|x+2| + C$

   (c). (7.5 points) $\int \sin^3 x\, dx$ **Answer:** $-\cos x + \frac{\cos^3 x}{3} + C$

5. (10 minutes) Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+1}$ converges or diverges. If it converges, is the convergence absolute or conditional?
   Answer: Conditionally convergent

6. (10 minutes) Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$. Where is the convergence absolute? **Answer:** $[2, 4)$, with absolute convergence on $(2, 4)$

7. (5 minutes) Use any valid method of your choice to give the Taylor series about $x = 0$ for the function $\cosh x = \frac{1}{2}(e^x + e^{-x})$. On what interval does that series converge to $\cosh x$? **Answer:** $1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2n}}{(2n)!} + \cdots$ for all $x \in \mathbf{R}$

8. (2.5 minutes per part)

   (a) Find the projection of $\mathbf{v} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$ onto the vector $\mathbf{w} = -2\mathbf{i} + \mathbf{j} + \mathbf{k}$. **Answer:** $-\frac{1}{3}\mathbf{i} + \frac{1}{6}\mathbf{j} + \frac{1}{6}\mathbf{k}$

   (b) What is the angle between $\mathbf{v}$ and $\mathbf{w}$? **Answer:** $\arccos(1/3) \approx .1.23$ RAD

(c) Find a unit vector in the direction of $\mathbf{v}$. **Answer:** $\frac{1}{\sqrt{6}}\mathbf{i}+\frac{2}{\sqrt{6}}\mathbf{j}+\frac{1}{\sqrt{6}}\mathbf{k}$

(d) Find a vector that is perpendicular to both $\mathbf{v}=\mathbf{i}+2\,\mathbf{j}+3\,\mathbf{k}$ and $\mathbf{w}=4\,\mathbf{i}+5\,\mathbf{j}+5\,\mathbf{k}$.
Answer: $\mathbf{v}\times\mathbf{w}=-5\,\mathbf{i}+7\,\mathbf{j}-3\,\mathbf{k}$.

9. (5 minutes) If f is continuous over the interval $[a,b]$, then show that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$. **Answer:** In the right integral, change variable to $u=a-x$

10. (5 minutes)

(a) Find the Maclaurin (Taylor series about $x=0$) for h if $h(x)=x^2e^x$. For which values of x does this series converge to $h(x)$? How do you know that?

(b) Obtain a power series for the function H if $H(x)=\int_0^x t^2e^t\,dt$.

(c) What is the sum of the series

$$2+\frac{1}{3}+\frac{1}{4}+\frac{1}{5\cdot 2!}+\frac{1}{6\cdot 3!}+\cdots+\frac{1}{(n+3)\cdot n!}+\cdots ?$$

Answers:

(a) $x^2+x^3+\frac{x^4}{2!}+\frac{x^5}{5!}+\cdots\frac{x^{n+2}}{n!}+\cdots$, which converges for all x since that is true of the series for e^x.

(b) $\frac{x^3}{3}+\frac{x^4}{4}+\frac{x^5}{5\cdot 2!}+\frac{x^6}{6\cdot 3!}+\cdots+\frac{x^{n+3}}{(n+3)n!}+\cdots$, which also converges to $H(x)$ for all real numbers x

(c) e