Reverse mathematics: an introduction

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S.I.G.M.A. Seminar
A motivating question

“What are the appropriate axioms for mathematics?”
A motivating question

“What axioms are sufficient and necessary for a given fragment of mathematics?”
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Determining sufficiency

Given an axiom system $\mathcal{B}$ and a mathematical theorem $\xi$. 

Example: 

$\text{ZFC} \vdash \text{Zorn's lemma}$ 

$\text{ZF} \not\vdash \text{Zorn's lemma}$.

So set theory with choice is sufficient for Zorn's lemma while set theory without choice is not.
Determining sufficiency

Given an axiom system $\mathcal{B}$ and a mathematical theorem $\xi$. How do we determine if $\mathcal{B}$ is sufficient to prove $\xi$?

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Given an axiom system $B$ and a mathematical theorem $\xi$. How do we determine if $B$ is sufficient to prove $\xi$?

Prove $\xi$ from $B$!
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Prove $\xi$ from $\mathcal{B}$!

If we can do this, we write

$$ \mathcal{B} \vdash \xi $$

and say $\mathcal{B}$ is sufficient for $\xi$.

Example:

$$ ZFC \vdash \text{Zorn's lemma} \quad \quad \quad \quad \quad \quad ZF \nvdash \text{Zorn's lemma}. $$
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So set theory with choice is sufficient for Zorn’s lemma while set theory without choice is not.
Determining necessity ...

Given an axiom system $\mathcal{B}$ and a mathematical theorem $\xi$. 

Suppose now $\mathcal{B} \nvdash \xi$. But an additional axiom $\mathcal{A}$ is sufficient for $\xi$, i.e. $\mathcal{B} + \mathcal{A} \vdash \xi$.

How do we determine if $\mathcal{A}$ was necessary to prove $\xi$ and not simply sufficient?

Example: $\text{ZF} \nvdash \text{Zorn's lemma} \quad \text{ZF} + \text{Axiom of choice} \vdash \text{Zorn's lemma}$
Determining necessity ...

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Determining necessity ...

Given an axiom system $\mathcal{B}$ and a mathematical theorem $\xi$. Suppose now

\[ \mathcal{B} \not\vdash \xi. \]

But an additional axiom $A$ is sufficient for $\xi$, i.e.

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Determining necessity ...

Given an axiom system $\mathcal{B}$ and a mathematical theorem $\xi$. Suppose now

$$\mathcal{B} \not\vdash \xi.$$  

But an additional axiom $A$ is sufficient for $\xi$, i.e.

$$\mathcal{B} + A \vdash \xi.$$  

How do we determine if $A$ was necessary to prove $\xi$ and not simply sufficient?

**Example:**

$ZF \not\vdash \text{Zorn's lemma} \quad ZF + \text{Axiom of choice} \vdash \text{Zorn’s lemma}$
Determining necessity ...

\[ \text{Note: } B + A \vdash \xi \text{ is equivalent to } B \vdash A \rightarrow \xi. \]
Note: $\mathcal{B} + A \vdash \xi$ is equivalent to $\mathcal{B} \vdash A \rightarrow \xi$.

Suppose we could show that the theorem was sufficient to prove the axiom

$$\mathcal{B} \vdash \xi \rightarrow A.$$
Note: $B + A \vdash \xi$ is equivalent to $B \vdash A \rightarrow \xi$.

Suppose we could show that the theorem was sufficient to prove the axiom

$$B \vdash \xi \rightarrow A.$$  

This shows that $A$ is necessary to prove $\xi$ as

$$B \vdash A \leftrightarrow \xi.$$  

Relative to $B$ the axiom $A$ and the theorem $\xi$ are provably equivalent.
To show $A$ is necessary for proving $\xi$ over $\mathcal{B}$, we prove

$$\mathcal{B} \vdash \xi \rightarrow A.$$
... by “reversing” mathematics

To show $A$ is necessary for proving $\xi$ over $B$, we prove

$$B \vdash \xi \rightarrow A.$$ 

We call this reversing $\xi$ to $A$ and such a proof is called a reversal.

Example:

$\text{ZF} \vdash \underbrace{\text{Axiom of choice} \rightarrow \text{Zorn’s lemma}}_{\text{forward for sufficiency}}$

$\text{ZF} \vdash \underbrace{\text{Zorn’s lemma} \rightarrow \text{Axiom of choice}}_{\text{reverse for necessity}}$
So, an axiom $A$ is sufficient to prove a theorem $\xi$ over a base theory $\mathcal{B}$ if

$$\mathcal{B} \vdash A \rightarrow \xi.$$
Reverse mathematics

So, an axiom $A$ is sufficient to prove a theorem $\xi$ over a base theory $B$ if

$$B \vdash A \rightarrow \xi.$$ 

And necessary if we can reverse $\xi$ to $A$:

$$B \vdash \xi \rightarrow A.$$
Reverse mathematics

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And necessary if we can reverse $\xi$ to $A$:

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Reverse mathematics is the program of determining which axioms are both sufficient and necessary for proving large fragments of mathematics via this strategy.

Example:

$$ZF \vdash \text{Axiom of choice} \leftrightarrow \text{Zorn’s lemma}$$
Will this work?

Possible issues:

- The axioms worth studying are hard to find or unnatural.
- The various branches of mathematics may require different and disconnected axioms.
- Each axiom may account for only a small portion of the desired fragment of mathematics.

Remarkably, a vast amount of mathematics can be shown equivalent to one of four axioms $A_1, A_2, A_3$ and $A_4$ over a single base theory $B$. The axioms themselves regard set comprehension and are naturally nested in an increasing order. The goal of this talk is to introduce the resulting 5 axiom systems.
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Two sorts of **variables**:

number variables $x, y, z \ldots$ and set variables $X, Y, Z, \ldots$. 
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Distinguished **constants**: 0 and 1.
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Formulas are built by combining the three atomic strings

\[ x = y \quad x < y \quad x \in X \]

using logical connectives and quantifiers.
Formal language

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   $\to, \leftrightarrow, \neg, \land, \lor$
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Logical connectives:
- $\rightarrow, \leftrightarrow, \neg, \land, \lor$

Distinguished quantifiers for each sort of variable:
- $\exists x, \forall y, \exists X, \forall Y$
Example:

$$\exists X \forall x \left( x \in X \iff \exists y \left( x = 3y \right) \right)$$

asserts the existence of the set of multiples of three.
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asserts the existence of the set of multiples of three.

$$\exists X \forall x (x \in X \leftrightarrow \neg (x \in X))$$

is Russel's paradox.
Second order arithmetic

A weak alternative to ZFC set theory.

Axiomatizes the natural numbers and their subsets. Is usually written $\mathbb{Z}^2$.

The following axioms:

1. $\forall x \neg (x + 1 = 0)$
2. $\forall x \forall y (x + 1 = y + 1 \rightarrow x = y)$
3. $\forall x x + 0 = x$
4. $\forall x \forall y (x + (y + 1) = (x + y) + 1)$
5. $\forall x x \cdot 0 = 0$
6. $\forall x \forall y (x \cdot (y + 1) = (x \cdot y) + x)$
7. $\forall x \neg (x < 0)$
8. $\forall x \forall y (x < y + 1 \leftrightarrow (x < y \lor x = y))$
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5. $\forall x x \cdot 0 = 0$
6. $\forall x \forall y x \cdot (y + 1) = (x \cdot y) + x$
7. $\forall x \neg (x < 0)$
8. $\forall x \forall y x < y + 1 \leftrightarrow (x < y \lor x = y)$
Second order arithmetic

A weak alternative to ZFC set theory.
Axiomatizes the natural numbers and their subsets.
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Is the collection of the following axioms:

- The **basic axioms of arithmetic**
  1. $\forall x \; \neg(x + 1 = 0)$
  2. $\forall x \forall y \; x + 1 = y + 1 \rightarrow x = y$
  3. $\forall x \; x + 0 = x$
  4. $\forall x \forall y \; x + (y + 1) = (x + y) + 1$
  5. $\forall x \; x \cdot 0 = 0$
  6. $\forall x \forall y \; x \cdot (y + 1) = (x \cdot y) + x$
  7. $\forall x \; \neg(x < 0)$
  8. $\forall x \forall y \; x < y + 1 \leftrightarrow (x < y \vee x = y)$
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Is the collection of the following axioms:

- The **basic axioms of arithmetic.**
- The **second order induction scheme**

$$\psi(0) \land \forall x(\psi(x) \rightarrow \psi(x + 1)) \rightarrow \forall x \ \psi(x)$$

where $\psi(x)$ is any formula in $Z_2$. 
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where $\psi(x)$ is any formula in $Z_2$.
- The **second order comprehension scheme**

$$\exists X \forall x (x \in X \leftrightarrow \varphi(x))$$

where $\varphi(x)$ is any formula of $Z_2$ in which $X$ does not occur freely.
The base system: $\text{RCA}_0$

The axiom system $\text{RCA}_0$ is the subsystem of $\mathbb{Z}_2$ consisting of the following axioms.

- The basic axioms of arithmetic

- The induction scheme

$$\psi(0) \land \forall x (\psi(x) \rightarrow \psi(x + 1)) \rightarrow \forall x \psi(x)$$

where $\psi(x)$ is any formula in $\text{RCA}_0$ with (at most) one number quantifier.

- The recursive comprehension scheme

$$\forall x (\phi(x) \leftrightarrow \psi(x)) \rightarrow \exists X \forall x (x \in X \leftrightarrow \phi(x))$$

where $\phi(x)$ is any formula with at most one existential quantifier and no other quantifiers and $\psi(x)$ is any formula with at most one universal quantifier and no others.
The base system: RCA₀

The axiom system RCA₀ is the subsystem of Z₂ consisting of the following axioms.

- The basic axioms of arithmetic
- The induction scheme

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where \( \psi(x) \) is any formula in that has (at most) one number quantifier.
The base system: \( \text{RCA}_0 \)

The axiom system \( \text{RCA}_0 \) is the subsystem of \( \mathbb{Z}_2 \) consisting of the following axioms.

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- The **induction scheme**

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- The **recursive comprehension scheme**

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Suppose we have an injective function $f : \mathbb{N} \rightarrow \mathbb{N}$. 

A non-example of recursive comprehension
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Suppose we have an injective function \( f : \mathbb{N} \to \mathbb{N} \).

To assert the existence of a set \( X \) which is the range of \( f \), we need one existential quantifier

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Thus, in RCA$_0$, we do not necessarily have the range of a given function.

RCA$_0$ is truly a weak axiom system.
An example of recursive comprehension

What can we obtain?

Suppose we have a strictly increasing function $g : \mathbb{N} \rightarrow \mathbb{N}$.

Define the range $Y$ with one existential quantifier:

$$\exists Y \forall y (y \in Y \leftrightarrow \exists x (f(x) = y)).$$

Define the compliment of the range with one existential quantifier:

$$\exists Y \forall y (y \not\in Y \leftrightarrow \exists x (f(x) > y) \land \forall z < x (f(z) \neq y)).$$

Membership in $Y$ can be defined via an existential or universal quantifier, so RCA$^0$ proves that $Y$ exists.
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Membership in $Y$ can be defined via an existential or universal quantifier, so $\text{RCA}_0$ proves that $Y$ exists.
While RCA$_0$ is a weak axiom system, we can do a modest amount of mathematics. For example,

**Theorem**

*The following are provable in RCA$_0$.*

1. *The system $\mathbb{Z}, +, -, \cdot, 0, 1, <$ is an ordered integral domain, Euclidean, etc.*
2. *The system $\mathbb{Q}, +, -, \cdot, 0, 1, <$ is an ordered field.*
3. *The system $\mathbb{R}, +, -, \cdot, 0, 1, <, =$ is an Archimedean ordered field.*
4. *The uncountability of $\mathbb{R}$.*
5. *The system $\mathbb{C}, +, -, \cdot, 0, 1, =$ is a field.*
For a first example, we code an ordered pair of natural numbers \((m, n)\) as follows:

\[
(m, n) \mapsto m + \frac{n}{2} + \frac{m^2}{2}.
\]

Note the last summand well-defines the ordering of \((m, n)\).

So \((2, 3) = 25 + 4 = 29\) and \((3, 2) = 25 + 9 = 34\).

To code finite sequences, we may simply nest this pairing map:

\[
(\ell, m, n) = ((\ell, (m, n))) = (\ell + (m + \frac{n}{2} + \frac{m^2}{2})) + \frac{n}{2} + \frac{m^2}{2}.
\]

\((n_0, n_1, \ldots, n_k) = ((n_0, (n_1, \ldots, n_k))) = (n_0 + (n_1 + \frac{n_2}{2} + \frac{n_1^2}{2} + \ldots + \frac{n_0^2}{2})).\)
For a first example, we code an ordered pair of natural numbers \((m, n)\) as follows

\[(m, n) \mapsto (m + n)^2 + m^2.\]

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To code finite sequences, we may simply nest this pairing map

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\[= (\ell + (m + n)^2 + m^2)^2 + \ell^2\]

\[(n_0, n_1, \ldots, n_k) = (n_0, (n_1, \ldots, n_k)).\]
To obtain the integers $\mathbb{Z}$, we use a (code for a) pair of natural numbers $(m, n)$ for the code of the integer $m - n$. 
Coding the number systems

To obtain the integers \( \mathbb{Z} \), we use a (code for a) pair of natural numbers \((m, n)\) for the code of the integer \(m - n\). Defining arithmetic on (codes of) integers then is straightforward.

\[
\begin{align*}
(m, n) +_{\mathbb{Z}} (p, q) &= (m + p, n + q) \\
(m, n) -_{\mathbb{Z}} (p, q) &= (m + q, n + p) \\
(m, n) \cdot_{\mathbb{Z}} (p, q) &= (m \cdot p + n \cdot q, m \cdot q + n \cdot p) \\
(m, n) <_{\mathbb{Z}} (p, q) &\iff m + q < n + p \\
(m, n) =_{\mathbb{Z}} (p, q) &\iff m + q = n + p
\end{align*}
\]
Coding the number systems

We then code the rationals \( \mathbb{Q} \) via pairs of (codes of) integers \((a, b)\)

\[
q = \frac{a}{b} = (a, b) = ((m_1, n_2), (m_2, n_2)) = ((m_1, n_1) + (m_2, n_2))^2 + (m_1, n_1)^2.
\]
Coding the number systems

We then code the rationals $\mathbb{Q}$ via pairs of (codes of) integers $(a, b)$

$q = \frac{a}{b} = (a, b)$

$= ((m_1, n_2), (m_2, n_2)) = ((m_1, n_1) + (m_2, n_2))^2 + (m_1, n_1)^2$.

$(a, b) +_{\mathbb{Q}} (c, d) = (a \cdot d + b \cdot c, b \cdot d)$

$(a, b) -_{\mathbb{Q}} (c, d) = (a \cdot d - b \cdot c, b \cdot d)$

$(a, b) \cdot_{\mathbb{Q}} (c, d) = (a \cdot c, b \cdot d)$

$(a, b) <_{\mathbb{Q}} (c, d) \iff a \cdot d < b \cdot c$

$(a, b) =_{\mathbb{Q}} (c, d) \iff a \cdot d = b \cdot c$
Coding the number systems

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Coding the reals $\mathbb{R}$ is a much more intricate affair. We code an infinite sequence of rationals $\langle q_0, q_1, \ldots \rangle$ by a function $f : \mathbb{N} \to \mathbb{Q}$ such that $f(k) = q_k$. Now $f$ maps $\mathbb{N}$ to codes for $\mathbb{Q}$ so $f$ really maps $\mathbb{N}$ to $\mathbb{N}$. As such $f \subset \mathbb{N} \times \mathbb{N} \subset \mathbb{N}$.
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And two real numbers $x = \langle q_k : k \in \mathbb{N} \rangle$ and $y = \langle q'_k : k \in \mathbb{N} \rangle$ equal, written $x = y$, if

$$\forall k \ |q_k - q'_k| \leq 2^{-k+1}.$$
We can continue in this way to code
- complete separable metric spaces;
- continuous functions;
- and countable algebraic structures (groups, rings, vector spaces, etc.).

using natural numbers and sets of natural numbers.

This implies that all of the mathematics we see today will really be happening within the natural numbers.
RCA₀ suffices to prove some less trivial facts from countable algebra, real and complex analysis . . .

**Theorem**

The following are provable in RCA₀.

7. **Basics of real linear algebra, including Gaussian Elimination.**
8. **Every countable abelian group has a divisible closure.**
9. **Every countable field has an algebraic closure.**
10. **The intermediate value theorem for continuous real-valued functions:** If \( f(x) \) is a continuous real-valued function on the unit interval \( 0 \leq x \leq 1 \) and \( f(0) < 0 < f(1) \), then there exists \( c \) such that \( 0 < c < 1 \) and \( f(c) = 0 \).
11. **Every holomorphic function is analytic.**
More mathematics in RCA₀

...the topology of complete separable metric spaces and mathematical logic.

**Theorem**

The following are provable in RCA₀.

12. *The Baire category theorem for complete separable metric spaces*: Let \(\langle U_k : k ∈ \mathbb{N}\rangle\) be a sequence of dense open sets in \(\hat{A}\). Then \(\bigcap_{k∈\mathbb{N}} U_k\) is dense in \(\hat{A}\).

13. *Urysohn’s lemma for complete separable metric spaces*: Given (codes for) disjoint closed sets \(C_0\) and \(C_1\) in \(X\), we can effectively find a (code for a) continuous function \(g : X → [0, 1]\) such that, for all \(x ∈ X\) and \(i ∈ \{0, 1\}\), \(x ∈ C_i\) if and only if \(g(x) = i\).

14. *The soundness theorem for predicate logic*: If \(X ⊂ SNT\) and there exists a countable model \(M\) such that \(M(σ) = 1\) for all \(σ ∈ X\), then \(X\) is consistent.
There is a lot of mathematics $\text{RCA}_0$ is not sufficient for.
There is a lot of mathematics $\text{RCA}_0$ is not sufficient for. This is a good thing.
Mathematics “out of” $\text{RCA}_0$

There is a lot of mathematics $\text{RCA}_0$ is not sufficient for. This is a good thing.

**Theorem**

The following are not provable in $\text{RCA}_0$

1. *The Heine/Borel covering lemma*: Every covering of the closed interval $[0, 1]$ by a sequence of open intervals has a finite subcovering.

2. *The Bolzano/Weierstraß theorem*: Every bounded sequence of real numbers contains a convergent subsequence.

3. *The perfect set theorem*: Every uncountable closed, or analytic, set has a perfect subset.

4. *The Cantor/Bendixson theorem*: Every closed subset of $\mathbb{R}$, or of any complete separable metric space, is the union of a countable set and a perfect set.
In RCA₀, we guaranteed the existence of sets who, along with their compliment, were definable with one number quantifier.
In RCA\(_0\), we guaranteed the existence of sets who, along with their compliment, were definable with one number quantifier. To strengthen this, let us allow any set who is definable by a formula any number of number quantifiers.
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**Definition**

The **arithmetical comprehension schema** are the axioms

$$\exists X \forall n \ (n \in X \leftrightarrow \varphi(n))$$

where $\varphi$ is any formula with no set quantifiers.
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**Definition**
The **arithmetical comprehension schema** are the axioms

$$\exists X \forall n \,(n \in X \leftrightarrow \varphi(n))$$

where $\varphi$ if any formula with no set quantifiers.

**Definition**
The axiom system ACA$_0$ consists of RCA$_0$ along with the axioms given in the arithmetical comprehension schema. Here ACA stands for “arithmetical comprehension axiom.”
An example of reverse mathematics

Our base theory $\mathcal{B}$ is $\text{RCA}_0$. 
Our base theory $\mathcal{B}$ is RCA$_0$.
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To do reverse mathematics, we need a known theorem $\xi$ and to show

$$\text{RCA}_0 \vdash \text{ACA}_0 \iff \xi.$$
Our base theory $\mathcal{B}$ is $\text{RCA}_0$.  
Our “additional axiom” $A$ is $\text{ACA}_0$.  
To do reverse mathematics, we need a known theorem $\xi$ and to show

$$\text{RCA}_0 \vdash \text{ACA}_0 \leftrightarrow \xi.$$ 

Here is an example. 

**Theorem**

*Over $\text{RCA}_0$, the following are equivalent*

1. $\text{ACA}_0$
2. *For all injective functions $f : \mathbb{N} \to \mathbb{N}$ there exists a set $X \subset \mathbb{N}$ such that $X$ is the range of $f$.***
Theorem
Over $\text{RCA}_0$, the following are equivalent

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2. For all injective functions $f : \mathbb{N} \to \mathbb{N}$ there exists a set $X \subseteq \mathbb{N}$ such that $X$ is the range of $f$.

Strategy:

Prove $\text{ACA}_0$ is sufficient: $\text{RCA}_0 \vdash \text{ACA}_0 \rightarrow \text{Item 2}$

Prove $\text{ACA}_0$ is necessary: $\text{RCA}_0 \vdash \text{Item 2} \rightarrow \text{ACA}_0$
Theorem

Over RCA\(_0\), the following are equivalent

1. ACA\(_0\)

2. For all injective functions \( f : \mathbb{N} \to \mathbb{N} \) there exists a set \( X \subseteq \mathbb{N} \) such that \( X \) is the range of \( f \).

Strategy:

Prove ACA\(_0\) is sufficient: RCA\(_0\) ⊢ ACA\(_0\) → Item 2

Prove ACA\(_0\) is necessary: RCA\(_0\) ⊬ Item 2 → ACA\(_0\)
Proof. (Forward direction or sufficiency).
An example of reverse mathematics

Proof. (Forward direction or sufficiency).

Let \( \varphi(n) \) be the formula \( (\exists m (f(m) = n)) \) and note that \( \varphi(n) \) is arithmetical.
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By arithmetical comprehension the set \( X \) defined by \( \varphi(n) \) exists. That is to say, we have

\[
\exists X \forall n \ (n \in X \iff \varphi(n)).
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An example of reverse mathematics

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$$\exists X \forall n (n \in X \iff \varphi(n)).$$

Clearly, $X$ is the range of $f$. 
An example of reverse mathematics

Proof. (Reverse direction).
To begin, let \( \varphi(n) \) be an arithmetical formula of the form \( \exists j \theta(j, n) \) where \( \theta \) has no quantifiers. (Extend by induction.)
An example of reverse mathematics

Proof. (Reverse direction).
To begin, let $\varphi(n)$ be an arithmetical formula of the form $\exists j \, \theta(j, n)$ where $\theta$ has no quantifiers. (Extend by induction.)
Within RCA$_0$, we can define the set

$$Y = \{(j, n) : \theta(j, n) \land \neg(\exists i < j) \theta(i, n)\},$$
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a function $\pi_Y : \mathbb{N} \to \mathbb{N}$ which enumerates the elements in strictly increasing order, and the second projection function

$$p_2 : (j, n) \mapsto n.$$
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p_2 : (j, n) \mapsto n.
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Then the function \( f : \mathbb{N} \to \mathbb{N} \) defined by \( f(m) = p_2(\pi_Y(m)) \).
An example of reverse mathematics

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Then the function \( f : \mathbb{N} \to \mathbb{N} \) defined by \( f(m) = p_2(\pi_Y(m)) \).
The definition of \( Y \) implies that \( f \) is injective.
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Then the function \( f : \mathbb{N} \rightarrow \mathbb{N} \) defined by \( f(m) = p_2(\pi_Y(m)) \).
The definition of \( Y \) implies that \( f \) is injective.
By item 2, there is a set such that

\[
\exists X \forall n (n \in X \leftrightarrow \exists m(f(m) = n) \leftrightarrow \exists j (j, n) \in Y \leftrightarrow \varphi(n))
\]
Another example of reverse mathematics
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**Theorem**

*Over RCA₀, the following are equivalent*

1. ACA₀

2. *Every countable abelian group has a subgroup consisting of the torsion elements.*
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We work in ACA₀ and let G be a countable abelian group.
Another example of reverse mathematics

Theorem
Over RCA$_0$, the following are equivalent

1. ACA$_0$

2. Every countable abelian group has a subgroup consisting of the torsion elements.

Proof. (Forward direction).
We work in ACA$_0$ and let $G$ be a countable abelian group.
Via arithmetical comprehension, we can form the set

$$T = \{ a \in G : \exists n (a^n = 1) \}.$$ 

It is then straight-forward to show $T$ is a subgroup of $G$. 
Another example of reverse mathematics

Proof. (The reversal).
Another example of reverse mathematics

*Proof. (The reversal).* Working over $\text{RCA}_0$, we assume Item 2 and seek to derive arithmetical comprehension.
Another example of reverse mathematics

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Working over RCA$_0$, we assume Item 2 and seek to derive arithmetical comprehension.
It will suffice to show that the range of an arbitrary injection $f : \mathbb{N} \to \mathbb{N}$ exists.
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Toward that end, let \( f : \mathbb{N} \to \mathbb{N} \) be an arbitrary injection.
We build a countable Abelian group \( G \) whose torsion subgroup determines the range of \( f \).
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Toward that end, let $f : \mathbb{N} \to \mathbb{N}$ be an arbitrary injection.
We build a countable Abelian group $G$ whose torsion subgroup determines the range of $f$.
Build $G$ using the generators $x_i, i \in \mathbb{N}$ and the relations $x_{f(m)}^{(2m+1)} = 1$ for all $m \in \mathbb{N}$. 
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\[
\forall m \left( m < |n_i| \rightarrow f(m) \neq i \right).
\]
Another example of reverse mathematics

**Proof.** (The reversal).

Working over RCA₀, we assume Item 2 and seek to derive arithmetical comprehension.

It will suffice to show that the range of an arbitrary injection \( f : \mathbb{N} \to \mathbb{N} \) exists.

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\forall m \left( m < |n_i| \rightarrow f(m) \neq i \right).
\]

As we only need a bounded quantifier, \( G \) exists by recursive comprehension.
Another example of reverse mathematics

Proof. (The reversal).
By Item 2, $G$ has a torsion subgroup $T$. 

$\forall i (i \in X \iff \exists m (f(m) = i))$.
So $X$ is the range of $f$.
By the previous theorem, Item 2 implies arithmetical comprehension and the reversal is complete.
Another example of reverse mathematics

Proof. (The reversal).
By Item 2, $G$ has a torsion subgroup $T$.

Using recursive comprehension once more, we can define the set

$$X = \{ i \in \mathbb{N} : x_i \in T \}.$$
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By the previous theorem, Item 2 implies arithmetical comprehension and the reversal is complete.
Theorem
Over \( \text{RCA}_0 \), the following are equivalent

1. \( \text{ACA}_0 \)
2. Every countable Abelian group has a unique divisible closure.
3. Every countable commutative ring has a maximal ideal.
4. Every countable vector space over a countable field has a basis.
5. Every countable field (of characteristic 0) has a transcendence basis.
Theorem

Over \( \text{RCA}_0 \), the following are equivalent

1. \( \text{ACA}_0 \)

6. Every Cauchy sequence of real numbers is convergent.

7. The Bolzano/Weierstraß theorem: Every bounded sequence of real numbers contains a convergent subsequence.

8. The Ascoli lemma: Every bounded equicontinuous sequence of real-valued continuous functions on a bounded interval has a uniformly convergent subsequence.
A few more results and ACA₀

**Theorem**

*Over RCA₀, the following are equivalent*

1. ACA₀

9. König’s lemma: Every infinite, finitely branching tree has an infinite path.

10. Ramsey’s theorem for colorings of $[\mathbb{N}]^k$, $k > 2$: For all finite colorings of increasing sequences of length $k$ of $\mathbb{N}$, there is an infinite subset $X \subset \mathbb{N}$ such that $[X]^k$ is homogeneous in color.
The $\Pi^1_1$ comprehension schema are the axioms

$$\exists X \forall n (n \in X \leftrightarrow \phi(n))$$

where $\phi$ is any formula of the form $\forall Y \theta$ where $\theta$ has no set quantifiers.

In the broader classification of formulas, we say $\phi$ is $\Pi^1_1$.

Definition The axiom systems $\Pi^1_1 - CA_0$ consists of $RCA_0$ along with the axioms given in the $\Pi^1_1$ comprehension schema.
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In the broader classification of formulas, we say $\varphi$ is $\Pi^1_1$.

Definition
The axiom systems $\Pi^1_1-CA_0$ consists of RCA$_0$ along with the axioms given in the $\Pi^1_1$ comprehension schema.
The reverse mathematics of $\Pi^1_1 - \text{CA}_0$

**Theorem**

Over $\text{RCA}_0$, the following are equivalent:

1. $\Pi^1_1 - \text{CA}_0$

2. Every countable Abelian group is the direct sum of a divisible group and a reduced group.

3. The Cantor/Bendixson theorem: Every closed subset of $\mathbb{R}$, or of any complete separable metric space, is the union of a countable set and a perfect set.

4. Silver’s theorem: For every Borel equivalence relation with uncountably many equivalence classes, there exists a nonempty perfect set of inequivalent elements.

5. Every tree has a largest perfect subtree.

6. Every $G_δ$ set in $[\mathbb{N}]^\mathbb{N}$ has the Ramsey property.
An equivalent characterization of the compactness of Cantor space $2^\mathbb{N}$ is known as weak König’s lemma.

**Definition.** Weak König’s lemma is the statement: Every infinite subtree of Cantor space has an infinite path.
Weak König’s Lemma and $\text{WKL}_0$

An equivalent characterization of the compactness of Cantor space $2^\mathbb{N}$ is known as *weak König’s lemma*.

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*Weak König’s lemma* is the statement:

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Every infinite subtree of Cantor space has an infinite path.
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Weak König’s Lemma and $\text{WKL}_0$

An equivalent characterization of the compactness of Cantor space $2^\mathbb{N}$ is known as *weak König’s lemma*.

**Definition**

Weak König’s lemma is the statement:

*Every infinite subtree of Cantor space has an infinite path.*

**Definition**

The axiom system $\text{WKL}_0$ consists of the axioms of $\text{RCA}_0$ along with weak König’s lemma.
The reverse mathematics of $\text{WKL}_0$

**Theorem**

Over $\text{RCA}_0$, the following are equivalent:

1. $\text{WKL}_0$

2. *The Heine/Borel covering lemma*: Every covering of the closed interval $[0, 1]$ by a sequence of open intervals has a finite subcovering.

3. *The maximum principle*: Every continuous real-valued function on $[0, 1]$ attains a supremum.

4. *Every continuous real-valued function on $[0, 1]$ is Riemann integrable.*
The reverse mathematics of \( \text{WKL}_0 \)

**Theorem**

Over \( \text{RCA}_0 \), the following are equivalent:

1. \( \text{WKL}_0 \)

5. Cauchy’s integral theorem: If \( f \) is holomorphic on an open set \( D \subset \mathbb{C} \), and \( \gamma \) is a triangular path in \( D \), then

\[
\int_{\gamma} f(z) \, dz = 0
\]


7. Brouwer’s fixed point theorem: Every uniformly continuous function \( \phi : [0, 1]^n \rightarrow [0, 1]^n \) has a fixed point.
The reverse mathematics of \( \text{WKL}_0 \)

**Theorem**

Over \( \text{RCA}_0 \), the following are equivalent:

1. \( \text{WKL}_0 \)

8. **The separable Hahn/Banach theorem:** If \( f \) is a bounded linear functional on a subspace of a separable Banach space, and if \( \|f\| \leq 1 \), then \( f \) has an extension \( \hat{f} \) to the whole space such that \( \|\hat{f}\| \leq 1 \).

9. **Every countable commutative ring has a prime ideal.**

10. **Every countable field (of characteristic 0) has a unique algebraic closure.**

11. **Gödel’s completeness theorem:** Every countable set of sentences in the predicate calculus has a countable model.
We have seen four axiom systems: $\text{RCA}_0$, $\text{ACA}_0$, $\Pi^1_1-\text{CA}_0$, $\text{WKL}_0$. 

**WKL\textsubscript{0} and ACA\textsubscript{0}**
We have seen four axiom systems: $\text{RCA}_0$, $\text{ACA}_0$, $\Pi^1_1-\text{CA}_0$, $\text{WKL}_0$.

How do these relate to one another?
We have seen four axiom systems: \( \text{RCA}_0, \text{ACA}_0, \Pi^1_1 \neg \text{CA}_0, \text{WKL}_0 \).

How do these relate to one another?

Clearly, by increasing set comprehension

\[
\text{RCA}_0 \not\vdash \quad \text{ACA}_0 \not\vdash \quad \Pi^1_1 \neg \text{CA}_0
\]

but

\[
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We have seen four axiom systems: $\text{RCA}_0$, $\text{ACA}_0$, $\Pi^1_1-\text{CA}_0$, $\text{WKL}_0$.

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\end{align*}
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So where does $\text{WKL}_0$ fit into this picture?
We have seen four axiom systems: RCA\(_0\), ACA\(_0\), \(\Pi_1^1 - CA_0\), WKL\(_0\).

How do these relate to one another?

Clearly, by increasing set comprehension

\[
\begin{align*}
\text{RCA}_0 & \not\vdash \text{WKL}_0 \not\vdash \text{ACA}_0 \not\vdash \Pi_1^1 - CA_0 \\
\text{but} & \\
\text{RCA}_0 & \vdash \text{WKL}_0 \vdash \text{ACA}_0 \vdash \Pi_1^1 - CA_0
\end{align*}
\]

So where does WKL\(_0\) fit into this picture?
The acronym ATR abbreviates “arithmetical transfinite recursion.”
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**Arithmetical transfinite recursion** is the axiom scheme which permits the iteration of arithmetical comprehension along any countable well-order.

This allows for transfinite constructions, where at each stage we define a new set from the last arithmetically.
The acronym ATR abbreviates “arithmetical transfinite recursion.”

**Arithmetical transfinite recursion** is the axiom scheme which permits the iteration of arithmetical comprehension along any countable well-order.

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The formal definition of these axioms is quite technical so we suggest the curious reader to see [4] for the actual definition.
The acronym ATR abbreviates “arithmetical transfinite recursion.”

**Arithmetical transfinite recursion** is the axiom scheme which permits the iteration of arithmetical comprehension along any countable well-order.

This allows for transfinite constructions, where at each stage we define a new set from the last arithmetically.

The formal definition of these axioms is quite technical so we suggest the curious reader to see [4] for the actual definition.

**Definition**
The axiom system ATR\textsubscript{0} consists of the axioms of RCA\textsubscript{0} along with axioms for arithmetical transfinite recursion.
The reverse mathematics of $\text{ATR}_0$

**Theorem**

Over $\text{RCA}_0$, the following are equivalent:

1. $\text{ATR}_0$
2. Any two countable well orderings are comparable.
3. The perfect set theorem: Every uncountable closed, or analytic, set has a perfect subset.
4. Lusin’s separation theorem: Any two disjoint analytic sets can be separated by a Borel set.
5. The domain of any single-valued Borel relation is Borel.
6. Ulm’s theorem: Any two countable reduced Abelian $p$-groups which have the same Ulm invariants are isomorphic.
7. The open Ramsey theorem: Every open subset of $[\mathbb{N}]^\mathbb{N}$ has the Ramsey property.
The big five

We now have seen five subsystems of second order arithmetic which serve as appropriate axiomatizations of substantial portions of mathematics.

These systems are known as *the big five*:

\[
\text{RCA}_0 \quad \text{WKL}_0 \quad \text{ACA}_0 \quad \text{ATR}_0 \quad \Pi^1_1-\text{CA}_0
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\]

Though we have shown many theorems equivalent to one of these, many more theorems have been shown to fit nicely into this hierarchy in the 40+ years since their introduction.

Because of this, we consider reverse mathematics to be an important partial answer to the motivating question

*what are the appropriate axioms of reverse mathematics?*
The big five

\[ \Pi^1_1-CA_0 \iff \text{The Cantor/Bendixson theorem} \]
\[ \downarrow \]
\[ ATR_0 \iff \text{The perfect set theorem} \]
\[ \downarrow \]
\[ ACA_0 \iff \text{The Bolzano/Weierstraß theorem} \]
\[ \downarrow \]
\[ WKL_0 \iff \text{The Heine/Borel covering lemma} \]
\[ \downarrow \]
\[ RCA_0 \iff \text{The intermediate value theorem} \]


Thank you!