Marriage Problems and Reverse Mathematics

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Agenda

- I: Marriage Problems
- I: Previous Results
- I: New Results
- II: Reverse Mathematics
I: Marriage Problems
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Some Notation

A marriage problem $M$ consists of three sets $B$, $G$ and $R$.

- $B$ is the set of boys,
- $G$ is the set of girls, and
- $R$ is the relation between the boys and girls.

$R \subseteq B \times G$ where $(b, g) \in R$ means “$b$ knows $g$”.
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A solution to $M = (B, G, R)$ is an injection

$$f : B \rightarrow G$$

such that $(b, f(b)) \in R$ for every $b \in B$. 
Some More Notation

$G(b)$ is convenient shorthand for the set of girls $b$ knows, i.e.

$$G(b) = \{ g \in G \mid (b, g) \in R \}.$$

$G(b)$ is not a function.
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$G_M(b)$ denotes the set of girls $b$ knows relative to the relation in $M$. 

$M$ is a:

- finite marriage problem if $|B|$ is finite.
- infinite marriage problem if $|B|$ is not finite.
- bounded marriage problem if there is a function $h: B \rightarrow G$ so that for each $b \in B$, $G(b) \subseteq \{0, 1, \ldots, h(b)\}$. 

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Examples of Marriage Theorems

Theorem
If \( M = (B, G, R) \) is a finite marriage problem such that
\[ |G(B_0)| \geq |B_0| \]
for every \( B_0 \subset B \), then \( M \) has a solution.
Due to Philip Hall.

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A New Result: Unique Solutions

What are the necessary and sufficient conditions for a marriage problem to have a *unique* solution?

Theorem

If $M = (B, G, R)$ is a finite marriage problem with $n$ boys and a unique solution $f$, then there is an enumeration of the boys $\langle b_i \rangle_{i \leq n}$ such that for every $1 \leq m \leq n$, $|G(\{b_1, b_2, \ldots, b_m\})| = m$. 
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Sketch of the proof

Lemma
If $M = (B, G, R)$ is a finite marriage problem with a unique solution $f$, then some boy knows exactly one girl.
Sketch of the proof

**Proof:** Suppose we have $M = (B, G, R)$ as stated above with some initial enumeration of $B$. Apply the lemma and let $b_1$ be the first boy such that $|G(b_1)| = 1$. After the $n$th iteration we have $(b_1, b_2, \ldots, b_n)$ where for every $1 \leq m \leq n$, $|G(\{b_1, b_2, \ldots, b_m\})| = m$. ■
Sketch of the proof

**Proof:** Suppose we have $M = (B, G, R)$ as stated above with some initial enumeration of $B$. Apply the lemma and let $b_1$ be the first boy such that $|G(b_1)| = 1$.

Define $M_2 = (B - \{b_1\}, G - G(b_1), R_2)$. Because $M$ has a unique solution, $M_2$ has a unique solution, namely the restriction of $f$ to the sets of $M_2$. Apply the lemma once more and let $b_2$ be the first boy in $B - \{b_1\}$ such that $|G_{M_2}(b_2)| = 1$. 

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Continuing this process inductively yields the \( j^{th} \) boy in our desired enumeration from \( M_j = (B - \{b_1, b_2, \ldots, b_{j-1}\}, G - G(b_1, b_2, \ldots, b_{j-1}), R_j) \).
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After the $n^{th}$ iteration we have $(b_1, b_2, \ldots, b_n)$ where for every $1 \leq m \leq n$, $|G(\{b_1, b_2, \ldots, b_m\})| = m$. ■
The statement regarding finite marriage problems with unique solutions can be generalized to the *infinite* case. Paralleling the previous work we have:

**Theorem**

*If* $M = (B, G, R)$ *is an infinite marriage problem with a unique solution* $f$, *then there is an enumeration of the boys* $\langle b_i \rangle_{i \geq 1}$ *such that for every* $n \geq 1$, $|G(\{b_1, b_2, \ldots, b_n\})| = n$. 
II: Reverse Mathematics
Reverse Mathematics

*Reverse mathematics* is the subfield of mathematical logic dedicated to classifying the *logical strength* of mathematical theorems.

This is done by proving theorems equivalent to a hierarchy of axioms over a weak base axiom system.

\[
\text{RCA}_0 \quad \text{WKL}_0 \quad \text{ACA}_0 \quad \text{ATR}_0 \quad \Pi^1_1 – \text{CA}_0
\]

\(\text{RCA}_0\) proves the *intermediate value theorem* and the *uncountability of } \mathbb{R}.

\(\text{RCA}_0\) does **not** prove the *existence of Riemann integrals.*
Equivalences

Theorem
The following are provable in RCA$_0$.

(i) WKL$_0$ $\iff$ For every continuous function $f(x)$ on a closed and bounded interval $a \leq x \leq b$, the Riemann integral $\int_a^b f(x)dx$ exists and is finite. (Simpson)

(ii) ACA$_0$ $\iff$ For all one-to-one functions $f : \mathbb{N} \to \mathbb{N}$ there exists a set $X \subseteq \mathbb{N}$ such that Ran($f$) $= X$. (Simpson)

(iii) ATR$_0$ $\iff$ Any two well orderings are comparable. (Friedman)

(iv) $\Pi^1_1 -$ CA$_0$ $\iff$ The Cantor/Bendixson theorem for $\mathbb{N}^\mathbb{N}$: Every closed set in $\mathbb{N}^\mathbb{N}$ is the union of a perfect closed set and a countable set. (Simpson)
Marriage Theorems and Reverse Mathematics

Jeff Hirst proved the following equivalence results:

Theorem

(RCA$_0$) If $M = (B, G, R)$ is a finite marriage problem such that $|G(B_0)| \geq |B_0|$ for every $B_0 \subset B$, then $M$ has a solution.
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(RCA$_0$) The following are equivalent:
1. ACA$_0$
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**Theorem**

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1. WKL$_0$
2. If $M = (B, G, R)$ is a bounded marriage problem such that $|G(B_0)| \geq |B_0|$ for every $B_0 \subseteq B$, then $M$ has a solution.*
Our new results echoed the previous work:

**Theorem**

(RCA\(_0\)) If \( M = (B, G, R) \) is a finite marriage problem with \( n \) boys a unique solution \( f \), then there is an enumeration of the boys \( \langle b_i \rangle_{i \leq n} \) such that for every \( 1 \leq m \leq n \), \( |G(\{b_1, b_2, \ldots, b_m\})| = m \).
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2. If $M = (B, G, R)$ is a bounded marriage problem with a unique solution $f$, then there is an enumeration of the boys $\langle b_i \rangle_{i \geq 1}$ such that for every $n \geq 1$, $|G(\{b_1, b_2, \ldots, b_n\})| = n$. 
References


