AGENDA

I: Marriage Problems

II: Previous Results

I: New Results

II: Reverse Mathematics
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Some Notation

A marriage problem $M$ consists of three sets $B$, $G$ and $R$.

$B$ is the set of boys,

$G$ is the set of girls, and

$R$ is the relation between the boys and girls.

$R \subseteq B \times G$ where $(b, g) \in R$ means “$b$ knows $g$”.

$G(b)$ is convenient shorthand for the set of girls $b$ knows, i.e. $G(b) = \{g \in G \mid (b, g) \in R\}$.

$G(M(b))$ denotes the set of girls $b$ knows relative to the relation in $M$. 

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Some More Notation

A solution to $M = (B, G, R)$ is an injection

$$f : B \rightarrow G$$

such that $(b, f(b)) \in R$ for every $b \in B$. 

$M$ is a:
- finite marriage problem if $|B|$ is finite.
- infinite marriage problem if $|B|$ is not finite.
- bounded marriage problem if there is a function $h : B \rightarrow G$ so that for each $b \in B$, $G(b) \subseteq \{0, 1, \ldots, h(b)\}$.
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Examples of Marriage Theorems

Theorem

If $M = (B, G, R)$ is a finite marriage problem such that $|G(B_0)| \geq |B_0|$ for every $B_0 \subset B$, then $M$ has a solution. Due to Philip Hall.

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A New Result: Unique Solutions

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In the finite case, we found the following necessary and sufficient condition.

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(RCA₀) If $M = (B, G, R)$ is a finite marriage problem with $n$ boys and a unique solution $f$, then there is an enumeration of the boys $\langle b_i \rangle_{i \leq n}$ such that for every $1 \leq m \leq n$, $|G(\{b_1, b_2, \ldots, b_m\})| = m$. 
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(RCA \(_0\)) *If* \( M = (B, G, R) \) *is a finite marriage problem with* \( n \) *boys and a unique solution* \( f \), *then there is an enumeration of the boys* \( \langle b_i \rangle_{i \leq n} \) *such that for every* \( 1 \leq m \leq n \), \( |G(\{b_1, b_2, \ldots, b_m\})| = m \).
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![Graph](image)
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Sketch of the proof

Lemma

(RCA\(_0\)) If \(M = (B, G, R)\) is a finite marriage problem with a unique solution \(f\), then some boy knows exactly one girl.
Sketch of the proof

**Proof:** Suppose we have $M = (B, G, R)$ as stated above with some initial enumeration of $B$. Apply the lemma and let $b_1$ be the first boy such that $|G(b_1)| = 1$.
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Define $M_2 = (B - \{b_1\}, G - G(b_1), R_2)$. Because $M$ has a unique solution, $M_2$ has a unique solution, namely the restriction of $f$ to the sets of $M_2$. Apply the lemma once more and let $b_2$ be the first boy in $B - \{b_1\}$ such that $|G_{M_2}(b_2)| = 1$. 
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Continuing this process inductively yields the \( j^{th} \) boy in our desired enumeration from \( M_j = (B - \{b_1, b_2, \ldots, b_{j-1}\}, G - G(b_1, b_2, \ldots, b_{j-1}), R_j) \).
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Continuing this process inductively yields the $j^{\text{th}}$ boy in our desired enumeration from $M_j = (B - \{b_1, b_2, \ldots, b_{j-1}\}, G - G(b_1, b_2, \ldots, b_{j-1}), R_j)$.

After the $n^{\text{th}}$ iteration we have $(b_1, b_2, \ldots, b_n)$ where for every $1 \leq m \leq n, |G(\{b_1, b_2, \ldots, b_m\})| = m$. ■
Generalizing this result

The statement regarding finite marriage problems with unique solutions can be generalized to the infinite case. Paralleling the previous work we have:

**Theorem**

*If* $M = (B, G, R)$ *is an infinite marriage problem with a unique solution* $f$, *then there is an enumeration of the boys* $\langle b_i \rangle_{i \geq 1}$ *such that for every* $n \geq 1$, $|G(\{b_1, b_2, \ldots, b_n\})| = n$. 


II: Reverse Mathematics
Reverse Mathematics

*Reverse mathematics* is the subfield of mathematical logic dedicated to classifying the logical strength of mathematical theorems.

This is done by proving theorems equivalent to a hierarchy of axioms over a weak base axiom system.

\[
\text{RCA}_0 \quad \text{WKL}_0 \quad \text{ACA}_0 \quad \text{ATR}_0 \quad \Pi^1_1 - \text{CA}_0
\]

\(\text{RCA}_0\) proves the *intermediate value theorem* and the *uncountability of \(\mathbb{R}\).*

\(\text{RCA}_0\) does *not* prove the *existence of Riemann integrals.*
Equivalences

Theorem
The following are provable in RCA₀.

(i) WKL₀ ⇔ For every continuous function f(x) on a closed and bounded interval a ≤ x ≤ b, the Riemann integral
\[ \int_a^b f(x)dx \] exists and is finite. (Simpson)

(ii) ACA₀ ⇔ For all one-to-one functions f : \( \mathbb{N} \to \mathbb{N} \) there exists a set \( X \subseteq \mathbb{N} \) such that Ran(f) = X. (Simpson)

(iii) ATR₀ ⇔ Any two well orderings are comparable. (Friedman)

(iv) \( \Pi^1_1 - CA₀ \) ⇔ The Cantor/Bendixson theorem for \( \mathbb{N}^\mathbb{N} \):
Every closed set in \( \mathbb{N}^\mathbb{N} \) is the union of a perfect closed set and a countable set. (Simpson)
Jeff Hirst proved the following equivalence results:

**Theorem**

(RCA$_0$) If $M = (B, G, R)$ is a finite marriage problem such that $|G(B_0)| \geq |B_0|$ for every $B_0 \subseteq B$, then $M$ has a solution.
Marriage Theorems and Reverse Mathematics

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**Theorem**

$$(\text{RCA}_0) \text{ The following are equivalent:}$$

1. ACA$_0$

2. If $M = (B, G, R)$ is an infinite marriage problem such that $|G(B_0)| \geq |B_0|$ for every $B_0 \subset B$, then $M$ has a solution.
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**Theorem**
(RCA$_0$) *The following are equivalent:*
1. WKL$_0$
2. *If* $M = (B, G, R)$ *is a bounded marriage problem such that* $|G(B_0)| \geq |B_0|$ *for every* $B_0 \subset B$, *then* $M$ *has a solution.*
Our new results echoed the previous work:

**Theorem**

(RCA$_0$) If $M = (B, G, R)$ is a finite marriage problem with $n$ boys a unique solution $f$, then there is an enumeration of the boys $\langle b_i\rangle_{i \leq n}$ such that for every $1 \leq m \leq n$, $|G(\{b_1, b_2, \ldots, b_m\})| = m$. 
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Theorem
(RCA$_0$) The following are equivalent:

1. WKL$_0$

2. If $M = (B, G, R)$ is a bounded marriage problem with a unique solution $f$, then there is an enumeration of the boys $\langle b_i \rangle_{i \geq 1}$ such that for every $n \geq 1$, $|G(\{b_1, b_2, \ldots, b_n\})| = n$. 
Future Work

- Marriage problems with any fixed finite number of solutions.

- “Entangled societies”
References


