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COVERING LEMMAS AND AN APPLICATION TO NODAL GEOMETRY ON RIEMANNIAN MANIFOLDS

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ABSTRACT. The main part of this note is to show a general covering lemma in R^n, $n \geq 2$, with the aim to obtain the estimate for BMO norm and the volume of a nodal set of eigenfunctions on Riemannian manifolds.

1. INTRODUCTION

This article is a continuation of our previous work [L]. In [L] we proved a covering lemma in R^2 and applied it to the BMO norm estimates for eigenfunctions on Riemannian surfaces. The principal part of this article is to prove a general covering lemma in R^n for $n \geq 2$. As applications, we can obtain the BMO estimate for eigenfunctions and the volume estimate for the nodal set.

Let M^n be a smooth, compact, and connected Riemannian manifold with no boundary. Let Δ denote the Laplacian on M^n. Let $-\Delta u = \lambda u$, u an eigenfunction with eigenvalue λ, $\lambda > 1$.

Our main results can be stated as follows

Theorem A (BMO estimate for $\log|u|$). *For u, λ as above and $n \geq 3$,*

$$\|\log|u|\|_{\mathrm{BMO}} \leq C\lambda^{n-1/8}(\log\lambda)^2,$$

where C is independent of λ and u and is only dependent on n and M^n.

Theorem B (geometry of nodal domains). *Let $n \geq 3$ and u, λ as above. Let $B \subset M^n$ be any ball, and let $\Omega \subset B$ be any of the connected components of $\{x \in B : u(x) \neq 0\}$. If Ω intersects the middle half of B, then*

$$|\Omega| \geq C\lambda^{-2n^2-n/4}(\log\lambda)^{-4n}|B|,$$

where C is independent of λ and u.

Donnelly and Fefferman [DF1, DF2] and Chanillo and Muckenhoupt [CM] proved Theorem A with $\lambda^{n-1/8}(\log\lambda)^2$ replaced by $\lambda^{n(n+2)/4}$ and $\lambda^n\log\lambda$, respectively, and Theorem B with $\lambda^{-2n^2-n/4}(\log\lambda)^{-4n}$ replaced by $\lambda^{-(n+n^2(n+2))/2}$ and $\lambda^{-2n^2-n/2}(\log\lambda)^{-2n}$, respectively.

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In the case $n = 2$, the following has been proved in [L]:

$$\|\log u\|_{\mathrm{BMO}} \le C\lambda^{15/8+\varepsilon},$$

where $\varepsilon > 0$ and $C = C(\varepsilon)$ is independent of λ and u.

The proof of Theorems A and B is based on the following covering lemma, which is of its own right and is really the main result of this paper.

Lemma C (covering lemma in R^n). *Let $\delta > 0$ be small enough. Let $\{B_\alpha\}_{\alpha\in I}$ be any finite collection of balls in R^n $(n \ge 2)$. Then one can select a subcollection $B_1, B_2, \dots, B_N$ such that*

(1) $\bigcup_\alpha B_\alpha \subset \bigcup_{i=1}^N (1+\delta)B_i$,

(2) $\sum_{i=1}^N \chi_{B_i}(x) \le C(\log\frac{1}{\delta})\delta^{-n+1/4}$ *for all* $x \in R^n$,

where C depends only on n but is independent of δ and the given balls.

From the proof of Lemma C (see §3), we can see that we also have covering lemmas for any finite collection of balls with some restrictions on the lower and upper bounds for the radii of the given balls. We state here these results for the interested reader.

Lemma D (for balls with almost equal radii). *Let δ be small enough. Let $\{B_\alpha\}_{\alpha\in I}$ be any finite collection of balls in R^n $(n \ge 2)$ with $r \le \rho(B_\alpha) \le r + 2^k\delta$ for some $2^k \le r \le 2^{k+1}$, where k is an integer. Then one can select a subcollection of balls $B_1, \dots, B_N$ such that*

$$\bigcup_\alpha B_\alpha \subset \bigcup_{i=1}^N (1+\delta)B_i,$$

$$\sum_{i=1}^N \chi_{B_i}(x) \le C\delta^{-(n-3/4)} \quad \textit{for all } x \in R^n,$$

where C only depends on the dimension n but is independent of k, δ, and the given balls.

The proof of Lemma D will be based on Lemma 3.1 in §3. The method of proof is similar to the one of Lemma 3.7.

Lemma E (for balls with radii of lower and upper bounds). *Let δ be small enough and k be any integer. Let $\{B_\alpha\}_{\alpha\in I}$ be any finite collection of balls in R^n, $n \ge 2$, with $2^k \le \rho(B_\alpha) \le 2^{k+1}$. Then one can select a subcollection $B_1, \dots, B_N$ such that*

$$\bigcup_\alpha B_\alpha \subset \bigcup_{i=1}^N (1+\delta)B_i,$$

$$\sum_{i=1}^N \chi_{B_i}(x) \le C\delta^{-(n-1/4)} \quad \textit{for all } x \in R^n,$$

where C depends only on the dimension n but is independent of k, δ, and the given balls.

Lemma E is just a restatement of Lemma 3.7 in §3 by replacing $(1 + C_n)\delta$ by $(1+\delta)$.

This covering Lemma C is an improvement of the one in [CM], which was δ^{-n} on the right-hand side of (2). This type of covering lemma may be useful since the Vitali covering lemma is not good enough in many cases and the Besicovitch covering lemma does not apply sometimes (see [SW]). The proof of the corresponding covering lemma in [CM] is very elegant, but the covering lemma in [CM] does not have the implications of Lemmas D and E. Obviously, from the proof of Lemma C, we can see that this covering lemma is not the best possible.

Once we have Lemma C, we can just modify the proof given in [CM], and thus the proof of Theorems A and B will be omitted here. The interested reader should refer to [CM]. Instead, we will concentrate on the proof of the covering lemmas, which will be given in the next two sections.

Notation. Throughout this paper, we will denote by c or C the generic constants not exactly equal at each occurrence and which depend on the dimension only. We will also use $\rho(B)$ to denote the radius of the ball B. If B is an $(n+1)$-dimensional ball in R^{n+1}, then we denote by B^* the projection of B onto the n-dimensional hyperplane $\{x_{n+1}=0\}$. Obviously, B^* is an n-dimensional ball with $\rho(B)=\rho(B^*)$.

2. A basic covering lemma in R^n

The main goal of this section is to prove a basic covering lemma in R^n for balls whose radii are close to one another and centered in an n-dimensional cube with sidelength $\sqrt{\delta}$. As mentioned in §1, we will denote by B^* the projected ball of the $(n+1)$-dim ball $B \subset R^{n+1}$ to the hyperplane $\{x_{n+1}=0\}$.

Since the proof of Lemma C adapts the method of induction on the dimension n on each cube with sidelength $\sqrt{\delta}$ and is based on a basic covering lemma in R^2 proved in [L, Lemma 4.1], we recall this essential lemma first.

Lemma 2.1. *Let $\delta > 0$ be given small enough. Given any cube $\mathcal{Q}$ in R^2 with sidelength $\sqrt{\delta}$ and given any finite collection of balls $\{B_\alpha\}_{\alpha\in I}$ in R^2 with $r \le \rho(B_\alpha) \le r+\delta$, for some $1 \le r \le 2$, and centered in this cube $\mathcal{Q}$, one can select a subcollection of balls $B_1, \dots, B_N$ such that*

$$\bigcup_\alpha B_\alpha \subset \bigcup_{i=1}^{N}(1+\delta)B_i, \tag{2.2}$$

$$N \le c\delta^{-1/4}, \tag{2.3}$$

where c is an absolute constant independent of δ and the given balls.

We also need the following.

Lemma 2.4. *Let $\delta > 0$ be given small enough. Let B_1 and B_2 be two $(m+1)$-dimensional balls in R^{m+1} with radius $r \le \rho(B_i) \le r+\delta$ for some $1 \le r \le 2$ $(i=1, 2)$. Assume that B_1 is centered at the origin in R^{m+1} and B_2 is centered at $O_2 = (s_1, s_2, \dots, s_m, s_{m+1})$ such that $\sqrt{\sum_{i=1}^{m+1} s_i^2} \le \sqrt{(m+1)\delta}$ and $|s_{m+1}| \le \delta$. If the point, lying on the boundary of B_1, $A = (0, \dots, 0, \rho(B_1), 0) \in (1+C_m\delta)B_2^*$, then all those points inside the ball B_1 of the form $(0, \dots, 0, t_m, t_{m+1})$ with $t_m \ge 0$ (we call the set of these points P) are in* *the ball* $(1+C_{m+1}\delta)B_2$, *where* C_m *and* C_{m+1} *are two constants only depending on* m.

Proof. Since $A \in (1+C_m\delta)B_2^*$, the distance between A and the center of B_2^*, i.e., $(s_1, s_2, \dots, s_m, 0)$, is no more than $(1+C_m\delta)\rho(B_2)$. This implies that

$$s_1^2 + \cdots + s_{m-1}^2 + (s_m - \rho(B_1))^2 \leq [(1+C_m\delta)\rho(B_2)]^2. \tag{2.5}$$

By the hypothesis, we have $s_1^2+\cdots+s_{m+1}^2 \leq (m+1)\delta$, $|s_{m+1}| \leq \delta$. We also have that $t_m^2+t_{m+1}^2 \leq \rho(B_1)^2$ and $t_m \geq 0$ for the points $(0, \dots, 0, t_m, t_{m+1})$ in P. We now claim that $s_m \geq -C\delta$ for some $C = C(m)$. In fact, from (2.5), noting $|s_m - \rho(B_1)| \leq (1+C_m\delta)\rho(B_2)$, it follows that $s_m \geq -(1+C_m)\delta\rho(B_2)+\rho(B_1) \geq -C\delta$ by the hypothesis that $r \leq \rho(B_i) \leq r+\delta$ for some $1 \leq r \leq 2$ $(i = 1, 2)$.

Thus the distance between O_2 and the points in P are not more than

$$\begin{aligned} &s_1^2 + s_2^2 + \cdots + s_{m-1}^2 + (s_m - t_m)^2 + (s_{m+1} - t_{m+1})^2 \\ &= \sum_{i=1}^{m-1} s_i^2 + s_m^2 + s_{m+1}^2 + t_m^2 + t_{m+1}^2 - 2s_m t_m - 2s_{m+1}t_{m+1} \\ &\leq (m+1)\delta + \rho(B_1)^2 + C\delta \leq [(1+C_{m+1}\delta)\rho(B_2)]^2. \end{aligned}$$

Thus $P \subset (1+C_{m+1}\delta)B_2$. Q.E.D.

Remark. The above set P is actually the intersection points between the ball B_1 and the hyperplane $\{x_1 = 0, \dots, x_{m-1} = 0\}$.

Lemma 2.6. *Let* $\delta > 0$ *be given small enough. Let* $\mathscr{Q} = \{(x_1, \dots, x_n, x_{n+1}) : 0 \leq x_i \leq \sqrt{\delta}$, *for* $1 \leq i \leq n$, $0 \leq x_{n+1} \leq \delta\}$ *be the parallelopiped in* R^{n+1}. *Assume as given any finite collection of* $(n+1)$*-dimensional balls* $\{B_\alpha\}_{\alpha\in I}$ *in* R^{n+1} *with* $r \leq \rho(B_\alpha) \leq r+\delta$, *for some* $1 \leq r \leq 2$, *centered in this parallelopiped* $\mathscr{Q}$. *Assume that there exists a subcollection of balls* $B_1, \dots, B_N$ *such that the projected balls* $\{B_i^*\}_{i=1}^N$ *onto the hyperplane* $\{x_{n+1} = 0\}$ *satisfy*

$$\bigcup_\alpha B_\alpha^* \subset \bigcup_{i=1}^N (1+C_n\delta)B_i^*. \tag{2.7}$$

Then we have

$$\bigcup_\alpha B_\alpha \subset \bigcup_{i=1}^N (1+C_{n+1}\delta)B_i,$$

where C_{n+1} *only depends on the dimension* n *and is independent of* δ *and the given balls.*

Proof. Fix B_α and let $x \in B_\alpha$. Let $O_\alpha = (t_1, \dots, t_{n+1})$ be the center of B_α and O_α^* and x^* be the projections of O_α and x onto the hyperplane $x_{n+1} = 0$ respectively. Let l_α be the ray originating from O_α and passing through x^*, and let A_α be the intersection point between l_α and the boundary ∂B_α^* of B_α^*. On account of (2.7), we have $A_\alpha \in (1+C_n\delta)B_i^*$ for some i, and we assume the coordinate of the center of B_i is $O_i = (s_1, \dots, s_{n+1})$. Then we claim $x \in (1+C_{n+1}\delta)B_i$ for the same i. To show the claim, we adapt the new Cartesian coordinates $(x_1', \dots, x_n', x_{n+1}')$ such that

(1) $(x_1', \dots, x_n', x_{n+1}')$ is derived from $(x_1, \dots, x_n, x_{n+1})$ by an orthogonal transformation and a translation.

(2) The origin of the new coordinate is at O_α and the coordinate of A_α is of the form $(0, \dots, 0, \rho(B_\alpha), 0)$. Then by the rigid invariance of the distances, all the $(n+1)$-dimensional balls B_α are still balls with the same radii under the new coordinates $(x'_1, \dots, x'_n, x'_{n+1})$.

(3) Since the distance between O_α and O_i is no more than $\sqrt{(n+1)\delta}$ and $|s_{n+1} - t_{n+1}| \le \delta$, the new coordinate of O_i, $O_i = (s'_1, \dots, s'_{n+1})$, satisfies $\sqrt{\sum_{i=1}^{n+1} (s'_i)^2} \le \sqrt{(n+1)\delta}$ and $|s'_{n+1}| \le \delta$.

Thus, by Lemma 2.4, there exists a constant C_{n+1} independent of δ and the given balls, only dependent on the dimension n, such that $x \in (1 + C_{n+1}\delta)B_i$. Since $x \in B_\alpha$ is arbitrary and B_α is an arbitrary ball in $\{B_\alpha\}$, we are done. Q.E.D.

By using Lemma 2.6, we can show the following basic covering lemma in R^n, $n \ge 2$.

Lemma 2.8. *Let $\delta > 0$ be given small enough. Given any cube $\mathscr{Q}_n$ in R^n with sidelength $\sqrt{\delta}$ and any finite collection of balls $\{B_\alpha\}_{\alpha \in I}$ in R^n with $r \le \rho(B_\alpha) \le r + \delta$, for some $1 \le r \le 2$, and centered in this cube $\mathscr{Q}_n$, one can select a subcollection of balls $B_1, \dots, B_N$ such that*

$$\bigcup_\alpha B_\alpha \subset \bigcup_{i=1}^{N} (1 + C_n\delta)B_i, \tag{2.9}$$

$$N \le c\delta^{-(2n-3)/4}, \tag{2.10}$$

where C_n only depends on the dimension n and is independent of δ and the given balls and c is a fixed constant which is equal to the constant c in (2.3) *of Lemma* 2.1.

Proof. We have proved this lemma for the case $n = 2$. Assume that (2.9) and (2.10) are true when $n = m$. We wish to prove the result for the case $n = m + 1$.

Given a cube $Q_{m+1} = \{0 \le x_i \le \sqrt{\delta} : 1 \le i \le m+1\} \subset R^{m+1}$, we subdivide Q_{m+1} into equal parallelopipeds

$$Q^k_{m+1} = \{0 \le x_i \le \sqrt{\delta} \text{ for } 1 \le i \le m,\ k\delta \le x_{m+1} \le (k+1)\delta\}$$

for $k = 0, 1, \dots, l$. Then $l \approx \delta^{-1/2}$.

Let $J_k = \{\alpha \in I : O_\alpha \in Q^k_{m+1}\}$, where O_α is the center of B_α. We claim that given any finite collection of $(m+1)$-dimensional balls $\{B_\alpha\}$ with centers O_α in Q^k_{m+1}, we can select a subcollection of balls $\{B_{k_j}\}_{j=1}^{N_k}$ such that

$$\bigcup_{\alpha \in J_k} B_\alpha \subset \bigcup_{j=1}^{N_k} (1 + C_{m+1}\delta)B_{k_j}, \tag{2.11}$$

$$N_k \le c\delta^{-(2m-3)/4}. \tag{2.12}$$

Without loss of generality, we only need to prove our claim for Q^0_{m+1}. We project the balls B_α to the m-dimensional hyperplane $x_{m+1} = 0$. Then we obtain the corresponding m-dimensional balls B^*_α with centers O^*_α contained in the cube $Q_m = \{0 \le x_i \le \sqrt{\delta},\ x_{m+1} = 0\}$. The induction hypothesis in the case $n = m$ lets us select $B_1^*, \ldots, B_{N_0}^*$ such that

$$\bigcup_{\alpha \in J_0} B_\alpha^* \subset \bigcup_{i=1}^{N_0} (1 + C_m\delta) B_i^*, \tag{2.13}$$

$$N_0 \le c\delta^{-(2m-3)/4}. \tag{2.14}$$

Then by Lemmas 2.6, 2.13, and 2.14, we have proved (2.11) and (2.12) and then the claim for Q_{m+1}^0. Since we can do the same selection in each Q_{m+1}^k as we did in Q_{m+1}^0, we can select $\{B_{k_j}\}$ such that

$$\bigcup_{\alpha \in J_k} B_\alpha \subset \bigcup_{j=1}^{N_k} (1 + C_{m+1}\delta) B_{k_j}, \tag{2.15}$$

$$N_k \le c\delta^{-(2m-3)/4}. \tag{2.16}$$

By (2.15) and (2.16) we can select $\{B_{k_j}\}$ such that

$$\bigcup_{\alpha \in I} B_\alpha \subset \bigcup_{k=1}^{l} \bigcup_{j=1}^{N_k} (1 + C_{m+1}\delta) B_{k_j}$$

and

$$\sum_{k=1}^{l} N_k \le l \cdot c\delta^{-(2m-3)/4} \le \delta^{-1/2} \cdot c\delta^{-(2m-3)/4} \le c\delta^{-(2(m+1)-3)/4},$$

which proves Lemma 2.8 by induction. Q.E.D.

3. Sketch of the proof of Lemma C

Once we have Lemma 2.8, we can do exactly the same proof as in the case $n = 2$ (see [L]), and then we have the following lemmas.

Lemma 3.1. *Let δ be given small enough. Let any cube $\mathcal{Q}$ in R^n, $n \ge 2$, with sidelength $2^k\sqrt{\delta}$ be given, and let $\{B_\alpha\}_{\alpha \in I}$ be any finite collection of balls with $r \le \rho(B_\alpha) \le r + 2^k\delta$ for some $2^k \le r \le 2^{k+1}$ and centered in $\mathcal{Q}$, where k is an integer. Then one can select a subcollection of balls $B_1, \ldots, B_N$ such that*

$$\bigcup_{\alpha} B_\alpha \subset \bigcup_{i=1}^{N} (1 + C_n\delta) B_i, \tag{3.2}$$

$$N \le c\delta^{-(2n-3)/4}, \tag{3.3}$$

where C_n only depends on the dimension n and c is a fixed constant (as in Lemma 2.1).

The proof of Lemma 3.1 is straightforward if we use Lemma 2.8 and the scaling property. Actually, Lemma 3.1 can be reduced to Lemma 2.8 by dilating R^n by 2^{-k}.

Lemma 3.4. *Let* Δ *be given small enough. Let any cube* $\mathscr{Q}$ *in* R^n, $n \geq 2$, *with sidelength* $2^k\sqrt{\delta}$ *be given, and let* $\{B_\alpha\}_{\alpha\in I}$ *be any finite collection of balls with* $2^k \leq \rho(B_\alpha) \leq 2^{k+1}$. *Then one can select a subcollection* $B_1, \dots, B_N$ *such that*

$$\bigcup_\alpha B_\alpha \subset \bigcup_{i=1}^N (1 + C_n\delta)B_i, \tag{3.5}$$

$$N \leq c\delta^{-(2n-1)/4}, \tag{3.6}$$

where C_n *depends only on the dimension* n *and* c *is a fixed constant* (*as in Lemma* 2.1).

The proof of Lemma 3.4 follows the routine of the proof of Lemma 5.2 in [L].

Lemma 3.7. *Let* δ *be given small enough and* $\{B_\alpha\}_{\alpha\in I}$ *be a finite collection of balls in* R^n, $n \geq 2$, *with* $2^k \leq \rho(B_\alpha) \leq 2^{k+1}$, *where* k *are integers. Then one can select balls* $B_1, \dots, B_N$ *such that*

$$\bigcup_\alpha B_\alpha \subset \bigcup_{i=1}^N (1 + C_n\delta)B_i, \tag{3.8}$$

$$\sum_{i=1}^N \chi_{B_i}(x) \leq c\delta^{-n+1/4} \tag{3.9}$$

for all $x \in R^n$, *where* C_n *only depends on the dimension* n *and* c *is some fixed constant* (*as in Lemma* 2.1).

The proof follows the proof of Lemma 5.3 in [L]. We subdivide R^n into a dyadic grid of $\{Q_j\}_{j=1}^\infty$ whose sidelengths are $2^k\sqrt{\delta}$. The only difference here is that the cardinality of those special j is $\delta^{-n/2}$ (which was δ^{-1} in Lemma 5.3 in [L]).

Finally, we can prove Lemma C by methods similar to the proof of Lemma 1 in [L]. Since the method of proof has nothing to do with the dimension, everything is the same, except we change R^2 to R^n.

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