Poincaré Inequalities, Isoperimetric Estimates, and Representation Formulas on Product Spaces

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Abstract. We derive Sobolev–Poincaré inequalities for product balls in case the component spaces are metric spaces associated with appropriate collections of vector fields on Euclidean space. We also consider weighted versions of such inequalities, representation formulas which express a function in terms of integrals of potential type of its component vector field gradients in product spaces, and isoperimetric estimates involving product balls.

1. Poincaré estimates. One purpose of this paper is to derive Poincaré inequalities of the form

$$
\begin{aligned}
(1.1)\qquad &\left(\frac{1}{|B|}\int_B \left|f(x)-f_B\right|^q dx\right)^{1/q} \\
&\le c\rho_1(B_1)\left(\frac{1}{|B|}\int_B \big(\sum_j \left|\langle X_j^{(1)},\nabla_1 f(x)\rangle\right|^2\big)^{p/2} dx\right)^{1/p} \\
&\quad + c\rho_2(B_2)\left(\frac{1}{|B|}\int_B \big(\sum_k \left|\langle X_k^{(2)},\nabla_2 f(x)\rangle\right|^2\big)^{p/2} dx\right)^{1/p}
\end{aligned}
$$

for product balls $B = B_1 \times B_2$ in $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$, $1 \le p < \infty$, and certain values $q \ge p$. Here $\{X_j^{(1)}\}$ and $\{X_k^{(2)}\}$ denote two appropriate collections of vector fields in $\mathbb{R}^{N_1}, \mathbb{R}^{N_2}$ respectively. Also, for $i = 1, 2, B_i$ is a suitably small ball in $\mathbb{R}^{N_i}$ of radius $\rho_i(B_i)$ relative to a metric ρ_i in $\mathbb{R}^{N_i}$ which is naturally associated

Indiana University Mathematics Journal ©, Vol. 47, No. 1 (1998)

with $\{X_l^{(i)}\}_l$, $f_B = |B|^{-1}\int_B f(x)\,dx$, ∇_i is the gradient in $\mathbb{R}^{N_i}$, $|B| = |B_1||B_2|$ is the Lebesgue measure of B, and c is a constant which is independent of f and B. There are analogous estimates for the product of any finite number of balls.

Poincaré inequalities have been studied extensively in the non-product case: see, for example, [CW, F, FGaW, FGuW, FLW1, GN, HK, J, JS, L1, L2, MS, SW] and the references cited there. For the product case, (1.1) has been studied in weighted situations for the case of the ordinary gradient in [ST] and [Ch]. The principal results in [Ch] involve mixed norm estimates, which we shall not treat in this paper. Our Poincaré results in the case of the ordinary gradient are somewhat sharper than those listed in [ST] and [Ch].

For the ordinary gradient, even in the weighted case, sharp Poincaré results can easily be derived by translation and dilation of f from the corresponding results when B_1 and B_2 are unit balls centered at the origin. This is not the technique used in [ST], and it is not an effective method for the case of more general vector fields. However, we will show that an almost equally simple derivation can be given by instead scaling the vector fields appropriately. Scaling the vector fields leads naturally to scaling the corresponding metrics and allows us to deduce results like (1.1) from known results for spaces of homogeneous type. Before describing the method, let us first give an example of the type of result which it implies.

We will show that (1.1) holds for $1 \le p < q < \infty$ if p and q are related by a condition which involves the local doubling order of Lebesgue measure for product balls. By a product ball, we always mean the Cartesian product $B = B_1 \times B_2$ of two metric balls. In the case of measures (weights) other than Lebesgue measure, p and q are related by a natural balance condition which involves the weighted measures of product balls; this condition reduces to a doubling condition when the measures are Lebesgue measure.

In order to state a result for Lebesgue measure, we need some additional notation. Let Ω_i be an open, connected set in $\mathbb{R}^{N_i}$, $i = 1,2$. Let $\{X_j^{(i)}\}_j$ be real C^∞ vector fields on Ω_i which satisfy the Hörmander condition [H]. Other classes of vector fields can be used, such as those studied in [F] and [FL] (see also [FS]), which are not smooth and so cannot satisfy the Hörmander condition (although they have some compensating structure). Thus it is possible for example to consider a product with Hörmander vector fields in one component space and Franchi-Lanconelli ones in the other. In fact, what is really important for our results is the existence of the L^1 (i.e., $p = q = 1$) version of (1.1). Thus, any condition on vector fields which implies an L^1 Poincaré estimate is acceptable as an initial hypothesis. For the sake of definiteness, we will focus on the Hörmander case in our first theorem, but more general situations are discussed later. For $i = 1,2$, we will denote points in Ω_i by x_i, y_i, and we will denote points in $\Omega = \Omega_1 \times \Omega_2$ by $x = (x_1,x_2), y = (y_1,y_2)$.

It is well known that one can naturally associate with $\{X_j^{(i)}\}_j$ a metric $\rho_i(x_i,y_i)$ for $x_i,y_i \in \Omega_i$ $(i = 1,2)$. The geometry of the metric space (Ω_i,ρ_i) is described in [SW, FP, S-Cal]. In particular, the ρ_i-topology and the Euclidean topology are equivalent in Ω_i, each metric ball

$$B_i(x_i,r) = \{y_i \in \Omega_i : \rho_i(x_i,y_i) < r\},\ x_i \in \Omega_i,\ r > 0,$$

contains some Euclidean ball with center x_i, and if K_i is a compact subset of Ω_i and $r_0^{(i)} > 0$, there is a constant c such that

$$(1.2) \qquad |B_i(x_i,2r)| \le c|B_i(x_i,r)|, \quad x_i \in K_i, \quad 0 < r < r_0^{(i)},$$

where $|E|$ denotes the Lebesgue measure of a measurable set E. This doubling property of Lebesgue measure is crucial for our results. If $B_i = B(x_i,r)$, we will use the notation $\rho_i(B_i)$ for the radius r of B_i.

By [NSW], given a ball $B_i = B_i(x_i,r_i)$, $x_i \in K_i \subset \Omega_i \subset \mathbb{R}^{N_i}$, $r_i < r_0^{(i)}$, there exist positive constants γ_i and c_i, with c_i depending only on K_i and $r_0^{(i)}$ but γ_i depending also on B_i, so that

$$(1.3) \qquad |J_i| \le c_i \left(\frac{\rho_i(J_i)}{\rho_i(I_i)}\right)^{N_i\gamma_i} |I_i|$$

for all balls I_i, J_i with $I_i \subset J_i \subset B_i$. We will call γ_i the (local) doubling order of Lebesgue measure for B_i. In fact, for Hörmander vector fields, by [NSW], $N_i\gamma_i$ lies somewhere in the range $N_i \le N_i\gamma_i \le Q_i$, where Q_i is the homogeneous dimension. We can always choose $N_i\gamma_i = Q_i$, but smaller values may arise for particular vector fields, and these values may vary with $B_i(x_i,r_i)$. The values of Q_i do not vary with B_i. See [FLW1] for some detailed observations and comments about (1.3).

In the theorem below, we assume that for a given compact set K in $\Omega_1 \times \Omega_2$, there exist positive exponents $q_i, i = 1,2$, so that for any two product balls I, J centered in K with small radii and with $I = I_1 \times I_2 \subset J = J_1 \times J_2$, we have

$$(1.4) \qquad |J_i| \le c \left(\frac{\rho_i(J_i)}{\rho_i(I_i)}\right)^{q_i} |I_i|, \quad i = 1,2.$$

The constants q_i and c may depend on K but are independent of I and J.

The Poincaré estimate that we obtain for Hörmander vector fields in the unweighted case is as follows.

Theorem 1. *Let K be a compact subset of $\Omega_1 \times \Omega_2$. There exists r_0 depending on K, Ω_i and $\{X_j^{(i)}\}$, $i = 1,2$, such that if $B = B_1(x_1,r_1) \times B_2(x_2,r_2)$ is the product of balls with $(x_1,x_2) \in K$ and $0 < r_i < r_0$, and if $1 \le p < Q$ and $1/q = 1/p - 1/Q$, where $Q = q_1 + q_2$ with q_i defined by* (1.4), *then*

$$\left(\frac{1}{|B|}\int_B \left|f(x) - f_B\right|^q dx\right)^{1/q} \le cr_1 \left(\frac{1}{|B|}\int_B \left|X^{(1)}f(x)\right|^p dx\right)^{1/p} + cr_2 \left(\frac{1}{|B|}\int_B \left|X^{(2)}f(x)\right|^p dx\right)^{1/p}$$

for any $f \in \mathrm{Lip}(\bar{B})$. The constant c depends on K, Ω_i, $\{X_j^{(i)}\}$ and the constant c in (1.4). *Also, f_B may be taken to be the Lebesgue average of f, $f_B = |B|^{-1}\int_B f(x)\,dx$.*

When we have Hörmander vector fields in each component space, we shall see that a simple process of scaling the vector fields and metrics leads to a Hörmander structure in the product space. Thus Theorem 1 and its weighted versions for Hörmander vector fields may be viewed as immediate corollaries of the scaling process combined with Poincaré results known to hold in the Hörmander setting. It is worth noting that such Poincaré results, including their weighted versions, are known to be ultimate corollaries of the basic case $p = q = 1$ of the conclusion of Theorem 1.

However, in more general situations such as the mixed product of Hörmander and Franchi-Lanconelli vector fields mentioned earlier, the appropriate Poincaré results in the product space are not already known. In such situations, our approach is slightly different: we first derive an L^1 Poincaré estimate in the product space by using Fubini's theorem and the corresponding estimates in the component spaces, and we then apply results from [FPW] (or, alternately, [FLW2] together with boundedness estimates for integral operators of potential type) to deduce improved Poincaré estimates in the product space. Results like Theorem 1 for Hörmander vector fields can also be derived in this (somewhat longer) way.

In any case, we need to work with scaled vector fields. In fact, in order to prove estimates like the one in Theorem 1 for a specific product ball B, rather than for all product balls, we only need to deal with smaller product balls which have the same eccentricity as B, where by the "eccentricity" of $B = B_1 \times B_2$, we mean the number $\rho_1(B_1)/\rho_2(B_2)$. Thus, in order to prove the Poincaré estimate for B, we only need to consider product balls $I = I_1 \times I_2$ with

$$\frac{\rho_1(I_1)}{\rho_1(B_1)} = \frac{\rho_2(I_2)}{\rho_2(B_2)}.$$

This simple observation allows us to replace the product situation with that of a space of homogeneous type.

We define a "local auxiliary metric" $\rho = \rho_{B_1 \times B_2}$ for each fixed product ball $B_1 \times B_2$ by

$$\rho(x,y) = \rho_{B_1 \times B_2}(x,y) = \max\left\{\frac{\rho_1(x_1,y_1)}{\rho_1(B_1)}, \frac{\rho_2(x_2,y_2)}{\rho_2(B_2)}\right\} \tag{1.5}$$

where $x = (x_1,x_2), y = (y_1,y_2)$. It is not difficult to see that ρ is a metric on the product space and that the ρ-ball $B(x,r)$ (which we also denote by $B_\rho(x,r)$) with center $x = (x_1,x_2)$ and radius r has the form

$$B(x,r) = B_1(x_1,r\rho_1(B_1)) \times B(x_2,r\rho_2(B_2)).$$

In particular, every ρ-ball is a product ball with the same eccentricity as $B_1 \times B_2$. We shall also use the notation $\rho(I)$ for the radius of a $\rho-$ball I. If I is a $\rho-$ball and $I = I_1 \times I_2$, then

$$\rho(I) = \frac{\rho_1(I_1)}{\rho_1(B_1)} = \frac{\rho_2(I_2)}{\rho_2(B_2)}. \tag{1.6}$$

Note that $\rho_{B_1 \times B_2}(B_1 \times B_2) = 1$.

It follows from (1.4) and (1.6) that if $B = B_1 \times B_2$ and $\rho = \rho_{B_1 \times B_2}$, then

$$|J| \le c\left\{\frac{\rho(J)}{\rho(I)}\right\}^Q |I|, \quad Q = q_1 + q_2, \tag{1.7}$$

whenever I, J are ρ-balls centered in K with small radii and with $I \subset J$. This hypothesis for $\rho-$balls I, J with $I \subset J \subset B$ is all that is actually needed in order to prove that the conclusion of Theorem 1 holds for a specific product ball B.

The local auxiliary metric arises naturally if we attempt to make a given product ball $B_1 \times B_2$ become a unit ball. To understand another reason why it is natural, let us consider the following situation involving two (finite) Carnot-Carathéodory metrics d_1, d_2 associated with collections $Y^{(1)}, Y^{(2)}$ of Lipschitz continuous vector fields in $\mathbb{R}^{N_1}, \mathbb{R}^{N_2}$ respectively. This means that the metric

$d_1(x_1, y_1)$ on $\mathbb{R}^{N_1}$ associated with $Y^{(1)}$ (and similarly the metric $d_2(x_2, y_2)$ on $\mathbb{R}^{N_2}$ for $Y^{(2)}$) is defined as follows. An absolutely continuous curve $\gamma_1(t) : [0, T_1] \to \mathbb{R}^{N_1}$ is called a sub-unit curve for $Y^{(1)} = \{Y_j^{(1)}\}_j$ if

$$\langle \gamma_1'(t), \xi_1 \rangle^2 \leq \sum_j \langle Y_j^{(1)}(\gamma_1(t)), \xi_1 \rangle^2$$

for all $\xi_1 \in \mathbb{R}^{N_1}$ and a.e. $t \in [0, T_1]$. If $x_1, y_1 \in \mathbb{R}^{N_1}$, then $d_1(x_1, y_1)$ is defined by

$$\begin{aligned} d_1(x_1, y_1) = \inf\{T_1 > 0 : &\text{there exists a sub-unit curve } \gamma_1 : [0, T_1] \to \mathbb{R}^{N_1} \\ &\text{with } \gamma_1(0) = x_1, \gamma_1(T_1) = y_1\}. \end{aligned}$$

See, for example, [FP].

We have the following simple lemma.

Lemma 1. *Let d_1, d_2 be Carnot-Carathéodory metrics associated with vector fields $Y^{(1)}$, $Y^{(2)}$ in $\mathbb{R}^{N_1}$, $\mathbb{R}^{N_2}$ respectively, and let d be the metric in $\mathbb{R}^{N_1+N_2}$ associated with the union Y of the two collections $Y^{(1)}, Y^{(2)}$ (we adjoin zero coordinates appropriately to the vectors in $Y^{(1)}$ and $Y^{(2)}$ to obtain vectors in $\mathbb{R}^{N_1+N_2}$). Then if $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are any two points in $\mathbb{R}^{N_1+N_2}$,*

$$d(x, y) = \max\{d_1(x_1, y_1), d_2(x_2, y_2)\}.$$

Taking the lemma momentarily for granted, we next observe that if d is the metric induced by a collection Y of vector fields and $r > 0$, then d/r is the metric corresponding to the collection rY. In fact, if $\gamma(t)$ is a sub-unit curve for Y, it is easy to see that $\gamma(t/r)$ is a sub-unit curve for rY, and our observation then follows immediately from the definition of the metric.

Thus, given a product ball $B = B_1 \times B_2$ and collections $X^{(1)}, X^{(2)}$ of vector fields in the respective component spaces, with corresponding metrics ρ_1, ρ_2, it follows by combining the observation above with Lemma 1 that the vector fields $X = X_B$ in the product space obtained by taking the union of $\rho_1(B_1)X^{(1)}$ and $\rho_2(B_2)X^{(2)}$ correspond to the metric $\rho(x, y)$ defined in the product space by

$$\rho(x, y) = \max\left\{\frac{\rho_1(x_1, y_1)}{\rho_1(B_1)}, \frac{\rho_2(x_2, y_2)}{\rho_2(B_2)}\right\},$$

which is exactly the local auxiliary metric for B defined in (1.5). This shows how the local metric enters naturally in product space considerations.

In case the vector fields $X^{(1)}, X^{(2)}$ also satisfy the Hörmander condition in their respective component spaces, it is now a simple matter to use these facts together with known Poincaré estimates for Hörmander vector fields to prove Theorem 1. Before doing so, we first prove the lemma.

Proof. Let us first show that $d(x,y) \le \max\{d_1(x_1,y_1), d_2(x_2,y_2)\}$. We may assume without loss of generality that $d_1(x_1,y_1) \le d_2(x_2,y_2)$. We then need to show that $d(x,y) \le d_2(x_2,y_2)$. Pick a sub-unit curve γ_2 for $Y^{(2)}$ which is nearly optimal; i.e., given $\varepsilon > 0$, pick $T_2 < d_2(x_2,y_2) + \varepsilon$ and a curve γ_2 in $\mathbb{R}^{N_2}$ such that for any $\xi_2 \in \mathbb{R}^{N_2}$,

$$\text{(1.8)} \qquad \langle \gamma_2'(t), \xi_2 \rangle^2 \le \sum_j \langle Y_j^{(2)}(\gamma_2(t)), \xi_2 \rangle^2, \quad \gamma_2(0) = x_2, \gamma_2(T_2) = y_2.$$

Clearly, $T_2 \ge d_2(x_2,y_2)$.

Let us show that we may assume that $T_2 > d_2(x_2,y_2)$. In fact, given a number $\tilde{T}_2 > T_2$, the curve $\tilde{\gamma}_2$ defined by $\tilde{\gamma}_2(t) = \gamma_2(T_2 t/\tilde{T}_2)$ satisfies $\tilde{\gamma}_2(0) = x_2$ and $\tilde{\gamma}_2(\tilde{T}_2) = y_2$, and it is also a sub-unit curve for $Y^{(2)}$ since for any $\xi_2 \in \mathbb{R}^{N_2}$,

$$\begin{aligned}
\langle \tilde{\gamma}_2'(t), \xi_2 \rangle^2 &= \left(\frac{T_2}{\tilde{T}_2}\right)^2 \langle \gamma_2' \left(\frac{T_2}{\tilde{T}_2} t\right), \xi_2 \rangle^2 \\
&\le \left(\frac{T_2}{\tilde{T}_2}\right)^2 \sum_j \langle Y_j^{(2)}(\gamma_2 \left(\frac{T_2}{\tilde{T}_2} t\right)), \xi_2 \rangle^2 \\
&= \left(\frac{T_2}{\tilde{T}_2}\right)^2 \sum_j \langle Y_j^{(2)}(\tilde{\gamma}_2(t)), \xi_2 \rangle^2 \\
&\le \sum_j \langle Y_j^{(2)}(\tilde{\gamma}_2(t)), \xi_2 \rangle^2 \quad \text{since } T_2 \le \tilde{T}_2.
\end{aligned}$$

Choosing $\tilde{T}_2$ with $T_2 < \tilde{T}_2 < d_2(x_2,y_2) + \varepsilon$ shows that we may assume $T_2 > d_2(x_2,y_2)$.

Since $d_1(x_1,y_1) \le d_2(x_2,y_2)$, we then obtain that $T_2 > d_1(x_1,y_1)$. By definition of d_1, we may choose T_1 and a curve γ_1 in $\mathbb{R}^{N_1}$ such that $T_1 \le T_2$ and

$$\text{(1.9)} \qquad \langle \gamma_1'(t), \xi_1 \rangle^2 \le \sum_j \langle Y_j^{(1)}(\gamma_1(t)), \xi_1 \rangle^2, \quad \gamma_1(0) = x_1, \gamma_1(T_1) = y_1$$

for all $\xi_1 \in \mathbb{R}^{N_1}$. Rescale γ_1 so that it reaches y_1 at time T_2, i.e., let $\tilde{\gamma}_1(t) = \gamma(T_1 t/T_2)$. Then $\tilde{\gamma}_1(0) = x_1$, $\tilde{\gamma}_1(T_2) = \gamma(T_1) = y_1$, and $\tilde{\gamma}_1$ is still a sub-unit curve for $Y^{(1)}$ by using the same sort of argument as above since $T_1 \le T_2$. Thus,

$$\langle \tilde{\gamma}_1'(t), \xi_1 \rangle^2 \le \sum_j \langle Y_j^{(1)}(\tilde{\gamma}_1(t)), \xi_1 \rangle^2 \tag{1.10}$$

for all $\xi_1 \in \mathbb{R}^{N_1}$.

Consider the curve $\gamma(t) = (\tilde{\gamma}_1(t), \gamma_2(t))$ in the product space. By adding (1.8) and (1.10), we see that γ is a sub-unit curve for Y. Clearly, $\gamma(0) = (x_1, x_2) = x$ and $\gamma(T_2) = (\tilde{\gamma}_1(T_2), \gamma_2(T_2)) = (y_1, y_2) = y$. Thus,

$$d(x,y) \le T_2 < d_2(x_2, y_2) + \varepsilon,$$

and the desired inequality follows by letting $\varepsilon \to 0$.

To prove the opposite inequality, let $\gamma(t)$ be a sub-unit curve for Y in the product space which is nearly optimal for $d(x,y)$; i.e., given $\varepsilon > 0$, let T and γ be such that for all $\xi = (\xi_1, \xi_2) \in \mathbb{R}^{N_1+N_2}$,

$$\langle \gamma'(t), \xi \rangle^2 \le \sum_j \langle Y_j^{(1)}(\gamma(t)), \xi_1 \rangle^2 + \sum_j \langle Y_j^{(2)}(\gamma(t)), \xi_2 \rangle^2,$$

$$\gamma(0) = x, \quad \gamma(T) = y, \quad \text{and } T \le d(x,y) + \varepsilon.$$

Write $\gamma(t) = (\gamma_1(t), \gamma_2(t))$. Choosing $\xi = (\xi_1, 0)$ and noting that $Y_j^{(1)}$ only depends on the first set of variables, we see that $\gamma_1(t)$ is a sub-unit curve for $Y^{(1)}$ with $\gamma_1(0) = x_1$ and $\gamma_1(T) = y_1$. Similarly, γ_2 is a sub-unit curve for $Y^{(2)}$ with $\gamma_2(0) = x_2, \gamma_2(T) = y_2$. Hence, $d_1(x_1, y_1) \le T$ and $d_2(x_2, y_2) \le T$, and the inequality $\max\{d_1(x_1,y_1), d_2(x_2,y_2)\} \le d(x,y)$ follows by letting $\varepsilon \to 0$. This completes the proof of the lemma. □

Proof of Theorem 1. We can easily deduce Theorem 1 from the observations above combined with known Poincaré results for Hörmander vector fields. In fact, fix a product ball $B = B_1 \times B_2$ and note that since the collections $X^{(1)}$ and $X^{(2)}$ satisfy the Hörmander condition in their respective component spaces, then the collection $X_B = \rho_1(B_1)X^{(1)}, \rho_2(B_2)X^{(2)}$ (with zero components adjoined appropriately) satisfies the Hörmander condition in the product space. Since we noted above that the auxiliary metric $\rho = \rho_{B_1 \times B_2}$ is the metric induced by X_B in the product space, we can apply the results of [FLW1] (see also [GN]) to obtain the conclusion. More precisely, using Theorem 1 of [FLW1] we have that if $1 \le p < Q, 1/q = 1/p - 1/Q$ and (1.7) holds, then

$$\left(\frac{1}{|B|} \int_B \left| f(x) - f_B \right|^q dx \right)^{1/q} \le c \rho_B(B) \left(\frac{1}{|B|} \int_B \left| X_B f(x) \right|^p dx \right)^{1/p}$$

with c independent of f and B. Since $\rho_B(B) = 1$ and

$$|X_B f(x)|^2 = |\rho_1(B_1)X^{(1)}f(x)|^2 + |\rho_2(B_2)X^{(2)}f(x)|^2,$$

Theorem 1 follows immediately. Weighted versions (for Hörmander vector fields) can be obtained in the same way by applying Theorem 2 of [FLW1].

The proof above relies on the existence of known Poincaré results for vector fields which satisfy the Hörmander condition, and so does not work in more general situations. We now give a different proof of Theorem 1 which avoids this difficulty and so can be used in other situations.

We begin by showing that the L^1, L^1 Poincaré estimate in the product space follows from the corresponding ones in the component spaces. For Hörmander vector fields, for example, the estimate in the component spaces follows from [J]. It will be convenient to work in a more general setting. Let $(\Omega_1, \rho_1, d\mu_1)$ and $(\Omega_2, \rho_2, d\mu_2)$ be two spaces of homogeneous type in the sense of [CoW], that is, ρ_i is a quasimetric on Ω_i and μ_i is a doubling measure on Ω_i relative to ρ_i-balls:

$$\begin{aligned} &\rho_i(x_i, y_i) \le K[\rho_i(x_i, z_i) + \rho(z_i, y_i)], \quad && x_i,\ y_i,\ z_i \in \Omega_i, \quad \text{and} \\ &\mu_i(B_i(x_i, 2r)) \le C\mu_i(B_i(x_i, r)), && x_i \in \Omega_i, r > 0, \end{aligned}$$

for $i = 1, 2$. We use the notation $\mu = \mu_1 \times \mu_2$ for the product measure associated with μ_1, μ_2. It follows that μ is a doubling measure with respect to product balls, i.e., $\mu(2B_1 \times 2B_2) \le C^2 \mu(B_1 \times B_2)$ for any product ball $B_1 \times B_2$.

For simplicity, we state the following lemma in a global form.

Lemma 2. *Let $(\Omega_i, \rho_i, \mu_i), i = 1, 2$, be two spaces of homogeneous type and let $\mu = \mu_1 \times \mu_2$ denote the corresponding product measure. Suppose that*

$$\int_{I_i} \left|f - f_{I_i}\right| d\mu_i \le c\rho_i(I_i) \int_{I_i} \left|X^{(i)} f\right| d\mu_i,$$

$i = 1, 2$, for all balls $I_i \subset \Omega_i$ and all $f \in \mathrm{Lip}(I_i)$. Then

$$\int_{I_1 \times I_2} \left|f - f_{I_1 \times I_2}\right| d\mu \le c \int_{I_1 \times I_2} \left\{\rho_1(I_1)\left|X^{(1)} f\right| + \rho_2(I_2)\left|X^{(2)} f\right|\right\} d\mu \tag{1.11}$$

for all product balls $I_1 \times I_2$ and all $f \in \mathrm{Lip}(I_1 \times I_2)$. Moreover, the constant c in (1.11) *is the same as the one in the hypothesis.*

Proof. Fix $I_1 \times I_2$ and a function $f \in \mathrm{Lip}(I_1 \times I_2)$. For $(x_1, x_2) \in I_1 \times I_2$, write

$$\begin{aligned}
&\left| f(x_1,x_2) - \frac{1}{\mu_1(I_1)\mu_2(I_2)} \int_{I_1}\int_{I_2} f(y_1,y_2)\, d\mu_1(y_1) d\mu_2(y_2) \right| \\
&\le \left| f(x_1,x_2) - \frac{1}{\mu_1(I_1)} \int_{I_1} f(y_1,x_2)\, d\mu_1(y_1) \right| + \left| \frac{1}{\mu_1(I_1)} \int_{I_1} f(y_1,x_2)\, d\mu(y_1) \right. \\
&\qquad\qquad \left. - \frac{1}{\mu_1(I_1)\mu_2(I_2)} \int_{I_1}\int_{I_2} f(y_1,y_2)\, d\mu_1(y_1) d\mu_2(y_2) \right| \\
&= S + T.
\end{aligned}$$

By integrating first with respect to $d\mu_1$ and using the hypothesis of the lemma, we have

$$\iint_{I_1 \times I_2} S\, d\mu_1(x_1) d\mu_2(x_2) \le c \iint_{I_1 \times I_2} \left| X^{(1)} f(x_1,x_2) \right| d\mu_1(x_1) d\mu_2(x_2).$$

For T, we have

$$T \le \frac{1}{\mu_1(I_1)} \int_{I_1} \left| f(y_1,x_2) - \frac{1}{\mu_2(I_2)} \int_{I_2} f(y_1,y_2)\, d\mu(y_2) \right| d\mu(y_1).$$

We integrate $\iint_{I_1 \times I_2} T\, d\mu_1 d\mu_2$ first with respect to $d\mu_2$, and by Fubini's theorem and the hypothesis of the lemma, we see this is at most

$$\begin{aligned}
&c \frac{1}{\mu(I_1)} \int_{I_1}\int_{I_1} \left[\rho_2(I_2) \int_{I_2} \left| X^{(2)} f(y_1,x_2) \right| d\mu_2(x_2) \right] d\mu_1(y_1)\, d\mu_1(x_1) \\
&= c\rho_2(I_2) \int_{I_1 \times I_2} \left| X^{(2)} f(y_1,x_2) \right| d\mu_1(y_2)\, d\mu_2(x_2),
\end{aligned}$$

which completes the proof. □

In passing, we mention that the analogue of Lemma 2 for the product of any finite number of component spaces can be used in conjunction with the fundamental theorem of calculus to derive an L^1, L^1 Poincaré estimate for arbitrary rectangles in n-dimensional Euclidean space with Lebesgue measure and the usual distance. Moreover, this can be accomplished with explicit and precise control of the constant which appears on the right side of the estimate. We make some further comments about this near the end of the paper (see Section 4).

There is a useful connection between (1.11) and the auxiliary metric. In fact, fix a product ball $B = B_1 \times B_2$ and let $\rho = \rho_{B_1 \times B_2}$ be the corresponding local metric defined by (1.5). If $I = I_1 \times I_2$ is any product ball with the same eccentricity as B, it follows from (1.6) that (1.11) may be rewritten in the form

$$(1.12) \qquad \int_I |f - f_I|\, d\mu \le C\, \rho(I) \int_I \left\{ |X^{(1)} f| \rho_1(B_1) + |X^{(2)} f| \rho_2(B_2) \right\} d\mu.$$

This inequality, however, is an L^1, L^1 Poincaré inequality for the metric ρ and the vector fields X defined so that

$$|Xf|^2 = (|X^{(1)} f| \rho_1(B_1))^2 + (|X^{(2)} f| \rho_2(B_2))^2.$$

Its validity for all ρ-balls $I \subset cB$ is enough to let us apply known results for spaces of homogeneous type. At this point, there are two ways to proceed. The quickest is to apply the results in [FPW], and we will do this now. The second way is based on obtaining representation formulas in a product space, and we will discuss this in the next section.

Given $1 \le p < \infty$, we first divide both sides of (1.12) by $\mu(I)$ and then apply Hölder's inequality to the right side to obtain

$$(1.13) \qquad \frac{1}{\mu(I)} \int_I |f - f_I|\, d\mu \le C\, b(I, f)$$

for all ρ-balls $I \subset cB$, where $b(I, f)$ is the functional defined by

$$b(I, f) = \rho(I) \left(\frac{1}{\mu(I)} \int_I |Xf|^p\, d\mu \right)^{1/p}.$$

Since X is a differential operator, given $q \ge p$, Theorem 3.1 of [FPW] immediately implies the improved inequality

$$\left(\frac{1}{\mu(B)} \int_B |f - f_B|^q\, d\mu \right)^{1/q} \le C b(B, f),$$

which reduces for example to the conclusion of Theorem 1 in case μ is Lebesgue measure, provided we verify that b satisfies the following condition of discrete type uniformly for every ρ-ball $I \subset cB$, every family $\{I^k\}$ of pairwise disjoint ρ-subballs of I and every $f \in \mathrm{Lip}(cB)$:

$$(1.14) \qquad \sum_k b(I^k, f)^q \mu(I^k) \le C\, b(I, f)^q \mu(I).$$

To verify (1.14), write

$$\sum_k b(I^k,f)^q \mu(I^k) = \sum_k \left(\frac{\rho(I^k)}{\mu(I^k)^{1/p}}\right)^q \mu(I^k) \left(\int_{I^k} |Xf|^p \, d\mu\right)^{q/p}.$$

Assuming that

$$\left(\frac{\rho(J)}{\mu(J)^{1/p}}\right)^q \mu(J) \le c \left(\frac{\rho(I)}{\mu(I)^{1/p}}\right)^q \mu(I) \quad \text{if } J \subset I \subset cB, \tag{1.15}$$

where J and I have the same eccentricity as B, and noting that $q \ge p$, we see the sum is at most

$$\begin{aligned} & c \left(\frac{\rho(I)}{\mu(I)^{1/p}}\right)^q \mu(I) \left(\sum_k \int_{I^k} |Xf|^p \, d\mu\right)^{q/p} \\ \le \;& c \left(\frac{\rho(I)}{\mu(I)^{1/p}}\right)^q \mu(I) \left(\int_I |Xf|^p \, d\mu\right)^{q/p} \\ = \;& c b(I,f)^q \mu(I), \end{aligned}$$

as desired.

Note that (1.15) is the same as the local balance condition

$$\frac{\rho(J)}{\rho(I)} \left(\frac{\mu(J)}{\mu(I)}\right)^{1/q} \le c \left(\frac{\mu(J)}{\mu(I)}\right)^{1/p}, \quad J \subset I. \tag{1.16}$$

In case μ is Lebesgue measure, condition (1.16) with $1/q = 1/p - 1/Q$ is the same as (1.7), and we thus obtain a second proof of Theorem 1.

Weighted versions of Theorem 1 can be obtained in essentially the same way. In fact, given a product ball $B = B_1 \times B_2$, two weight functions w, v on B and $1 \le p < q < \infty$, we assume that the following local balance condition holds for all ρ_B-balls I, J with $I \subset J \subset cB$:

$$\frac{\rho(I)}{\rho(J)} \left(\frac{w(I)}{w(J)}\right)^{1/q} \le c \left(\frac{v(I)}{v(J)}\right)^{1/p}, \tag{1.17}$$

where $\rho(I), \rho(J)$ are the radii relative to $B = B_1 \times B_2$, and $w(I) = \int_I w \, d\mu$. Note that (1.17) is the same as (1.16) if $w = v = 1$ and further reduces in the case of Lebesgue measure to (1.7) with $1/q = 1/p - 1/Q$.

Also, given a product ball B, we say that a weight $v \in A_p(\mu)$ relative to B, $1 \le p < \infty$, if

$$\left(\frac{1}{|I|}\int\limits_I v\; d\mu\right)\left(\frac{1}{|I|}\int\limits_I v^{-1/(p-1)} d\mu\right)^{p-1} \le C \quad \text{when } 1 < p < \infty$$

$$\frac{1}{|I|}\int\limits_I v\; d\mu \le C \operatorname*{ess\,inf}_I v \qquad \text{when } p = 1$$

for all product balls I with the same eccentricity as B and with $I \subset B$ (i.e., for all $\rho-$balls $I \subset B$). We say that $v \in A_\infty(\mu)$ relative to B if $v \in A_p(\mu)$ relative to B for some p. The fact that μ satisfies the doubling condition (1.2) allows us to develop the usual theory of such weight classes as in [Ca]. It follows easily from the definition and (1.2) that if $v \in A_p(\mu)$ relative to B, then for any product ball I with the same eccentricity as B and with $2I \subset B$,

$$v(2I) \le Cv(I)$$

with C depending on both the A_p constant of v relative to B and the doubling constant of Lebesgue measure. We say that any such weight is doubling relative to B. All the weights we will consider will be doubling relative to B.

Our weighted result of Poincaré type for $p \le q$ is then as follows.

Theorem 2. *Let $B = B_1 \times B_2$ be a product ball in $\Omega_1 \times \Omega_2$, and let μ be the corresponding product measure. Suppose that* (1.11) *holds for all product balls $I \subset cB$ with the same eccentricity as B. If $1 \le p \le q < \infty$ and w, v are weights satisfying the balance condition* (1.17) *relative to B, with $v \in A_p(\mu)$ and $w \in A_\infty(\mu)$ relative to B, then*

$$\left(\frac{1}{w(B)}\int\limits_B \left|f(x) - f_B\right|^q w(x)\, d\mu\right)^{1/q}$$

$$\le Cr_1\left(\frac{1}{v(B)}\int\limits_{cB} \left|X^{(1)}f(x)\right|^p v(x)\, d\mu\right)^{1/p} + Cr_2\left(\frac{1}{v(B)}\int\limits_{cB} \left|X^{(2)}f(x)\right|^p v(x)\, d\mu\right)^{1/p}$$

for any $f \in \mathrm{Lip}\ (\bar{B})$, with $f_B = w(B)^{-1}\int\limits_B f(x)w(x)d\mu$. The constant c depends only on the homogeneous spaces Ω_i, and C depends only on the constants in (1.11) *and in the conditions imposed on w and v (which may depend on B).*

One can also derive results in which there are different weights in the two terms on the right side of the conclusion of the theorem. The constant c can be taken to be 1 in the case of Hörmander or Franchi-Lanconelli vector fields by standard arguments involving Boman chain domains. Moreover, the restriction that $w \in A_\infty(\mu)$ can be weakened to just assuming that w is doubling by using the representation formula in the next section together with known weighted results for integral operators of potential type.

To prove the result as stated requires first noting that (1.11) and the A_p condition imply that (1.13) holds for the functional b defined by

$$b(I,f) = C\,\rho(I)\left(\frac{1}{v(I)}\int_I |Xf|^p v\,d\mu\right)^{1/p},$$

where ρ is the local auxiliary metric for B and X is the differential operator with $|Xf| = |X^{(1)}f|\rho_1(B_1) + |X^{(2)}f|\rho_2(B_2)$. To complete the proof, by [FPW] we only need to check the following analogue of (1.14):

$$\sum_k b(I^k,f)^q w(I^k) \le c\,B(I,f)^q w(I)$$

for every ρ-ball $I \subset cB$ and every collection $\{I^k\}$ of pairwise disjoint ρ-balls contained in I. This estimate follows from (1.17) by the same sort of argument we used before, which proves Theorem 2.

We mention in passing that it is possible to use the Poincaré estimates above to derive analogous estimates for domains other than products of metric balls. In particular, this can be done for a product of two domains each of which satisfies the Boman chain condition.

2. Representation formulas in product spaces. In [ST], Shi and Torchinsky derive the following representation formula for a Lipschitz continuous function f in the product $P = I \times J$ of two cubes $I \subset \mathbb{R}^n$ and $J \subset \mathbb{R}^m$:

$$|f(x_1,x_2) - f_P| \le$$

$$\frac{c}{|P|}\iint_P \big(\left|\nabla_1 f(y_1,y_2)\right|\,\left|x_1-y_1\right| + \left|\nabla_2 f(y_1,y_2)\right|\,\left|x_2-y_2\right|\big)\mathcal{K}(x,y)\,dy_1\,dy_2,$$

where $\mathcal{K}$ is the kernel defined by

$$\mathcal{K}(x,y) = \min\left(\frac{|I|^{1/n}}{|x_1-y_1|}, \frac{|J|^{1/m}}{|x_2-y_2|}\right)^{n+m}.$$

Our purpose in this section is to derive some similar formulas for products of balls in homogeneous spaces. The formulas vary somewhat depending on whether the spaces have any extra structure. As usual, let $(\Omega_1, \rho_1, d\mu_1)$ and $(\Omega_2, \rho_2, d\mu_2)$ be two spaces of homogeneous type, and let $B = B_1 \times B_2$ be a fixed product ball. In order to obtain a formula similar to the one above, we shall impose some additional assumptions on the spaces relative to B; each assumption is satisfied for the special classes of vector fields in [H] or [F]. We again use the notation $\mu = \mu_1 \times \mu_2$ for the product measure.

Our first assumption is that given any point of B, there exists a suitable chain of sets in the product space which approach the point. More precisely, we assume that given a point $x = (x_1, x_2) \in B$, $B = B_1 \times B_2$, there exist sets $\{E^k, k = 0, 1, \cdots\}$ and product balls $B^k = B_1^k \times B_2^k$ all with the same eccentricity as B and contained in cB such that

(2.1)

- (i) $\mu(E^0) \approx \mu(B)$ uniformly in x;
- (ii) $E^k \cup E^{k+1} \subset B^k$ and $\mu(E^k) \approx \mu(E^{k+1}) \approx \mu(B^k)$ uniformly in k, x;
- (iii) $\rho_1(B_1^k), \rho_2(B_2^k) \to 0$ as $k \to \infty$;
- (iv) For $y = (y_1, y_2) \in B^k$, $\rho_1(x_1, y_1) \approx \rho_1(B_1^k)$, $\rho_2(x_2, y_2) \approx \rho_2(B_2^k)$ and $\dfrac{\rho_1(x_1, y_1)}{\rho_1(B_1)} \approx \dfrac{\rho_2(x_2, y_2)}{\rho_2(B_2)}$ uniformly in x, y and k;
- (v) The balls $\{B^k\}$ have bounded overlaps.

Property (iv) means roughly that the component sizes of the product balls are comparable to their respective component distances to x_1, x_2, and that the product balls have a scaled "diagonal" position relative to $B_1 \times B_2$. The sets E^k, B^k in (2.1) depend on x.

For example, in the simplest case, when both spaces are the real line with Lebesgue measure and the usual distance, given a rectangle $B = B_1 \times B_2$ with dimensions r_1, r_2 and a point $x = (x_1, x_2) \in B$, we pick

$$E^0 = I_1^0 \times I_2^0 = \left(x_{B_1} + \frac{3r_1}{2},\ x_{B_1} + \frac{5r_1}{2} \right) \times \left(x_{B_2} + \frac{3r_2}{2},\ x_{B_2} + \frac{5r_2}{2} \right),$$

and for $k \geq 1$, we pick

$$E^k = I_1^k \times I_2^k = \left(x_1 + \frac{r_1}{2^k},\ x_1 + \frac{2r_1}{2^k} \right) \times \left(x_2 + \frac{r_2}{2^k},\ x_2 + \frac{2r_2}{2^k} \right).$$

Then choosing $B^k = B_1^k \times B_2^k$ where B_1^k is the smallest interval containing $I_1^k \cup I_1^{k+1}$, and similarly for B_2^k, it is easy to see that all the conditions are satisfied. Note that in this case the first set E^0 is independent of x.

More generally, if each $\Omega_i, i = 1, 2$, is a metric space and satisfies the "segment property" (we say that a metric space with metric d satisfies the segment property if for each pair of points there exists a continuous curve γ connecting the points such that $d(\gamma(s), \gamma(t)) = |s - t|$), then the properties above are satisfied by the Cartesian products, taken one-by-one in order, of the balls constructed in [FW] for each component space. We refer the reader to [FW] for details. We also note that the segment property holds in a space where there are Carnot-Carathéodory vector fields for which the associated metric is continuous and for which Lebesgue measure is doubling for metric balls: see [GN] for the global case; the local version was used in [FGuW].

We also assume that there is an L^1, L^1 Poincaré inequality of the same form as in §1 for product balls $I = I_1 \times I_2 \subset cB$ which have the same eccentricity as B:

$$\int_I \left|f - f_I\right| d\mu \leq c \int_I \{\rho_1(I_1)\left|X^{(1)} f\right| + \rho_2(I_2)\left|X^{(2)} f\right|\} \, d\mu \tag{2.2}$$

for all $f \in \mathrm{Lip}(I)$.

We now state our first (and most precise) representation formula. Some weaker but useful formulas are considered in the remarks below.

Theorem 3. *For $i = 1, 2$, let $(\Omega_i, \rho_i, \mu_i)$ be two spaces of homogeneous type which satisfy hypotheses (2.1) and (2.2) relative to a product ball $B = B_1 \times B_2$. Let ρ be the local auxiliary metric defined by*

$$\rho(x, y) = \rho_{B_1 \times B_2}(x, y) = \max\left\{\frac{\rho_1(x_1, y_1)}{\rho_1(B_1)}, \frac{\rho_2(x_2, y_2)}{\rho_2(B_2)}\right\},$$

where $x = (x_1, x_2)$ and $y = (y_1, y_2)$, and let k_1, k_2 be the kernels defined by

$$\begin{aligned} k_i(x, y) = k_{i, B_1 \times B_2}(x, y) &= \frac{\rho_i(x_i, y_i)}{\mu(B(x, \rho(x, y)))} \\ &= \frac{\rho_i(x_i, y_i)}{\mu_1(B_1(x_1, \rho_1(B_1)\rho(x, y)))\, \mu_2(B_2(x_2, \rho_2(B_2)\rho(x, y)))}, \end{aligned}$$

$i = 1, 2$, where μ denotes the product measure $\mu_1 \times \mu_2$. Then for any Lipschitz continuous function f and every $x \in B$,

$$\begin{aligned} |f(x) - f_B| \leq C \int_{cB} &\left\{\left|X^{(1)} f(y)\right| k_1(x, y) + \left|X^{(2)} f(y)\right| k_2(x, y)\right\} d\mu(y) \\ &+ C \frac{1}{\mu(B)} \int_{cB} \left\{\left|X^{(1)} f\right| \rho_1(B_1) + \left|X^{(2)} f\right| \rho_2(B_2)\right\} d\mu. \end{aligned} \tag{2.3}$$

Moreover, if we replace f_B by f_{E^0}, where E^0 is the first set in (2.1), which may depend on x, then the conclusion holds with the second term on the right deleted.

Proof. The proof is an adaptation of the one in [FW] (where instead of a product structure there is a single space with a Carnot-Carathéodory structure). Fix $x = (x_1, x_2) \in B$ and let $\{E^k\}$ and $\{B^k\}$ be the sets described in (2.1). Then

$$
\begin{aligned}
&\left| f(x_1,x_2) - \frac{1}{\mu(E^0)} \int_{E^0} f\,d\mu \right| \\
&\qquad \le \sum_{k=0}^{\infty} \left| \frac{1}{\mu(E^{k+1})} \int_{E^{k+1}} f\,d\mu - \frac{1}{\mu(E^k)} \int_{E^k} f\,d\mu \right| \\
&\qquad\qquad \text{(by Lebesgue's differentiation theorem and (2.1)(iii) and (iv))} \\
&\qquad \le \sum_{k=0}^{\infty} \left(\left| \frac{1}{\mu(E^{k+1})} \int_{E^{k+1}} f\,d\mu - f_{B^k} \right| + \left| f_{B^k} - \frac{1}{\mu(E^k)} \int_{E^k} f\,d\mu \right| \right) \\
&\qquad \le \sum_{k=0}^{\infty} \left(\frac{1}{\mu(E^{k+1})} \int_{E^{k+1}} |f - f_{B^k}|\,d\mu + \frac{1}{\mu(E^k)} \int_{E^k} |f - f_{B^k}|\,d\mu \right) \\
&\qquad \le \sum_{k=0}^{\infty} c \frac{1}{\mu(B^k)} \int_{B^k} |f - f_{B^k}|\,d\mu \\
&\qquad \le c \sum_{k=0}^{\infty} \left(\frac{\rho_1(B_1^k)}{\mu(B^k)} \int_{B^k} |X^{(1)} f|\,d\mu + \frac{\rho_2(B_2^k)}{\mu(B^k)} \int_{B^k} |X^{(2)} f|\,d\mu \right),
\end{aligned}
\tag{2.4}
$$

where $B^k = B_1^k \times B_2^k$ and $\rho_1(B_1^k), \rho_2(B_2^k)$ are the radii of B_1^k, B_2^k, respectively. To obtain the last two inequalities, we have used (2.1)(ii) and (2.2).

The sum arising from the first term on the right in (2.4) equals

$$
\int_{\bigcup B^k} |X^{(1)} f(y)| \Big(\sum_{k=0}^{\infty} \chi_{B^k}(y) \frac{\rho_1(B_1^k)}{\mu_1(B_1^k)\mu_2(B_2^k)} \Big)\, d\mu(y).
\tag{2.5}
$$

We claim that the sum in the integrand in (2.5) is bounded by a constant multiple of $k_1(x, y)$. To show this, first note that given $y = (y_1, y_2)$, by (2.1)(v), there are at most a fixed finite number of nonzero terms in the sum. Moreover, if $B_1^k \times B_2^k$

contains (y_1, y_2), then since $y_1 \in B_1^k$, we have by (2.1)(iv) that

$$\begin{aligned}\rho_1(B_1^k) &\approx \rho_1(x_1,y_1)\\ &\approx \rho_1(x_1,y_1) + \frac{\rho_1(B_1)}{\rho_2(B_2)}\rho_2(x_2,y_2) \approx \rho_1(B_1)\,\rho(x,y).\end{aligned}$$

Also, by doubling,

$$\mu_1(B_1^k) \approx \mu_1(B_1(y_1,\rho_1(x_1,y_1))) \approx \mu_1(B_1(x_1,\rho_1(x_1,y_1))),$$

and therefore,

$$\mu(B_1^k) \approx \mu_1(B_1(x_1,\rho_1(B_1)\rho(x,y))).$$

Similarly, since $y_2 \in B_2^k$,

$$\mu_2(B_2^k) \approx \mu_2(B_2(x_2,\rho_2(B_2)\rho(x,y))).$$

Our claim now follows by combining estimates. This shows that (2.5) is at most a constant times

$$\int_{cB} \left|X^{(1)} f(y)\right| k_1(x,y)\, d\mu(y).$$

Similarly, the sum arising from the second term on the right in (2.4) is bounded by a multiple of

$$\int_{cB} \left|X^{(2)} f(y)\right| k_2(x,y)\, d\mu(y),$$

and the second statement of the theorem follows. Note that we have not used (2.1)(i) yet.

To prove the first statement of the theorem, write

$$|f(x) - f_B| \le |f(x) - f_{E^0}| + |f_{E^0} - f_B|.$$

The first term on the right is what we estimated above. For the second term, we have

$$\begin{aligned}|f_{E^0} - f_B| &\le |f_{E^0} - f_{cB}| + |f_{cB} - f_B|\\ &\le \frac{1}{\mu(E^0)}\int_{E^0} \left|f(y) - f_{cB}\right| d\mu(y) + \frac{1}{\mu(B)}\int_B \left|f(y) - f_{cB}\right| d\mu(y).\end{aligned}$$

Since $E^0 \subset cB$ and $\mu(E^0)$ is comparable to $\mu(cB)$ (by (2.1)(i)), each of the last two terms is bounded by

$$C\,\frac{1}{\mu(cB)}\int_{cB}\left|f(y)-f_{cB}\right|d\mu(y),$$

which by (2.2) is in turn bounded by the second term on the right in the conclusion of Theorem 3. The theorem now follows by combining estimates.

There is a weaker form of the representation formula that can be used to derive Poincaré estimates. In fact, if we define

$$\begin{aligned}(2.6)\qquad k(x,y)=k_{B_1\times B_2}(x,y)&=\frac{\rho(x,y)}{\mu(B(x,\rho(x,y)))}\\ &=\frac{\rho(x,y)}{\mu_1(B_1(x_1,\rho_1(B_1)\rho(x,y)))\,\mu_2(B_2(x_2,\rho_2(B_2)\rho(x,y)))},\end{aligned}$$

where $x=(x_1,x_2)$ and $\rho=\rho_{B_1\times B_2}$, and then note that the definition of $\rho(x,y)$ gives both $\rho_1(x_1,y_1)\le\rho_1(B_1)\rho(x,y)$ and $\rho_2(x_2,y_2)\le\rho(B_2)\rho(x,y)$, we immediately obtain

$$k_1(x,y)\le\rho_1(B_1)k(x,y)\text{ and }k_2(x,y)\le\rho_2(B_2)k(x,y).$$

Consequently, we have the following representation formula.

Corollary 1. *With the same hypotheses as in Theorem 3 and with $k(x,y)$ defined by (2.6), if $x\in B=B_1\times B_2$, then*

$$\begin{aligned}(2.7)\quad &\left|f(x)-f_B\right|\\ &\le C\int_{cB}\left\{\left|X^{(1)}f(y)\right|\rho_1(B_1)+\left|X^{(2)}f(y)\right|\rho_2(B_2)\right\}k(x,y)\,d\mu(y)\\ &\quad+C\frac{1}{\mu(B)}\int_{cB}\left\{\left|X^{(1)}f(y)\right|\rho_1(B_1)+\left|X^{(2)}f(y)\right|\rho_2(B_2)\right\}d\mu(y).\end{aligned}$$

We now make several remarks related to Corollary 1.

Remark 1. We will show that the conclusion of Corollary 1 is valid without the second term on the right in (2.7), i.e., that

$$\begin{aligned}(2.8)\quad &\left|f(x)-f_B\right|\\ &\le C\int_{cB}\left\{\left|X^{(1)}f(y)\right|\rho_1(B_1)+\left|X^{(2)}f(y)\right|\rho_2(B_2)\right\}k(x,y)\,d\mu(y),\end{aligned}$$

$x \in B$, if we make an additional natural assumption about the *reverse* doubling orders of the measures μ_i (or, instead, about the product measure μ). In fact, suppose that for *either* $i = 1$ or $i = 2$,

$$\frac{\rho_i(B_i)}{\mu_i(B_i)} \leq c\,\frac{\rho_i(I_i)}{\mu_i(I_i)} \quad \text{for every ball } I_i \subset CB_i,$$

or equivalently that for either $i = 1$ or $i = 2$,

$$\mu_i(B_i) \geq c\left(\frac{\rho_i(B_i)}{\rho_i(I_i)}\right)\mu_i(I_i) \quad \text{for every ball } I_i \subset CB_i. \tag{2.9}$$

With this assumption, we will show that (2.8) holds in B.

To explain (2.9), we note that any doubling measure μ_i on a space of homogeneous type satisfies a reverse doubling condition of some order $\alpha > 0$, i.e., there exists $\alpha > 0$ such that

$$\mu_i(B_i) \geq c\left(\frac{\rho_i(B_i)}{\rho_i(I_i)}\right)^{\alpha}\mu_i(I_i) \quad \text{for every ball } I_i \subset CB_i.$$

(See [W], (3.21).) Thus condition (2.9) means that at least one of the measures μ_i satisfies the reverse doubling condition of order 1, at least locally. For Hörmander vector fields in R^n for example, by [NSW], Lebesgue measure even satisfies a reverse doubling condition of order n with respect to the corresponding metric, for suitably small balls. Note that the condition is also satisfied by Lebesgue measure on the real line with the standard metric. Moreover, by [FW], (2.9) is true for Lebesgue measure for the Carnot-Carathéodory metric induced by any collection of Lipschitz continuous vector fields, provided the metric is continuous in the Euclidean topology and Lebesgue measure is a doubling measure for metric balls.

To show that the second term on the right in (2.7) can be absorbed in the first if (2.9) holds for $i = 1$ or 2, it is enough to show that

$$\frac{1}{\mu(B)} \leq C\,k(x,y) \quad \text{if } x,\, y \in cB. \tag{2.10}$$

Without loss of generality, we may suppose that (2.9) holds for $i = 1$. Note that if $x, y \in cB$ then $\rho(x,y) \leq C\rho(B) = C$, and therefore $\rho(x,y)\rho_1(B_1) \leq C\rho_1(B_1)$. Then by (2.9), if $x, y \in cB$ and $x = (x_1, x_2)$,

$$\frac{\rho_1(B_1)}{\mu_1(B_1)} \leq C\,\frac{\rho(x,y)\rho_1(B_1)}{\mu_1(B_1(x_1, \rho(x,y)\rho_1(B_1)))},$$

or, dividing by $\rho_1(B_1)$,

$$\frac{1}{\mu_1(B_1)} \leq C \frac{\rho(x,y)}{\mu_1(B_1(x_1,\rho(x,y)\rho_1(B_1)))}.$$

Moreover, since $x_2 \in cB_2$, we have by doubling and the fact that $\rho(x,y) \leq C$ that

$$\frac{1}{\mu_2(B_2)} \leq C \frac{1}{\mu_2(B_2(x_2,\rho_2(B_2)))} \leq C \frac{1}{\mu_2(B_2(x_2,\rho(x,y)\rho_2(B_2)))}.$$

The desired estimate (2.10) follows by taking the product of the last two, which verifies Remark 1. In fact, instead of assuming that (2.9) holds for one of the component spaces, it would be enough to assume the following reverse doubling condition for μ relative to ρ_B: for all ρ-balls $I \subset CB$,

$$\mu(B) \geq c \frac{\rho(B)}{\rho(I)} \mu(I) \quad \left(= c \frac{\mu(I)}{\rho(I)} \right),$$

since this easily implies (2.10).

We note in passing that in the Hörmander case, (2.8) also follows from [FLW1], Proposition 2.12. See also [CDG].

Remark 2. Inequality (2.8) is true even if we do not assume that hypothesis (2.1) holds provided we assume instead a stronger version of the reverse doubling condition (2.9). (We always assume that (2.2) holds.) This stronger assumption is that a reverse doubling condition of order strictly larger than 1 holds in at least one of the component spaces, i.e., that there exists $\varepsilon > 0$ such that for either $i = 1$ or $i = 2$,

$$\mu_i(B_i) \geq C \left(\frac{\rho_i(B_i)}{\rho_i(I_i)} \right)^{1+\varepsilon} \mu_i(I_i) \quad \text{for every ball } I_i \subset cB_i. \tag{2.11}$$

For example, if $X^{(i)}$ is a collection of Hörmander vector fields in $\mathbb{R}^{N_i}$, then as mentioned earlier, (2.11) holds for Lebesgue measure with $\varepsilon = N_i - 1$ by [NSW]. Of course, (2.11) is not valid for Lebesgue measure if $X^{(i)}$ is the ordinary first derivative in $\mathbb{R}^1$.

The fact that (2.8) holds if we assume only (2.2) and (2.11) for either i follows immediately from the main result of [FLW2] applied to the homogeneous space $\Omega_1 \times \Omega_2$ with the auxiliary metric ρ and product measure μ. In fact, instead of (2.11), we only need to assume that there exists $\varepsilon > 0$ such that for every ρ-ball $I \subset cB$,

$$\mu(B) \geq C\left(\frac{\rho(B)}{\rho(I)}\right)^{1+\varepsilon} \mu(I) \quad \left(= C\left(\frac{1}{\rho(I)}\right)^{1+\varepsilon} \mu(I)\right).$$

Remark 3. The representation formulas (2.7) or (2.8) can be combined with known results about the boundednesss of potential operators on weighted L^p, L^q spaces to obtain an improvement in the statement of Theorem 2. In fact, Theorem 2 holds for $p < q$ if we only assume that the weight w there is a doubling weight for ρ_B-balls rather than being in the class $A_\infty(\mu)$. This result follows immediately from the boundedness estimates in [SW] or [FGuW]. Similarly, the conclusion is valid for $p = q$ and $1 < p < \infty$, if we replace the A_∞ condition by the assumption that there exists $s > 1$ such that w^s is a doubling weight relative to B and the balance condition (1.17) is replaced by the condition

$$\left(\frac{\rho(I)}{\rho(J)}\right)^p \frac{\mathcal{A}_s(I,w)}{w(J)} \leq c\frac{v(I)}{v(J)}$$

for all product balls I, J with the same eccentricity as B and $I \subset J \subset cB$, where

$$\mathcal{A}_s(I,w) = \mu(I)\left(\frac{1}{\mu(I)}\int_I w^s \, d\mu\right)^{1/s}.$$

Note that $w(I) \leq \mathcal{A}_s(I,w)$ for $s > 1$ by Hölder's inequality, and, as is well-known, $w(I)$ and $\mathcal{A}_s(I,w)$ are equivalent if $w \in A_\infty(\mu)$.

3. Isoperimetric inequalities in product spaces. We will use the Poincaré estimates for $p = 1$ to derive analogues of the relative isoperimetric inequality. The classical relative isoperimetric inequality for a bounded open set $E \subset \mathbb{R}^N$ with sufficiently regular boundary ∂E and a Euclidean ball B is

$$\min\{|B \cap E|, |B \backslash E|\}^{1-\frac{1}{N}} \leq cH_{N-1}(B \cap \partial E),$$

where H_{N-1} denotes $(N-1)$-dimensional Hausdorff measure. Some analogues of this estimate which are related to either Hörmander vector fields or vector fields of the type [FL], including weighted versions, are derived in [FLW1]. See also [GN] for unweighted results. By adapting the arguments in [FLW1], we can obtain the following result of relative type. The result is interesting in comparison to the corresponding one for a single space Ω since it allows product balls of any eccentricity, for example, long and thin rectangles.

Theorem 4. *Let $\Omega_i \subset \mathbb{R}_i^N$ $(i = 1,2)$ and $\{X_j^{(i)}\}$ be Carnot-Carathéodory vector fields in Ω_i with associated metric ρ_i. Assume that in each Ω_i, Lebesgue measure is a doubling measure with respect to metric balls. Let $B = B_1 \times B_2$ be a product ball in $\Omega_1 \times \Omega_2$. Assume* (1.11) *holds for Lebesgue measure for all product balls $I \subset cB$ with the same eccentricity as B. Let w and v be weights satisfying the balance condition* (1.17) *(μ is now taken to be Lebesgue measure) for $p = 1$ and some q, $1 < q < \infty$, with v continuous, $v \in A_1$ and w doubling relative to B with respect to Lebesgue measure. If E is an open, bounded, connected subset of $\Omega_1 \times \Omega_2$ whose boundary ∂E is an oriented C^1 manifold such that E lies locally on one side of ∂E, then*

$$\begin{aligned}
&\frac{v(B)}{w(B)^{1/q}} \min\{w(B_\cap E), w(B\backslash E)\}^{1/q} \\
&\qquad \le \; c\rho_1(B_1) \int\limits_{\partial E_\cap B} \Big(\sum_j \langle X_j^{(1)}, \nu\rangle^2\Big)^{1/2} v\, dH_{N_1+N_2-1} \\
&\qquad\qquad + c\rho_2(B_2) \int\limits_{\partial E_\cap B} \Big(\sum_k \langle X_k^{(2)}, \nu\rangle^2\Big)^{1/2} v\, dH_{N_1+N_2-1},
\end{aligned}$$

where ν is the unit normal to ∂E,

$$v(B) = \int_B v(x)\, dx,$$

and the constants c, r_0 are independent of E and B.

To prove Theorem 4, note that the assumptions made in Theorem 4 on the vector fields and weights imply that the conclusion of Theorem 2 holds with $\mu =$ Lebesgue measure for $p = 1$ and q as in the hypothesis of Theorem 4; see Remark 3 in §2 concerning the assumption that $w\,dx$ is a doubling measure. The proof of Theorem 4 is then essentially identical to that of Theorem 3 of [FLW1] with one exception, namely, instead of introducing the balls B_j before (4.3) of [FLW1] to control the factors $\rho_i(B_i)w(B)^{1/q}/v(B)$, $i = 1,2$, we simply leave these factors unchanged throughout the proof. We refer to the reader to [FLW1] for the remaining details.

In the particular case of Lebesgue measure (i.e., when $v = w = 1$) and when the vector fields satisfy the Hörmander condition, the conclusion of Theorem 4 holds with $q = Q/(Q-1)$, where $Q = q_1 + q_2$ and q_1, q_2 satisfy condition (1.4). Since this case is of special interest and importance we state it separately:

Theorem 5. *Let $\Omega_i \subset \mathbb{R}^{N_i}$ $(i = 1,2)$ and $\{X_j^{(i)}\}$ be Hörmander vector fields in Ω_i with metric ρ_i. Let E be an open, bounded, connected subset of $\Omega_1 \times \Omega_2$ whose boundary ∂E is an oriented C^1 manifold such that E lies locally on one side of ∂E. Let K be a compact subset of $\Omega_1 \times \Omega_2$. There exists r_0 depending on K, Ω_i and $\{X_j^{(i)}\}, i = 1,2$, such that if $B = B_1(x_1,r_1) \times B_2(x_2,r_2)$ is the product of balls with $(x_1,x_2) \in K$ and $0 < r_i < r_0$, and if $Q = q_1 + q_2$ with q_i defined by* (1.4), *then*

$$\begin{aligned} &|B|^{1/Q} \min\{|B\cap E| \,,\, |B\backslash E|\}^{(Q-1)/Q} \\ &\quad \le c\rho_1(B_1) \int_{\partial E \cap B} \Big(\sum_j \langle X_j^{(1)}, \nu\rangle^2\Big)^{1/2} dH_{N_1+N_2-1} \\ &\qquad + c\rho_2(B_2) \int_{\partial E \cap B} \Big(\sum_k \langle X_k^{(2)}, \nu\rangle^2\Big)^{1/2} dH_{N_1+N_2-1}, \end{aligned}$$

where ν is the unit normal to ∂E, and the constants c, r_0 are independent of E and B.

As mentioned earlier, all the theorems proved in this paper hold as well for multiple product spaces $\Omega_1 \times \cdots \times \Omega_m$ for any finite $m \ge 3$. The proofs are essentially identical to those given here.

4. Appendix 1: An elementary Poincaré inequality with exact constant. As we pointed out before, we can use Lemma 2 to derive an L^1 Poincaré estimate for arbitrary rectangles in Euclidean space, with an explicit and sharp constant, by iterating the corresponding estimates for intervals on the real line. Since we have not been able to find an existing proof for the one-dimensional case, we include one here.

Lemma 4.1. *Let $f \in \mathrm{Lip}_1([a,b])$ and $-\infty < a < b < \infty$. Then*

$$\int_a^b \left| f(x) - \frac{1}{b-a}\int_a^b f(y)\,dy \right| dx \le \frac{b-a}{2}\int_a^b |f'(x)|\,dx.$$

The constant $(b-a)/2$ is the smallest constant for which the inequality holds.

Proof. To see that $(b-a)/2$ is the best possible constant, it is enough for example to pick $(a,b) = (0,1)$ and consider the Lipschitz continuous functions $f_\varepsilon(x)$, $0 < \varepsilon < 1/2$, defined as follows: $f_\varepsilon(x) = -1$ on $[0, \frac{1}{2} - \varepsilon]$, $+1$ on $[\frac{1}{2} + \varepsilon, 1]$, and linear on $[\frac{1}{2}-\varepsilon, \frac{1}{2}+\varepsilon]$. If we choose $f = f_\varepsilon$ in the inequality above, we see that the right side is 1 for each ε and the left side converges to 1 as $\varepsilon \to 0$. We would like to thank S.-K. Chua for pointing out an error in our original example of this type, and for suggesting the present example.

To prove the inequality itself, fix f and (a, b), and write

$$f(x) - f(y) = \int_y^x f'(z)dz.$$

Integrating with respect to y from a to b and taking the average, we get

$$\begin{aligned}
&f(x) - \frac{1}{b-a}\int_a^b f(y)\,dy \\
&= \frac{1}{b-a}\int_a^b \int_y^x f'(z)\,dz\,dy \\
&= \frac{1}{b-a}\left[\int_a^x \int_y^x f'(z)\,dz\,dy - \int_x^b \int_x^y f'(z)\,dz\,dy\right] \\
&= \frac{1}{b-a}\left[\int_a^x (z-a)f'(z)\,dz - \int_x^b (b-z)f'(z)\,dz\right].
\end{aligned}$$

Then

$$\begin{aligned}
&(b-a)\int_a^b \left| f(x) - \frac{1}{b-a}\int_a^b f(y)\,dy\right| dx \\
&\leq \int_a^b \int_a^x (z-a)\left|f'(z)\right| dz\,dx + \int_a^b \int_x^b (b-z)\left|f'(z)\right| dz\,dx \\
&= \int_a^b \left(\int_z^b dx\right)(z-a)\left|f'(z)\right| dz + \int_a^b \left(\int_a^z dx\right)(b-z)\left|f'(z)\right| dz \\
&= 2\int_a^b (z-a)(b-z)\left|f'(z)\right| dz \\
&\leq \frac{(b-a)^2}{2}\int_a^b \left|f'(z)\right| dz,
\end{aligned}$$

where the last inequality follows from

$$\max_{a\le z\le b}(b-z)(z-a)=\frac{(b-a)^2}{4}.$$

Therefore

$$\int_a^b\left|f(x)-\frac{1}{b-a}\int_a^b f(y)\,dy\right|dx\le\frac{b-a}{2}\int_a^b\left|f'(z)\right|dz,$$

which proves the lemma. □

If we combine the result above with Lemma 2 (or rather with the analogue of Lemma 2 for products of n spaces), we obtain the following sharp L^1 Poincaré estimate for rectangles in $\mathbb{R}^n$:

Let

$$f(x)=f(x_1,\cdots,x_n)\in \mathrm{Lip}_1([a_1,b_1]\times\cdots\times[a_n,b_n]).$$

Then

$$\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\left|f(x)-\frac{1}{\Pi_{i=1}^n(b_i-a_i)}\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}f(y)\,dy\right|dx \tag{4.2}$$
$$\le\sum_{i=1}^n\frac{(b_i-a_i)}{2}\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\left|\frac{\partial f}{\partial x_i}(x)\right|dx.$$

The factor $\frac{1}{2}$ appears on the right above because of the comment we made in the statement of Lemma 2 about the relation of the constants there. This factor is sharp for the given inequality, as can be seen by considering the unit cube and using the functions $f(x)=f_\varepsilon(x_i)$ for any fixed i as before.

5. Appendix 2: An elementary proof of the representation formula in Euclidean space. In $\mathbb{R}^n\times\mathbb{R}^m$ with the usual Euclidean structure, we now present a second proof of the representation formula in Corollary 1 (without the second term there). We include the proof because of its simplicity. It is based on applying a scaling argument to the corresponding formula in a single space. The weighted Poincaré results proved in [ST] can also be obtained by using this kind of scaling (on both the weights and the functions) applied to the corresponding Poincaré results for cubes. This scaling argument also works for product spaces of several homogeneous groups, but we prove it here only for the usual Euclidean case.

Theorem 6. *Let B_1, B_2 be Euclidean balls with radii δ_1, δ_2 and with $B_1 \subset \mathbb{R}^n$, $B_2 \subset \mathbb{R}^m$. If $f(x_1,x_2) \in \mathrm{Lip}(B_1 \times B_2)$, then*

$$\begin{aligned}
&|f(x_1,x_2) - f_{B_1\times B_2}| \\
&\quad \le \frac{C}{|B_1|\;|B_2|} \iint_{B_1\times B_2} \frac{\delta_1|\nabla_1 f(y_1,y_2)| + \delta_2|\nabla_2 f(y_1,y_2)|}{\left(\dfrac{|x_1-y_1|^2}{\delta_1^2} + \dfrac{|x_2-y_2|^2}{\delta_2^2}\right)^{(n+m-1)/2}}\, dy_1\, dy_2.
\end{aligned}$$

Proof. By translation, it is enough to prove the estimate for balls centered at the origin. Let Q_1 and Q_2 denote the unit balls in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. Define $f_\delta(x_1,x_2)$ on $Q_1 \times Q_2$ by $f_\delta(x_1,x_2) = f(\delta_1 x_1, \delta_2 x_2)$. By using the known representation formula on $Q_1 \times Q_2$, if $(x_1,x_2) \in Q_1 \times Q_2$ then

$$\begin{aligned}
&|f_\delta(x_1,x_2) - (f_\delta)_{Q_1\times Q_2}| \\
&\quad \le \iint_{Q_1\times Q_2} \frac{|\nabla_1 f_\delta(y_1,y_2)| + |\nabla_2 f_\delta(y_1,y_2)|}{|(x_1,x_2)-(y_1,y_2)|^{n+m-1}}\, dy_1 dy_2 \\
&\quad = \iint_{Q_1\times Q_2} \frac{\delta_1|\nabla_1 f(\delta_1 y_1,\delta_2 y_2)| + \delta_2|\nabla_2 f(\delta_1 y_1,\delta_2 y_2)|}{|(x_1,x_2)-(y_1,y_2)|}\, dy_1\, dy_2 \\
&\quad = C\delta_1^{-n}\delta_2^{-m} \iint_{\delta_1 Q_1\times \delta_2 Q_2} \frac{\delta_1|\nabla_1 f(y_1,y_2)| + \delta_2|\nabla_2 f(y_1,y_2)|}{|(x_1,x_2)-(\delta_1^{-1}y_1,\delta_2^{-1}y_2)|^{n+m-1}}\, dy_1\, dy_2.
\end{aligned}$$

Note that

$$\begin{aligned}
&|(x_1,x_2)-(\delta_1^{-1}y_1,\delta_2^{-1}y_2)|^{n+m-1} \\
&\qquad = \left[|x_1-\delta_1^{-1}y_1|^2 + |x_2-\delta_2^{-1}y_2|^2\right]^{(n+m-1)/2} \\
&\qquad = \left[\frac{|\delta_1 x_1 - y_1|^2}{\delta_1^2} + \frac{|\delta_2 x_2 - y_2|^2}{\delta_2^2}\right]^{(n+m-1)/2}.
\end{aligned}$$

Since $B_1 = \delta_1 Q_1$ and $B_2 = \delta_2 Q_2$, we get

$$\begin{aligned}
&|f(\delta_1 x_1,\delta_2 x_2) - (f_\delta)_{Q_1\times Q_2}| \\
&\quad \le \frac{C}{|B_1||B_2|} \iint_{B_1\times B_2} \frac{\delta_1|\nabla_1 f(y_1,y_2)| + \delta_2|\nabla_2 f(y_1,y_2)|}{\left[\dfrac{|\delta_1 x_1-y_1|^2}{\delta_1^2} + \dfrac{|\delta_2 x_2-y_2|^2}{\delta_2^2}\right]^{(n+m-1)/2}}\, dy_1\, dy_2
\end{aligned}$$

if $x_1 \in Q_1$ and $x_2 \in Q_2$. Replacing $\delta_1 x_1$ by x_1 and $\delta_2 x_2$ by x_2, we obtain that for $(x_1, x_2) \in B_1 \times B_2$,

$$|f(x_1,x_2) - (f_\delta)_{Q_1\times Q_2}| \\ \leq \frac{C}{|B_1||B_2|} \iint_{B_1\times B_2} \frac{\delta_1|\nabla_1 f(y_1,y_2)| + \delta_2|\nabla_2 f(y_1,y_2)|}{\left[\dfrac{|x_1-y_1|^2}{\delta_1^2} + \dfrac{|x_2-y_2|^2}{\delta_2^2}\right]^{(n+m-1)/2}} \, dy_1\, dy_2.$$

Since $(f_\delta)_{Q_1\times Q_2} = f_{B_1\times B_2}$, the result follows. □

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Guozhen Lu was partially supported by NSF grant DMS#93-15963 and DMS#96-22996, and Richard L. Wheeden was partially supported by NSF grant DMS#95-00799.

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Received: July 29th, 1997; revised: October 21st, 1997.