Constr. Approx. (2004) 20: 647–668
DOI: 10.1007/s00365-003-0553-2

CONSTRUCTIVE APPROXIMATION

Simultaneous Representation and Approximation Formulas and High-Order Sobolev Embedding Theorems on Stratified Groups

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Abstract. We give various integral representation formulas simultaneously for a function and its derivatives in terms of vector field gradients of the function of appropriately high order. When the function has compact support, simpler formulas can be derived. Many of the results proved here appear to be new even in the special case of classical Euclidean space. For instance, Theorem 2.2 below reduces to the following result in the usual Euclidean case:

Let B be a ball in $\mathbf{R}^N$ with radius $r(B)$, let m be a positive integer, and let $f \in W^{m,1}(B)$. Then there is a polynomial $P = \pi_m(B, f)$ of degree $m-1$ such that for any integers i, j with $0 \le j < i \le m$ and a.e. $x \in B$,

$$|\nabla^j(f-P)(x)| \le C \int_B \frac{|\nabla^i f(y)|}{|x-y|^{N-(i-j)}}\, dy + Cr(B)^{i-j-N} \int_B |\nabla^i f(y)|\, dy.$$

Moreover, if $0 < i - j \le N$, then for a.e. $x \in B$ we have the more refined formula

$$|\nabla^j(f-P)(x)| \le C \int_B \frac{|\nabla^i f(y)|}{|x-y|^{N-i+j}}\, dy.$$

1. Introduction

In recent papers [17] and [11] by the authors, a relationship between higher-order Poincaré inequalities and representation formulas has been established in fairly general spaces of homogeneous type. This extends the first-order result derived in [10]. In particular, an appropriate notion of polynomials in metric spaces is introduced in [17] and [11], and it is shown that the existence of polynomials satisfying L^1 to L^1 Poincaré inequalities implies higher-order representation formulas (see [16] for a more refined result which assumes only an L^1 to L^p Poincaré inequality for some $0 < p < 1$). In the case of stratified Lie groups (also known as Carnot groups), where polynomials do exist,

Date received: October 31, 2002. Date revised: July 8, 2003. Date accepted: July 17, 2003. Communicated by Robert C. Sharpley. Online publication: March 3, 2004.
AMS classification: 46E35.
Key words and phrases: Representation formulas, Poincaré inequalities, Embedding theorems, Polynomials, Stratified groups, Higher-order vector fields.

higher-order representation formulas are derived in [17] for functions in the Sobolev spaces defined by Folland and Stein in [8]. An example of such a formula (actually, an inequality) is

$$|f(x) - P(x)| \le C \int_B |X^m f(y)| \frac{\rho(x, y)^m}{|B(x, \rho(x, y))|}\, dy, \qquad x \in B,$$

where P denotes a "polynomial" depending on f and B of degree $m-1$, X^m denotes an appropriate differential operator of order m associated naturally with the metric $\rho(x, y)$, B is a metric ball, and $B(x, r)$ is the metric ball with center x and radius r. The constant C here should be independent of f, x, and B.

In order to be more precise, we now review some preliminaries concerning stratified Lie groups. We refer the reader to the books [8] and [26] for further details. Let $\mathcal{G}$ be a finite-dimensional, stratified, nilpotent Lie algebra. Assume that

$$\mathcal{G} = \bigoplus_{i=1}^{s} V_i,$$

with $[V_i, V_j] \subset V_{i+j}$ for $i + j \le s$ and $[V_i, V_j] = 0$ for $i + j > s$. Let $X_1, \ldots, X_l$ be a basis for V_1 and suppose that $X_1, \ldots, X_l$ generate $\mathcal{G}$ as a Lie algebra. Then for $2 \le j \le s$, we can choose a basis $\{X_{ij}\}$, $1 \le i \le k_j$, for V_j consisting of commutators of length j. We set $k_1 = l$ and $X_{i1} = X_i$, $i = 1, \ldots, l$, and we call X_{i1} a commutator of length 1.

If $\mathbf{G}$ is the simply connected Lie group associated with $\mathcal{G}$, then the exponential mapping is a global diffeomorphism from $\mathcal{G}$ to $\mathbf{G}$. Thus, for each $g \in \mathbf{G}$, there is $x = (x_{ij}) \in \mathbf{R}^N$, $1 \le i \le k_j$, $1 \le j \le s$, $N = \sum_{j=1}^{s} k_j$, such that

$$g = \exp\left(\sum x_{ij} X_{ij}\right).$$

A homogeneous norm function $|\cdot|$ on $\mathbf{G}$ is defined by

$$|g| = \left(\sum |x_{ij}|^{2s!/j}\right)^{1/2s!},$$

and $Q = \sum_{j=1}^{s} j k_j$ is said to be the **homogeneous dimension** of $\mathbf{G}$. The dilation δ_r, $r > 0$, on $\mathbf{G}$ is defined by

$$\delta_r(g) = \exp\left(\sum r^j x_{ij} X_{ij}\right) \qquad \text{if} \quad g = \exp\left(\sum x_{ij} X_{ij}\right).$$

We call a curve $\gamma : [a, b] \to \mathbf{G}$ a horizontal curve connecting two points $x, y \in \mathbf{G}$ if $\gamma(a) = x$, $\gamma(b) = y$, and $\gamma'(t) \in V_1$ for all t. Then the Carnot–Carathéodory distance between x, y is defined as

$$\rho_{cc}(x, y) = \inf_{\gamma} \int_a^b \langle \gamma'(t), \gamma'(t) \rangle^{1/2}\, dt,$$

where the infimum is taken over all horizontal curves γ connecting x and y. It is known that any two points x, y on $\mathbf{G}$ can be joined by a horizontal curve of finite length and then ρ_{cc} is a left-invariant metric on $\mathbf{G}$. Associated with this metric, we can define the metric ball centered at x and with radius r associated with this metric by

$$B_{cc}(x, r) = \{y : \rho_{cc}(x, y) < r\}.$$

We note that the metric ρ_{cc} is equivalent to the metric $d(x, y) = |x^{-1}y|$ defined by the homogeneous norm $|\cdot|$ in the following sense:

$$C_1 d(x, y) \leq \rho_{cc}(x, y) \leq C_2 d(x, y),$$

where C_1 and C_2 are positive constants which are independent of x, y. We denote the metric ball associated with d by $D(x, r) = \{y \in \mathbf{G} : d(x, y) < r\}$. An important feature of both of these distance functions is that they, and thus the associated metric balls, are left-invariant, namely,

$$\rho_{cc}(zx, zy) = \rho_{cc}(x, y), \qquad B_{cc}(x, r) = xB_{cc}(0, r),$$

and

$$d(zx, zy) = d(x, y), \qquad D(x, r) = xD(0, r).$$

However, we will use the metric ρ_{cc} and the metric balls $B_{cc}(x, r)$ in this paper, and we will drop the subscript "cc" from both ρ_{cc} and $B_{cc}(x, r)$, and write simply $\rho(x, y)$ and $B(x, r)$. We use $r(B)$ to denote the radius of a ball B. The Lebesgue measure of a ball B is known to satisfy $|B| = C_Q r(B)^Q$.

We now recall the definition of the class of polynomials on $\mathbf{G}$ given by Folland and Stein [8]. Let $X_1, \ldots, X_l$ in V_1 be the generators of the Lie algebra $\mathcal{G}$, and let $X_1, \ldots, X_l, \ldots, X_N$ be a basis of $\mathcal{G}$. We define $d(X_j) = d_j$ to be the length of X_j as a commutator, and we arrange the order so that $1 \leq d_1 \leq \cdots \leq d_N$. Then it is easy to see that $d_j = 1$ for $j = 1, \ldots, l$. Let $\xi_1, \ldots, \xi_N$ be the dual basis for $\mathcal{G}^*$, and let $\eta_i = \xi_i \circ \exp^{-1}$. Each η_i is a real-valued function on $\mathbf{G}$, and $\eta_1, \ldots, \eta_N$ gives a system of global coordinates on $\mathbf{G}$. A function P on $\mathbf{G}$ is said to be a polynomial on $\mathbf{G}$ if $P \circ \exp$ is a polynomial on $\mathcal{G}$. Every polynomial on $\mathbf{G}$ can be written uniquely as

$$P(x) = \sum_I a_I \eta^I(x), \qquad \eta^I = \eta_1^{i_1} \cdots \eta_N^{i_N}, \quad a_I \in \mathbf{R},$$

where all but finitely many of the coefficients a_I vanish. Clearly η^I is homogeneous of degree $d(I) = \sum_{j=1}^N i_j d_j$, i.e., $\eta^I(\delta_r x) = r^{d(I)} \eta_i(x)$. If $P = \sum_I a_I \eta^I$, then we define the homogeneous degree (or order) of P to be $\max\{d(I) : a_I \neq 0\}$.

Throughout this paper, for each positive integer m, we use $\mathcal{P}_m$ to denote the class of polynomials of homogeneous degree strictly less than m.

We adopt the following multi-index notation for higher-order derivatives. If $I = (i_1, \ldots, i_N) \in \mathbf{N}^N$, we set

$$X^I = X_1^{i_1} \cdot X_2^{i_2} \cdots X_N^{i_N}.$$

By the Poincaré–Birkhoff–Witt theorem (see Bourbaki [3, I.3.7]), the differential operators X^I form a basis for the algebra of left-invariant differential operators in $\mathbf{G}$. Furthermore, we set

$$|I| = i_1 + i_2 + \cdots + i_N, \qquad d(I) = d_1 i_1 + d_2 i_2 + \cdots + d_N i_N.$$

Thus, $|I|$ is the order of the differential operator X^I, and $d(I)$ is its degree of homogeneity; $d(I)$ is called the homogeneous degree of X^I. We also denote

$$|X^m f| = \left(\sum_{I:d(I)=m} |X^I f|^2 \right)^{1/2}$$

for any positive integer m.

Let m be a positive integer, $1 \leq p < \infty$, and let Ω be an open set in $\mathbf{G}$. The Folland–Stein Sobolev space $W^{m,p}(\Omega)$ associated with the vector fields $X_1, \ldots, X_l$ is defined to consist of all functions $f \in L^p(\Omega)$ with distributional derivatives $X^I f \in L^p(\Omega)$ for every X^I defined above with $d(I) \leq m$. Here we say that the distributional derivative $X^I f$ exists and equals a locally integrable function g_I if, for every $\varphi \in C_0^\infty(\Omega)$,

$$\int_\Omega f X^I \varphi \, dx = (-1)^{d(I)} \int_\Omega g_I \varphi \, dx.$$

$W^{m,p}(\Omega)$ is equipped with the norm

$$\|f\|_{W^{m,p}(\Omega)} = \|f\|_{L^p(\Omega)} + \sum_{1 \leq d(I) \leq m} \|X^I f\|_{L^p(\Omega)}. \tag{1.1}$$

When $\Omega = \mathbf{G}$, we sometimes use $\|f\|_{m,p}$ to denote $\|f\|_{W^{m,p}(\mathbf{G})}$. We also sometimes use $\|f\|_{m,p;\Omega}$ to denote $\|f\|_{W^{m,p}(\Omega)}$. If $f \in W^{m,p}(\Omega)$ for some m and p, we will refer to f as a Sobolev function.

We now recall some possible choices of polynomials associated with any given Sobolev function $f \in W^{m,p}(\Omega)$. These polynomials are useful in obtaining higher-order Poincaré inequalities for f. The following result is Theorem 3.7 in [14]:

Theorem 1.2. *Let $\Omega \subset \mathbf{G}$ be an open set of finite Lebesgue measure. Then given any positive integer m and $f \in W^{m,1}(\Omega)$, there exists a unique polynomial $P = P_m(\Omega, f)$ on $\mathbf{G}$ of degree less than m such that*

$$\int_\Omega X^I(f - P) = 0 \qquad \text{for all } I \text{ with} \quad 0 \leq d(I) < m. \tag{1.3}$$

On the Heisenberg group, the existence of polynomials satisfying (1.3) was proved in [20]. See also [21] for general Carnot groups.

In any case, let us show that

$$X^J P_m(B, f) = P_{m-d(J)}(B, X^J f)$$

for each J. In fact, $P_m(B, f)$ is the unique polynomial P with

$$\int_B X^I(f - P) = 0 \qquad \text{for all } I \text{ with} \quad d(I) < m,$$

and, consequently,

$$\int_B X^K(X^J(f - P_m(B, f))) = 0 \qquad \text{if} \quad d(K) < m - d(J),$$

i.e.,

$$\int_B X^K(X^J f - X^J P_m(B, f)) = 0 \qquad \text{if} \quad d(K) < m - d(J),$$

and the formula follows from the uniqueness property.

In [14], a second class of polynomials associated with Sobolev functions is considered. Polynomials in this class are called "projection polynomials" and are described in the next definition.

Definition 1.4. For each $m \in \mathbf{N}$ and ball $B \subset \mathbf{G}$, a projection of order m associated with B is defined to be a linear map

$$\pi_m(B, \cdot) : W^{m,1}(B) \to \mathcal{P}_m$$

such that the following two properties hold:

$$\sup_{x \in B} |\pi_m(B, f)(x)| \le C r(B)^{-Q} \|f\|_{L^1(B)}, \tag{1.5}$$

with C independent of f and B and

$$\pi_m(B, P) = P \qquad \text{for all} \quad P \in \mathcal{P}_m. \tag{1.6}$$

We will refer to $\pi_m(B, f)$ as a projection polynomial of order $m - 1$ associated with B and f.

The polynomials constructed in Theorem 1.2 may not satisfy (1.5). The existence of projection polynomials is proved in Theorem 3.6 in [14]. It is also shown there (see Theorem 3.8) that the following result holds:

Theorem 1.7. *Let $m \in \mathbf{N}$ and let B be a ball in Ω. Then for any projection $\pi_m(B, \cdot) : W^{m,1}(B) \to \mathcal{P}_m$, any q with $1 \le q \le \infty$, and any multiple index I with $d(I) = i \ge 0$,*

$$\|X^I \pi_m(B, f)\|_{L^q(B)} \le C \|X^i f\|_{L^q(B)}, \tag{1.8}$$

with C independent of f and B.

This shows that a subelliptic derivative of $\pi_m(B, f)$ is controlled by the same order subelliptic derivative of f.

We also recall the notions of a Boman chain domain and (see [17]) a "weak" Boman chain domain.

Definition 1.9. A domain (i.e., an open connected set) Ω in $\mathbf{G}$ is said to satisfy the Boman chain condition of type σ, M, or to be a member of $\mathcal{F}(\sigma, M)$, if there exist constants $\sigma > 1$, $M > 0$, and a family $\mathcal{F}$ of metric balls $B \subset \Omega$ such that:

(i) $\Omega = \bigcup_{B \in \mathcal{F}} B$.
(ii) $\sum_{B \in \mathcal{F}} \chi_{\sigma B}(x) \le M \, \chi_{\Omega}(x)$ for all $x \in \mathcal{S}$.

(iii) There is a "central ball" $B_0 \in \mathcal{F}$ such that for each ball $B \in \mathcal{F}$, there is a positive integer $k = k(B)$ and a chain of balls $\{B_j\}_{j=0}^{k}$ for which $B_k = B$ and each $B_j \cap B_{j+1}$ contains a ball D_j with $B_j \cup B_{j+1} \subset MD_j$.

(iv) $B \subset MB_j$ for all $j = 0, \ldots, k(B)$ with B and B_j as in (iii).

If we replace the hypothesis that $\sigma > 1$ by $\sigma = 1$, we say that Ω satisfies the weak Boman chain condition.

It follows from condition (iv) that such domains are bounded.

To motivate the results of this paper, we need to review some known facts about high-order representation formulas on stratified groups. The following two theorems are special cases of more general ones in [17].

Theorem 1.10. *Let B be a ball in $\mathbf{G}$, let m be a positive integer, and let $f \in W^{m,1}(B)$. Then for either of the polynomials $P = P_m(B, f)$ or $P = \pi_m(B, f)$ of order less than m and a.e. $x \in B$,*

$$|f(x) - P(x)| \le C \int_B |X^m f(y)| \frac{\rho(x, y)^m}{|B(x, \rho(x, y))|}\, dy + C \frac{r(B)^m}{|B|} \int_B |X^m f(y)|\, dy.$$

Moreover, if $m \le Q$, then for a.e. $x \in B$,

$$|f(x) - P(x)| \le C \int_B |X^m f(y)| \frac{\rho(x, y)^m}{|B(x, \rho(x, y))|}\, dy.$$

The constant C is independent of f, x, and B.

For a weak Boman chain domain, we have also proved the next result in [17].

Theorem 1.11. *Let Ω be a weak Boman chain domain in $\mathbf{G}$ with a central ball B_0, and let $f \in W^{m,1}(\Omega)$. If $1 \le m \le Q$, then for a.e. $x \in \Omega$ and either of the polynomials $P = P_m(B_0, f)$ or $P = \pi_m(B_0, f)$ of order less than m,*

$$|f(x) - P(x)| \le C \int_\Omega |X^m f(y)| \frac{\rho(x, y)^m}{|B(x, \rho(x, y))|}\, dy.$$

These two theorems essentially say that every Sobolev function can be approximated pointwise by polynomials over metric balls or Boman chain domains, and the remainders are controlled by the fractional integral of higher-order derivatives of the function.

We remark that, even in the special case of ordinary Euclidean space $\mathbf{R}^N$ (i.e., $\mathbf{R}^N$ with the usual Euclidean metric), results of the above type were previously known only for domains Ω which are star-shaped with respect to an open set $D \subset \Omega$ (see [22], [23], [19], or [1, p. 217]). The choice of polynomial in [1] is the usual Taylor polynomial and thus may not have the properties of the polynomials $P_m(B, f)$ or $\pi_m(B, f)$ given in Theorems 1.2 and 1.7. When the function has compact support in a domain $\Omega \subset \mathbf{R}^N$, the polynomial in Theorem 1.11 can be dropped (see [22], [23], [19], [1]). We also mention that polynomials are used to measure the smoothness of functions by considering their maximal functions in Euclidean space (see [6]).

Motivated by Theorems 1.10 and 1.11 for Carnot groups and by the work in the special case of Euclidean space, we ask the following natural questions for Sobolev functions on Carnot groups:

Question 1. Can we approximate a Sobolev function f and its derivatives simultaneously? More precisely, can we approximate not only f by the associated polynomial but also approximate the derivatives of f by the corresponding derivatives of the same polynomial?

Question 2. When f has compact support, can we replace the polynomial by zero in the representation inequality?

The main purpose of this paper is to answer these questions affirmatively and give some applications. We remark that our results appear to be new even in the usual Euclidean case. Concerning the second question above, we will show that a representation formula for higher-order Sobolev functions with compact support (with the polynomials $P_m(B, f)$ and $\pi_m(B, f)$ that we used in Theorems 1.2 and 1.7 then replaced by 0) follows by a limit argument from the formula for functions without compact support. High-order representation formulas for functions with compact support cannot presently be derived by the process of integration by parts against the fundamental solutions of arbitrarily high-order subelliptic operators because such fundamental solutions and their estimates are not known to exist. However, by using a limit argument and the polynomials $\pi_m(B, f)$, we will be able to derive the pointwise representation formula without any polynomial on the left side for functions with compact support. This is done by first deriving a formula for global Sobolev functions (i.e., Sobolev functions in the whole space **G**), from which the formula for functions with compact support follows easily (see Section 3). Such an argument can also be used to derive higher-order Sobolev inequalities for functions with compact support from higher-order Poincaré inequalities (see Section 4). For functions without compact support, we need the simultaneous representation formulas to derive the simultaneous Poincaré and exponential estimates in Section 6.

The organization of the paper is as follows. In Section 2 we state and prove the simultaneous representation formulas. If we allow the derivatives on the right-hand side of the formula to be of top order (i.e., $|X^m f|$), then we can choose either of the polynomials $P_m(B, f)$ or $\pi_m(B, f)$ on the left-hand side, i.e., we can approximate $X^J f$ by either $X^J P_m(B, f)$ or $X^J \pi_m(B, f)$ for all J with $d(J) < m$. This is Theorem 2.1. If we choose to use the polynomials $\pi_m(B, f)$, then we can approximate $X^J f$ by $X^J \pi_m(B, f)$ for $d(J) < m$ by instead using on the right-hand side only $|X^i f|$ for any i with $d(J) < i \leq m$. This is Theorem 2.2. In Section 3 we answer Questions 2 and 3 above by proving representation formulas of Sobolev type, i.e., formulas for compactly supported f and with no polynomial on the left-hand side. Theorem 3.1 is a representation formula for globally defined functions with sufficiently small growth at ∞, and Theorem 3.2 treats the special case of functions with compact support. Section 4 deals with higher-order Sobolev norm inequalities as opposed to pointwise estimates. We mention here that Sobolev norm inequalities of the same type have also been obtained in [5] for functions vanishing on a set of positive Lebesgue measure or, more generally, vanishing on a set of positive Bessel capacity, by using the Bessel potential estimates for stratified groups given in [15]. However, the derivation here is of independent interest and simpler. In Sections 5 and 6 we give simultaneous weighted higher-order Poincaré and Sobolev inequalities as well as exponential estimates and L^∞ estimates. Their derivation uses the simultaneous pointwise representation formulas in Sections 2 and 3.

2. Simultaneous Representation Formulas

The main results of this section are given in the next three theorems.

Theorem 2.1. *Let B be a ball, let m be a positive integer, and let $f \in W^{m,1}(B)$. Then for either of the polynomials $P = P_m(B, f)$ or $P = \pi_m(B, f)$, any integer j with $0 \le j < m$, and a.e. $x \in B$,*

$$|X^j(f - P)(x)| \le C \int_B |X^m f(y)| \frac{\rho(x, y)^{m-j}}{|B(x, \rho(x, y))|}\, dy + C \frac{r(B)^{m-j}}{|B|} \int_B |X^m f(y)|\, dy,$$

where as usual $|X^j f| = (\sum_{d(I)=j} |X^I f|^2)^{1/2}$.

Moreover, if Q is the homogeneous dimension of **G** *and $0 < m - j \le Q$, then for a.e $x \in B$ we have the more refined formula*

$$|X^j(f - P)(x)| \le C \int_B |X^m f(y)| \frac{\rho(x, y)^{m-j}}{|B(x, \rho(x, y))|}\, dy.$$

Theorem 2.2. *Let B be a ball, let m be a positive integer, and let $f \in W^{m,1}(B)$. Let i, j be integers with $0 \le j < i \le m$. Then for the polynomial $P = \pi_m(B, f)$ and a.e. $x \in B$,*

$$|X^j(f - P)(x)| \le C \int_B |X^i f(y)| \frac{\rho(x, y)^{i-j}}{|B(x, \rho(x, y))|}\, dy + C \frac{r(B)^{i-j}}{|B|} \int_B |X^i f(y)|\, dy.$$

Moreover, if $0 < i - j \le Q$, then for a.e. $x \in B$ we have the more refined formula

$$|X^j(f - P)(x)| \le C \int_B |X^i f(y)| \frac{\rho(x, y)^{i-j}}{|B(x, \rho(x, y))|}\, dy.$$

In the case $i = m$, Theorem 2.2 is included in Theorem 2.1 and, in that case, is the same as the part of Theorem 2.1 for $P = \pi_m(B, f)$. The case $j = 0$ of Theorem 2.1 is contained in Theorem A of [17]; see Theorem 2.6 below and the discussion which precedes it.

We can also show that the second estimate in Theorem 2.2 remains valid when $i - j \le Q$ if the ball B is replaced by a weak Boman chain domain in **G**. This is stated in the next theorem.

Theorem 2.3. *Let m be a positive integer, let Ω be a weak Boman chain domain in* **G** *with a central ball B_0, and let $f \in W^{m,1}(\Omega)$. Let i, j be integers with $0 \le j < i \le m$ and $i - j \le Q$. Then for the polynomial $P = \pi_m(B_0, f)$ and a.e. $x \in \Omega$,*

$$|X^j(f - P)(x)| \le C \int_\Omega |X^i f(y)| \frac{\rho(x, y)^{i-j}}{|B(x, \rho(x, y))|}\, dy.$$

To prove these theorems, we will use the next three results from [13], [14], and [17] concerning higher-order Poincaré inequalities.

Theorem 2.4. *Let m be a positive integer, $p \geq 1$, let B be a ball, and let $f \in W^{m,p}(B)$. Then for either of the polynomials $P = P_m(B, f)$ or $P = \pi_m(B, f)$ and any integer j with $0 \leq j < m$,*

$$\left(\frac{1}{|B|}\int_B |X^j(f-P)(x)|^{q_{mj}}\,dx\right)^{1/q_{mj}} \leq Cr(B)^{m-j}\left(\frac{1}{|B|}\int_B |X^m f(x)|^p\,dx\right)^{1/p}$$

for all $1 \leq p < Q/(m-j)$ and $q_{mj} = pQ/[Q-(m-j)p]$, where C is independent of B and f.

In fact, more general L^p, L^q analogues of Theorem 2.4 are proved in [13], [14], and they follow from repeated use of the Poincaré inequalities of first order proved, e.g., in [18], [9], [12]. The proofs also use the vanishing integral property (1.3) of the polynomial $P_m(B, f)$.

If we choose the projection polynomial $\pi_m(B, f)$, then Theorem 2.4 can be improved as follows (see [14, Theorem 6.3]):

Theorem 2.5. *Let m be a positive integer, $p \geq 1$, let B be a ball, and let $f \in W^{m,p}(B)$. Then for any integers i, j with $0 \leq j < i \leq m$,*

$$\left(\frac{1}{|B|}\int_B |X^j(f-\pi_m(B,f))(x)|^{q_{ij}}\,dx\right)^{1/q_{ij}} \leq Cr(B)^{i-j}\left(\frac{1}{|B|}\int_B |X^i f(x)|^p\,dx\right)^{1/p}$$

for all $1 \leq p < Q/(i-j)$ and $q_{ij} = pQ/[Q-(i-j)p]$, where C is independent of B and f.

In fact, we only need the special cases of Theorems 2.4 and 2.5 when $p = 1$ (and with q_{mj}, q_{ij} replaced by 1) in order to prove Theorems 2.1–2.3, but the general cases will be used in Section 4.

Since we may not have the vanishing integral property (1.3) for the projection polynomial $\pi_m(B, f)$, the proof of Theorem 2.5 does not follow immediately by iteration from the Poincaré inequality of first order. The interesting feature of the theorem is that even for $i < m$ (thus the degree of $\pi_m(B, f)$ is larger than $i-1$), the left-hand side is controlled by the ith-order derivatives of f alone. For the convenience of the reader, we now reproduce a proof given in [14].

Proof of Theorem 2.5. We will use Theorem 2.4 in the proof. Given $0 \leq j < i \leq m$, let $P_i(B, f)$ be the polynomial of degree less than i guaranteed by Theorem 1.2. Then

$$\left(\frac{1}{|B|}\int_B |X^j(f-\pi_m(B,f))(x)|^{q_{ij}}\,dx\right)^{1/q_{ij}}$$
$$\leq \left(\frac{1}{|B|}\int_B |X^j(f-P_i(B,f))(x)|^{q_{ij}}\,dx\right)^{1/q_{ij}}$$

$$
\begin{aligned}
&+ \left(\frac{1}{|B|} \int_B |X^j (P_i(B, f) - \pi_m(B, f))(x)|^{q_{ij}} \, dx \right)^{1/q_{ij}} \\
&= I_1 + I_2,
\end{aligned}
$$

where I_1 and I_2 are defined by the last equality. The term I_1 is bounded by the expression on the right-hand side of the conclusion of the theorem by Theorem 2.4 with m replaced by i there. Thus we only need to estimate I_2. By Bernstein's inequality (see, e.g., [14]),

$$
\begin{aligned}
I_2 &\le Cr(B)^{-j} \frac{1}{|B|} \int_B |P_i(B, f)(x) - \pi_m(B, f)(x)| \, dx \\
&= Cr(B)^{-j} \frac{1}{|B|} \int_B |\pi_m(B, P_i(B, f) - f)(x)| \, dx \\
&\le Cr(B)^{-j} \sup_{x \in B} |\pi_m(B, P_i(B, f) - f)(x)| \\
&\le C \frac{r(B)^{-j}}{|B|} \int_B |f(x) - P_i(B, f)(x)| \, dx,
\end{aligned}
$$

where in the equality we have used (1.6), and in the last inequality we have used (1.5). We can estimate the last term by using the Poincaré inequality given in Theorem 2.4: in fact, the last term is at most

$$
Cr(B)^{i-j} \left(\frac{1}{|B|} \int_B |X^i f(x)|^p \, dx \right)^{1/p}, \qquad 1 \le p < \frac{Q}{i},
$$

which finishes the proof of Theorem 2.5. ■

To prove Theorems 2.1, 2.2, and 2.3, we will also need some results from [17, Theorems A, C], including a relationship between higher-order Poincaré inequalities and representation formulas. However, we need these results only in the special case of stratified groups; more general metric spaces are considered in [17]. For stratified groups, the properties required of polynomials in [17, Theorems A, C] are true by [8]. The result that we need is stated in the next theorem.

Theorem 2.6. *Let B_0 be a ball in $\mathbf{G}$, let m be a positive integer, and let f, g be integrable functions on B_0. Suppose that for each ball $B \subset B_0$, there is a polynomial $p(B) = p(B, f, g, m)$ of order less than m such that*

$$
\int_B |f(x) - p(B)(x)| \le cr(B)^m \int_B |g(x)| \, dx \tag{2.7}
$$

for an absolute constant c. Then for a.e. $x \in B_0$,

$$
|f(x) - p(B_0)(x)| \le C \int_{B_0} |g(y)| \frac{\rho(x, y)^m}{|B(x, \rho(x, y))|} \, dy + C \frac{r(B_0)^m}{|B_0|} \int_{B_0} |g(y)| \, dy,
$$

and if $m \le Q$, *then for a.e.* $x \in B_0$,

$$|f(x) - p(B_0)(x)| \le C \int_{B_0} |g(y)| \frac{\rho(x, y)^m}{|B(x, \rho(x, y))|}\, dy.$$

The constant C *is independent of* f, g, x, *and* B_0.

Moreover, if Ω *is a weak Boman chain domain in* $\mathbf{G}$ *with a central ball* B_0, *and* (2.7) *holds for each ball* $B \subset \Omega$, *then for a.e.* $x \in \Omega$,

$$|f(x) - p(B_0)(x)| \le C \int_{\Omega} |g(y)| \frac{\rho(x, y)^m}{|B(x, \rho(x, y))|}\, dy.$$

In particular, since (2.7) is true with $g = |X^m f|$ for either of the polynomials $P_m(B, f)$ or $\pi_m(B, f)$, we obtain Theorems 1.10 and 1.11 as immediate corollaries of Theorem 2.6.

We now turn to the proofs of the main theorems in this section. The proofs are simple corollaries of Theorem 2.6.

Proof of Theorem 2.1. Fix m, B_0 and a function $f \in W^{m,1}(B_0)$. Given J with $d(J) = j < m$, let $F = X^J f$ and $g = |X^m f|$. By Theorem 2.4, for each ball $B \subset B_0$, and either of the polynomials $p(m, F, B) = X^J P_m(B, f)$ or $p(m, F, B) = X^J \pi_m(B, f)$, we have

$$\int_B |F(x) - p(m, F, B)|\, dx \le C r(B)^{m-j} \int_B g(x)\, dx,$$

where C is independent of B, f, and F. Thus, by Theorem 2.6 with m replaced by $m - j$, it follows that, for a.e. $x \in B_0$,

$$|F(x) - p(m, F, B_0)(x)| \le C \int_{B_0} g(y) \frac{\rho(x, y)^{m-j}}{|B(x, \rho(x, y))|}\, dy + C \frac{r(B_0)^{m-j}}{|B_0|} \int_{B_0} g(y)\, dy.$$

Similarly, if $m - j \le Q$, then for a.e. $x \in B_0$,

$$|F(x) - p(m, F, B_0)(x)| \le C \int_{B_0} g(y) \frac{\rho(x, y)^{m-j}}{|B(x, \rho(x, y))|}\, dy.$$

The constants C are independent of f, F, x, and B_0. This is equivalent to saying that for $P = P_m(B_0, f)$ or $P = \pi_m(B_0, f)$ and a.e. $x \in B_0$,

$$\begin{aligned} &|X^J f(x) - X^J P(x)| \\ &\qquad \le C \int_{B_0} |X^m f(y)| \frac{\rho(x, y)^{m-j}}{|B(x, \rho(x, y))|}\, dy + C \frac{r(B_0)^{m-j}}{|B_0|} \int_{B_0} |X^m f(y)|\, dy, \end{aligned}$$

and if, in addition, $m - j \le Q$, then

$$|X^J f(x) - X^J P(x)| \le C \int_{B_0} |X^m f(y)| \frac{\rho(x, y)^{m-j}}{|B(x, \rho(x, y))|}\, dy$$

for the same constants C as above, and the result follows. ∎

It is possible to give an alternate proof of Theorem 2.1 based on the known case $j = 0$ and the identity $X^J P_m(B_0, f) = P_{m-d(J)}(B_0, X^J f)$, $d(J) < m$, noted earlier. In fact, in the case the polynomial P in Theorem 2.1 is chosen to be $P_m(B_0, f)$, then the first statement of Theorem 2.1 for $1 \le j < m$ follows immediately from this identity and the case $j = 0$ of Theorem 2.1 applied to the functions $X^J f$ with $d(J) = j$ and with m replaced by $m - j = m - d(J)$. If P is instead chosen to be $\pi_m(B_0, f)$, the corresponding estimates in the first statement of Theorem 2.1 can then be deduced as follows. Since

$$|X^j(f - \pi_m(B_0, f))| \le |X^j(f - P_m(B_0, f))| + |X^j(P_m(B_0, f) - \pi_m(B_0, f))|,$$

it is enough to show that

$$|X^j(P_m(B_0, f) - \pi_m(B_0, f))(x)| \le C\frac{r(B_0)^{m-j}}{|B_0|}\int_{B_0}|X^m f|\,dy, \qquad x \in B_0.$$

In the case $j = 0$, for all $x \in B_0$,

$$\begin{aligned} |P_m(B_0, f)(x) - \pi_m(B_0, f)(x)| &= |\pi_m(B_0, f - P_m(B_0, f))(x)| \quad \text{by (1.6)} \\ &\le Cr(B_0)^{-Q}\|f - P_m(B_0, f)\|_{L^1(B_0)} \quad \text{by (1.5)} \\ &\le C\frac{r(B_0)^m}{|B_0|}\int_{B_0}|X^m f|\,dy \end{aligned}$$

by the L^1, L^1 Poincaré inequality in [17]. For any j and all $x \in B_0$, by Bernstein's inequality,

$$|X^j(P_m(B_0, f) - \pi_m(B_0, f))(x)| \le Cr(B_0)^{-j}\frac{1}{|B_0|}\int_{B_0}|P_m(B_0, f) - \pi_m(B_0, f)|\,dy,$$

and the desired estimate then follows from the one just established for $j = 0$. A similar proof can be given for the second statement in Theorem 2.1.

In passing, we note that the estimate

$$|P_m(B_0, f)(x) - \pi_m(B_0, f)(x)| \le C\frac{r(B_0)^m}{|B_0|}\int_{B_0}|X^m f|\,dy, \qquad x \in B_0,$$

shown above implies that the polynomials $P_m(B_0, f)$ satisfy the property of the polynomials $\pi_m(B_0, f)$ in (1.6). In fact, if we apply the estimate to any polynomial p of degree at most $m - 1$, then since $X^m p$ is identically 0, we obtain that $P_m(B_0, p) = \pi_m(B_0, p)$ and, therefore, by (1.6) that $P_m(B_0, p) = p$.

Proof of Theorem 2.2. Fix m, B_0, a function $f \in W^{m,1}(B_0)$, and integers i, j with $0 \le j < i \le m$. Given J with $d(J) = j$, let $F = X^J f$ and $g = |X^i f|$. By Theorem 2.5, for each ball B and the polynomial $p = X^J \pi_m(B, f)$,

$$\int_B |F(x) - p(x)|\,dx \le cr(B)^{i-j}\int_B g(x)\,dx$$

with c independent of B, f, and F. Thus, by Theorem 2.6 with m there replaced by $i - j$, we have for a.e. $x \in B_0$ that

$$|F(x) - p(x)| \leq C \int_{B_0} g(y) \frac{\rho(x, y)^{i-j}}{|B(x, \rho(x, y))|} \, dy + C \frac{r(B_0)^{i-j}}{|B_0|} \int_{B_0} g(y) \, dy,$$

and if, in addition, $i - j \leq Q$, then

$$|F(x) - p(x)| \leq C \int_{B_0} g(y) \frac{\rho(x, y)^{i-j}}{|B(x, \rho(x, y))|} \, dy.$$

The constants C are independent of f, F, x, and B_0. This is equivalent to the conclusion of the theorem. We do not know how to give a second proof of Theorem 2.2 analogous to the second proof of Theorem 2.1. ■

The proof of Theorem 2.3 is similar and we omit it.

3. Representation Formulas of Sobolev Type

This section answers Question 2 and deals with higher-order representation formulas of Sobolev type, i.e., formulas for functions with compact support. As we mentioned in the Introduction, the results generalize known ones in the classical Euclidean case ([22], [23], [19], [1]). The main idea of our proofs is to choose the projection polynomial $\pi_m(B, f)$ in the simultaneous representation formulas. In this way, the proofs will follow very easily from Theorems 2.1 and 2.2.

Theorem 3.1. *Suppose that m is any positive integer and $f \in W^{m,1}_{\text{loc}}(\mathbf{G})$. Let j be an integer with $0 \leq j < m$ and $m - j \leq Q$. Then for a.e. $x \in \mathbf{G}$,*

$$|X^j f(x)| \leq C \int_{\mathbf{G}} |X^m f(y)| \frac{\rho(x, y)^{m-j}}{|B(x, \rho(x, y))|} \, dy$$

provided $\int_{B(0,r)} |f(y)| \, dy = o(r^m)$ as $r \to \infty$.

Remark. The term on the right-hand side is finite a.e. in $\mathbf{G}$ under appropriate assumptions. For instance, if $f \in W^{m,p}(\mathbf{G})$ (globally) for some $p \geq 1$ and $m - j < Q$, the right-hand side is a fractional integral of order $m - j$ and thus maps $L^p(\mathbf{G})$ into $L^q(\mathbf{G})$ provided $p > 1$ and $1/q = 1/p - (m - j)/Q$, and it maps $L^1(\mathbf{G})$ into weak $L^q(\mathbf{G})$, $q = Q/[Q - (m - j)]$. In either case, the right-hand side above is clearly finite a.e. in $\mathbf{G}$. When $m - j = Q$, it equals $\int_{\mathbf{G}} |X^m f(y)| \, dy$ and is finite as long as $f \in W^{m,1}(\mathbf{G})$. Moreover, as is easy to see by Hölder's inequality, the requirement that $\int_{B(0,r)} |f| = o(r^m)$ as $r \to \infty$ is met if $f \in L^p(\mathbf{G})$ for some $p' > Q/m$, where $1/p + 1/p' = 1$.

Corollary 3.2. *Let Ω be any domain in $\mathbf{G}$, and let m, j be integers with $0 \leq j < m$ and $m - j \leq Q$. If $f \in W^{m,1}_0(\Omega)$, then for a.e. $x \in \Omega$,*

$$|X^j f(x)| \leq C \int_{\Omega} |X^m f(y)| \frac{\rho(x, y)^{m-j}}{|B(x, \rho(x, y))|} \, dy.$$

Proof of Theorem 3.1. Let $B_r = B(0, r)$. From Theorem 2.1, we have for a.e. $x \in B_r$ and any J with $d(J) = j$ that

$$|X^J f(x) - X^J \pi_m(B_r, f)(x)| \le C \int_{B_r} |X^m f(y)| \frac{\rho(x, y)^{m-j}}{|B(x, \rho(x, y))|} \, dy.$$

By Bernstein's inequality and (1.5),

$$\|X^j \pi_m(B_r) f\|_{L^\infty(B_r)} \le C r^{-j} \|\pi_m(B_r, f)\|_{L^\infty(B_r)} \le C r^{-j} \frac{1}{|B_r|} \int_{B_r} |f(y)| \, dy.$$

Since $r^{-j}/|B_r| \approx r^{-j-Q}$ and we have assumed that $m \le j + Q$ and that $\int_{B_r} |f| = o(r^m)$, we obtain

$$\lim_{r \to \infty} \|X^j \pi_m(B_r) f\|_{L^\infty(B_r)} = 0,$$

and the result follows easily. ■

Proof of Corollary 3.2. Clearly, $W_0^{m,1}(\Omega) \subset W^{m,1}(\mathbf{G})$, and if $f \in W_0^{m,1}(\Omega)$, then $\int_{B_r} |f(y)| \, dy = o(r^m)$ as $r \to \infty$. The result then follows from Theorem 3.1. ■

4. Higher-Order Sobolev Inequalities

In this section, we prove higher-order Sobolev inequalities for functions with compact support. There are several possible ways to proceed. One way is to use the representation formulas in Section 3 together with estimates about L^p to L^q boundedness for fractional integrals. A technical problem arises in this method in the case $p = 1$ since the corresponding fractional integral result concerns only the space weak L^q, but this technicality can be overcome by truncation arguments in the case of first order. In the case of higher orders, it is not known if such an argument is valid. Another way to proceed is to iterate known first-order Sobolev inequalities in order to derive higher-order results. However, we present a different method here. It is similar to the one used in Section 3 to derive representation formulas for functions with compact support from representation formulas for functions without compact support, namely, a passage to the limit. Thus we will pass from Poincaré inequalities to Sobolev inequalities by a limiting argument which works for all $p \ge 1$. The first result below is not restricted to functions with compact support.

Theorem 4.1. *Let $p \ge 1$ and let $f \in W^{m,p}(\mathbf{G})$. Then for any integers i, j with $0 \le j < i \le m$,*

$$\left(\int_{\mathbf{G}} |X^j f(x)|^{q_{ij}} \, dx \right)^{1/q_{ij}} \le C \left(\int_{\mathbf{G}} |X^i f(x)|^p dx \right)^{1/p},$$

provided $1 \le p < Q/(i - j)$ and $q_{ij} = pQ/[Q - (i - j)p]$. The constant C is independent of f.

Corollary 4.2. *Let Ω be an open subset of* $\mathbf{G}$, $p \geq 1$, *and let* $f \in W_0^{m,p}(\Omega)$. *Then for* $0 \leq j < i \leq m$,

$$\left(\int_\Omega |X^j f(x)|^{q_{ij}} \, dx \right)^{1/q_{ij}} \leq C \left(\int_\Omega |X^i f(x)|^p \, dx \right)^{1/p},$$

provided $1 \leq p < Q/(i-j)$ *and* $q_{ij} = pQ/[Q-(i-j)p]$. *The constant* C *is independent of* Ω *and* f.

Corollary 4.3. *Let* $B \subset \mathbf{G}$ *be a metric ball*, $p \geq 1$, *and let* $f \in W_0^{m,p}(B)$. *Then for* $0 \leq j < i \leq m$,

$$\left(\frac{1}{|B|} \int_B |X^j f(x)|^q \, dx \right)^{1/q} \leq C r(B)^{i-j} \left(\frac{1}{|B|} \int_B |X^i f(x)|^p \, dx \right)^{1/p},$$

provided $1 \leq p < Q/(i-j)$ *and* $1 \leq q \leq q_{ij} = pQ/[Q-(i-j)p]$. *The constant* C *is independent of* B *and* f.

Proof of Theorem 4.1. Let $B_r = B(0, r)$. By Theorem 2.5, for $0 \leq j < i \leq m$,

$$\left(\frac{1}{|B_r|} \int_{B_r} |X^j (f - \pi_m(B_r, f))(x)|^{q_{ij}} \, dx \right)^{1/q_{ij}} \leq C r^{i-j} \left(\frac{1}{|B_r|} \int_{B_r} |X^i f(x)|^p \, dx \right)^{1/p}$$

for $1 \leq p < Q/(i-j)$ and $q_{ij} = pQ/[Q-(i-j)p]$, where C is independent of r and f. By definition of q_{ij}, this estimate can be rewritten as

$$\left(\int_{B_r} |X^j (f(x) - \pi_m(B_r, f)(x))|^{q_{ij}} \, dx \right)^{1/q_{ij}} \leq C \left(\int_{B_r} |X^i f(x)|^p \, dx \right)^{1/p}.$$

Note that

$$\begin{aligned} \left(\int_{B_r} |X^j \pi_m(B_r, f)(x)|^{q_{ij}} \, dx \right)^{1/q_{ij}} &\leq C r^{Q/q_{ij}} \|X^j \pi_m(B_r, f)\|_{L^\infty(B_r)} \\ &\leq C r^{Q/q_{ij} - j - Q} \int_{B_r} |f(y)| \, dy \end{aligned}$$

as in the proof of Theorem 3.1. Since $Q/q_{ij} - j - Q < 0$, the last expression tends to 0 as $r \to +\infty$, and we obtain the desired estimate

$$\left(\int_{\mathbf{G}} |X^j f(x)|^{q_{ij}} \, dx \right)^{1/q_{ij}} \leq C \left(\int_{\mathbf{G}} |X^i f(x)|^p \, dx \right)^{1/p}. \qquad \blacksquare$$

Proof of Corollary 4.2. This follows from Theorem 4.1 because $W_0^{m,p}(\Omega) \subset W^{m,p}(\mathbf{G})$. ■

Proof of Corollary 4.3. Taking $\Omega = B$ in Theorem 4.2, we get

$$\left(\int_B |X^j f(x)|^{q_{ij}} \, dx \right)^{1/q_{ij}} \leq C \left(\int_B |X^i f(x)|^p \, dx \right)^{1/p}.$$

Thus, by Hölder's inequality and the definition of q_{ij},

$$\left(\frac{1}{|B|}\int_B |X^j f(x)|^q\,dx\right)^{1/q} \le Cr(B)^{i-j}\left(\frac{1}{|B|}\int_B |X^i f(x)|^p\,dx\right)^{1/p}$$

for all $1 \le p < Q/(i-j)$ and $1 \le q \le q_{ij}$ $(= pQ/[Q-(i-j)p])$, with C independent of B and f. ■

5. Simultaneous Embedding Theorems

5.1. *Weighted Simultaneous Poincaré Inequalities*

We begin by using the simultaneous representation formulas to derive weighted simultaneous Poincaré inequalities for high-order vector field gradients on a stratified group **G**, assuming a balance condition similar to the one in [4]. We adapt weighted results for integral operators of potential type derived in [24] and [25].

If $w(x) \in L^1_{\text{loc}}(\mathbf{G})$ and $w(x) \ge 0$, we say that w is a weight and use the notation $w(E) = \int_E w(x)\,dx$ for any measurable set E. If w is a weight, we say that $w \in A_p$, $1 < p < \infty$, if there is a constant C such that for all metric balls B,

$$\left(\frac{1}{|B|}\int_B w(x)\,dx\right)^{1/p}\left(\frac{1}{|B|}\int_B w(x)^{-p'/p}dx\right)^{1/p'} \le C,$$

where $p' = p/(p-1)$. A Borel measure μ on **G** is said to be doubling of order N if there is a constant $C > 0$ such that, for any balls B_1 and B_2 with $B_1 \subset B_2$,

$$\mu(B_2) \le C\left(\frac{r(B_2)}{r(B_1)}\right)^N \mu(B_1).$$

Clearly, Lebesgue measure is doubling of order Q. In the case the measure $d\mu = w\,dx$ is a doubling measure, we will say that w is doubling. It is not hard to see that w is doubling if $w \in A_p$ for some p.

Theorem 5.1. *Let B_0 be a ball in a stratified Lie group* **G**, *and let m, j be integers with $0 \le j < m$. Suppose that w_1, w_2 are weights satisfying the following balance conditions for some p, q_j with $1 < p < q_j < \infty$:*

$$\left(\frac{r(B)}{r(B_0)}\right)^{m-j}\left(\frac{w_2(B)}{w_2(B_0)}\right)^{1/q_j} \le C\left(\frac{w_1(B)}{w_1(B_0)}\right)^{1/p} \tag{5.2}$$

for all metric balls B with $B \subset cB_0$, where c is a suitably large geometric constant. Suppose also that $w_1 \in A_p$ and w_2 is doubling. If $f \in W^{m,p}(B_0)$, then for either of the polynomials $P = P_m(B_0, f)$ or $P = \pi_m(B_0, f)$,

$$\left(\frac{1}{w_2(B_0)}\int_{B_0} |X^j(f-P)|^{q_j} w_2\,dx\right)^{1/q_j} \le Cr(B_0)^{m-j}\left(\frac{1}{w_1(B_0)}\int_{B_0} |X^m f|^p w_1\,dx\right)^{1/p}. \tag{5.3}$$

The nonweighted case of (5.3) was already given in Theorem 2.4. The balance condition (5.2) leads to the restrictions on the indices in Theorem 2.4. The case $j = 0$ of Theorem 5.1 was proved in [17].

Proof of Theorem 5.1. There are several ways to proceed. First observe by Theorem 2.1 that for either of the polynomials $P = P_m(B_0, f)$ or $P = \pi_m(B_0, f)$ and a.e. $x \in B_0$,

(5.4)
$$|X^j(f - P)(x)| \le C \int_{B_0} |X^m f(y)| \frac{\rho(x, y)^{m-j}}{|B(x, \rho(x, y))|}\, dy + C \frac{r(B_0)^{m-j}}{|B_0|} \int_{B_0} |X^m f(y)|\, dy.$$

Using the integral operator T_j defined by

$$T_j g(x) = \int_{\mathbf{G}} g(y) \frac{\rho(x, y)^{m-j}}{|B(x, \rho(x, y))|}\, dy,$$

we may rewrite (5.4) as

$$|X^j(f - P)(x)|\chi_{B_0}(x) \le C T_j(|X^m f|\chi_{B_0})(x) + C \frac{r(B_0)^{m-j}}{|B_0|} \int_{B_0} |X^m f(y)|\, dy.$$

The second term on the right here is a constant, and by Hölder's inequality, it is bounded by the right side of (5.3) because $w_1 \in A_p$. Consequently, (5.3) will follow by verifying the norm estimate

$$\left(\int_{B_0} |T_j(g\chi_{B_0})(x)|^{q_j} w(x)\, dx \right)^{1/q_j} \le C \left(\int_{B_0} |g(x)|^p v(x)\, dx \right)^{1/p} \tag{5.5}$$

with weights w, v chosen to be

$$w(x) = \frac{1}{w_2(B_0)} w_2(x) \qquad \text{and} \qquad v(x) = \frac{r(B_0)^{(m-j)p}}{w_1(B_0)} w_1(x).$$

The rest of the proof is now identical to that of Theorem 5.1 in [17]. ∎

Another way to proceed is to deduce the result from the known case $j = 0$ proved in [17]. In fact, first choosing $P = P_m(B_0, f)$, we can combine the fact that $X^J P_m(B_0, f) = P_{m-j}(B_0, X^J f)$ if $d(J) = j$ with the known case $j = 0$ to immediately deduce (5.3) for $P = P_m(B_0, f)$. Next, in order to deduce (5.3) for the choice $P = \pi_m(B_0, f)$, we simply use the fact that (5.3) for any polynomial of degree less than m implies (5.3) for the polynomial $\pi_m(B_0, f)$. This can be seen by the sort of reasoning used in the second proof of Theorem 2.1 coupled with Hölder's inequality in order to pass from Lebesgue measure to w_1-measure, since $w_1 \in A_p$.

The first of these methods also leads to the next result.

Theorem 5.6. *Let B_0 be a ball in a stratified Lie group $\mathbf{G}$, and let m, i, and j be integers with $0 \le j < i \le m$. Suppose that w_1, w_2 are weights satisfying the following balance conditions for some p, q_{ij} with $1 < p < q_{ij} < \infty$:*

$$\left(\frac{r(B)}{r(B_0)} \right)^{i-j} \left(\frac{w_2(B)}{w_2(B_0)} \right)^{1/q_{ij}} \le C \left(\frac{w_1(B)}{w_1(B_0)} \right)^{1/p}$$

for all metric balls B with $B \subset cB_0$, where c is a suitably large geometric constant. Suppose also that $w_1 \in A_p$ and w_2 is doubling. If $f \in W^{m,p}(B_0)$, then

$$\left(\frac{1}{w_2(B_0)}\int_{B_0}|X^j(f-\pi_m(B_0,f))|^{q_{ij}}w_2\,dx\right)^{1/q_{ij}} \le Cr(B_0)^{i-j}\left(\frac{1}{w_1(B_0)}\int_{B_0}|X^i f|^p w_1\,dx\right)^{1/p}. \tag{5.7}$$

The nonweighted case of Theorem 5.6 was already given in Theorem 2.5.

The proof of Theorem 5.6 is similar to the first proof of Theorem 5.1, but uses Theorem 2.2 instead of Theorem 2.1.

5.2. *Simultaneous Exponential Inequalities*

We now fix $k > 0$, a ball $B \subset \mathbf{G}$ and a Borel measure μ, and then define

$$T_{B,k}g(x) = \int_B g(y)\frac{\rho(x,y)^k}{\mu(B(x,\rho(x,y)))}\,d\mu(y).$$

We will need the following special case of Theorem 5.8 in [17]:

Lemma 5.8. *Let μ be a doubling measure of order N and let $T_{B,k}g$ be defined as above for $k > 0$ and a fixed ball $B \subset \mathbf{G}$. Suppose that $pk = N$ and $p > 1$. Then there is a constant $C > 0$ independent of B and g such that*

$$\frac{1}{\mu(B)}\int_B \exp\left\{\left(\frac{r(B)^k}{C\mu(B)^{1/p}}\frac{|T_{B,k}g(x)|}{\|g\|_{L^p(B,d\mu)}}\right)^{p/(p-1)}\right\}d\mu(x) \le C.$$

Theorem 5.9. *Let B be a metric ball in a stratified group $\mathbf{G}$ of homogeneous dimension Q. Let m and j be integers with $0 \le j < m < Q$, and let p_j be defined by $p_j = Q/(m-j)$. If $f \in W^{m,Q}(B)$, then for either of the polynomials $P = P_m(B,f)$ or $\pi_m(B,f)$,*

$$\frac{1}{|B|}\int_B \exp\left\{\left(\frac{|X^j(f-P)(x)|}{C\|X^m f\|_{L^{p_j}(B,dx)}}\right)^{p_j/(p_j-1)}\right\}dx \le C,$$

with C independent of f and B. Moreover, for the same p_j, a similar result holds for any weak Boman domain Ω in $\mathbf{G}$: if B_0 is a central ball for Ω and P is either $P_m(B_0,f)$ or $\pi_m(B_0,f)$, then

$$\frac{1}{|\Omega|}\int_\Omega \exp\left\{\left(\frac{|X^j(f-P)(x)|}{C\|X^m f\|_{L^{p_j}(\Omega,dx)}}\right)^{p_j/(p_j-1)}\right\}dx \le C.$$

Proof. The first statement follows immediately from Lemma 5.8 and the simultaneous representation formulas in Theorem 2.1. Recall that on a stratified group $\mathbf{G}$, $|B| = C_Q r(B)^Q$. The second statement can be obtained from Theorem 2.3 by arguing as in the proof of Corollary 5.9 in [17]. ∎

Similarly, we can deduce the next theorem from the simultaneous representation formula in Theorem 2.2.

Theorem 5.10. *Let B be a metric ball in a stratified group $\mathbf{G}$ of homogeneous dimension Q. Let j, i, m be integers with $0 \le j < i \le m < Q$, and let p_{ij} be defined by $p_{ij} = Q/(i-j)$. If $f \in W^{m,Q}(B)$, then*

$$\frac{1}{|B|}\int_B \exp\left\{\left(\frac{|X^j(f-\pi_m(B,f))(x)|}{C\|X^i f\|_{L^{p_{ij}}(B,dx)}}\right)^{p_{ij}/(p_{ij}-1)}\right\}dx \le C,$$

with C independent of f and B. Moreover, for the same p_{ij}, a similar result holds for any weak Boman domain Ω in $\mathbf{G}$: if B_0 is a central ball for Ω, then

$$\frac{1}{|\Omega|}\int_\Omega \exp\left\{\left(\frac{|X^j(f-\pi_m(B_0,f))(x)|}{C\|X^i f\|_{L^{p_{ij}}(\Omega,dx)}}\right)^{p_{ij}/(p_{ij}-1)}\right\}dx \le C.$$

5.3. *Simultaneous L^∞ Estimates and the Hölder Continuity*

We now prove some estimates on stratified groups in the case either $p = 1$ and $m - j \ge Q$ or $p > 1$ and $p(m-j) > Q$; these complement the results in Subsection 5.2 where $p > 1$ and $p(m-j) = Q$.

Theorem 5.11. *Let B be a metric ball in a stratified group $\mathbf{G}$ of homogeneous dimension Q. Let $0 \le j < m$, $p \ge 1$, $(m-j)p > Q$ if $p > 1$ and $m - j \ge Q$ if $p = 1$. If $f \in W^{m,p}(B)$, then for either of the polynomials $P = P_m(B, f)$ or $P = \pi_m(B, f)$,*

$$\|X^j(f-P)\|_{L^\infty(B,dx)} \le Cr(B)^{m-j-Q/p}\|X^m f\|_{L^p(B,dx)} \tag{5.12}$$

with C independent of f and B. In particular,

$$\|X^j f\|_{L^\infty(B,dx)} \le \frac{C}{r(B)^{j+Q}}\int_B |f(y)|\,dy + Cr(B)^{m-j-Q/p}\|X^m f\|_{L^p(B,dx)}. \tag{5.13}$$

Moreover, if $p > 1$, $m - j \le Q$, and $(m-j)p > Q$, a similar result holds for any weak Boman domain Ω in $\mathbf{G}$: if B_0 is a central ball, then for either $P = P_m(B_0, f)$ or $P = \pi_m(B_0, f)$,

$$\|X^j(f-P)\|_{L^\infty(\Omega,dx)} \le C|\Omega|^{(m-j)/Q-1/p}\|X^m f\|_{L^p(\Omega,dx)}. \tag{5.14}$$

In particular, for the central ball B_0,

$$\|X^j f\|_{L^\infty(\Omega,dx)} \le \frac{C(\Omega)}{r(B_0)^j|B_0|}\int_{B_0} |f(y)|\,dy + C|\Omega|^{(m-j)/Q-1/p}\|X^m f\|_{L^p(\Omega,dx)}. \tag{5.15}$$

Proof. If $P = P_m(B, f)$ or $P = \pi_m(B, f)$, then by Theorem 2.1,

$$|X^j(f-P)(x)| \le C\int_B \rho(x,y)^{m-j-Q}|X^m f(y)|\,dy + Cr(B)^{m-j-Q}\int_B |X^m f(y)|\,dy$$

for a.e. $x \in B$. If $p = 1$ and $m - j \geq Q$, the right side is at most

$$Cr(B)^{m-j-Q} \int_B |X^m f(y)|\, dy,$$

and (5.12) follows. If $p > 1$ and $(m - j)p > Q$, then by the Hölder inequality, both terms on the right are easily seen to be bounded by $Cr(B)^{m-j-(Q/p)} \|X^m f\|_{L^p(B,dx)}$, which proves (5.12). If we choose P to be $\pi_m(B, f)$, namely, to satisfy

$$\|\pi_m(B, f)\|_{L^\infty(B,dx)} \leq \frac{C}{|B|} \int_B |f(y)|\, dy, \tag{5.16}$$

then (5.13) follows from (5.12) by the triangle inequality.

If $m - j \leq Q$ and B_0 is a central ball for Ω, by Theorem 2.1 and then the Hölder inequality, we obtain that, for either $P = P_m(B_0, f)$ or $P = \pi_m(B, f)$ and a.e. $x \in \Omega$,

$$|X^j(f - P)(x)| \leq C \left(\int_\Omega \rho(x, y)^{(m-j-Q)p'} dy \right)^{1/p'} \|X^m f\|_{L^p(\Omega,dx)}. \tag{5.17}$$

By selecting R with $|B(x, R)| = |\Omega|$ and using the fact that $-Q < (m - j - Q)p' \leq 0$, we see that the first factor on the right in (5.17) is bounded by

$$\left(\int_{B(x,R)} \rho(x, y)^{(m-j-Q)p'} dy + \int_{\rho(x,y)>R;\, \Omega} R^{(m-j-Q)p'} dy \right)^{1/p'} \leq C|\Omega|^{(m-j)/Q-1/p}.$$

This completes the proof of (5.14). To prove (5.15), note that since $\pi_m(B_0, f)$ satisfies (5.16) for B_0, then (5.14) implies

$$\|X^j f\|_{L^\infty(\Omega,dx)} \leq \|X^j \pi_m(B_0, f)\|_{L^\infty(\Omega,dx)} + C|\Omega|^{(m-j)/Q-1/p} \|X^m f\|_{L^p(\Omega,dx)}.$$

Since $\Omega \subset MB_0$ by Definition 1.9(iv), the first term on the right is at most

$$\begin{aligned} \|X^j \pi_m(B_0, f)\|_{L^\infty(MB_0,dx)} &\leq C(M) \|X^j \pi_m(B_0, f)\|_{L^\infty(B_0,dx)} \\ &\leq \frac{C(M)}{r(B_0)^j |B_0|} \int_{B_0} |f(y)|\, dy, \end{aligned}$$

where the penultimate estimate follows from Bernstein's inequality and (1.5). We emphasize that $C(M)$ is independent of f and B_0. This finishes the proof of (5.15). ■

6. Other Embedding Theorems of Sobolev Type

By using the representation formulas of Sobolev type proved in Section 3, we can obtain the following weighted Sobolev inequalities by methods like those used in Section 5.

Theorem 6.1. *Let* **G** *be a stratified Lie group, and let m be any positive integer less than Q. Let $1 < p < q < \infty$, let B_0 be a metric ball, and let w_1, w_2 be weights which satisfy the balance condition* (5.2) *for $j = 0$ and all metric balls $B \subset cB_0$. Suppose also that $w_1 \in A_p$ and w_2 is doubling. If $f \in W_0^{m,p}(B_0)$, then*

$$\left(\frac{1}{w_2(B_0)} \int_{B_0} |f|^q w_2\, dx \right)^{1/q} \leq Cr(B_0)^m \left(\frac{1}{w_1(B_0)} \int_{B_0} |X^m f|^p w_1\, dx \right)^{1/p}.$$

Proof of Theorem 6.1. First observe that, by Theorem 3.2,

$$|f(x)| \le C \int_{B_0} |X^m f(y)| \frac{\rho(x, y)^m}{|B(x, \rho(x, y))|}\, dy, \quad \text{a.e. } x \in B_0.$$

The result then follows from (5.5) with $j = 0$, in the same way that Theorem 5.1 was proved. ■

Similarly, using the representation formula in Theorem 3.2, we can derive the next result.

Theorem 6.2. *Let B be a metric ball in a stratified group $\mathbf{G}$ of homogeneous dimension Q, and let $p > 1$ and m be a positive integer with $pm = Q$. If $f \in W_0^{m,p}(B)$, then*

$$\frac{1}{|B|} \int_B \exp\left\{ \left(\frac{|f(x)|}{C\|X^m f\|_{L^p(B,dx)}} \right)^{p/(p-1)} \right\} dx \le C$$

with C independent of f and B. Moreover, for the same p and m, a similar result holds for any domain Ω in $\mathbf{G}$ and $f \in W_0^{m,p}(\Omega)$:

$$\frac{1}{|\Omega|} \int_\Omega \exp\left\{ \left(\frac{|f(x)|}{C\|X^m f\|_{L^p(\Omega,dx)}} \right)^{p/(p-1)} \right\} dx \le C.$$

Acknowledgments. Research partly supported by NSF for Guozhen Lu. Results of this paper were presented in the conference of “Integral Inequalities and Applications” in Cortona, Italy, in June, 1999.

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