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Unique Continuation with Weak Type Lower Order Terms*

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Abstract. This paper proves a unique continuation property for the elliptic differential inequality

$$|\Delta u| \leqslant A|u| + B|\nabla u|,$$

where the coefficients A and B are functions in the Lorentz space with small weak type norm.

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1. Introduction

We will prove the following result.

THEOREM 1. *If $d \geqslant 3$, then there are $p < 2d/(d+2)$ and a constant $\varepsilon_d > 0$ making the following true. Assume that $\Omega \subset \mathbb{R}^d$ is a domain, $A : \Omega \to \mathbb{R}$ and $B : \Omega \to \mathbb{R}$ are functions such that*

$$\lim_{r\to 0} \|A\|_{L^{d/2\infty}(D(a,r))} \leqslant \varepsilon_d \tag{1.1}$$

$$\lim_{r\to 0} \|B\|_{L^{d\infty}(D(a,r))} \leqslant \varepsilon_d \tag{1.2}$$

for each $a \varepsilon \Omega$. Assume also that $u \varepsilon W^{2p}_{\text{loc}}(\Omega)$ satisfies

$$| \triangle u| \leqslant A|u| + B| \bigtriangledown u|. \tag{1.3}$$

Then if u vanishes on an open set it vanishes identically.

Here W^{2p} is the Sobolev space, i.e., functions whose second derivatives are in L^p. Our proof will show (see the remark at the end of Section 2) that p can be taken

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to be any number greater than 1 if $d = 3$, any number greater than $\frac{8}{7}$ if $d = 4$ and any number greater than $2d(d-3)/(d^2-d-4)$ if $d \geqslant 5$. $\|A\|_{L^{q\infty}(D(a,r))}$ is the weak type norm defined as follows

$$\|A\|_{L^{q\infty}(D(a,r))} = \sup_{\lambda>0}(\lambda^q|\{x \in D(a,r) : |A(x)| > \lambda\}|)^{1/q}.$$

Theorem 1 improves on a result of [7] where A and B are assumed to be in $L^{d/2}_{\text{loc}}$ and L^d_{loc} respectively, in the same way as the result of [6] improves on the result of [2]. It should be pointed out, however, that in contrast to the result of [6] Theorem 1 is not known to be sharp.

A word about the proof: we will use the main lemma (Lemma 1) of [7], but the method used in [7] to derive unique continuation theorems from that lemma does not work here because it depends on the fact that

$$\sum_j \|A\|^q_{L^q(E_j)} \leqslant \|A\|^q_q,$$

if $\{E_j\}$ are disjoint sets, and this is clearly false if the L^q norm is replaced by the $L^{q\infty}$ norm.

Section 2 of this paper shows the Carleman inequalities needed for Theorem 1, and Section 3 is the proof of the theorem.

2. Carleman Inequalities

We first state a Carleman inequality due to [3].

LEMMA 2.1. *Let* $p_0 = 2d/(d+2), d \geqslant 3$. *Then for any* p *satisfying*

(i) $|1/p - 1/p_0| < 1/2d$ *and*
(ii) $1/p - 1/q = 2/d$,

we have for any $k \in \mathbb{R}^d$ *and* $u \in C_0^\infty(\mathbb{R}^d)$

$$\|e^{k\cdot x}u\|_{qp} \leqslant C\|e^{k\cdot x} \triangle u\|_p,$$

with $C = C(p,d)$.

We note in the above that $\|\cdot\|_{qp}$ is the Lorentz norm defined by

$$\|f\|_{qp} = \left(q\int_0^\infty s^{p-1}|\{x\colon |f(x)| > s\}|^{p/q}\,\mathrm{d}s\right)^{1/p}.$$

We now assume $m_k(\xi) = i\xi - k/(|\xi|^2 - i\xi\cdot k - |k|^2)$ for $k \in \mathbb{R}^d$, i.e., a multiplier such that

$$(e^{k\cdot x} \bigtriangledown u)^\wedge(\xi) = m_k(\xi)(e^{k\cdot x} \triangle u)^\wedge(\xi),$$

for $u \in C_0^\infty(R^d)$. We set $\varphi \in C_0^\infty(D(0,2))$ and $\varphi_k(\xi) = \varphi(|k|^{-1}\xi)$, where $D(0,2)$ is the disk centered at the origin with radius 2. Define two multiplier operators T_1 and T_2 by

$$\widehat{T_1 f} = (m_k(1 - \varphi_k))\hat{f},$$

$$\widehat{T_2 f} = (m_k \varphi_k)\hat{f}.$$

Also let s be the Stein–Tomas exponent, $s = 2d + 2/(d+3)$. Then we can show the following lemma.

LEMMA 2.2. *Let* $1 < p < s$ *when* $d = 3$ *and* $2d(d-3)/(d^2 - d - 4) < p < s$ *when* $d \geqslant 4$. *Assume* $k \in \mathbb{R}^d, E \subset \mathbb{R}^d, |E| \geqslant |k|^{-d}, u \in W^{2p}$ *has compact support. Then there exists some* $\theta(p)$ *with* $0 < \theta(p) < 1/d$ *such that for any* θ *with* $\theta(p) < \theta < 1/d$ *the following two inequalities hold:*

(i) $\|T_2(e^{k\cdot x} \triangle u)\|_{L^q(E)} \leqslant C_\theta(|k|^d|E|)^\theta |k|^{d/r-1} \|e^{k\cdot x} \triangle u\|_p$,
provided $1/d < 1/p - 1/q = 1/r < 1/d + \theta$.
(ii) $\|e^{k\cdot x} \bigtriangledown u\|_{L^{qp}(E)} \leqslant C_\theta(|k|^d|E|)^\theta \|e^{k\cdot x} \triangle u\|_p$
provided $1/p - 1/q = 1/d$.

Proof. First we show that with $k = e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^d$ the following holds

$$\|T_2(e^{x_1} \triangle u)\|_{L^q(E)} \leqslant C_\theta |E|^\theta \|e^{x_1} \triangle u\|_p.$$

It is shown in [7], page 264, that $\|T_2 f\|_{q_1} \leqslant C\|f\|_{p_1}$ provided

$$1 \leqslant p_1 \leqslant s \quad \text{and} \quad \frac{1}{q_1} < \frac{1}{p_1'} + \frac{1}{2} - \frac{s'}{2p_1'}, \quad \text{where } s' = \frac{s}{s-1}. \tag{2.3}$$

Note that (2.3) is equivalent to

$$1 \leqslant p_1 \leqslant s \quad \text{and} \quad \frac{1}{p_1} - \frac{1}{q_1} > \mu(p_1),$$

where

$$\mu(p_1) = \frac{1}{p_1}\left(\frac{d-3}{d-1}\right) - \frac{d-5}{2(d-1)}.$$

We also note that $\mu(p_1)$ is decreasing in p_1 and $1/d < \mu(p_1) < 2/d$ provided that $1 \leqslant p_1 \leqslant s$ when $d = 3$ and $2d(d-3)/(d^2 - d - 4) < p_1 < s$ when $d \geqslant 4$.

Set $\theta(p_1) = \mu(p_1) - 1/d$ and for any $\theta(p_1) < \theta < 1/d$ select q_1 such that $\mu(p_1) < 1/p_1 - 1/q_1 < \theta + 1/d$. Thus for any $q < q_1$ and $1/p_1 - 1/q > 1/d$ we get

$$\begin{aligned}\|T_2(e^{x_1} \triangle u)\|_{L^q(E)} &\leqslant |E|^{1/q - 1/q_1} \|T_2(e^{x_1} \triangle u)\|_{q_1} \\ &\leqslant C|E|^{1/q-1/q_1} \|e^{x_1} \triangle u\|_{p_1} \leqslant C|E|^\theta \|e^{x_1} \triangle u\|_{p_1},\end{aligned}$$

since $1/q - 1/q_1 < \theta$. By scaling, we thus have proved that

$$\|T_2(e^{k\cdot x} \triangle u)\|_{L^q(E)} \leqslant C_\theta(|k|^d|E|)^\theta|k|^{d/r-1}\|e^{k\cdot x} \triangle u\|_{p_1},$$

provided that $q > p_1$ satisfies $1/p_1 - 1/q = 1/r$ and $1/d < /1/r < \theta + 1/d$. The condition that p_1 satisfies here is the same as p in Lemma (2.2). This proves (i) of Lemma (2.2).

We now turn to the proof of Lemma 2.2, part (ii). We still first assume $k = e_1$. We split the multiplier and then define T_1 and T_2 as before. We note that the multiplier for T_1 can be written as $\left(1+|\xi|^2\right)^{-1/2}\mu$, where μ satisfies the Hörmander multiplier condition. Thus

$$\|T_1(e^{x_1} \triangle u)\|_{L^{qp}} \leqslant C\|e^{x_1} \triangle u\|_p \text{ for any } p > 1 \quad \text{and} \quad \frac{1}{p} - \frac{1}{q} = \frac{1}{d}. \tag{2.4}$$

In the proof of (i), we have actually shown for any given $\theta > \theta(p)$ that

$$\|T_2(e^{x_1} \triangle u)\|_{q_1p} \leqslant \|e^{x_1} \triangle u\|_p,$$

provided that p is as assumed in the statement of Lemma 2.2 and $\theta + 1/d > 1/p - 1/q_1 > \mu(p)$. Thus for such p and the corresponding q with $1/p - 1/q = 1/d$ we get

$$\|T_2(e^{x_1} \triangle u)\|_{L^{qp}(E)} \leqslant |E|^{1/q-1/q_1}\|T_2(e^{x_1} \triangle u)\|_{L^{q_1p}} \leqslant |E|^\theta\|e^{x_1} \triangle u\|_p.$$

Hence by scaling

$$\|e^{k\cdot x} \triangle u\|_{L^{qp}(E)} \leqslant \left(|E||k|^d\right)^\theta\|e^{k\cdot x} \triangle u\|_p,$$

this proves (ii) of Lemma 2.2. □

We remark here that the value of p in Lemma 2.2 satisfies $|1/p - 1/p_0| < 1/d(d-3)$ for $d \geqslant 4$ and $|1/p - 1/p_0| < \frac{1}{6}$ when $d = 3$. Thus for such p Lemma (2.1) holds when $d = 3$ and $d \geqslant 5$. In order for Lemma (2.1) to hold when $d = 4$, we further restrict in this case $\frac{8}{7} < p < \frac{10}{7}$. Thus if p is as discussed after the statement of Theorem 1 and sufficiently close to the lower bound there, both Lemmas (2.1) and (2.2) are applicable.

3. Proof of Theorem 1

We first make a reduction. Let $e_d = (0, \ldots, 0, 1)$.

LEMMA 3.1. *If Theorem 1 fails, then for every* $\tilde{\varepsilon} > 0$ *there is a function* $\tilde{u} : \tilde{\Omega} \mapsto \mathbb{R}$ *where*

$$\tilde{\Omega} = \mathbb{R}^d \setminus \overline{D(-e_d, 1/2)}, \tag{3.2}$$

such that

$$\text{supp } \tilde{u} \subset \overline{D(-e_d, 1)},\ 0 \in \text{ supp } \tilde{u}, \tag{3.3}$$

$$\tilde{u} \in W^{2p}_{\text{loc}}, \tag{3.4}$$

$$|\bigtriangleup \tilde{u}| \leqslant \tilde{A}|\tilde{u}| + \tilde{B}|\bigtriangledown \tilde{u}|, \tag{3.5}$$

where

$$\|\tilde{A}\|_{L^{d/2\infty}(\tilde{\Omega})} < \tilde{\varepsilon},\ \|\tilde{B}\|_{L^{d\infty}(\tilde{\Omega})} < \tilde{\varepsilon}. \tag{3.6}$$

Proof. Fix $\tilde{\varepsilon} > 0$. Let Ω and u satisfy (1.1)–(1.3) for a sufficiently small ε ($= C^{-1}\tilde{\varepsilon}$ for a suitable constant C). Assume u vanishes on an open set but not identically. Let D be an open disc contained in $\Omega \setminus \text{supp } u$ such that $\partial D \cap \text{ supp } u \neq \emptyset$. Fix $a \in \partial D \cap \text{ supp } u$. Considering points on the line segment connecting a to the center of D, it is clear that for every sufficiently small δ there is a point $w_\delta \in \Omega$ such that $\text{dist}(w_\delta,\ \text{supp } u) = \delta = |w_\delta - a|$. Let δ be small, let Γ be a rotation taking e_d to $(a - w_\delta)/|a - w_\delta|$ and consider the function

$$v(x) = u(w_\delta + \delta\Gamma(x)).$$

We regard v as a function on $D(0, 2)$ which is clearly possible for small δ. Then

$$v = 0 \text{ on } D(0, 1), \qquad e_d \in \text{ supp } v, \tag{3.7}$$

$$v \in W^{2p}_{\text{loc}}(D(0, 2)), \tag{3.8}$$

$$|\bigtriangleup v| \leqslant \overline{A}|v| + \overline{B}|\bigtriangledown v|, \tag{3.9}$$

where

$$\|\overline{A}\|_{L^{d/2\infty}(D(0,2))} < \varepsilon,\ \|\overline{B}\|_{L^{d\infty}(D(0,2))} < \varepsilon. \tag{3.10}$$

Here (3.7) follows from the defining property of w_δ, (3.8) is clear, (3.9) is also clear with

$$\overline{A}(x) = \delta^2 A(w_\delta + \delta\Gamma(x)), \qquad \overline{B}(x) = \delta B(w_\delta + \delta\Gamma(x))$$

and then (3.10) follows for small δ since

$$\|\overline{A}\|_{L^{d/2\infty}(D(0,2))} = \|A\|_{L^{d/2\infty}(D(w_\delta,2\delta))} \leqslant \|A\|_{L^{d/2\infty}(D(a,4\delta))} < \varepsilon,$$

by (1.1), and similarly with $\|\overline{B}\|_{L^{d\infty}(D(0,2))}$.

Now consider the Kelvin transform $\tilde{u}(x) = |x+e_d|^{2-d} v(\overline{x+e_d/|x+e_d|^2})$. Its domain is $\tilde{\Omega} \overset{\text{def}}{=} \{x\colon (x+e_d)/|x+e_d|^2 \in D(0,2)\} = \mathbb{R}^d \backslash \overline{D(-e_d, \frac{1}{2})}$, i.e., (3.2) holds. (3.3) follows from (3.7), and (3.4) follows from (3.8) since $-e_d \notin \tilde{\Omega}$. One checks using the chain rule that (3.5) holds with

$$\tilde{A}(x) \;=\; \text{const.} \cdot |x+e_d|^{-4} \overline{A}\left(\frac{x+e_d}{|x+e_d|^2}\right),$$

$$\tilde{B}(x) \;=\; \text{const.} \cdot |x+e_d|^{-2} \overline{B}\left(\frac{x+e_d}{|x+e_d|^2}\right),$$

then (3.10) clearly implies

$$\|\tilde{A}\|_{L^{d/2\infty}\left(\overline{D(-e_d,1)}\backslash\overline{D(-e_d,1/2)}\right)} \leqslant C\varepsilon,$$

$$\|\tilde{B}\|_{L^{d\infty}\left(\overline{D(-e_d,1)}\backslash\overline{D(-e_d,1/2)}\right)} \leqslant C\varepsilon.$$

Outside $\overline{D(-e_d,1)}$ we can replace $\tilde{A}$ and $\tilde{B}$ by 0 in view of (3.3), so the proof is complete. □

By Lemma 3.1, if Theorem 1 fails, we may assume u satisfies (3.2)–(3.6) (dropping the tilde's here).

We now let K be the convex hull of supp $u \bigcap \{x_d \geqslant -\frac{1}{4}\}$. Select $\phi\colon \mathbb{R}^d \to \mathbb{R}$ such that $\phi(x) = 0$ when $x_d \leqslant -\frac{1}{3}$ and $\phi = 1$ on a neighborhood of the boundary of K. Set $v = \phi u$, then

$$|\triangle v| \;\leqslant\; A|v| + B|\bigtriangledown v| + \big(Bu|\bigtriangledown \phi| + 2|\bigtriangledown \phi \cdot \bigtriangledown u + u \triangle \phi|\big)$$

$$= \; A|v| + B|\bigtriangledown v| + \chi,$$

where $\chi \in L^p$ and supp $\chi \subset A_1 \cup A_2$, where

$$A_1 = \overline{D(-e_d,1)} \bigcap \left\{x\colon -\frac{1}{3} \leqslant x_d \leqslant -\frac{1}{4}\right\},$$

$A_2 =$ a compact subset of Int K.

Let Γ be the cone $\{k \in \mathbb{R}^d : k_d \geqslant 4\sqrt{|k|^2 - k_d^2}\}$. Then the proof of Lemma 7.1 in [7] applies also here and we have

LEMMA 3.11. *If $k \in \Gamma$ and $|k|$ is sufficiently large, then* $\|e^{k\cdot x}\chi\|_p \leqslant \|e^{k\cdot x}(A|v| + B|\bigtriangledown v|)\|_p$.

We now let M be large enough such that Lemma 3.11 holds for $k \in \Gamma$ and $|k| > \frac{1}{2}M$. We apply Lemma 1 in [7] to the measure

$$\mu = (A|v| + B|\bigtriangledown v|)^p \,\mathrm{d}x,$$

as in the Theorem 1, and take $\mathcal{C} = D(pMe_d, pM/100)$. If $l \in \mathcal{C}$, then μ_l has the form $[e^{kx}(A|v| + B|\bigtriangledown v|)]^p \,\mathrm{d}x$, where $k = l/p$ satisfies $M/2 < |k| < 2M$, $k \in \Gamma$. Then we have the following.

LEMMA 3.12. *Under the assumptions* (3.2)–(3.6), *we can select* $\{k_j\}$ *and disjoint* $\{E_j\}$ *satisfying*

$$\tfrac{1}{2}M < |k_j| < 2M, \qquad k_j \in \Gamma, \tag{3.13}$$

$$\|e^{k_j\cdot x}(A|v| + B|\bigtriangledown v|)\|_{L^p(E_j)} \geqslant 2^{-(1/p)}\|e^{k_j\cdot x}(A|v| + B|\bigtriangledown v|)\|_p, \tag{3.14}$$

$$\sum_j |E_j|^{-1} \geqslant C^{-1}M^d, \tag{3.15}$$

$$|E_j| \geqslant M^{-d} \text{ for each } j, \quad \text{diam } E_j \leqslant CM^{-(1/2)} \text{ for each } j, \tag{3.16}$$

$$\|B\|_{L^{d\infty}(E_j)} \geqslant C_\theta^{-1}(M^d|E_j|)^{-\theta}, \tag{3.17}$$

for all $1/d > \theta > \theta(p)$, *and*

$$\|B\|_{L^r(E_j)} \geqslant C_\theta^{-1}(M^d|E_j|)^{-\theta}M^{1-(d/r)}, \tag{3.18}$$

provided $\theta(p) < \theta < 1/d$ *and* $1/d < 1/r < 1/d + \theta$.

Proof. (3.13)–(3.16) follow from Lemma 1' in [7]. For so selected $\{k_j\}$ and $\{E_j\}$, we apply Lemma 2.1 and Lemma 2.2, (ii), and we get for q with $1/p - 1/q = 2/d$ and $1/p - 1/q_1 = 1/d$,

$$\begin{aligned}
&\|e^{k_j\cdot x}(A|v| + B|\bigtriangledown v|)\|_{L^p(E_j)} \\
&\quad \leqslant \|e^{k_j\cdot x}v\|_{qp}\|A\|_{L^{d/2\infty}(E_j)} + \|e^{k_j\cdot x}\bigtriangledown v\|_{L^{q_1p}(E_j)}\|B\|_{L^{d\infty}(E_j)} \\
&\quad \leqslant C\|e^{k_j\cdot x}\bigtriangleup v\|_p\|A\|_{L^{d/2\infty}(E_j)} + C_\theta\|e^{k_j\cdot x}\bigtriangleup v\|_p(M^d|E_j|)^\theta\|B\|_{L^{d\infty}(E_j)} \\
&\quad \leqslant C_\theta[\|A\|_{L^{d/2\infty}(E_j)} + \|B\|_{L^{d\infty}(E_j)}(M^d|E_j|)^\theta]\|e^{k_j\cdot x}\bigtriangleup v\|_p.
\end{aligned}$$

Noticing by Lemma (3.11)

$$\begin{aligned}\|e^{k_j\cdot x} \triangle v\|_p &\leqslant \|e^{k_j\cdot x}(A|v| + B|\bigtriangledown v|) + e^{k_j\cdot x}\chi\|_p \\ &\leqslant 2\|e^{k_j\cdot x}(A|v| + B|\bigtriangledown v|)\|_p,\end{aligned}$$

which by (3.14) is

$$\leqslant C\|e^{k_j\cdot x}(A|v| + B|\bigtriangledown v|)\|_{L^p(E_j)}.$$

Thus we obtain,

$$\|A\|_{L^{d/2\infty}(E_j)} + (M^d|E_j|)^\theta \ \|B\|_{L^{d\infty}(E_j)} \geqslant C_\theta^{-1}.$$

By dropping $\|A\|_{L^{d/2\infty}(E_j)}$ since it is very small by (3.6) we get

$$\|B\|_{L^{d\infty}(E_j)} \geqslant C_\theta^{-1}(M^d|E_j|)^{-\theta},$$

for all $1/d > \theta > \theta(p)$.

On the other hand, by (2.4) we have $\|T_1(e^{k\cdot x} \triangle v)\|_{q_1 p} \leqslant \|e^{k\cdot x} \triangle v\|_p$ provided $1/p - 1/q_1 = 1/d$. Thus by Lemma (2.2), (i) and Lemma 2.1, we get

$$\begin{aligned}&\|e^{k_j\cdot x}(A|v| + B|\bigtriangledown v|)\|_{L^p(E_j)} \\ &\leqslant \|e^{k_j\cdot x}A|v| + B|T_1(e^{k_j\cdot x} \triangle v)|\|_{L^p(E_j)} + \|B|T_2(e^{k_j\cdot x} \triangle v)|\|_{L^p(E_j)} \\ &\leqslant C[\|A\|_{L^{d/2\infty}(E_j)} + \|B\|_{L^{d\infty}(E_j)} \\ &\quad +\|B\|_{L^r(E_j)}(M^d|E_i|)^\theta M^{d/r-1}]\|e^{k_j\cdot x} \triangle v\|_p \\ &\leqslant C[\|A\|_{L^{d/2\infty}(E_j)} + \|B\|_{L^{d\infty}(E_j)} + |B\|_{L^r(E_j)}(M^d|E_j|)^\theta M^{d/r-1}]\cdot \\ &\quad \|e^{k_j\cdot x}(A|v| + B|\bigtriangledown v|)\|_{L^p(E_j)}.\end{aligned}$$

The last inequality above follows from (3.14) and Lemma 3.11. Thus

$$\|A\|_{L^{d/2\infty}(E_j)} + \|B\|_{L^{d\infty}(E_j)} + \|B\|_{L^r(E_j)}(M^d|E_j|)^\theta M^{d/r-1} \geqslant C^{-1}.$$

We drop $\|A\|_{L^{d/2\infty}(E_j)}$ and $\|B\|_{L^{d\infty}(E_j)}$ again by (3.6), we get

$$\|B\|_{L^r(E_j)} \geqslant C^{-1}(M^d|E_j|)^{-\theta}M^{d/r-1},$$

provided $\theta(p) < \theta < 1/d$ and $1/d < 1/r < 1/d + \theta$. $\square$

We now prove

LEMMA 3.19. *Under the assumptions* (3.2)–(3.6)*, there exist disjoint sets* E_j *such that*

$$\sum_j |E_j|^{-1} \geqslant C^{-1} M^d, \tag{3.20}$$

$$C^{-1} M^{-d} \leqslant |E_j| \leqslant C M^{-d/2} \tag{3.21}$$

and there exist θ_1, θ_2 *with* $\theta(p) < \theta_1, \theta_2 < 1/d$ *and* $r < d$ *such that for each* j*, there exists some* λ_j *such that*

$$|\{x \in E_j : |B(x)| > \lambda_j\}| \geqslant C_{\theta_2}^{-1} \lambda_j^{-d} (M^d |E_j|)^{-d\theta_2} \tag{3.22}$$

and

$$C^{-1}(M^d|E_j|)^{(-\theta_1 d - 1)/d} M \leqslant \lambda_j \leqslant C(M^d|E_j|)^{\theta_2 r/(d-r)} M. \tag{3.23}$$

Proof. (3.20) and (3.21) follow immediately from (3.15) and (3.16) respectively. By taking $\tilde{\varepsilon}$ small we may actually assume that $M^d|E_j| \geqslant C_1$ for a sufficiently large constant $C_1 > 0$ for each j, since if there exists some j such that $M^d|E_j| \leqslant C_1$, then by (3.17) we get $\|B\|_{L^{d\infty}(E_j)} \geqslant$ const, which is impossible by the assumption (3.6). We now choose $\theta = \theta_1$ in (3.17), $\theta = \theta_2$ in (3.18), and $r < d$ such that $\theta_1 - \theta_2 + 1/d - 1/r > 0$.

By (3.17), there exists some $\lambda = \lambda_{E_j}$ such that

$$|\{x \in E_j : |B(x)| > \lambda\}| \geqslant C_{\theta_1}^{-1} \lambda^{-d} (M^d|E_j|)^{-d\theta_1}. \tag{3.24}$$

We note $|\{x \in E_j : |B(x)| > \lambda\}| \leqslant |E_j|$ for all λ, thus (3.24) leads to

$$\begin{aligned} \lambda &\geqslant C^{-1}[|E_j|^{-1}(M^d|E_j|)^{-d\theta_1}]^{1/d} \\ &= C^{-1}(M^d|E_j|)^{(-\theta_1 d - 1)/d} \cdot M \stackrel{\text{def}}{=} \lambda^*. \end{aligned} \tag{3.25}$$

Let now λ_0 be the smallest λ such that (3.24) holds. We now consider

$$\begin{aligned} \|B\|_{L^r(E_j)}^r &= r \int_0^\infty \lambda^{r-1} |\{x \in E_j : |B(x)| > \lambda\}| d\lambda \\ &= \left(r\int_0^{\lambda^*} + r\int_{\lambda^*}^{\lambda_0} + r\int_{\lambda_0}^{\infty} \right) (\lambda^{r-1}|\{x \in E_j : |B(x)| > \lambda\}|)\, \mathrm{d}\lambda \\ &\leqslant C \left[\int_0^{\lambda^*} \lambda^{r-1}|E_j|\, \mathrm{d}\lambda + \int_{\lambda^*}^{\lambda} \lambda^{r-1}\lambda^{-d}(M^d|E_j|)^{-d\theta_1}\, \mathrm{d}\lambda \right. \\ &\qquad \left. + \int_{\lambda_0}^{\infty} \lambda^{r-1}\lambda^{-d}\, \mathrm{d}\lambda \right] \\ &\leqslant C(r)[\lambda^{*^r}|E_j| + \lambda^{*r-d}(M^d|E_j|)^{-d\theta_1} + \lambda_0^{r-d}]. \end{aligned}$$

We used (3.6) here to estimate the integral over $\lambda > \lambda_0$. Next we note that $\lambda^{*^r}|E_j| \approx \lambda^{*^{r-d}}(M^d|E_j|)^{-d\theta_1}$. We want to show that

$$(C(r)\lambda^{*^r}|E_j|)^{1/r} \leqslant C^{-1}[(M^d|E_j|)^{\theta_2}M^{d/r-1}]^{-1} \tag{3.26}$$

for an arbitrarily prescribed constant C provided that $\tilde{\varepsilon}$ is small enough.

After some calculation (3.26) is equivalent to

$$(M^d|E_j|)^{\theta_1-\theta_2+1/d-1/r} \geqslant \text{ a large constant.} \tag{3.27}$$

But (3.27) is true since $\theta_1 - \theta_2 + 1/d - 1/r > 0$ and $M^d|E_j| \geqslant$ a large constant by assumption. We note that the right-hand side of (3.26) is the lower bound of $\|B\|_{L^r(E_j)}$ by (3.18), thus by (3.26) and the inequality preceding it we get

$$\lambda_0^{r-d} \geqslant C^{-1}\|B\|^r_{L^r(E_j)} \geqslant C[(M^d|E_j|)^{\theta_2}M^{d/r-1}]^{-r}$$

and by the selection of r, i.e., $r < d$ we obtain

$$\lambda_0 \leqslant C[(M^d|E_j|)^{\theta_2}M^{d/r-1}]^{r/(d-r)} = C(M^d|E_j|)^{\theta_2 r/(d-r)}M.$$

This completes the proof of Lemma 3.19. □

REMARK. As we pointed out in the introduction, $\|f\|^d_{L^{d\infty}(E)} \geqslant \Sigma_j \|f\|^d_{L^{d\infty}(E_j)}$ is not true in general even for the disjoint union $E = \bigcup_j E_j$. An easy example is $f(x) = |x|^{-1}$ since $\|f\|_{L^{d\infty}(R^d)} \leqslant C$ and $\|f\|_{L^{d\infty}(2^{-(k+1)} \leqslant |x| \leqslant 2^{-k})} \geqslant C^{-1}$ for each k. Though we have the lower bound $\|B\|_{L^{d\infty}(E_j)}$ for each E_j, we can not get control over the lower bound $\|B\|_{L^{d\infty}(\cup E_j)}$ by simply adding $\|B\|_{L^{d\infty}(E_j)}$. This is the reason we need the bounds (3.23) for the values of λ satisfying (3.22). These bounds together with the following combinatorial lemma will allow us to complete the proof of Theorem 1.

LEMMA 3.28. *Assume* $\{x_j\}$ *and* $\{\lambda_j\}$ *are two sequences of positive numbers satisfying the following conditions:*

(i) $x_j \geqslant 1$ *for each* j,
(ii) $\sum x_j^{-1} \geqslant 1$,
(iii) $x_j^{-\gamma_1} \leqslant \lambda_j \leqslant x_j^{\gamma_2}$ *for some* $\gamma_1 > 0, \gamma_2 > 0$.

Fix $0 < \alpha < 1$, *then there exists some* $\lambda > 0$ *such that*

$$\sum_{j:\lambda_j > \lambda} x_j^{-\alpha}\left(\frac{\lambda}{\lambda_j}\right)^d \geqslant C^{-1},$$

where C depends on α ,γ_1, γ_2.

Proof. Let $a_k = \{j : 2^k \leqslant \lambda_j \leqslant 2^{k+1}\}$,

$$I_k = \sum_{j \in a_k} x_j^{-\alpha}, \qquad B_k = \sum_{j \in a_k} x_j^{-1}.$$

Then

$$B_k = \sum_{j \in a_k} x_j^{-1} = \sum_{j \in a_k} x_j^{-\alpha} x_j^{-1+\alpha}.$$

We first consider the case $k \geqslant 0$. We note $\lambda_j \leqslant x_j^{\gamma_2}$, so $x_j^{-1+\alpha} \leqslant 2^{(k/\gamma_2)(-1+\alpha)}$. Hence, $B_k \leqslant (\Sigma_{j \in a_k} x_j^{-\alpha}) 2^{-(k/\gamma_2)(1-\alpha)} = I_k \cdot 2^{-(k/\gamma_2)(1-\alpha)}$.

Let now $k < 0$, we note $\lambda_j \geqslant x_j^{-\gamma_1}$ then $B_k = \Sigma_{j \in a_k} x_j^{-1} \leqslant I_k \cdot 2^{(k+1/\gamma_1)(1-\alpha)}$. By the assumption $\Sigma_k B_k \geqslant 1$, thus

$$\begin{aligned} 1 \leqslant \sum_k B_k &= \sum_{k \geqslant 0} B_k + \sum_{k<0} B_k \\ &\leqslant \sum_{k \geqslant 0} I_k 2^{-(k/\gamma_2)(1-\alpha)} + \sum_{k<0} I_k 2^{((k+1)/\gamma_1)(1-\alpha)} \\ &\leqslant \left(\sup_k I_k\right) \left(\sum_{k \geqslant 0} 2^{-(k/\gamma_2)(1-\alpha)} + \sum_{k<0} 2^{((k+1)/\gamma_1)(1-\alpha)}\right) \\ &\leqslant C \sup_k I_k, \end{aligned}$$

therefore $I_{k_0} \geqslant C^{-1}$ for some k_0. This proves the lemma. □

Proof of Theorem 1. Set $x_j = M^d |E_j|$. Then (i) and (ii) of Lemma 3.28 are immediate from the properties (3.20) and (3.21) of the sets $\{E_j\}$. By (3.23) we get

$$x_j^{-\gamma_1} \leqslant \frac{\lambda_j}{M} \leqslant x_j^{\gamma_2},$$

where $\gamma_1 = (d\theta_1 + 1)/d$ and $\gamma_2 = \theta_2 r/(d-r)$. By (3.22),

$$\begin{aligned} \lambda^d |\{x \in \bigcup E_j : |B(x)| > \lambda\}| &\geqslant C^{-1} \textstyle\sum_{j:\lambda_j > \lambda} \left(\dfrac{\lambda}{\lambda_j}\right)^d x_j^{-d\theta_2} \\ &= C^{-1} \textstyle\sum_{j:\lambda_j/M > \mu} \left(\dfrac{\mu}{\lambda_j/M}\right)^d x_j^{-d\theta_2}, \end{aligned}$$

where $\mu = \lambda/M$. Since $\alpha = d\theta_2 < 1$, it follows by Lemma (3.28) that for some μ the summation is bigger than a constant, which is a contradiction with (3.6).

REMARK. There is an alternate way of sharpening the result of [2], using Campanato–Morrey type conditions instead of weak type conditions. This was first done by [2]. We have nothing to add to the known results (e.g., [2], [5]) in this direction.

Added in Proof: The result of this paper has been extended to the variable coefficient case in [4].

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