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On Conformally Invariant Equation $(-\Delta)^p u - K(x)\, u^{\frac{N+2p}{N-2p}} = 0$ and Its Generalizations (*).

GUOZHEN LU - JUNCHENG WEI - XINGWANG XU

Abstract. – *We consider the question of existence and non-existence of positive entire solutions for conformally invariant equations involving polyharmonic operator. We obtain existence of infinitely many positive solutions if the potential decays sufficiently fast at infinity and the nonexistence of positive solutions if the potential grows too fast at infinity. We also establish a Kazdan-Warner type condition for non-existence of solutions decaying at infinity.*

1. – Introduction.

Let us start with the concept of conformally invariant operators. On a general Riemannian manifold M with metric g, a metrically defined operator A is said to be conformally invariant if metrics g_w and g are pointwise conformally related, i.e., if $g_w = e^{2w} g$, the pair of corresponding operators A_w and A are related by

$$A_w(\varphi) = e^{-bw} A(e^{aw}\varphi) \tag{1.1}$$

for all $\varphi \in C^\infty(M)$.

Conformal Laplacian $4(n-1)/(n-2)\,\Delta - k$, where k is the scalar curvature of the metric g, is a well known second order conformally invariant operator. Associated with this well understood operator, there is a prescribed scalar curvature problem. Given a smooth positive function K defined on a Riemannian manifold (M, g_0) of dimension $n \geq 2$, we ask whether there exists a metric g pointwise conformal to g_0 such that K is the scalar curvature of the new metric g. Let $g = e^{2u} g_0$ for $n = 2$ or $g = u^{\frac{4}{n-2}} g_0$ for $n \geq 3$, then the problem is reduced to find solutions of the following nonlinear elliptic equations:

$$\Delta_{g_0} u + Ke^u = k_0 \tag{1.2}$$

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Indirizzo degli AA.: GUOZHEN LU: Department of Mathematics, Wayne State University, Detroit, MI 48202. E-mail address: gzlu@math.wayne.edu; JUNCHENG WEI: Department of Mathematics, Chinese University of Hong Kong, Shatin, Hong Kong. E-mail address: wei@math.cuhk.edu.hk; XINGWANG XU: Department of Mathematics, National University of Singapore, Singapore 119260, Republic of Singapore. E-mail address: matxuxw@math.nus.edu.sg

for $n=2$, or

$$\begin{cases} \dfrac{4(n-1)}{n-2}\Delta_{g_0} u + K u^{\frac{n+2}{n-2}} = k_0 u \\ u > 0 \quad \text{on } M \end{cases} \tag{1.3}$$

for $n \geqslant 3$, where Δ_{g_0} denotes the Beltrami-Laplacian operator of (M, g_0) and k_0 is the scalar curvature of g_0. When $M = R^n$ or equivalently, S^n, this question has been received a lot of attention in the past two decades.

In this paper we consider the question of existence and nonexistence of positive solutions of the following polyharmonic equation

$$(-\Delta)^p u - K(x)\, u^q = 0 \tag{1.4}$$

in $\mathbb{R}^N$ with $N \geqslant 2p+1$, $q > 1$.

One of its characters is its conformal invariance. By comparing with (1.3), they are quite similar. In fact, it is known that $P_{2p} = (-\Delta)^p$ is a conformally invariant operator on $\mathbb{R}^{2p}$ which is a special form of the general operator discovered by Paneitz [17] when $p=2$, see also Branson [1] and Djadli-Hebey-Ledoux [6] for further related results. It is generally known for $p > 2$ just as being verified in [7]. But one does not know the exact form of this operator on a general Riemannian manifold unless $p=2$.

There is a $2p$th operator on $\mathbb{R}^N$ with $N > 2p$ which is similar to conformal Laplacian operator. In fact, when $p=2$, on a general Riemannian manifold (M, g_0), we can explicitly define the so-called Q-curvature as the following

$$Q_{g_0} = -\frac{1}{2(N-1)}\Delta R + d_N R^2 + c_N Ric^2 \tag{1.5}$$

where

$$c_N = -\frac{2}{(N-2)^2}, \quad d_N = \frac{N^3 - 4N^2 + 16N - 16}{8(N-1)^2(N-2)^2}.$$

Let

$$P_4 = (-\Delta)^2 + \delta(a_N R + b_N Ric)\, d$$

be the Paneitz operator in M with $a_N = \frac{N-2}{4(N-1)} + \frac{1}{(N-1)(N-2)}$ and $b_N = -\frac{4}{N-2}$.

Given a smooth function $Q(x)$ on M, the prescribing Q-curvature problem is the following: find a conformal metric g such that $Q_g = Q$. If we let $g = u^{\frac{4}{N-4}} g_0$, then the problem is reduced to find out solutions to the following equation

$$P_4 u + \frac{N-4}{2} Q_0 u = \frac{N-4}{2} Q u^{\frac{N+4}{N-4}}. \tag{1.6}$$

When $M = R^N$, then equation (1.6) is exactly (1.4) with $p=2$.

We will restrict our attention to the case $M = \mathbb{R}^N$ since one does not have much information about this operator in general case. However, for more recent results concerning this general case, one refers the reader to [3].

Theorems proved in this paper are in spirit motivated by and inspired to the earlier results for second order semilinear equations by W.-M. Ni [16] and the subsequent works by Kenig and Ni [8], Lin [11], Li and Ni [12], and many others. In particular, we will avoid using the comparison method with radial equations in order to use the sub-super solution scheme. We also refer the interested reader to the recent work by the first two authors [13] and [14] for this sort of argument, where the Yamabe-type problem was studied for subelliptic operators on the Heisenberg and stratified groups.

One of the most intriguing results in this paper is Theorem 1.3. The main difficulty in proving this lies in the lack of the Maximum Principle for higher order elliptic operators. As a substitute in this setting, we use Lemma 1.4 below which is really the key machinery to derive Theorem 1.3. We hope that this lemma will be found useful in studying other related problems for higher order operators. By adapting similar ideas of proving Lemma 1.4 given in this paper, the second and third authors have subsequently obtained analogous result when $K = 1$ in [18], which the authors use it to prove the following

THEOREM A. – Let $K = 1$. Then the equation (1.4) has no positive entire solution for $q < (N+2p)/(N-2p)$.

We should mention that, regarding to equation (1.4), the above theorem was derived earlier by Lin in [10], and the third author with different method, in [20] for $p = 2$.

We now turn to state the main results of this paper. Our first result is an existence theorem which is an extension of the results of [16].

THEOREM 1.1. – *Let $K(x)$ be a bounded locally Hölder continuous function in $\mathbb{R}^N$. Let $(x_1, x_2) \in R^{N-m} \times R^m$ and suppose that $m > 2p$ and*

$$|K(x_1, x_2)| \leqslant \frac{C}{|x_2|^l}$$

for $|x_2|$ large, uniformly in $x_1 \in \mathbb{R}^{N-m}$ for some constants $C > 0$ and $l > 2p$. Then the equation (1.4) has infinitely many bounded positive solutions with the property that $\lim_{|x| \to \infty} u(x) = C_0$ for some positive C_0. Furthermore, let $\mathcal{P}_k(R^N)$ be the set of nonnegative polynomials with degree k. If $m > 2p > k$ and $l > 2p + kq$ then for any sufficiently small positive constants $a_0 > 0$, $a_1 > 0$ and for any $P_k(x) \in \mathcal{P}_k(R^m)$, there exists a solution u of (1.4) such that

$$\lim_{|x_2| \to \infty} [u(x_1, x_2) - (a_0 + a_1 P_k(x_2))] = 0\,,$$

$$\lim_{|x_2| \to \infty} (-\Delta)^i [u(x_1, x_2) - (a_0 + a_1 P_k(x_2))] = 0, \; i = 1, \ldots, p-1\,.$$

REMARK. – Theorem (1.1) shows that the solution structure of (1.4) is somehow more complicated than that for the classical Yamabe-type problem

$$\Delta u + K(x)\, u^q = 0\,, \quad q > 1 \tag{1.7}$$

(see [16]).

Next we discuss some non-existence results.

THEOREM 1.2. – *Suppose that* $K(x) = O(|x|^{-l})$ *for some* $K \in C^1(\mathbb{R}^N)$ *and* $l > 2p$ *and that the function*

$$L(x) = \left[N - \frac{(N-2p)(q+1)}{2} \right] K(x) + x \cdot \nabla K(x)$$

never changes sign in $\mathbb{R}^N$ $(N > 2p)$. *Then the equation* (1.4) *does not possess any bounded positive solution* u *with* $\lim\limits_{|x| \to \infty} \inf u(x) = 0$.

Finally we show that if $K(x)$ grows too fast, then there are no positive solutions.

THEOREM 1.3. – *If* $K(x) \geq C|x|^l$ *for some* $l \geq q(N-2p) - N$. *Then the equation* (1.4) *has no positive solutions for* $q > 1$.

Theorems 1.1 and 1.2, in such general forms, are new for the equation (1.4). As we can see, there is a difference between the decay power in Theorem 1.2 and the growth power of $K(x)$ in Theorem 1.3. It will be an interesting open question to study the case when $K(x)$ lies in between. Our Theorem 1.3, under additional hypothesis that the inequality (1.8) holds and that u is radial, was derived in [15]. Our theorem does not require any of the aforementioned extra hypothesis given in [15].

As we have mentioned earlier, unlike the second order equation, the major difficulty in studying equation (1.5) is that the Maximum Principle can not be directly applied to u without any information of $(-\Delta)^i u$, $i = 1, \ldots, p-1$. Thus, we have to get sufficient information about $(-\Delta)^i u$ from equation (1.4).

The following is the key lemma.

LEMMA 1.4. – *Let* u *be a positive solution of* (1.4) *with* $K(x) \geq C|x|^l$ *with* $l \geq q(N-2p) - N$. *Then* u *must satisfy the following*

$$(-\Delta)^i u(x) > 0\,, \quad i = 1, \ldots, p-1\,. \tag{1.8}$$

We now sketch briefly the ideas employed to derive our theorems. We prove Theorem 1.1 by sub-super solution method. Here, we compare both u and $(-\Delta)^i u$, $i = 1, \ldots, p-1$ by using the sub-super solutions for elliptic systems.

Theorem 1.2 is proved via the Pohozaev identity. To apply Pohozaev identity, we have to obtain a priori estimates on the asymptotic behavior of u and $(-\Delta)^i u$, $i = 1, \ldots, p-1$. This is given in Theorem 2.4.

To obtain Theorem 1.3 we take the average on both sides of (1.4) and obtain a $2p$-th differential inequality. Then we use Lemma 1.4 and standard blow up arguments to conclude.

The organization of the paper is the following: In Section 2, we present some a priori estimates for some Newtonian potential. We prove Theorem 1.1 in Section 3. Section 4 contains the Pohozaev identity and the proof of Theorem 1.2. Finally we prove Lemma 1.4, which is stated as Theorem 5.2, and Theorem 1.3 in Section 5.

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2. – A priori estimates.

In this section, we shall study the asymptotic behavior of positive solutions of the equation (1.4), i.e.,

$$(-\Delta)^p u(x) - K(x)\, u^q(x) = 0 \qquad x \in \mathbb{R}^N$$

for $N > 2p$, $q > 1$ under various hypotheses on K.

Let us first estimate the Newtonian potential of f where f satisfies

$$|f(x)| \le C|x|^l, \quad l < -2p\,. \tag{2.1}$$

LEMMA 2.1. – *Let w be the Newtonian potential of f, i.e.,*

$$w(x) = C_N \int_{\mathbb{R}^N} \frac{f(y)}{|x-y|^{N-2p}}\, dy\,, \tag{2.2}$$

where f satisfies the assumption (2.1). *Then w is well-defined and at ∞, the following estimates hold true.*

$$|w(x)| \le \begin{cases} C|x|^{2p-N}, & \text{if } l < -N\,; \\ C|x|^{2p-N}\log|x|\,, & \text{if } l = -N\,; \\ C|x|^{2p+l}, & \text{if} -N < l < -2p\,; \end{cases}$$

PROOF. – The proof is rather standard. We include a proof here just for the sake of completeness.

It is easy to see that $w(x)$ is well-defined since by (2.2) that there exists a constant $C > 0$ such that

$$|w(x)| \leqslant C \int_{\mathbf{R}^N} \frac{(1 + |y|)^l}{|x-y|^{N-2p}} dy$$

for some $l < -2p$. Keep in mind that in the following the positive constant C may be varied from line to line.

Next we decompose the above integral as follows:

$$|w(x)| \leqslant \left(\int_{|x-y| < |x|/2} + \int_{|x|/2 \leqslant |x-y| \leqslant 2|x|} + \int_{2|x| \leqslant |x-y|} \right) \left(\frac{C(1+|y|)^l}{|x-y|^{N-2p}} \right) dy$$

$$= I_1 + I_2 + I_3 .$$

We shall estimate I_i, $(i = 1, 2, 3)$ separately.

$$I_1 = \int_{|x-y| \leqslant |x|/2} \frac{C(1+|x|)^l}{|x-y|^{N-2p}} dy$$

$$\leqslant C|x|^l \int_{|x-y| \leqslant |x|/2} \frac{C}{|x-y|^{N-2p}} dy .$$

By evaluating the integral, we get

$$I_1 \leqslant C|x|^{l+2p}.$$

To estimate I_3, we note that $|x-y| \geqslant 2|x|$ implies $|y| \geqslant |x|$ so that $|x-y| \leqslant |x| + |y| \leqslant 2|y|$, i.e., $|y| \geqslant |x-y|/2$. Hence

$$I_3 = C \int_{|x-y| \geqslant 2|x|} \frac{(1+|y|)^l}{|x-y|^{N-2p}} dy$$

$$\leqslant C \int_{|x-y| \geqslant 2|x|} \frac{(1+|x-y|/2)^l}{|x-y|^{N-2p}} dy$$

$$\leqslant C \int_{2|x|}^{\infty} r^{2p+l-1} dr$$

$$= C|x|^{2p+l}.$$

We now estimate I_2 as follows:

$$I_2 \leq C|x|^{2p-N} \int_{|x|/2 \leq |x-y| \leq 2|x|} (1+|y|)^l dy$$
$$\leq C|x|^{2p-N}\left(\int_{|y| \leq 1} (1+|y|)^l dy + \int_{1 \leq |y| \leq 3|x|} (1+|y|)^l dy \right).$$

Note

$$\int_{1 \leq |y| \leq 3|x|} |y|^l dy \leq \begin{cases} C & \text{if } N-1+l < -1\,; \\ C \log|x| & \text{if } N-1+l = -1\,; \\ C|x|^{N+l} & \text{if } N-1+l > -1\,. \end{cases}$$

Thus estimate for I_2 follows.

Similarly we have

LEMMA 2.2. – *If $f \geq 0$ in $\mathbb{R}^N$ and $|f(x)| \geq C|x|^l$ at infinity for some $l < -2p$, $C > 0$, then the Newtonian potential w, if it exists, defined by* (2.2) *has the following lower bounds*

$$|w(x)| \geq \begin{cases} C|x|^{2p-N} & \text{if } l < -N\,; \\ C|x|^{2p-N}\log|x| & \text{if } l = -N\,; \\ C|x|^{2p+l} & \text{if } -N < l < -2p\,. \end{cases}$$

The proof of this lemma is similar to that of lemma 2.1. We shall omit it here.

LEMMA 2.3. – *Let v be a bounded solution of*

$$(-\Delta)^p v - f = 0$$

in $\mathbb{R}^N$, where $N > 2p$, $|f(x)| \leq C|x|^l$, $l < -2p$.

Then $v_\infty = \lim_{|x| \to \infty} v(x)$ exists and

$$v(x) = v_\infty + C_N \int_{\mathbb{R}^N} \frac{f(y)}{|x-y|^{N-2p}} dy\,.$$

PROOF. – Let $w(x) = \int_{\mathbb{R}^N} \frac{f(y)}{|x-y|^{N-2p}} dy$. Then

$$(-\Delta)^p w(x) = f(x)$$

and

$$(-\Delta)^p (w - v) = 0\,.$$

Note that $w - v$ is bounded in $\mathbb{R}^N$ and then by the Liouville theorem for the polyharmonic operator we have

$$w - v = -C$$

for some constant C (we include a proof of it at the end of this section since we can not locate a reference for this well-known fact, see Lemma 2.5 in this section). Since $\lim_{|x|\to\infty} w(x) = 0$, we have $C = v_\infty$.

The following is the main estimate of this section.

THEOREM 2.4. – *Let u be a bounded positive solution of* (1.4) *in* $\mathbb{R}^N$, $N > 2p$, $q > 0$, *and* $\liminf_{|x|\to\infty} u(x) = 0$. *Suppose that K is a locally Hölder continuous function which satisfies the following decay*

$$|K(x)| \leq C|x|^l, \quad \text{at } \infty,$$

for some $l < -2p$. Then

$$u(x) \leq \begin{cases} C|x|^{2p-N} \text{ at } \infty, & \text{if } p > \dfrac{N+l}{N-2p}; \\ C_\varepsilon |x|^{\frac{(1-\varepsilon)(l+2p)}{1-q}} \text{ at } \infty, & \text{if } p \leq \dfrac{N+l}{N-2p} \end{cases}$$

where C_ε only depends on ε.

PROOF. – Let $f(x) = K(x)\, u^q(x)$, we then have

$$(-\Delta)^p u - f = 0 \quad \text{in } \mathbb{R}^N.$$

By Lemma 2.3, we have

$$u(x) = C_N \int_{\mathbb{R}^N} \frac{f(y)}{|x-y|^{N-2p}} dy.$$

Since $u_\infty = \lim_{|x|\to\infty} u(x)$ always exist and is thus equal to zero. We shall divide the proof into three cases.

Case 1. $l < -N$. In this case the result follows directly from Lemma 2.1.

Case 2. $l = -N$. Lemma 2.1 implies that for $|x|$ large enough

$$|u(x)| \leq C|x|^{2p-N} \log|x| \leq C|x|^{\frac{2p-N}{2}}$$

thus

$$|f(x)| = |K(x)| \cdot |u(x)|^p \leq C|x|^{l+q(2p-N)/2}$$

for $|x|$ large.

Since $l = -N$, $q > 0$, we have $l + q(2p - N)/2 < -N$, and thus our result follows from Lemma 2.1 again.

Case 3. $-2p > l > -N$. From Lemma 2.1 we have for $|x|$ large

$$|f(x)| = |K(x)| \cdot |u(x)|^q \leqslant C|x|^{l+q(l+2p)}.$$

If $l + q(l + 2p) \leqslant -N$, then we are done by Cases 1 and 2. Otherwise, apply Lemma 2.1 to get

$$|u(x)| \leqslant C|x|^{l+q(l+2p)+2p} = C|x|^{(l+2p)(q+1)}$$

for large $|x|$. We iterate this process to conclude that after k-th iteration

$$|u(x)| \leqslant \begin{cases} C|x|^{2p-N} & \text{if } l_k \leqslant -(N-2p) \\ C|x|^{l_k} & \text{if } l_k > -(N-2p), \end{cases}$$

where $l_k = (1 + q + q^2 + \ldots + q^k)\ (l + 2p)$.

If $q \geqslant 1$, then $l_k \to -\infty$ as $k \to \infty$ since $l < -2p$. For $q < 1$, when $q > \dfrac{N+l}{N-2p}$, since $l_{k+1} - l_k = q^{k+1}(l + 2p) < 0$ and $\lim\limits_{k\to\infty} l_k = (l + 2p)/(1 - q) < -(N - 2p)$, there exists a very large k such that $l_k \leqslant -(N - 2p)$ and then it follows that

$$|u(x)| \leqslant C|x|^{2p-N} \quad \text{at } \infty .$$

When $q \leqslant \dfrac{N+l}{N-2p}$, $l_k \downarrow \dfrac{l+2p}{1-q} \geqslant -(N-2p)$ and thus our result follows from lemma 2.1.

Finally, we state and prove a Liouville theorem for polyharmonic functions, which is of independent interest.

LEMMA 2.5. – *Let ϕ be a bounded function such that $(-\Delta)^p\phi = 0$. Then $\phi \equiv$ Constant.*

PROOF. – Let $\phi_0 = \phi$, $\phi_i = (-\Delta)^i\phi$, $i = 1, \ldots, p-1$. We first prove that

$$\phi_{p-1} \geqslant 0 .$$

Suppose not, there exists $x_0 \in R^N$ such that

$$\phi_{p-1}(x_0) < 0 .$$

Without loss of generality, we assume that $x_0 = 0$.

Let $\bar{u}(r)$ be the spherical average of $u(x)$, namely,

$$\bar{u}(r) = \frac{1}{w_{N-1} r^{N-1}} \int_{|x|=r} u(x)\, ds .$$

Then we have

$$\Delta\bar{\phi}_0 + \bar{\phi}_1 = 0 , \quad \Delta\bar{\phi}_1 + \bar{\phi}_2 = 0, \ldots, \Delta\bar{\phi}_{p-1} = 0 .$$

Since $\overline{\phi}_{p-1}(0) < 0$ and $\overline{\phi}'_{p-1} = 0$, we have

$$\overline{\phi}_{p-1}(r) = \overline{\phi}_{p-1}(0) < 0 \quad \text{for all } r > \bar{r}_1 = 0 .$$

Then it is easy to see that

$$\overline{\phi}'_{p-2} > \frac{(-\overline{\phi}_{p-1}(0))}{N} r .$$

Hence

$$\overline{\phi}_{p-2}(r) \geq c_2 r^2$$

for r large.

Same arguments show that

$$\overline{\phi}_{p-3}(r) \leq -c_3 r^4$$

and

$$(-1)^i \overline{\phi}_{p-i}(r) \geq c_i r^{2(i-1)}, \quad i = 1, \ldots, p$$

for r large, which is a contradiction to the fact that $\phi_0 = \phi(x)$ is bounded.

Hence $\phi_{p-1} \geq 0$. By changing ϕ to $-\phi$, we have $-\phi_{p-1} \geq 0$. So $\phi_{p-1} = 0$. Similarly we have

$$(-\Delta)^i \phi = 0 , \quad i = 1, \ldots, p-1 .$$

So ϕ is harmonic and the lemma is thus proved. ■

3. – Proof of Theorem 1.1.

Theorem 1.1 can be proved by using sub-super solution method. Let us first state a comparison theorem.

THEOREM 3.1. – *Let $(u^1, v_1^1, \ldots, v_{p-1}^1)$ and $(u^2, v_1^2, \ldots, v_{p-1}^2)$ be a pair of functions satisfying*

$$\Delta u^1 + v_1^1 \geq \Delta u^2 + v_1^2,$$

$$\Delta v_i^1 + v_{i+1}^1 \geq \Delta v_i^2 + v_{i+1}^2,\ i = 1, \ldots, p-2 ,$$

and

$$\Delta v_{p-1}^1 + K(x)(u^1)^q \geq \Delta v_{p-1}^2 + K(x)(u^2)^q$$

with $u^1 \leq u^2$, $v_i^1 \leq v_i^2$, $i = 1, \ldots, p-1$ and $q > 1$.

Then there exists a solution $(u, v_1, \ldots, v_{p-1})$ *of the following problem*

$$\begin{cases} \Delta u + v_1 = 0;\ \Delta v_i + v_{i+1} = 0,\ i = 1, \ldots, p-2\,; \\ \Delta v_{p-1} + K u^q = 0\,; \end{cases}$$

Moreover, $u^1 \leqslant u \leqslant u^2$ *and* $v_i^1 \leqslant v_i \leqslant v_i^2$, $i = 1, \ldots, p-1$.

PROOF. – Note that our system is quasi-monotone. On bounded domains with Dirichlet boundary condition, please see the proof of Theorem 1.2-5 of [9]. On $\mathbb{R}^N$, we can use an approximate procedure as in the proof of Lemma 2.7 of [5]. Since the proofs are standard, we omit the details.

We now use Theorem 3.1 to prove Theorem 1.1. Equation (1.4) can be rewritten as

$$\Delta u + v_1 = 0\,, \quad \Delta v_1 + v_2 = 0, \ldots, \Delta v_{p-1} + K(x)\, u^q = 0\,.$$

PROOF OF THEOREM 1.1. – By our assumption on K, there exists a continuous function $\tilde{K}(x_2)$ such that

$$|K(x)| \leqslant \tilde{K}(x_2)\,, \quad \tilde{K}(x_2) \leqslant (1 + |x_2|)^l.$$

Let $\alpha > 0$ and $M_1, M_2 > 0$ be numbers to be chosen later, Let $u_+(x) = \alpha(M_1 + M_2\phi + \phi_0)$ and $u_- = \alpha(M_1 + M_2\phi - \phi_0)$, where $\phi \in \mathcal{P}_k(R^m)$ and

$$(-\Delta_{x_2})^p \phi_0(x_2) - |\tilde{K}(x_2)|(1 + \phi(x_2))^q = 0\,,$$

$$(-\Delta_{x_2})^i \phi_0(x_2) \to 0 \quad \text{as } |x_2| \to \infty \quad \text{for all } i = 1, 2, \ldots, p-1\,.$$

(This implies that $(-\Delta_{x_2})^i \phi_0 > 0$, $i = 1, \ldots, p-1$.)

Note that when $\phi(x_2) = 0$ we need to require

$$|\tilde{K}(x_2)| \leqslant C|x_2|^l, \quad l < -2p\,,$$

while when $\phi \neq 0$ we require

$$|\tilde{K}(x_2)| \leqslant C|x_2|^l, \quad l < -2p - kq\,,$$

so that $|\tilde{K}(x_2)(1 + \phi(x_2))^q| \leqslant C|x_2|^l$, $l < -2p$.

Thus $\lim_{|x_2| \to \infty} |\phi_0(x_2)| = 0$ by Lemmas 2.1 and 2.4.

Define v_i^+ and v_i^-, $i = 1, \ldots, p-1$ so that

$$(-\Delta)^i u_+ = v_i^+\,, \quad (-\Delta)^i u_- = v_i^-\,, \quad i = 1, \ldots, p-1\,.$$

Thus

$$\begin{aligned} &\Delta u_+ + v_1^+ = 0 = \Delta u_- + v_1^- \\ &\Delta v_i^+ + v_{i+1}^+ = \Delta v_i^- + v_{i+1}^-, \quad i = 1, \ldots, p-2 \end{aligned}$$

and

$$\begin{aligned}
&\Delta v_{p-1}^{+} + K(x)\, u_+^q \\
=\ & -\alpha |\tilde{K}(x_2)| (1+\phi)^q + K(x)\, \alpha^q (M_1 + M_2 \phi + \phi_0)^q \\
\leqslant\ & |\tilde{K}(x_2)| (-\alpha(1+\phi)^q + \alpha^q (M_1 + M_2\phi + \phi_0)^q) \\
\leqslant\ & |\tilde{K}(x_2)| (\alpha^q (\max(2M_1, M_2))^q - \alpha)(1+\phi)^q \\
\leqslant\ & 0
\end{aligned}$$

provided that $\alpha \geqslant \alpha^q (\max(2M_1, M_2))^q$ and $-M_1 \leqslant \phi_0 \leqslant M_1$. Similarly we have

$$\begin{aligned}
&\Delta v_{p-1}^{-} + K(x) u_-^q \\
=\ & \alpha |\tilde{K}(x_2)| (1+\phi)^q + K(x)\, \alpha^q (M_1 + M_2\phi - \phi_0)^q \\
\geqslant\ & |\tilde{K}(x_2)| (\alpha - \alpha^q (\max(2M_1, M_2))^q)(1+\phi)^q \\
\geqslant\ & 0
\end{aligned}$$

provided that $\alpha > \alpha^q (\max(2M_1, M_2))^q$. Thus,

$$\Delta u_+ + v_1^+ = \Delta u_- + v_1^- = 0\,, \quad \Delta v_i^+ + v_{i+1}^+ = \Delta v_i^- + v_{i+1}^-\,, \quad i = 1, \ldots, p-2$$

and

$$\Delta v_{p-1}^{+} + K(x)\, u_+^q \leqslant 0 \leqslant \Delta v_{p-1}^{-} + K(x) u_-^q$$

and $u_+ \geqslant u_-$ and $v_i^+ - v_i^- = 2\alpha(-\Delta)^i \phi_0 > 0$. Therefore, by Theorem 3.1 there exists a solution to the equations

$$\Delta u + v_1 = 0\,, \quad \Delta v_i + v_{i+1} = 0\,, \quad \Delta v_{p-1} + K(x)\, u^q = 0\,.$$

Theorem 1.1 is thus proved.

4. – Proof of Theorem 1.2.

In this section we prove Theorem 1.2. To this end, we first state a Pohozaev identity.

LEMMA 4.1. – *Let u be a solution of $(-\Delta)^p u = f(x, u)$, then we have*

$$\int_\Omega \left[nF(x, u) - \frac{n-2p}{2} u f(x, u) + x \cdot \nabla F(x, u) \right] dx = - \int_{\partial\Omega} B_p(u)\, d\sigma \tag{4.1}$$

where when $p=2m$,

$$B_p(u)=\left(2-\frac{n}{2}\right)\sum_{k=1}^{m}\left[(-\Delta)^{2m-k}u\frac{\partial(-\Delta)^{k-1}u}{\partial\nu}-\frac{\partial(-\Delta)^{p-k}u}{\partial\nu}(-\Delta)^{k-1}u\right]+$$

$$+2\sum_{k=1}^{m-1}\sum_{j=1}^{k}\left[(-\Delta)^{p-j}u\frac{\partial(-\Delta)^{j-1}u}{\partial\nu}-\frac{\partial(-\Delta)^{p-j}u}{\partial\nu}(-\Delta)^{j-1}u\right]+$$

$$+\sum_{k=1}^{m}\left[\langle x,\nabla(-\Delta)^{k-1}u\rangle\frac{\partial(-\Delta)^{p-k}u}{\partial\nu}-[(-\Delta)^{p-k}u]\frac{\partial\langle x,\nabla(-\Delta)^{k}u\rangle}{\partial\nu}\right]+$$

$$+1/2((-\Delta)^m u)^2\langle x,\nu\rangle-F(x,u)\langle x,\nu\rangle;$$

when $p=2m+1$,

$$-B_p(u)=F(x,u)\langle x,\nu\rangle-2\sum_{k=1}^{m}\sum_{j=1}^{k}\left[(-\Delta)^{p-j}u\frac{\partial((-\Delta)^{j-1}u)}{\partial\nu}-((-\Delta)^{j-1}u)\frac{\partial(-\Delta)^{p-j}u}{\partial\nu}\right]+$$

$$+(1-n/2)\sum_{k=1}^{m}\left[((-\Delta)^{k-1}u)\frac{\partial(-\Delta)^{p-k}u}{\partial\nu}-(-\Delta)^m u\frac{\partial(-\Delta)^m u}{\partial\nu}\right]+$$

$$+\sum_{k=1}^{m}\left[\langle x,\nabla(-\Delta)^{k-1}u\rangle\frac{\partial(-\Delta)^{p-k}u}{\partial\nu}-[(-\Delta)^{p-k}u]\frac{\partial\langle x,\nabla(-\Delta)^{k-1}u\rangle}{\partial\nu}\right]-$$

$$-1/2\,|\nabla[(-\Delta)^m u]|^2\langle x,\nu\rangle+\langle x,\nabla(-\Delta)^m u\rangle\frac{\partial(-\Delta)^m u}{\partial\nu},$$

where $F(x,u)=\int_0^u f(x,s)\,ds$ *and* ν *is the unit outward normal vector along the boundary* $\partial\Omega$.

PROOF. – Notice that

$$(-\Delta)[\langle x,\nabla(-\Delta)^i u\rangle]=2(-\Delta)\,u+\langle x,\nabla(-\Delta)^{i+1}u\rangle.$$

By repeatedly using this fact and the second Green's identity, we can get the above formula easily.

Let u be a bounded positive solution of

$$(-\Delta)^p u-K(x)\,u^q=0$$

where $|K(x)|\le C|x|^l$ for some $l<-2p$. We set $f(x)=K(x)\,u^q$. Then we have

$$|u(x)|=O(|x|^{2p-N}),\quad at\ \infty,$$

and

$$|f(x)| \leq C(1+|x|)^{l-q(N-2p)}, \quad x \in \mathbb{R}^N.$$

Note that $l - q(N-2p) < -N$ and

$$\nabla^\alpha u(x) = C_N \int_{\mathbb{R}^N} \nabla^\alpha \left(\frac{1}{|x-y|^{N-2p}} \right) f(y)\, dy$$

for $|\alpha| \leq 2p-1$.

We have

$$|x|^{N-2p+|\alpha|}\, |\nabla^\alpha u(x)| \leq C \int_{\mathbb{R}^N} \frac{|x|^{N-2p+|\alpha|}}{|x-y|^{N-2p+|\alpha|}(1+|y|)^{q(N-2p)-l}} dy\,.$$

We can argue as in Lemma 2.1 to get

$$|x|^{N-2p+|\alpha|}\, |\nabla^\alpha u(x)| \leq C\,.$$

We now apply the Pohazaev identity for $f(x,u) = K(x)\, u^q$ on $\Omega = B_R$ to get

$$\int_{B_R} \left[\frac{N}{q+1} K(x) + \frac{1}{q+1} x \cdot \nabla K(x) - \frac{N-2p}{2} K(x) \right] u^{q+1}\, dx = -\int_{\partial B_R} B_p(u)\, d\sigma$$

where $B_p(u)$ is defined by Lemma 4.1.

We only prove the case when $p = 2m$. The odd case is similar.

We now estimate each term on the right hand side:

$$\left| \int_{\partial B_R} (x, \nu)\, K(x) u^{q+1} d\sigma \right| \leq C R^N \cdot R^l \cdot R^{-(q+1)(N-2p)} \to 0, \quad \text{since } l < -2p;$$

$$\left| \int_{\partial B_R} \sum_{i=1}^{m} (-\Delta)^{2m-k} u \frac{\partial (-\Delta)^{k-1} u}{\partial \nu} d\sigma \right| \leq C R^{N-1} R^{2p-N-2(2m-k)} R^{2p-N-2(k-1)-1} = C R^{2p-N} \to 0$$

$$\left| \int_{\partial B_R} \sum_{i=1}^{m} \frac{\partial (-\Delta)^{2m-k} u}{\partial \nu} (-\Delta)^{k-1} u d\sigma \right| \leq C R^{N-1} R^{2p-N-2(2m-k)-1} R^{2p-N-2(k-1)} = C R^{2p-N} \to 0$$

$$\left| \int_{\partial B_R} \sum_{k=1}^{m-1} \sum_{j=1}^{k} \left[(-\Delta)^{p-j} u \frac{\partial (-\Delta)^{j-1} u}{\partial \nu} - \frac{\partial (-\Delta)^{p-j} u}{\partial \nu} (-\Delta)^{j-1} u \right] d\sigma \right| \leq$$

$$\leq C R^{N-1} R^{2p-2N+1} \to 0$$

$$\left| \int_{\partial B_R} ((-\Delta)^m u)^2 \langle x, \nu \rangle \, d\sigma \right| \leq C R^{2p-N} \to 0$$

$$\Bigg| \int_{\partial B_R} \sum_{k=1}^{m} \langle x, \nabla(-\Delta)^{k-1} u \rangle \frac{\partial(-\Delta)^{p-k} u}{\partial \nu} - \sum_{k=1}^{m} [(-\Delta)^{p-k} u] \cdot$$

$$\cdot \frac{\partial \langle x, \nabla(-\Delta)^k u \rangle}{\partial \nu} \, d\sigma \Bigg| \leq C R^{2p-N} \to 0$$

Therefore we have

$$\int_{\mathbb{R}^N} \left[x \cdot \nabla K(x) + \left(N - \frac{N-2p}{2}(q+1) \right) K(x) \right] u^{q+1} dx = 0 .$$

The rest of the proof follows a method similar to the proof of Theorem 1.4 in Section 3.1 of [12].

5. - Proof of Theorem 1.3.

Throughout this section we assume that $K(x) \geq 0$ and $u(x)$ is a solution of (1.4). We define

$$\bar{u}(r) = \frac{1}{w_{N-1} r^{N-1}} \int_{|x|=r} u(x) \, d\sigma ,$$

and

$$\bar{\bar{K}} = \left(\frac{1}{w_{N-1} r^{N-1}} \int_{|x|=r} K(x)^{-\frac{1}{p-1}} d\sigma \right)^{-\frac{p-1}{p}} .$$

Then we have

LEMMA 5.1. – *Let u be a positive solution of* (1.4) *in* $\mathbb{R}^N$. *Then $\bar{u}(r)$ satisfies the following differential inequality*

$$(-\Delta)^p \bar{u} - \bar{\bar{K}}(r) \, \bar{u}^q \geq 0 \quad \textit{in } [0, \infty)$$

and

$$\bar{u}'(0) = 0, ((-\Delta)^i \bar{u})'(0) = 0 , \quad i = 1, \ldots, p-1 .$$

PROOF. – By a slight modification of the proof of Lemma 3.1 in [16].

The following theorem, which is stated as Lemma 1.4 in the introduction, is the key result in this section.

THEOREM 5.2. – *Let u be a positive solution of $(-\Delta)^p u = K(x)\, u^q$ with $K(x) \geq C|x|^l$ and $l \geq q(N-2p) - N$, $q > 1$. Then we have*

$$(-\Delta)^i u \geq 0\,, \quad i = 1, \dots, p-1\,.$$

PROOF. – Let $v_i = (-\Delta)^i u$, $i = 0, 1, 2, \dots, p-1$ with $v_0 = u$. We first prove the following

$$v_{p-1} \geq 0\,. \tag{5.1}$$

Suppose not, there exists $x_0 \in R^n$ such that

$$v_{p-1}(x_0) < 0\,.$$

Without loss of generality, we assume that $x_0 = 0$. By Lemma 5.1 and the assumptions on $K(x)$,

$$\Delta \bar{u} + \bar{v}_1 = 0\,, \quad \Delta \bar{v}_1 + \bar{v}_2 = 0, \dots, \Delta \bar{v}_{p-1} + Cr^l (\bar{u})^q \leq 0$$

Since $\bar{v}_{p-1}(0) < 0$ and $\bar{v}'_{p-1}(r) \leq 0$, we have

$$\bar{v}_{p-1}(r) \leq \bar{v}_{p-1}(0) < 0\,, \quad \text{for all } r > \bar{r}_1 = 0\,. \tag{5.2}$$

Then integrating the second last equation we have

$$\bar{v}'_{p-2} > \frac{(-\bar{v}_{p-1}(0))}{N} r\,.$$

Hence

$$\bar{v}_{p-2}(r) \geq c_2 r^2, \qquad \text{for } r \geq \bar{r}_2 > \bar{r}_1\,. \tag{5.3}$$

Same arguments show that

$$\bar{v}_{p-3}(r) \leq -c_3 r^4, \quad \text{for } r \geq \bar{r}_3 > \bar{r}_2 \tag{5.4}$$

and

$$(-1)^i \bar{v}_{p-i}(r) \geq c_i r^{2(i-1)}, \quad \text{for } r \geq \bar{r}_i,\ i = 1, \dots, p\,. \tag{5.5}$$

Hence if p is odd, we have a contradiction to the fact that $u > 0$.
So p must be even and we have

$$\bar{u}(r) \geq c_0 r^{\sigma_0}, \quad \sigma_0 = 2(p-1) \tag{5.6}$$

and

$$(-1)^i \bar{v}_{p-i} > 0$$

for $r > \bar{r}_0 > 0$.

We can now start the iteration. Setting $A = (2q(p-1) + l + N + 2p)$. Note that $A \geqslant 2q(p-1) + q(N-2p) + 2p = q(N-2) + 2p > 0$.

Suppose now that

$$\bar{u}(r) \geqslant c_0^{q^k} \frac{r^{\sigma_k}}{A^{b_k}}, \qquad \text{for } r \geqslant r_k. \tag{5.7}$$

Then we have

$$r^{N-1}(\bar{v}_{p-1})' \leqslant r_k^{N-1}(\bar{v}_{p-1})'(r_k) - \int_{r_k}^{r} s^{l+N-1} u^q(s)\, ds$$

$$\bar{v}'_{p-1} \leqslant -\frac{r^{q\sigma_k + l + 1} - r_k^{q\sigma_k + l + 1}}{A^{qb_k}(q\sigma_k + l + N)} c_0^{q^{k+1}}.$$

Hence

$$\bar{v}'_{p-1} \leqslant -\frac{c_0^{q^{k+1}} r^{q\sigma_k + l + 1}}{2A^{qb_k}(q\sigma_k + l + N)}$$

for $r \geqslant 2^{\frac{1}{q\sigma_k + l + 2}} r_k$.

Similarly

$$\bar{v}_{p-1} \leqslant -\frac{c_0^{q^{k+1}} r^{q\sigma_k + l + 2}}{4A^{qb_k}(q\sigma_k + l + N)(q\sigma_k + l + 2)}$$

for $r \geqslant 2^{\frac{1}{q\sigma_k + l + 2}} 2^{\frac{1}{q\sigma_k + l + 2}} r_k$.

Hence

$$\bar{v}_{p-1} \leqslant -\frac{c_0^{q^{k+1}} r^{q\sigma_k + l + 2}}{A^{qb_k} 4(q\sigma_k + l + N)^2}$$

for $r \geqslant 2^{\frac{2}{q\sigma_k + l + 2}} r_k$.

By induction, we have

$$(-1)^i \bar{v}_{p-i}(r) \geqslant \frac{c_0^{q^{k+1}} r^{q\sigma_k + l + 2i}}{(q\sigma_k + N + l + 2p)^{2i} A^{qb_k} 4^i}, \qquad r \geqslant 2^{\frac{2i}{q\sigma_k + l + 2}} r_k. \tag{5.8}$$

Hence

$$\bar{u}(r) \geqslant \frac{c_0^{q^{k+1}} r^{q\sigma_k + l + 2p}}{2^{2p} A^{qb_k}(q\sigma_k + l + N + 2p)^{2p}}, \qquad r \geqslant 2^{\frac{2p}{q\sigma_k + l + 1}} r_k. \tag{5.9}$$

Set

$$\sigma_0 = 2(p-1), \quad r_0 = \bar{r}_0$$

$$\sigma_{k+1} = q\sigma_k + l + 2p,$$

$$r_{k+1} = 2^{\frac{2p}{q\sigma_k + l + 1}} r_k.$$

(Note that the condition that $l \geq q(N-2p) - N$ ensures that

$$l + 2 + 2q(p-1) \geq q(N-2p) + 2 - N + 2q(p-1) = (q-1)(N-2) > 0.$$

Therefore it is easy to check that

$$q\sigma_k + l + 2 > 0$$

so that the previous arguments do work.)

First of all, by mathematical induction, it is easy to see that

$$2^{2p}(q\sigma_k + N + l + 2p)^{2p} \leq A^{2p(k+1)}$$

by noticing that

$$q\sigma_k + N + l + 2p \leq A(q\sigma_{k-1} + N + l + 2p).$$

Hence we also can set

$$b_0 = 0, \quad b_{k+1} = qb_k + 2p(k+1).$$

Then we have

$$\bar{u}(r) \geq \frac{c_0^{q^{k+1}} r^{\sigma_{k+1}}}{A^{b_{k+1}}}, \quad r \geq r_{k+1}.$$

Notice that

$$r_{k+1} \leq cr_0$$

where c can be chosen to be $2^{\sum_{k=0}^{\infty} \frac{2p}{q\sigma_k + l + 2}}$.

Also notice that, by using the iteration formulas above, we have

$$\sigma_k = 2\,\frac{p(q^{k+1}-1) - q + 1}{q-1},$$

and

$$b_k = 2p\,\frac{q^{k+1} - (k+1)q^2 + k}{(q-1)^2}.$$

Hence, if we take $M > 1$ is large enough so that $MA^{2/(q-1)} \geq 2cr_0$ if $c_0 \geq 1$ and $MA^{2/(q-1)} c_0^{-1} \geq 2cr_0$ if $c_0 < 1$, and then take $r_1 = MA^{2/(q-1)}$ or $MA^{2/(q-1)} c_0^{-1}$ depending on whether c_0 is greater than or less than 1, then we have

$$\bar{u}(r_1) \geqslant [A^{1/(q-1)^2}]^{2pq^{k+1} - 4(p+q) + 4 + 2p(k+1)q^2 - 2pk} \longrightarrow \infty \quad \text{as } k \longrightarrow \infty .$$

Since r_1 is independent of k, a contradiction is reached.

Hence

$$v_{p-1} \geqslant 0 .$$

Next we claim that

$$v_{p-2} \geqslant 0 .$$

The proof is exactly the same as before except now that we need take extra care about the case that p is odd. We omit the details.

Next we recall the following lemma.

LEMMA 5.3. – *Let $u \in C^{2m}(R^N)$, $m \geqslant 1$ be radially symmetric satisfying the inequalities*

$$(-\Delta)^k u \geqslant 0 \quad \text{in } R^N \text{ for } 0 \leqslant k \leqslant m$$

where $2m < N$. Then necessarily we have

$$(ru'(r) + (N-2m)\, u(r))' < 0 . \tag{5.10}$$

PROOF. – See example 2.3 in [2].

REMARK. – Note that (5.10) is equivalent to $(r^{N-2m} u(r))' \geqslant 0$. hence

$$\bar{u}(r) \geqslant C r^{2m-N}, \quad C > 0,\ r > r_0. \tag{5.11}$$

We are now ready to prove Theorem 1.3.

PROOF OF THEOREM 1.3. – As in the proof of Lemma 5.1, we apply the spherical mean operator to (1.1) and we obtain for $r \in (0, \infty)$

$$\Delta \bar{u} + \bar{v}_1 = 0,\ \ldots,\ \Delta \bar{v}_{p-1} + C r^l \bar{u}^q \leqslant 0 .$$

Hence we have

$$(r^{N-1}\bar{u}')' + r^{N-1}\bar{v}_1 = 0,\ \ldots, \tag{5.12}$$

$$(r^{N-1}\bar{v}_{p-1}')' + C r^{l+N-1}\bar{u}^p = 0 . \tag{5.13}$$

Integrating (5.12)-(5.13) on $(0, r)$ and taking into account that $\bar{u}, \bar{v}_i$ are non-increasing, we obtain

$$\bar{u}(r) \geqslant C r^2 \bar{v}_1(r),\ \bar{v}_1(r) \geqslant C r^2 \bar{v}_2(r),\ \ldots,\ \bar{v}_{p-1}(r) \geqslant C r^{2+l}\bar{u}^q(r) .$$

Therefore we have

$$\bar{u}(r) \geqslant Cr^{2(p-1)}\bar{v}_{p-1}(r)$$

and

$$-\Delta \bar{v}_{p-1}(r) \geqslant Cr^{l+2q(p-1)}(\bar{v}_{p-1})^q(r).$$

Hence

$$\bar{v}_{p-1}(r) \leqslant Cr^{-(l+2+2q(p-1))/(q-1)}. \tag{5.14}$$

On the other hand, it follows from (5.11) with $m=1$ that

$$\bar{v}_{p-1}(r) \geqslant Cr^{2-N}. \tag{5.15}$$

Hence we have

$$Cr^{2-N} \leqslant \bar{v}_{p-1}(r) \leqslant Cr^{-\frac{l+2+2q(p-1)}{q-1}}.$$

Thus if

$$l+2+2q(p-1) > q(N-2p) - N + 2 + 2q(p-1) = (N-2)(q-1)$$

we obtain a contradiction for large r.

For the case $l+2+2q(p-1) = (q-1)(N-2)$, we proceed as follows.

By equation (5.12) and (5.13), we have

$$-\Delta \bar{v}_{p-1}(r) \geqslant Cr^{l+2q(p-1)}\bar{v}^q_{p-1}(r). \tag{5.16}$$

Hence we have

$$-(r\bar{v}'_{p-1} + (N-2)\,\bar{v})' \geqslant Cr^{l+1+2q(p-1)}\bar{v}^q.$$

Let t be a fixed large number. Integrating the above equation from r to t, we have

$$r\bar{v}'_{p-1} + (N-2)\,\bar{v}_{p-1}(r) \geqslant C\int_r^t r^{l+1+2q(p-1)}\bar{v}^q_{p-1} \geqslant C(\bar{v}_{p-1}r^{N-2})^q\int_r^t \xi^{1+l+2q(p-1)+q(2-N)}\,d\xi.$$

Letting $t \to \infty$, we obtain

$$r\bar{v}'_{p-1} + (N-2)\,\bar{v}_{p-1}(r) \geqslant C(\bar{v}_{p-1}r^{N-2})^q r^{l+2+q(2p-N)} \geqslant C(\bar{v}_{p-1}r^{N-2})^q r^{2-N}$$

since $l+2p = (q-1)(N-2p)$.

Hence we obtain

$$(\bar{v}_{p-1}r^{N-2})' \geqslant C(\bar{v}_{p-1}r^{N-2})^q r^{-1}.$$

Integrating the last inequality from r to t and noting that $q>1$, we reach a contradiction.

REFERENCES

[1] T. BRANSON, *Group representations arising from the Lorentz conformal geometry*, J. Funct. Anal., **74** (1987), pp. 199-293.

[2] P. CLEMENT - R. MANASEVICH - E. MITIDIERI, *Positive solutions for a quasilinear system via blow up*, Comm. PDE, **18** (1993), pp. 2071-2106.

[3] S-Y A. CHANG - P. C. YANG, *Extremal metrics of zeta function determinants on 4-manifolds*, Ann. Math., **142** (1995), pp. 171-212.

[4] S-Y A. CHANG - P. C. YANG, *On uniqueness of solutions of n-th order differential equations in conformal geometry*, Math. Research Letters, **4** (1997), pp. 91-102.

[5] W.-Y. DING - W.-M. NI, *On the elliptic equation* $\Delta u + Ku^{(n+2)/(n-2)} = 0$ *and related topics*, Duke Math. J., **52** (1985), pp. 485-506.

[6] DJADLI - E. HERBEY - M. LEDOUX, *Paneitz type operators and applications*, preprint.

[7] C. R. GRAHAM - R. JENNE - L. MASON - G. SPARLING, *Conformally invariant powers of the Laplacian, I: Existence*, J. London Math. Soc., **46** (1992), pp. 557-565.

[8] C. KENIG - W.-M. NI, *An exterior Dirichlet problem with applications to some nonlinear equations arising in geometry*, Amer. J. Math., **106** (1984), pp. 689-702.

[9] A. LEUNG, *Systems of nonlinear partial differential equations: applications to biology and engineering*, Mathematics and its Applications, Kluwer Academic Publishers, 1989.

[10] C.-S. LIN, *A classification of solutions of a conformally invariant fourth order equation in* R^n, preprint (1996).

[11] F. H. LIN, *On the elliptic equation* $\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial U}{\partial x_j}\right) - k(x)\,U + K(x)u^p = 0$, Proc. AMS, **2** (1985), pp. 219-226.

[12] YI LI - W.-M. NI, *On conformal scalar curvature equations in* R^n, Duke Math. J., **57** (1988), pp. 895-924.

[13] G. LU - J. WEI, *On positive entire solutions to the Yamabe-type problem on the Heisenberg and stratified groups*, Electronic Research Announcements of Amer. Math. Soc., Vol. **3**, 1997.

[14] G. LU - J. WEI, *On positive entire solutions to CR Yamabe-type problem on the Heisenberg and stratified groups: Existence and non-existence results*, preprint (1997).

[15] E. MITIDIERI, *Nonexistence of positive solutions of semilinear elliptic systems in* R^N, Diff. Int. Eqns., **9** (1996), pp. 465-479.

[16] W.-M. NI, *On the elliptic equation* $\Delta u + Ke^{\frac{n+2}{n-2}} = 0$, *its generalizations and applications in geometry*, Indiana Univ. Math. J., **31** (1982), pp. 493-529.

[17] S. PANEITZ, *A quadratic conformally covariant differential operator for arbitrary pseudo-Riemannian manifolds*, preprint (1983).

[18] J. WEI - X. XU, *Classification of solutions of higher order conformally invariant equations*, To appear in Math. Ann.

[19] X. XU, *Classification of solutions of certain fourth order nonlinear elliptic equations in* R^4, preprint (1996).

[20] X. XU, *Uniqueness theorem for the entire positive solutions of biharmonic equations in* $\mathbb{R}^n$, preprint (1997).