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Advances in Mathematics 217 (2008) 1838–1868
ADVANCES IN Mathematics
www.elsevier.com/locate/aim

Inhomogeneous infinity Laplace equation ☆

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Received 25 August 2006; accepted 29 November 2007

Available online 3 January 2008

Communicated by Michael J. Hopkins

Abstract

We present the theory of the viscosity solutions of the inhomogeneous infinity Laplace equation $\partial_{x_i} u \partial_{x_j} u \partial^2_{x_i x_j} u = f$ in domains in $\mathbf{R}^n$. We show existence and uniqueness of a viscosity solution of the Dirichlet problem under the intrinsic condition f does not change its sign. We also discover a characteristic property, which we call the comparison with standard functions property, of the viscosity sub- and super-solutions of the equation with constant right-hand side. Applying these results and properties, we prove the stability of the inhomogeneous infinity Laplace equation with nonvanishing right-hand side, which states the uniform convergence of the viscosity solutions of the perturbed equations to that of the original inhomogeneous equation when both the right-hand side and boundary data are perturbed. In the end, we prove the stability of the well-known homogeneous infinity Laplace equation $\partial_{x_i} u \partial_{x_j} u \partial^2_{x_i x_j} u = 0$, which states the viscosity solutions of the perturbed equations converge uniformly to the unique viscosity solution of the homogeneous equation when its right-hand side and boundary data are perturbed simultaneously.

Keywords: Inhomogeneous equation; Infinity Laplace equation; Absolute minimizers; Viscosity solutions; Comparison with standard functions; Stability of solutions

0. Introduction

Since the introduction of absolute minimizers by G. Aronsson in his works [1–3] in the 1960s, the infinity Laplace equation has undergone several phases of extensive study. According to

☆ G. Lu is partially supported by an NSF Grant and P. Wang is supported by the University Research Grant from Wayne State University.

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0001-8708/$ – see front matter
doi:10.1016/j.aim.2007.11.020

Aronsson, an **absolute minimizer** in a domain $\Omega \subset \mathbf{R}^n$ is a continuous real-valued function which has the least possible Lipschitz constant in every open set whose closure is compactly contained in Ω. In the fundamental work [14] by R. Jensen, equivalence of the absolute minimizers and viscosity solutions of the homogeneous infinity Laplace equation was established and an original proof of the uniqueness of absolute minimizers was provided. A special property of absolute minimizers was discovered, namely, the difference of an absolute minimizer and a cone function verifies the weak maximum principle in any domain excluding the vertex of the cone and where they are defined. This is the so-called comparison with cone property.

Since then, many people have contributed to the theory of absolute minimizers which are also called infinity harmonic functions. To mention a few of such contributions which are of course far from a complete list, we refer to the works of [10,9,8] by Crandall, Evans and Gariepy, [16] by Lindqvist and Manfredi, [15] by Juutinen, and [7] by Barron, Jensen and Wang which help to complete the theory of absolute minimizers.

There are some further development in the theory of absolute minimizers. For instance, the work [18] of Manfredi, Petrosyan and Shahgholian dealt with a free boundary problem of the homogeneous infinity Laplace equation.

A systematic treatment of the theory of absolute minimizers can be found in the manuscript [5] by Aronsson, Crandall and Juutinen, and the references therein.

Uniqueness of absolute minimizers is worth special attention in the theory. After Jensen's fundamental work [14], Barles and Busca gave a second proof of the uniqueness of absolute minimizers in [6], which is quite different from Jensen's original one and works for a broad class of degenerate elliptic equations. Recently, Crandall, Gunnarsson and Wang provided a third proof of the uniqueness of absolute minimizers in bounded domains and they successfully applied their truncation method to many unbounded domains including all exterior domains, i.e. the domains obtained from the whole space $\mathbf{R}^n$ by deleting a compact set, and to some non-euclidean norms (see [11]).

This paper is our first attempt to analyze the inhomogeneous degenerate equations. The inhomogeneous infinity Laplace equation is the prototype of such highly degenerate nonlinear partial differential equations. Our motive to study the inhomogeneous infinity Laplace equation is not only for the theory's own good but also for the seeking of the connection between the homogeneous infinity Laplace equation and the inhomogeneous infinity Laplace equation, namely the property preserved under the perturbation of the homogeneous infinity Laplace equation.

We concentrate on the inhomogeneous ∞-Laplace equation

$$\Delta_\infty u := \sum_{i,j=1}^{n} \partial_{x_i} u \partial_{x_j} u \partial^2_{x_i x_j} u = f$$

(the notation is explained in Section 1), where the right-hand side function f is continuous but stays strictly away from 0.

In Section 2, a Perron's method is applied to establish the existence of a viscosity solution of the Dirichlet problem for the inhomogeneous ∞-Laplace equation. More precisely, a family of admissible super-solutions is constructed and the infimum of the family is shown to be a viscosity solution. A fact worth of noting is the nonexistence of classical (i.e. C^2) solutions, which follows from our uniqueness theorem in the coming section.

In Section 3, a penalization method initially introduced in the work of Crandall, Ishii and Lions, [12], for elliptic equations and later applied in [11] is employed to lead to a contradiction, if a comparison theorem were untrue. The uniqueness theorem is an immediate consequence of the comparison principle proved. A significant feature of the uniqueness theorem is the assumption the right-hand side f stays strictly away from 0. This is an intrinsic condition instead of a technical reason. The uniqueness theorem is invalid if f changes its sign. We provide a counter-example in this case in Appendix A.

In Section 4, we present the comparison with standard functions property for sub- and super-solutions of the inhomogeneous infinity Laplace equation $\Delta_\infty u = 1$. The proof bears the ideas in Crandall's work [8] and the joint work of Crandall and Wang, [13]. We found a special family of singular radial classical solutions, the standard functions with which every viscosity sub- or super-solution of the equation enjoys comparison in a sense to be made clear in the section, of the inhomogeneous infinity Laplace equation with nonzero constant right-hand side. We believe it is not accidental, as a well-known characteristic property of the infinity harmonic functions (i.e. viscosity solutions of the homogeneous infinity Laplace equation) is the comparison with cone functions. Unlike the homogeneous case, we need to separate the standard functions into two sub-families and formulate the comparison property for sub- and super-solutions of the inhomogeneous equation with the two sub-families separately. The two comparison properties thus obtained for sub- and super-solutions, called the comparison with standard functions from above and from below, characterize the viscosity sub- and super-solutions of $\Delta_\infty u = 1$ completely. A closely related parabolic version of the comparison principle can be found in the second part of [13].

In Section 5, we perturb the right-hand side f and boundary data g of the Dirichlet problem

$$\begin{cases} \Delta_\infty u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$$

We assume f and g are continuous functions in their respective domains, and the values of f are kept strictly away from 0. Our analysis shows that the viscosity solutions of the perturbed equations with perturbed boundary data converge uniformly to the viscosity solution of the original inhomogeneous Dirichlet problem, provided that the perturbations converge uniformly to 0. It is a surprise to us as the equation is highly degenerate.

In Section 6, we establish a connection between the inhomogeneous infinity Laplace equation with its well-studied homogeneous counterpart. As we did to the inhomogeneous infinity Laplace equation in Section 5, we perturb the homogeneous infinity Laplace equation and the boundary data, and we prove the uniform convergence of the viscosity solutions of the perturbed equations to the viscosity solution of the homogeneous infinity Laplace equation, if the perturbations converge uniformly to 0 in their respective domains.

At last, we provide a counter-example of the uniqueness of a viscosity solution of the Dirichlet problem for the inhomogeneous equation $\Delta_\infty u = f$, if f is allowed to change its sign, in Appendix A. It was modified from a counter-example constructed in [19].

We end this introduction by pointing out that existence, uniqueness and stability results, and comparison property with cone-like functions have been recently established for the normalized infinity Laplace equation using PDE methods in [17].

1. Definitions and notations

For two vectors $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n) \in \mathbf{R}^n$,

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i$$

is the inner product of x and y, while $x \otimes y$ is the tensor product $y^t x$, or $[y_i x_j]_{n\times n}$ in the matrix form, of the vectors x and y. For $x \in \mathbf{R}^n$, $|x|$ denotes the Euclidean norm $\langle x, x\rangle^{\frac{1}{2}}$ of x and $\hat{x} = \frac{x}{|x|}$ denotes the normalized vector for $x \neq 0$.

The standard notations in the set theory and analysis are adopted here. For example, $\partial\Omega$ and $\bar{\Omega}$ mean the boundary and closure of a set Ω respectively, while $\partial_{x_i} u$ denotes the partial derivative of u with respect to x_i. $V \Subset \Omega$ means V is compactly contained in Ω, i.e. V is a subset of Ω whose closure is also contained in Ω. Also, for two positive numbers λ and μ, $\lambda \ll \mu$ means λ is bounded above by a sufficiently small multiple of μ. $\circ(\varepsilon)$ denotes quantities whose quotients by ε approach 0 as ε does, while $O(\varepsilon)$ denotes quantities that are comparable to ε.

Suppose S is a subset of $\mathbf{R}^n$. A function $f : S \to \mathbf{R}$ is said to be **Lipschitz continuous** on S if there is a constant L such that

$$\left|f(x) - f(y)\right| \leqslant L|x - y|,$$

for any x and y in S. The least of such constant is denoted by $L_f(S)$. If S is an open subset Ω of $\mathbf{R}^n$, we use the symbol $\mathrm{Lip}(\Omega)$ to denote the set of all Lipschitz continuous functions on Ω. If instead $S = \partial\Omega$ is the boundary of an open subset Ω of $\mathbf{R}^n$, we use the symbol $\mathrm{Lip}_\partial(\Omega)$ to denote the set of all Lipschitz continuous functions on $\partial\Omega$. Ω always denotes an open subset of $\mathbf{R}^n$ and is usually bounded. $C(\Omega)$ denotes the set of continuous functions defined on Ω and $C(\bar{\Omega})$ denotes the set of continuous functions on $\bar{\Omega}$. $C^2(\Omega)$ denotes the set of functions which are continuously twice differentiable on Ω. A smooth function usually means a C^2 function in this paper. If $f \in C(\Omega)$, then $\|f\|_{L^\infty(\Omega)} := \sup_{x\in\Omega} |f(x)|$ denotes the L^∞-norm of f on Ω.

Throughout this paper, the infinity Laplace operator Δ_∞ is the highly degenerate nonlinear partial differential operator defined on C^2 functions u by

$$\Delta_\infty u = \partial_{x_i} u \partial_{x_j} u \partial^2_{x_i x_j} u,$$

where the right-hand side is the sum over $i, j = 1, 2, \ldots, n$. $\Delta_\infty u$ is usually called the infinity Laplacian of u.

$\mathcal{S}_{n\times n}$ denotes the set of all $n \times n$ symmetric matrices with real entries. We use I to denote the identity matrix in $\mathcal{S}_{n\times n}$. For an element $S \in \mathcal{S}_{n\times n}$, $\|S\|$ denotes its operator norm, namely $\sup_{x\in\mathbf{R}^n\setminus\{0\}} \frac{\langle Sx,x\rangle}{|x|^2}$.

$u \prec_{x_0} \varphi$ means $u - \varphi$ has a local maximum at x_0. In this case, we say φ touches u by above at x_0. Almost always in this paper, $u \prec_{x_0} \varphi$ is understood as $u(x) \leqslant \varphi(x)$ for all $x \in \Omega$ in interest and $u(x_0) = \varphi(x_0)$, as subtracting a constant from φ does not cause any problem in the standard viscosity solution argument applied in the paper. On the other hand, if $\varphi \prec_{x_0} u$, we say φ touches u by below at x_0.

Definition 1. A continuous function u defined in an open subset Ω of $\mathbf{R}^n$ is called a **viscosity sub-solution**, or simply abbreviated **sub-solution**, of the inhomogeneous infinity Laplace equation $\Delta_\infty w = f(x)$, if

$$\Delta_\infty \varphi(x_0) \geqslant f(x_0),$$

whenever $u \prec_{x_0} \varphi$ for any $x_0 \in \Omega$ and any C^2 test function φ. Occasionally, we use the phrase '$\Delta_\infty u(x) \geqslant f(x)$ is verified in the viscosity sense' instead.

The second-order **superjet** of u at x_0 is defined to be the set

$$J_{\Omega}^{2,+}u(x_0)=\left\{\left(D\varphi(x_0),\,D^2\varphi(x_0)\right)\colon\ \varphi \text{ is } C^2 \text{ and } u \prec_{x_0} \varphi\right\},$$

whose closure is defined to be

$$\begin{aligned}\bar{J}_{\Omega}^{2,+}u(x_0)=\Big\{(p,X)\in \mathbf{R}^n\times \mathcal{S}_{n\times n}\colon\ &\exists(x_n,p_n,X_n)\in\Omega\times\mathbf{R}^n\times\mathcal{S}_{n\times n} \text{ such that}\\ &(p_n,X_n)\in J_{\Omega}^{2,+}u(x_n) \text{ and } \big(x_n,u(x_n),p_n,X_n\big)\to\big(x_0,u(x_0),p,X\big)\Big\}.\end{aligned}$$

On the other hand, u is called a **viscosity super-solution**, or simply **super-solution**, of the inhomogeneous infinity Laplace equation $\Delta_\infty w = f(x)$, if

$$\Delta_\infty\varphi(x_0)\leqslant f(x_0),$$

whenever $\varphi \prec_{x_0} u$ for any $x_0\in\Omega$ and any C^2 test function φ.

The second-order **subjet** of u at x_0 is defined to be the set

$$J_{\Omega}^{2,-}u(x_0)=\left\{\left(D\varphi(x_0),\,D^2\varphi(x_0)\right)\colon\ \varphi \text{ is } C^2 \text{ and } \varphi \prec_{x_0} u\right\},$$

whose closure is defined to be

$$\begin{aligned}\bar{J}_{\Omega}^{2,-}u(x_0)=\Big\{(p,X)\in \mathbf{R}^n\times \mathcal{S}_{n\times n}\colon\ &\exists(x_n,p_n,X_n)\in\Omega\times\mathbf{R}^n\times\mathcal{S}_{n\times n} \text{ such that}\\ &(p_n,X_n)\in J_{\Omega}^{2,-}u(x_n) \text{ and } \big(x_n,u(x_n),p_n,X_n\big)\to\big(x_0,u(x_0),p,X\big)\Big\}.\end{aligned}$$

A **viscosity solution**, or simply **solution**, of the inhomogeneous infinity Laplace equation $\Delta_\infty w = f(x)$, is both a sub-solution and a super-solution.

When $f(x)\equiv 0$, the sub- and super-solutions are called **the infinity sub-** and **super-harmonic functions** in Ω respectively.

A similar definition of a strict differential inequality is the following

Definition 2. Suppose $u\in C(\Omega)$. We say u verifies the differential inequality

$$|Du|>0$$

in Ω in the viscosity sense if

$$\left|D\varphi(x_0)\right|>0$$

for every C^2 function φ such that $u\prec_{x_0}\varphi$ and $x_0\in\Omega$.

Let us caution that the negative of a sub-solution of the equation $\Delta_\infty u = f(x)$ is a super-solution of the equation $\Delta_\infty u = -f(x)$ instead of $\Delta_\infty u = f(x)$.

In this paper, whenever we consider the inhomogeneous infinity Laplace equation $\Delta_\infty w = C$ with constant right-hand side C, for simplicity, we always take $C=1$ in the statements of the theorems.

In the end, we give an example to justify our conclusion about the nonexistence of classical solutions of the equation $\Delta_\infty w = 1$. An example is $u(x, y, z) = x^{\frac{4}{3}} - y^{\frac{4}{3}} + c_0 z^{\frac{4}{3}}$ in any open subset of $\mathbf{R}^3$ which intersects with both the xz- and yz-planes but does not intersect with the xy-plane. Here and anywhere else in this paper the constant $c_0 = \frac{3\sqrt[3]{3}}{4}$. This is a nonclassical viscosity solution in such open sets. Assuming the uniqueness of solutions of the Dirichlet problem which is proved in Section 3, we can see that no classical solution to the Dirichlet problem with the same continuous boundary data exists.

2. Existence

We prove the existence of a viscosity solution of the inhomogeneous infinity Laplace equation by constructing a solution as the infimum of a family of admissible super-solutions.

Theorem 1. *Suppose Ω is a bounded open subset of $\mathbf{R}^n$, $f \in C(\Omega)$ with $\inf_\Omega f > 0$ and $g \in C(\partial\Omega)$.*

Then there exists $u \in C(\bar{\Omega})$ such that $u = g$ on $\partial\Omega$ and

$$\Delta_\infty u(x) = f(x)$$

in Ω in the viscosity sense.

Proof. We define the admissible set to be

$$\mathcal{A}_{f,g} = \bigl\{v \in C(\bar{\Omega}):\ \Delta_\infty v \leqslant f(x) \text{ in } \Omega,\ \text{and } v \geqslant g \text{ on } \partial\Omega\bigr\}.$$

Here the differential inequality $\Delta_\infty v(x) \leqslant f(x)$ is verified in the viscosity sense as introduced in Section 1.

Take

$$u(x) = \inf_{v \in \mathcal{A}_{f,g}} v(x), \quad x \in \bar{\Omega}.$$

We may take a constant function which is bigger than the supremum of g on $\partial\Omega$. This constant function is clearly an element of $\mathcal{A}_{f,g}$. So the admissible set $\mathcal{A}_{f,g}$ is nonempty.

As the infimum of a family of continuous functions, u is upper-semicontinuous on $\bar{\Omega}$.

According to the standard theory of viscosity solutions, u, as the infimum of viscosity super-solutions, is clearly a viscosity super-solution of $\Delta_\infty u(x) = f(x)$ in Ω and the inequality $u \geqslant g$ holds on $\partial\Omega$.

We prove $\Delta_\infty u(x) \geqslant f(x)$ in Ω in the viscosity sense. Suppose not, there exists a C^2 function φ and a point $x_0 \in \Omega$ such that

$$u \prec_{x_0} \varphi,$$

but $\Delta_\infty \varphi(x_0) < f(x_0)$.

We write

$$\varphi(x) = \varphi(x_0) + \nabla\varphi(x_0) \cdot (x - x_0) + \frac{1}{2}\bigl\langle D^2\varphi(x_0)(x - x_0), x - x_0\bigr\rangle + \circ\bigl(|x - x_0|^2\bigr).$$

For any small $\varepsilon > 0$, we define

$$\varphi_\varepsilon(x) = \varphi(x_0) + \nabla\varphi(x_0)\cdot(x - x_0) + \frac{1}{2}\langle D^2\varphi(x_0)(x - x_0), x - x_0\rangle + \varepsilon|x - x_0|^2.$$

Clearly, $u \prec_{x_0} \varphi \prec_{x_0} \varphi_\varepsilon$, and $\Delta_\infty\varphi_\varepsilon(x) < f(x)$ for all x close to x_0, if ε is taken small enough, thanks to the continuity of f. Moreover, x_0 is a strict local maximum point of $u - \varphi_\varepsilon$. In other words, $\varphi_\varepsilon > u$ for all x near but other than x_0 and $\varphi_\varepsilon(x_0) = u(x_0)$.

We define $\hat{\varphi}(x) = \varphi_\varepsilon(x) - \delta$ for a small positive number δ. Then $\hat{\varphi} < u$ in a small neighborhood, contained in the set $\{x\colon \Delta_\infty\varphi_\varepsilon(x) < f(x)\}$, of x_0 but $\hat{\varphi} \geqslant u$ outside this neighborhood, if we take δ small enough.

Take $\hat{v} = \min\{\hat{\varphi}, u\}$. Because u is a viscosity super-solution in Ω and $\hat{\varphi}$ also is in the small neighborhood of x_0, $\hat{v}$ is a viscosity super-solution of $\Delta_\infty w(x) = f(x)$ in Ω, and along $\partial\Omega$, $\hat{v} = u \geqslant g$. This implies $\hat{v} \in \mathcal{A}_{f,g}$, but $\hat{v} = \hat{\varphi} < u$ near x_0, which is a contradiction to the definition of u as the infimum of all elements of $\mathcal{A}_{f,g}$. Therefore,

$$\Delta_\infty u(x) \geqslant f(x)$$

in Ω in the viscosity sense.

We now show $u = g$ on $\partial\Omega$. For any point $z \in \partial\Omega$, and any $\varepsilon > 0$, there is a neighborhood $B_r(z)$ of z such that $|g(x) - g(z)| < \varepsilon$ for all $x \in B_r(z)$. Take a large number $C > 0$ such that $Cr > 2\|g\|_{L^\infty(\partial\Omega)}$. We define

$$v(x) = g(z) + \varepsilon + C|x - z|,$$

for $x \in \bar{\Omega}$. For $|x - z| < r$ and $x \in \partial\Omega$, $v(x) \geqslant g(z) + \varepsilon \geqslant g(x)$; while for $|x - z| \geqslant r$ and $x \in \partial\Omega$, $v(x) \geqslant g(z) + \varepsilon + Cr \geqslant \|g\|_{L^\infty(\partial\Omega)} \geqslant g(x)$. In addition, $\Delta_\infty v = 0 \leqslant f(x)$ in Ω, as $\inf_\Omega f > 0$. So $v \in \mathcal{A}_{f,g}$ and $v(z) = g(z) + \varepsilon$. So

$$g(z) \leqslant u(z) \leqslant g(z) + \varepsilon,$$

$\forall\varepsilon > 0$. So $u(z) = g(z)$, $\forall z \in \partial\Omega$.

Let us construct another set of admissible functions by defining

$$\mathcal{S}_{f,g} = \{w \in C(\bar{\Omega})\colon\ \Delta_\infty w \geqslant f(x) \text{ in } \Omega, \text{ and } w \leqslant g \text{ on } \partial\Omega\}.$$

Again $\Delta_\infty w \geqslant f(x)$ is satisfied in the viscosity sense. $\mathcal{S}_{f,g}$ is nonempty with a particular element $C\psi_{z,bd}(x) := C(c_0|x - z|^{\frac{4}{3}} + d)$ for a constant C such that $C^3 > \|f\|_{L^\infty(\Omega)}$, any fixed point $z \in \partial\Omega$ and some negative number d with sufficiently large absolute value, because $\Delta_\infty(C\psi_{z,bd}(x)) = C^3 > \|f\|_{L^\infty(\Omega)} \geqslant f(x)$ for $x \in \Omega$ and $C\psi_{z,bd} \leqslant g$ on $\partial\Omega$. We refer the reader to the computation for $\psi_{z,bd}$ in Section 4.

We take

$$\bar{u}(x) = \sup_{w \in \mathcal{S}_{f,g}} w(x)$$

for every $x \in \bar{\Omega}$. Clearly, $\bar{u}$ is lower-semicontinuous in $\bar{\Omega}$ and $\bar{u}(z) \leqslant g(z)$ for any $z \in \partial\Omega$.

Fix a point $z \in \partial\Omega$ and a positive number ε. Since g is continuous on $\partial\Omega$, there exists a positive number r such that $|g(x) - g(z)| < \varepsilon$ for all $x \in B_r(z)$. As Ω is a bounded domain, the values of $|x - z|$ are bounded above and bounded below from zero for all $x \in \Omega\backslash B_r(z)$. We take positive numbers A and B such that $A > 3|x - z|$ for all $x \in \bar{\Omega}$ and $B = \frac{1}{4}A^{\frac{4}{3}}$. So particularly

$$B - \frac{1}{4}(A - 3r)^{\frac{4}{3}} > 0.$$

We take a positive number $C \geqslant 1$ such that

$$C\left(B - \frac{1}{4}(A - 3r)^{\frac{4}{3}}\right) \geqslant 2\|g\|_{L^\infty(\partial\Omega)}$$

and $C^3 \geqslant \|f\|_{L^\infty(\Omega)}$. We define

$$w(x) = g(z) - \varepsilon - C\left\{B - \frac{1}{4}\big(A - 3|x - z|\big)^{\frac{4}{3}}\right\}$$

with A, B and C as chosen.

Computation shows that

$$\Delta_\infty w(x) = C^3 \geqslant \|f\|_{L^\infty(\Omega)} \geqslant f(x)$$

for all $x \in \Omega$. Furthermore, on $\partial\Omega \cap B_r(z)$, $w(x) \leqslant g(z) - \varepsilon < g(x)$; while on $\partial\Omega \backslash B_r(z)$, $w(x) \leqslant g(z) - \varepsilon - C(B - \frac{1}{4}(A - 3r)^{\frac{4}{3}}) < g(z) - 2\|g\|_{L^\infty(\partial\Omega)} \leqslant -\|g\|_{L^\infty(\Omega)} \leqslant g(x)$. So the function w defined above is in the family $\mathcal{S}_{f,g}$. Note that $w(z) = g(z) - \varepsilon$ according to our choice of A and B. So $\bar{u}(z) \geqslant g(z) - \varepsilon$ for any $\varepsilon > 0$, which implies that $\bar{u}(z) \geqslant g(z)$ for any $z \in \partial\Omega$. As $\bar{u}$ is lower-semicontinuous on $\bar{\Omega}$, we know that

$$\liminf_{x\in\Omega\to z} \bar{u}(x) \geqslant g(z)$$

for any $z \in \partial\Omega$.

In the end, we prove $u \in C(\bar{\Omega})$.

Indeed, as $\Delta_\infty u = f(x) \geqslant \inf_{\partial\Omega} f \geqslant 0$ in Ω, it is well known that u which is ∞-subharmonic is locally Lipschitz continuous in Ω (see e.g. [5, Lemma 2.9]). Therefore all we need to prove is that for $\forall z \in \partial\Omega$,

$$\lim_{x\in\Omega\to z} u(x) = g(z).$$

About this matter, as u is upper-semicontinuous on $\bar{\Omega}$ and $u = g$ on $\partial\Omega$, we know

$$\limsup_{x\in\Omega\to z} u(x) \leqslant g(z)$$

for any $z \in \partial\Omega$.

On the other hand, the comparison theorem, Theorem 3, in the next section implies that $w \leqslant v$ on $\bar{\Omega}$ for every $v \in \mathcal{A}_{f,g}$ and every $w \in \mathcal{S}_{f,g}$. As a result, $\bar{u} \leqslant u$ on $\bar{\Omega}$. In particular,

$$\liminf_{x\in\Omega\to z} u(x) \geqslant \liminf_{x\in\Omega\to z} \bar{u}(x) \geqslant g(z)$$

for every point $z \in \partial\Omega$.

Thus we have shown

$$\lim_{x\in\Omega\to z} u(x) = g(z),$$

for $\forall z \in \partial\Omega$. $\square$

Remark 1. We applied the comparison theorem, Theorem 3, from the next section in the above proof. The proof of the comparison theorem is independent of the existence result.

The following theorem is obtained from the above one by considering $v = -u$ and the proof is clear.

Theorem 2. *Suppose Ω is a bounded open subset of $\mathbf{R}^n$, $f \in C(\Omega)$ with $\sup_\Omega f < 0$ and $g \in C(\partial\Omega)$.*

Then there exists $u \in C(\bar{\Omega})$ such that $u = g$ on $\partial\Omega$ and

$$\Delta_\infty u(x) = f(x)$$

in Ω in the viscosity sense.

In the following Sections 5 and 6, we only apply the theorems proved in this section in the cases $f(x) \equiv c$ or $f(x) \equiv -c$ for positive constants c on most occasions.

3. Uniqueness

Ω always denotes a bounded open subset of $\mathbf{R}^n$.

We first prove a strict version of a comparison principle.

Lemma 1. *For $j = 1, 2$, suppose $u_j \in C(\bar{\Omega})$ and*

$$\Delta_\infty u_1 \leqslant f_1 \quad \textit{and} \quad \Delta_\infty u_2 \geqslant f_2$$

in Ω, where $f_1 < f_2$ in Ω, and $f_j \in C(\Omega)$. Assume also $u_1 \geqslant u_2$ on $\partial\Omega$.

Then $u_1 \geqslant u_2$ in Ω.

Proof. Suppose $u_1(x^*) < u_2(x^*)$ for certain $x^* \in \Omega$. For any small $\varepsilon > 0$, we define

$$w_\varepsilon(x, y) = u_2(x) - u_1(y) - \frac{1}{2\varepsilon}|x - y|^2, \quad \forall (x, y) \in \bar{\Omega} \times \bar{\Omega}.$$

We define $M_0 = \max_{\bar{\Omega}}(u_2 - u_1)$ and

$$M_\varepsilon = \max_{\bar{\Omega}\times\bar{\Omega}} w_\varepsilon = u_2(x_\varepsilon) - u_1(y_\varepsilon) - \frac{1}{2\varepsilon}|x_\varepsilon - y_\varepsilon|^2 \quad \text{for some } (x_\varepsilon, y_\varepsilon) \in \bar{\Omega} \times \bar{\Omega}.$$

Our assumption implies $M_0 > 0$.

By Lemma 3.1 of [12], we know

$$\lim_{\varepsilon\downarrow 0} M_\varepsilon = M_0,$$
$$\lim_{\varepsilon\downarrow 0} \frac{1}{2\varepsilon}|x_\varepsilon - y_\varepsilon|^2 = 0$$

and

$$\lim_{\varepsilon\downarrow 0}\big(u_2(x_\varepsilon) - u_1(y_\varepsilon)\big) = M_0.$$

As a result of the second equality, $\lim_{\varepsilon\downarrow 0}|x_\varepsilon - y_\varepsilon| = 0$.

As $M_0 > 0 \geqslant \max_{\partial\Omega}(u_2 - u_1)$, we know $x_\varepsilon, y_\varepsilon \in \Omega_1$ for some $\Omega_1 \Subset \Omega$ and all small ε.

Theorem 3.2 of [12] implies that there exist $X, Y \in \mathcal{S}_{n\times n}$ such that $(\frac{x_\varepsilon - y_\varepsilon}{\varepsilon}, X) \in \bar{J}^{2,+}_\Omega u_2(x_\varepsilon)$, $(\frac{y_\varepsilon - x_\varepsilon}{\varepsilon}, Y) \in \bar{J}^{2,-}_\Omega u_1(y_\varepsilon)$ and

$$-\frac{3}{\varepsilon}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leqslant \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leqslant \frac{3}{\varepsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

In particular, $X \leqslant Y$.

The meaning of $\bar{J}^{2,+}_\Omega u_2(x_\varepsilon)$ and $\bar{J}^{2,-}_\Omega u_1(y_\varepsilon)$ implies that

$$\begin{aligned} f_2(x_\varepsilon) &\leqslant \left\langle X\left(\frac{x_\varepsilon - y_\varepsilon}{\varepsilon}\right), \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}\right\rangle \\ &\leqslant \left\langle Y\left(\frac{x_\varepsilon - y_\varepsilon}{\varepsilon}\right), \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}\right\rangle = \left\langle Y\left(\frac{y_\varepsilon - x_\varepsilon}{\varepsilon}\right), \frac{y_\varepsilon - x_\varepsilon}{\varepsilon}\right\rangle \leqslant f_1(y_\varepsilon). \end{aligned}$$

On the other hand, for certain subsequences x_{ε_j} and y_{ε_j} of x_ε and y_ε and some $x_0 \in \bar{\Omega}_1 \subset \Omega$, $x_{\varepsilon_j}, y_{\varepsilon_j} \to x_0$, as a result of $x_\varepsilon, y_\varepsilon \in \Omega_1 \Subset \Omega$ and $\lim_{\varepsilon\downarrow 0}|x_\varepsilon - y_\varepsilon| = 0$. If we send ε to 0 in $f_2(x_{\varepsilon_j}) \leqslant f_1(y_{\varepsilon_j})$, we get $f_2(x_0) \leqslant f_1(x_0)$ which is in contradiction with $f_1(x_0) < f_2(x_0)$ given in the hypothesis. The proof is complete. □

To prove the uniqueness of viscosity solutions to the Dirichlet problem, we need to prove the following comparison principle.

Theorem 3. *Suppose $u, v \in C(\bar{\Omega})$ satisfy*

$$\Delta_\infty u \geqslant f(x)$$

and

$$\Delta_\infty v \leqslant f(x)$$

in the viscosity sense in the domain Ω, where f is a continuous function defined on Ω and $\inf_\Omega f(x) > 0$.

Then $u \leqslant v$ on $\partial\Omega$ implies $u \leqslant v$ in Ω.

Proof. For any $\delta > 0$, we define $u_\delta = (1+\delta)u - \delta\|u\|_{L^\infty(\partial\Omega)}$ on $\bar{\Omega}$. Then

$$\Delta_\infty u_\delta \geqslant (1+\delta)^3 f > f \geqslant \Delta_\infty v$$

in Ω and $u_\delta \leqslant u \leqslant v$ on $\partial\Omega$. Then apply the preceding Lemma 1 to conclude that $u_\delta \leqslant v$ in Ω for all $\delta > 0$. Sending δ to 0, we have $u \leqslant v$ in Ω. □

It is obvious that the theorem is true if the condition $\inf_\Omega f > 0$ is replaced by $\sup_\Omega f < 0$.

We may write the previous comparison principle in the form of a maximum principle as follows.

Theorem 4. *Suppose $u, v \in C(\bar{\Omega})$ satisfy*

$$\Delta_\infty u \geqslant f(x)$$

and

$$\Delta_\infty v \leqslant f(x)$$

in the viscosity sense in the domain Ω, where f is a continuous function defined in Ω and either $\inf_\Omega f(x) > 0$ or $\sup_\Omega f < 0$ holds.

Then

$$\sup_\Omega (u - v) \leqslant \max_{\partial\Omega} (u - v).$$

As a direct corollary of this theorem, the uniqueness result is stated below.

Theorem 5. *Suppose Ω is a bounded open subset of $\mathbf{R}^n$, and u and $v \in C(\bar{\Omega})$ are both viscosity solutions of the inhomogeneous infinity Laplace equation $\Delta_\infty w = f(x)$ in Ω, where f is a continuous function defined on Ω such that either $\inf_\Omega f > 0$ or $\sup_\Omega f < 0$ holds. If, in addition, $u = v$ on $\partial\Omega$, then $u = v$ in Ω.*

The condition $\inf_\Omega f > 0$ in the above theorems is necessary and intrinsic. The uniqueness theorem is untrue if this condition is omitted, though the strict comparison principle, Lemma 1, does not require the condition. A counter-example is provided in Appendix A to justify our conclusion.

4. Comparison with standard functions

In this section, we demonstrate comparison properties of sub-solutions and super-solutions of the inhomogeneous equation $\Delta_\infty u = 1$ with special classes of standard functions. Those comparison properties characterize the sub- and super-solutions of this equation completely. On some occasions, they may also be regarded as the maximum and minimum principles for this nonlinear inhomogeneous degenerate elliptic equation.

For any $x_0 \in \mathbf{R}^n$, b and $d \in \mathbf{R}$, we define

$$\psi_{x_0,bd}(x) = c_0\big(|x - x_0| + b\big)^{\frac{4}{3}} + d,$$

which we will call a **standard function** in the following. Here and in the following the constant $c_0 = \frac{3\sqrt[3]{3}}{4}$. We define the **domain** $\mathcal{D}(x_0, b)$ **of differentiability** of $\psi_{x_0,bd}$ as

$$\mathcal{D}(x_0, b) = \begin{cases} \mathbf{R}\backslash\{x_0\}, & \text{if } b \geqslant 0, \\ \mathbf{R}\backslash\{x\colon\ x = x_0 \text{ or } |x - x_0| = -b\}, & \text{if } b < 0. \end{cases}$$

We call those values in $\mathcal{D}(x_0, b)$ **admissible points** of $\psi_{x_0,bd}$. In the following, we will use ψ for $\psi_{x_0,bd}$ quite often when there is no ambiguity.

We first carry out a computation for the standard function $\psi_{x_0,bd}$ in $\mathcal{D}(x_0, b)$.

Write $r = |x - x_0|$. Then $\psi(x) = c_0(r + b)^{\frac{4}{3}} + d$.

If we differentiate ψ and denote $D\psi(x) = p$ at any admissible point, then it is easy to find that

$$D^2\psi(x) = \frac{1}{|p|^2}\hat{p} \otimes \hat{p} + \frac{|p|}{r}(I - \hat{p} \otimes \hat{p}).$$

So

$$\Delta_\infty \psi = \langle D^2\psi\, D\psi, D\psi\rangle = \frac{1}{|p|^2}|p|^2 + \frac{|p|}{r}\big(|p|^2 - |p|^2\big) = 1$$

at any admissible point.

Lemma 2. *For any $x_0 \in \mathbf{R}^n$, $b \in \mathbf{R}$ and $d \in \mathbf{R}$, $\psi_{x_0,bd}$ is a classical solution, and hence a viscosity solution, of $\Delta_\infty \psi = 1$ in $\mathcal{D}(x_0, b)$.*

Proof. The fact that a classical solution is a viscosity solution follows easily from the definition of a viscosity solution. $\square$

For a continuous function u defined in Ω, we use the notation $u \in \operatorname{Max} P(\Omega)$ to denote the fact that u verifies the weak maximum principle

$$\sup_V u = \max_{\partial V} u$$

for any compact set $V \subseteq \Omega$. Similarly, $u \in \operatorname{Min} P(\Omega)$ means u verifies the weak minimum principle

$$\inf_V u = \min_{\partial V} u$$

for any compact set $V \subseteq \Omega$.

Though the following lemma is a direct corollary of the maximum principle, Theorem 4, we would like to give an elementary proof to make the comparison property an independent part of the theory.

Lemma 3. *Let Ω and Σ be two open subsets of $\mathbf{R}$. Assume $u \in C(\Omega)$ is a viscosity sub-solution of*

$$\Delta_\infty u = 1$$

in Ω and $v \in C^2(\Sigma)$ is a classical solution of

$$\Delta_\infty u = 1$$

in $\Omega \cap \Sigma$ and v is bounded on $\overline{\Omega \cap \Sigma}$. Then $u - v \in \operatorname{Max} P(\Omega \cap \Sigma)$.

Similarly, if $u \in C(\Omega)$ is a viscosity super-solution of

$$\Delta_\infty u = 1$$

in Ω and $v \in C^2(\Sigma)$ is a classical solution of

$$\Delta_\infty u = 1$$

in $\Omega \cap \Sigma$ and v is bounded on $\overline{\Omega \cap \Sigma}$. Then $u - v \in \operatorname{Min} P(\Omega \cap \Sigma)$.

Proof. In the first case, $\Delta_\infty u \geqslant 1$ in the viscosity sense in Ω. Suppose $V \subset (\Omega \cap \Sigma)$ is a compact set and $\exists x_* \in V$ such that

$$u(x_*) - v(x_*) > \max_{\partial V}(u - v),$$

say

$$u(x_*) - v(x_*) = \max_{\partial V}(u - v) + \delta$$

for some $\delta > 0$.

For small $\varepsilon > 0$ to be taken, we define

$$w(x) = (1 - \varepsilon)v(x).$$

Then

$$\begin{aligned} u(x_*) - w(x_*) &= u(x_*) - v(x_*) + \varepsilon v(x_*) \\ &\geqslant \max_{\partial V}(u - v) + \delta - \varepsilon\|v\|_{L^\infty(V)} \\ &\geqslant \max_{\partial V}(u - w) + \delta - 2\varepsilon\|v\|_{L^\infty(V)} \\ &\geqslant \max_{\partial V}(u - w) + \frac{\delta}{2}, \quad \text{for } \varepsilon > 0 \text{ small enough} \\ &> \max_{\partial V}(u - w). \end{aligned}$$

Without loss of generality, we assume $u - w$ assumes its maximum on V at x_*, i.e.

$$(u - w)(x_*) = \max_{V}(u - w).$$

In particular, we know $u \prec_{x_*} w$.

By the definition of viscosity sub-solutions, $\Delta_\infty w(x_*) \geqslant 1$.

However,

$$\Delta_\infty w(x_*) = (1-\varepsilon)^3 \Delta_\infty v(x_*) = (1-\varepsilon)^3 < 1,$$

as $\varepsilon > 0$. We obtain a contradiction.

To prove the second half for super-solutions, one needs only to modify the above argument by taking $w = (1+\varepsilon)v$ instead of $w = (1-\varepsilon)v$ and change max to min and reverse the direction of the inequalities accordingly. We omit the detailed proof. □

The following comparison principles with standard functions for viscosity sub-solutions and super-solutions of the equation $\Delta_\infty u = 1$ are the main results in this section. We want to point out that if u is replaced by $-u$ in the above lemma, we can obtain parallel comparison principles with standard functions of the dual equation $\Delta_\infty u = -1$.

The idea of the following comparison principles can be traced back to a parallel comparison principle for ∞-heat equation established in a joint work of one of the authors with M. Crandall, [13].

Theorem 6. *Assume $u \in C(\Omega)$ verifies $|Du| > 0$ in Ω in the viscosity sense. Then*

$$\Delta_\infty u \geqslant 1$$

in the viscosity sense in Ω if and only if $u - \psi_{x_0,bd} \in \operatorname{Max} P(\Omega\backslash\{x_0\})$ for any $x_0 \in \mathbf{R}^n$, $b \geqslant 0$ and $d \in \mathbf{R}$.

We say u **enjoys comparison with standard functions from above** in Ω if the condition $u - \psi_{x_0,bd} \in \operatorname{Max} P(\Omega\backslash\{x_0\})$ for all $x_0 \in \mathbf{R}^n$, $b \geqslant 0$ and $d \in \mathbf{R}$ stated in the theorem holds.

Remark 2. Without the additional assumption $|Du| > 0$ in the viscosity sense, we may have constant functions as counter-examples of the sufficiency in the theorem.

Now we prove the comparison principle with standard functions.

Proof. "Only if": One simply apply the previous Lemmas 2 and 3.

"If": Assume u enjoys comparison with standard functions from above in Ω. Suppose u is not a viscosity subsolution of $\Delta_\infty u = 1$ in Ω. Then at some point x_* in Ω, $\exists \varphi \in C^2(\Omega)$ that touches u by above at x_* and $\Delta_\infty \varphi(x_*) < 1$.

Without loss of generality, we assume $x_* = 0$. Denote $p = D\varphi(0)$ and $S = D^2\varphi(0)$. Then $\langle Sp, p\rangle < 1$. Note that $|p| > 0$ as $|Du| > 0$ in the viscosity sense in Ω.

We will construct a standard function

$$\psi_{x_0,b}(x) = c_0\big(|x-x_0| + b\big)^{\frac{4}{3}}$$

such that $u - \psi_{x_0,b} \notin \operatorname{Max} P(\Omega\backslash\{x_0\})$ with $x_0 \neq 0$ and $b \geqslant 0$.

It suffices to construct $\psi = \psi_{x_0,b}$ such that 0 is a strict local maximum point of $\varphi - \psi$. Then 0 is also a strict local maximum point of $u - \psi$. In a small neighborhood of 0, $u - \psi_{x_0,b}$ violates the maximum principle, i.e. $u - \psi_{x_0,b} \notin \operatorname{Max} P(\Omega\backslash\{x_0\})$.

It is sufficient to construct $\psi_{x_0,b}$ such that

$$p = D\varphi(0) = D\psi_{x_0,b}(0)$$

and

$$S = D^2\varphi(0) < D^2\psi_{x_0,b}(0).$$

Recall that

$$\psi(x) = \psi_{x_0,b}(x) = c_0\big(|x-x_0|+b\big)^{\frac{4}{3}}$$

so that

$$D\psi(x) = c_0\frac{4}{3}\big(|x-x_0|+b\big)^{\frac{1}{3}}\frac{x-x_0}{|x-x_0|}$$

and

$$D^2\psi(x) = \frac{1}{|D\psi(x)|^2}D\hat{\psi}(x)\otimes D\hat{\psi}(x) + \frac{|D\psi(x)|}{|x-x_0|}\big(I - D\hat{\psi}(x)\otimes D\hat{\psi}(x)\big),$$

at any $x \in \mathcal{D}(x_0, b)$.

So $D\psi(0) = -c_0\frac{4}{3}(r+b)^{\frac{1}{3}}\hat{x}_0$ and

$$D^2\psi(0) = \frac{1}{|D\psi(0)|^2}\hat{x}_0\otimes\hat{x}_0 + \frac{|D\psi(0)|}{r}(I - \hat{x}_0\otimes\hat{x}_0),$$

where $r = |x_0|$.

Write $a = c_0\frac{4}{3} = \sqrt[3]{3}$. The sufficiency conditions become

$$D\psi(0) = -a(r+b)^{\frac{1}{3}}\hat{x}_0 = p \tag{1}$$

and

$$\frac{1}{|D\psi(0)|^2}\hat{x}_0\otimes\hat{x}_0 + \frac{|D\psi(0)|}{r}(I - \hat{x}_0\otimes\hat{x}_0) > S. \tag{2}$$

Condition (1) implies

$$\hat{x}_0 = -\hat{p}$$

and

$$a(r+b)^{\frac{1}{3}} = |p|.$$

We rewrite condition (2) as

$$D^2\psi(0) = \frac{1}{|p|^2}\hat{p}\otimes\hat{p} + \frac{|p|}{r}(I - \hat{p}\otimes\hat{p}) > S.$$

It suffices to prove, for $r > 0$ small,

$$\langle D^2\psi(0)x, x\rangle > \langle Sx, x\rangle \quad \text{for any } x \in \mathbf{R}^n \backslash \{0\}.$$

If we write $x = \alpha\hat{p} + y^1$ with $\langle \hat{p}, y^1\rangle = 0$, then

$$\langle D^2\psi(0)x, x\rangle = \frac{\alpha^2}{|p|^2} + \frac{|p|}{r}\left|y^1\right|^2$$

and

$$\langle Sx, x\rangle = \alpha^2\langle S\hat{p}, \hat{p}\rangle + 2\alpha\langle S\hat{p}, y^1\rangle + \langle Sy^1, y^1\rangle.$$

For any $\varepsilon > 0$ small,

$$\begin{aligned}
&\alpha^2\langle S\hat{p}, \hat{p}\rangle + 2\alpha\langle S\hat{p}, y^1\rangle + \langle Sy^1, y^1\rangle \\
&\quad\leqslant \frac{\alpha^2}{|p|^2}\langle Sp, p\rangle + \alpha^2\varepsilon|S\hat{p}|^2 + \frac{1}{\varepsilon}\left|y^1\right|^2 + \langle Sy^1, y^1\rangle \\
&\quad\leqslant \frac{\alpha^2}{|p|^2}\big(\langle Sp, p\rangle + \varepsilon|Sp|^2\big) + \left(\frac{1}{\varepsilon} + \|S\|\right)\left|y^1\right|^2 \\
&\quad< \frac{\alpha^2}{|p|^2} + \frac{|p|^3}{r}\left|y^1\right|^2
\end{aligned}$$

for $\varepsilon > 0$ small and $r > 0$ small, as $\langle Sp, p\rangle < 1$. The condition (2) is proved.

$b = \frac{|p|^3}{3} - r$ is determined as well. The proof is complete. □

Theorem 6 clearly implies the following theorem.

Theorem 7. *Assume $u \in C(\Omega)$ verifies $|Du| > 0$ in the viscosity sense. Suppose, in addition, $f \in C(\Omega)$ satisfies $\inf_{x\in\Omega} f(x) > 0$.*

If u is a viscosity sub-solution of the equation

$$\Delta_\infty u = f(x)$$

in Ω then $u - C\psi_{x_0,bd} \in \operatorname{Max} P(\tilde{\Omega}\backslash\{x_0\})$ for any $\tilde{\Omega} \Subset \Omega$, positive constant C with $C^3 \leqslant \inf_{\tilde{\Omega}} f(x)$, $x_0 \in \mathbf{R}^n$, $b \geqslant 0$ and $d \in \mathbf{R}$.

On the other hand, if $u - C\psi_{x_0,bd} \in \operatorname{Max} P(\tilde{\Omega}\backslash\{x_0\})$ for any $\tilde{\Omega} \Subset \Omega$, positive constant C with $C^3 \geqslant \sup_{\tilde{\Omega}} f(x)$, $x_0 \in \mathbf{R}^n$, $b \geqslant 0$ and $d \in \mathbf{R}$, then u is a viscosity sub-solution of the equation

$$\Delta_\infty u = f(x)$$

in Ω.

We now state a comparison principle for super-solutions.

Theorem 8. *Assume $u \in C(\Omega)$. Then*

$$\Delta_\infty u \leqslant 1$$

in the viscosity sense in Ω if and only if $u - \psi_{x_0,bd} \in \operatorname{Min} P(\Omega \cap \mathcal{D}(x_0, b))$ for any $x_0 \in \mathbf{R}^n$, $b < 0$ and $d \in \mathbf{R}$, where $\mathcal{D}(x_0, b)$ denotes the set of admissible points of $\psi_{x_0,bd}$.

We say u **enjoys comparison with standard functions from below** in Ω if the condition $u - \psi_{x_0,bd} \in \operatorname{Min} P(\Omega \cap \mathcal{D}(x_0, b))$, for any $x_0 \in \mathbf{R}^n$, $b < 0$, and $d \in \mathbf{R}$, stated in the theorem holds.

Proof. Again the necessity is given by Lemmas 2 and 3.

Now we assume u enjoys comparison with standard functions from below in Ω.

Suppose u is not a viscosity super-solution in Ω. Then $\exists x_* \in \Omega$ and $\varphi \in C^2(\Omega)$ such that φ touches u by below at x_* and $\Delta_\infty \varphi(x_*) > 1$. Without loss of generality, we may assume $x_* = 0$.

Denote $p = D\varphi(0)$ and $S = D^2\varphi(0)$. $\langle Sp, p\rangle > 1$ and it clearly implies $|p| > 0$.

We will construct a standard function

$$\psi_{x_0,b}(x) = c_0\big(|x - x_0| + b\big)^{\frac{4}{3}}$$

such that $u - \psi_{x_0,b} \notin \operatorname{Min} P(\Omega \cap \mathcal{D}(x_0, b))$ with $x_0 \neq 0$ and $b < -|x_0| < 0$.

It suffices to construct $\psi_{x_0,b}$ such that 0 is a strict local minimum point of $\varphi - \psi_{x_0,b}$. Then 0 is also a strict local minimum point of $u - \psi$. In a small neighborhood of 0, $u - \psi_{x_0,b}$ violates the minimum principle.

It is sufficient to construct $\psi_{x_0,b}$ such that

$$D\varphi(0) = D\psi_{x_0,b}(0)$$

and

$$D^2\varphi(0) > D^2\psi_{x_0,b}(0).$$

One can express the above two conditions explicitly as follows:

$$D\psi(0) = -a(r + b)^{\frac{1}{3}}\hat{x}_0 = p \tag{3}$$

and

$$\frac{1}{|D\psi(0)|^2}\hat{x}_0 \otimes \hat{x}_0 + \frac{|D\psi(0)|}{r}(I - \hat{x}_0 \otimes \hat{x}_0) < S, \tag{4}$$

where $a = \frac{4}{3}c_0$ and $r = |x_0|$, the second of which in turn is equivalent to, as a result of $b + r < 0$,

$$I > \frac{r}{a(r + b)^{\frac{1}{3}}}S - \frac{r}{3(r + b)}\hat{x}_0 \otimes \hat{x}_0 + \hat{x}_0 \otimes \hat{x}_0.$$

We must take $\hat{x}_0 = \hat{p}$ and enforce $a(r+b)^{\frac{1}{3}} = -|p|$. Denote $q = \hat{p} = \hat{x}_0$ and $\mu = \frac{r}{a|(r+b)^{1/3}|} = \frac{r}{|p|} > 0$. In order to secure the second condition, we also need to establish

$$I > -\mu S - \frac{r}{3(r+b)} q \otimes q + q \otimes q$$

under the condition $\langle Sp, p\rangle = \frac{1}{|p|^2}\langle Sq, q\rangle > 1$, by taking suitable values of $b < 0$ and $r > 0$.

Denote the right-hand side matrix by M, i.e.

$$M = -\mu S - \frac{r}{3(r+b)} q \otimes q + q \otimes q.$$

For our purpose, it suffices to show

$$\langle Mx, x\rangle < |x|^2,$$

for any nonzero vector $x \in \mathbf{R}^n$.

We can write $x = \alpha q + y^1$ for any nonzero $x \in \mathbf{R}^n$, where $\langle q, y^1\rangle = 0$ and $\alpha \in \mathbf{R}$. Then

$$\begin{aligned}
\langle Mx, x\rangle &= \alpha^2 \langle Mq, q\rangle + \alpha\big\{\big\langle Mq, y^1\big\rangle + \big\langle My^1, q\big\rangle\big\} + \big\langle My^1, y^1\big\rangle \\
&= \alpha^2 \left\{-\mu\langle Sq, q\rangle - \frac{r}{3(r+b)} + 1\right\} - \mu\big\{2\alpha\big\langle Sq, y^1\big\rangle + \big\langle Sy^1, y^1\big\rangle\big\} \\
&\leqslant \alpha^2 \left\{-\mu\langle Sq, q\rangle - \frac{r}{3(r+b)} + 1\right\} + \mu\left\{\alpha^2\varepsilon|Sq|^2 + \frac{1}{\varepsilon}\big|y^1\big|^2 + \|S\|\big|y^1\big|^2\right\} \\
&= \alpha^2 \left\{1 - \frac{r}{3(r+b)} - \mu\big(\langle Sq, q\rangle - \varepsilon|Sq|^2\big)\right\} + \mu\left\{\frac{1}{\varepsilon} + \|S\|\right\}\big|y^1\big|^2,
\end{aligned}$$

where $\varepsilon > 0$ is an interpolation constant whose value will be taken in the following.

Note that

$$\begin{aligned}
\mu\langle Sq, q\rangle &= \frac{r}{|p|^3}\langle Sp, p\rangle \\
&= \frac{r}{a^3|r+b|}\langle Sp, p\rangle \\
&= -\frac{r}{3(r+b)}\langle Sp, p\rangle, \quad \text{as } a^3 = 3 \\
&> -\frac{r}{3(r+b)}, \quad \text{as } \langle Sp, p\rangle > 1.
\end{aligned}$$

Therefore

$$\begin{aligned}
&1 - \frac{r}{3(r+b)} - \mu\big(\langle Sq, q\rangle - \varepsilon|Sq|^2\big) \\
&\quad = 1 - \frac{\mu}{|p|^2} + \frac{\mu}{|p|^2}\langle Sp, p\rangle + \mu\varepsilon|Sq|^2 \quad \text{as } \frac{r}{3(r+b)} = -\frac{\mu}{|p|^2} \\
&\quad = 1 - \frac{\mu}{|p|^2}\big(\langle Sp, p\rangle - 1 - \varepsilon|Sq|^2\big).
\end{aligned}$$

Taking $\varepsilon > 0$ small enough while keeping the value of $\mu > 0$ free, we have

$$\begin{aligned}
&1 - \frac{r}{3(r+b)} - \mu\big(\langle Sq, q\rangle - \varepsilon|Sq|^2\big) \\
&\quad = 1 - \frac{\mu}{|p|^2}\big(\langle Sp, p\rangle - 1 - \varepsilon|Sq|^2\big) \\
&\quad < 1.
\end{aligned}$$

Then we take $\mu > 0$ small enough so that

$$\mu\left\{\frac{1}{\varepsilon} + \|S\|\right\} < 1.$$

Therefore $\langle Mx, x\rangle < \alpha^2 + |y^1|^2 = |x|^2$ if $x \neq 0$.

So $r = \mu|p|$ is determined and b is determined by $b = -\frac{|p|^3}{3} - r < -r$. The proof is now complete. $\square$

A generalized form of the comparison principle from below is the following theorem which follows from the previous theorem directly.

Theorem 9. *Assume $u \in C(\Omega)$ and $f \in C(\Omega)$ satisfies $\inf_{x\in\Omega} f(x) > 0$.*

If u is a viscosity super-solution of the equation

$$\Delta_\infty u = f(x)$$

in Ω then $u - C\psi_{x_0,bd} \in \operatorname{Min} P(\tilde{\Omega}\backslash\{x_0\})$ for any $\tilde{\Omega} \Subset \Omega$, positive constant C with $C^3 \geqslant \sup_{\tilde{\Omega}} f(x)$, $x_0 \in \mathbf{R}^n$, $b < 0$ and $d \in \mathbf{R}$.

On the other hand, if $u - C\psi_{x_0,bd} \in \operatorname{Min} P(\tilde{\Omega}\backslash\{x_0\})$ for any $\tilde{\Omega} \Subset \Omega$, positive constant C with $C^3 \leqslant \inf_{\tilde{\Omega}} f(x)$, $x_0 \in \mathbf{R}^n$, $b < 0$ and $d \in \mathbf{R}$, then u is a viscosity super-solution of the equation

$$\Delta_\infty u = f(x)$$

in Ω.

5. Stability of $\Delta_\infty u = f$ with $f \neq 0$

In this section, Ω again denotes a bounded open subset of $\mathbf{R}^n$.

We need the strict comparison principle, Lemma 1, and the following lemmas to prove the perturbation theorem.

Lemma 4. *Assume $f \in C(\Omega)$ such that either $\inf_\Omega f > 0$ or $\sup_\Omega f < 0$. For $j = 1, 2$, suppose $c_j > 0$, $g_j \in C(\partial\Omega)$ and $u_j \in C(\bar{\Omega})$ is the viscosity solution of the Dirichlet problem*

$$\begin{cases} \Delta_\infty u_j = c_j f & \text{in } \Omega, \\ u_j = g_j & \text{on } \partial\Omega. \end{cases}$$

Then

$$\left\| \frac{u_1}{\sqrt[3]{c_1}} - \frac{u_2}{\sqrt[3]{c_2}} \right\|_{L^\infty(\Omega)} \leqslant \left\| \frac{g_1}{\sqrt[3]{c_1}} - \frac{g_2}{\sqrt[3]{c_2}} \right\|_{L^\infty(\partial\Omega)}.$$

If, in particular $g_1 = g_2 = g \in C(\partial\Omega)$, *then*

$$\left\| \frac{u_1}{\sqrt[3]{c_1}} - \frac{u_2}{\sqrt[3]{c_2}} \right\|_{L^\infty(\Omega)} \leqslant \left| \frac{1}{\sqrt[3]{c_1}} - \frac{1}{\sqrt[3]{c_2}} \right| \|g\|_{L^\infty(\partial\Omega)}.$$

Proof. Let

$$v_j = \frac{1}{\sqrt[3]{c_j}} u_j.$$

Then v_j is the viscosity solution of the Dirichlet problem

$$\begin{cases} \Delta_\infty v_j = f & \text{in } \Omega, \\ v_j = \frac{1}{\sqrt[3]{c_j}} g_j & \text{on } \partial\Omega, \end{cases}$$

$j = 1, 2$. Applying the maximum principle, Theorem 4, one obtains

$$\|v_1 - v_2\|_{L^\infty(\Omega)} \leqslant \left\| \frac{g_1}{\sqrt[3]{c_1}} - \frac{g_2}{\sqrt[3]{c_2}} \right\|_{L^\infty(\partial\Omega)},$$

which implies the desired inequality. □

Lemma 5. *Assume* $f \in C(\Omega)$ *such that either* $\inf_\Omega f > 0$ *or* $\sup_\Omega f < 0$. *Suppose* $c_k \to 0$, g_k, $g \in C(\partial\Omega)$ *such that* $\|g_k - g\|_{L^\infty(\partial\Omega)} \to 0$, *and* u_k *and* u *in* $C(\bar{\Omega})$ *are the viscosity solutions of the following Dirichlet problems respectively*

$$\begin{cases} \Delta_\infty u_k = (1 + c_k) f & \textit{in } \Omega, \\ u_k = g_k & \textit{on } \partial\Omega \end{cases}$$

and

$$\begin{cases} \Delta_\infty u = f & \textit{in } \Omega, \\ u = g & \textit{on } \partial\Omega. \end{cases}$$

Then

$$\sup_\Omega (u_k - u) \to 0$$

as $k \to \infty$.

Proof. The previous lemma implies

$$\left\| \frac{1}{\sqrt[3]{1+c_k}} u_k - u \right\|_{L^\infty(\Omega)} \leqslant \left\| \frac{g_k}{\sqrt[3]{1+c_k}} - g \right\|_{L^\infty(\partial\Omega)},$$

which in turn implies

$$\begin{aligned} &\frac{1}{\sqrt[3]{1+c_k}} \|u_k - u\|_{L^\infty(\Omega)} - \left| \frac{1}{\sqrt[3]{1+c_k}} - 1 \right| \|u\|_{L^\infty(\Omega)} \\ &\quad \leqslant \frac{1}{\sqrt[3]{1+c_k}} \|g_k - g\|_{L^\infty} + \left| 1 - \frac{1}{\sqrt[3]{1+c_k}} \right| \|g\|_{L^\infty(\partial\Omega)}. \end{aligned}$$

Therefore

$$\|u_k - u\|_{L^\infty(\Omega)} \leqslant \left| \sqrt[3]{1+c_k} - 1 \right| \left(\|u\|_{L^\infty(\Omega)} + \|g\|_{L^\infty(\Omega)} \right) + \|g_k - g\|_{L^\infty(\partial\Omega)}.$$

So $\lim_{k\to\infty} \|u_k - u\|_{L^\infty(\Omega)} = 0$. □

The main result in this section is the following perturbation theorem.

Theorem 10. *Suppose $\{f_k\}$ is a sequence of continuous functions in $C(\Omega)$ which converges uniformly in Ω to $f \in C(\Omega)$ and either $\inf_\Omega f > 0$ or $\sup_\Omega f < 0$. Furthermore, $\{g_k\}$ is a sequence of functions in $C(\partial\Omega)$ which converges uniformly on $\partial\Omega$ to $g \in C(\partial\Omega)$. Assume $u_k \in C(\bar{\Omega})$ is a viscosity solution of the Dirichlet problem*

$$\begin{cases} \Delta_\infty u_k = f_k & \text{in } \Omega, \\ u_k = g_k & \text{on } \partial\Omega, \end{cases}$$

while $u \in C(\bar{\Omega})$ is the unique viscosity solution of the Dirichlet problem

$$\begin{cases} \Delta_\infty u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$$

Then $\sup_\Omega |u_k - u| \to 0$ as $k \to \infty$.

Proof. Without loss of generality, we assume $\inf_\Omega f > 0$.

Let $\varepsilon_k = \sup_\Omega |f_k - f|$. Then $\varepsilon_k \to 0$ as $k \to \infty$ and

$$f(x) - \varepsilon_k \leqslant f_k(x) \leqslant f(x) + \varepsilon_k \quad \text{for all } x \in \Omega.$$

To forbid $\varepsilon_k = 0$, we replace ε_k by $\varepsilon_k + \frac{1}{k}$ and still denote the new quantity by ε_k, as the new $\varepsilon_k \to 0$. And now

$$f(x) - \varepsilon_k < f_k(x) < f(x) + \varepsilon_k \quad \text{for all } x \in \Omega.$$

Since $\inf_\Omega f > 0$, the sequence $\{c_k\}$ defined by $c_k = \frac{\varepsilon_k}{\inf_\Omega f}$ converges to 0 but never equals 0. So, for all $x \in \Omega$,

$$(1 - c_k) f(x) < f_k(x) < (1 + c_k) f(x),$$

as a result of $\varepsilon_k \leqslant c_k f(x)$.

We define u_k^1 and u_k^2 to be the viscosity solutions of the following Dirichlet problems respectively

$$\begin{cases} \Delta_\infty u_k^1 = (1 - c_k) f & \text{in } \Omega, \\ u_k^1 = g_k & \text{on } \partial\Omega \end{cases}$$

and

$$\begin{cases} \Delta_\infty u_k^2 = (1 + c_k) f & \text{in } \Omega, \\ u_k^2 = g_k & \text{on } \partial\Omega. \end{cases}$$

By Lemma 1, we know that $u_k^2 \leqslant u_k \leqslant u_k^1$ on $\bar{\Omega}$, since $(1 - c_k) f(x) < f_k(x) < (1 + c_k) f(x)$ for all $x \in \Omega$. In addition, the previous Lemma 5 implies $\sup_\Omega |u_k^j - u| \to 0$ for $j = 1, 2$. Consequently,

$$\sup_\Omega |u_k - u| \to 0$$

as $k \to \infty$. $\square$

6. Stability of $\Delta_\infty u = 0$

Now we are at a position to prove the main theorem of this paper stated below.

Theorem 11. *Ω is a bounded open subset of $\mathbf{R}^n$. Suppose $g \in \mathrm{Lip}_\partial(\Omega)$ and $\{f_k\}$ is a sequence of continuous functions on Ω which converges uniformly to 0 in Ω. If $u_k \in C(\bar{\Omega})$ is a viscosity solution of the Dirichlet problem*

$$\begin{cases} \Delta_\infty u_k = f_k & \text{in } \Omega, \\ u_k = g & \text{on } \partial\Omega, \end{cases}$$

and $u \in C(\bar{\Omega})$ is the unique viscosity solution of the Dirichlet problem

$$\begin{cases} \Delta_\infty u = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases}$$

then u_k converges to u uniformly on $\bar{\Omega}$, i.e.

$$\sup_\Omega |u_k - u| \to 0$$

as $k \to \infty$.

Proof. Let $c_k = \| f_k \|_{L^\infty(\Omega)}$ and $\{\varepsilon_k\}$ denotes a sequence of positive numbers that converges to 0.

Let u_k^1 and $u_k^2 \in C(\bar{\Omega})$ be the respective viscosity solutions of the following Dirichlet problems

$$\begin{cases} \Delta_\infty u_k^1 = -c_k - \varepsilon_k & \text{in } \Omega, \\ u_k^1 = g & \text{on } \partial\Omega \end{cases}$$

and

$$\begin{cases} \Delta_\infty u_k^2 = c_k + \varepsilon_k & \text{in } \Omega, \\ u_k^2 = g & \text{on } \partial\Omega. \end{cases}$$

By Lemma 1, we know that

$$u_k^2 \leqslant u_k \leqslant u_k^1$$

on $\bar{\Omega}$.

So it suffices to show that $\sup_\Omega |u_k^j - u| \to 0$ as $k \to \infty$, for both $j = 1$ and 2. As the proof of either case of the above convergence implies that of the other, we will only prove $\sup_\Omega (u - u_k^2) \to 0$ as $k \to \infty$. The proof of $\sup_\Omega (u_k^1 - u) \to 0$ follows when one considers $-u_k^1$ and $-u$. In other words, we reduce the problem to the case in which u_k is a viscosity solution of the Dirichlet problem

$$\begin{cases} \Delta_\infty u_k = \delta_k & \text{in } \Omega, \\ u_k = g & \text{on } \partial\Omega \end{cases}$$

where $\delta_k > 0$ and $\delta_k \to 0$ as $k \to \infty$, and our goal is to prove

$$\sup_\Omega (u - u_k) \to 0$$

as $k \to \infty$, since $u_k \leqslant u$ is clear. For simplicity, we omitted the superscript 2 in the above and will do the same in the following.

We use argument by contradiction. Suppose there is an $\varepsilon_0 > 0$ and a subsequence $\{u_{k_j}\}$ such that $\sup_\Omega (u - u_{k_j}) \geqslant \varepsilon_0$, for all $j = 1, 2, 3, \ldots$. In addition, we may assume $\{\delta_{k_j}\}$ is a strictly decreasing sequence that converges to 0.

Without further confusion, we will abuse our notation by using $\{u_k\}$ for the subsequence $\{u_{k_j}\}$ and δ_k for δ_{k_j}.

So we will derive a contradiction from the fact

$$\sup_\Omega (u - u_k) \geqslant \varepsilon_0 > 0, \quad \forall k,$$

where $u_k \in C(\bar{\Omega})$ is a viscosity solution of the Dirichlet problem

$$\begin{cases} \Delta_\infty u_k = \delta_k & \text{in } \Omega, \\ u_k = g & \text{on } \partial\Omega \end{cases}$$

and $\{\delta_k\}$ decreases to 0.

By Lemma 1, one obtains

$$u_k \leqslant u_{k+1} \leqslant u$$

in Ω, $\forall k$. So $\{u_k\}$ converges pointwise on $\bar{\Omega}$, as $u_k = g$ on $\partial\Omega$.

Moreover, $\{u_k\}$ is equicontinuous on any compact subset of Ω. In fact, let K be any compact subset of Ω. Then the distance from K to $\partial\Omega$ defined by

$$\operatorname{dist}(K, \partial\Omega) = \inf\bigl\{\operatorname{dist}(x, \partial\Omega)\colon\ x \in K\bigr\},$$

must equal to some positive number ε. Take $R > 0$ such that $4R < \varepsilon$. Then for any $z \in K$, $B_{4R}(z) \subset \Omega$. Since u_k is infinity sub-harmonic in Ω, i.e. $\Delta_\infty u_k \geqslant 0$ in the viscosity sense, it is well known, e.g. [5, Lemma 2.9], that

$$\bigl|u_k(x) - u_k(y)\bigr| \leqslant \Bigl(\sup_{B_{4R}(z)} u_k - \sup_{B_R(z)} u_k\Bigr)\frac{|x-y|}{R},$$

for any x, $y \in B_{4R}(z)$. As $u_1 \leqslant u_k \leqslant u$ in Ω, we have

$$\bigl|u_k(x) - u_k(y)\bigr| \leqslant \Bigl(\sup_{B_{4R}(z)} u - \sup_{B_R(z)} u_1\Bigr)\frac{|x-y|}{R} \leqslant L_R\frac{|x-y|}{R},$$

where $L_R = \sup_\Omega u - \inf_\Omega u_1 \geqslant 0$, which is independent of k. As K can be covered by finitely many balls $B_{4R}(z)$, $z \in K$, $\{u_k\}$ must be equicontinuous on K.

Therefore a subsequence of $\{u_k\}$ converges locally uniformly to some function $\bar{u} \in C(\Omega)$ in Ω. We once again abuse our notation by denoting the convergent subsequence by $\{u_k\}$.

We claim that $\bar{u}$ verifies

(i) $\Delta_\infty \bar{u} = 0$ in the viscosity sense in Ω,
(ii) $\forall x_0 \in \partial\Omega$, $\lim_{x\in\Omega\to x_0} \bar{u}(x) = g(x_0)$, and
(iii) $\bar{u} \in C(\bar{\Omega})$ if we extend the definition of $\bar{u}$ to $\partial\Omega$ by defining $\bar{u}|_{\partial\Omega} = g$.

(i) is proved by a standard viscosity solution approach. In fact, suppose $\varphi \in C^2(\Omega)$ touches u by above at $x_0 \in \Omega$. Then, for any small $\varepsilon > 0$, the function $x \mapsto u(x) - (\varphi(x) + \frac{\varepsilon}{2}|x - x_0|^2)$ has a strict maximum at x_0. In particular,

$$u(x_0) - \varphi(x_0) > \max_{y\in\partial B_r(x_0)}\biggl(u(y) - \biggl(\varphi(y) + \frac{\varepsilon}{2}|y - x_0|^2\biggr)\biggr)$$

for all small $r > 0$ and $B_r(x_0) \Subset \Omega$.

As $\{u_k\}$ converges to u uniformly on $\bar{B}_r(x_0)$, for all large k,

$$\begin{aligned}
&\sup_{x\in B_r(x_0)}\biggl(u_k(x) - \biggl(\varphi(x) + \frac{\varepsilon}{2}|x - x_0|^2\biggr)\biggr)\\
&\quad\geqslant u_k(x_0) - \varphi(x_0) > \max_{y\in\partial B_r(x_0)}\biggl(u_k(y) - \biggl(\varphi(y) + \frac{\varepsilon}{2}|y - x_0|^2\biggr)\biggr).
\end{aligned}$$

So the function $x \mapsto u_k(x) - (\varphi(x) + \frac{\varepsilon}{2}|x - x_0|^2)$ assumes its maximum over $\bar{B}_r(x_0)$ at some point $x_k \in B_r(x_0)$.

By the definition of viscosity solutions,

$$\Delta_\infty\left(\varphi(x) + \frac{\varepsilon}{2}|x - x_0|^2\right) \geqslant \delta_k$$

at $x = x_k$, i.e.

$$\Delta_\infty\varphi(x_k) + O(\varepsilon) \geqslant \delta_k,$$

$\forall\varepsilon > 0$ and $\forall k \geqslant k(r)$, where $k(r) \uparrow \infty$ as $r \downarrow 0$. If we send r to 0, we obtain $\Delta_\infty\varphi(x_0) \geqslant O(\varepsilon)$ for any $\varepsilon > 0$, which implies $\Delta_\infty\varphi(x_0) \geqslant 0$, i.e. $\Delta_\infty u(x_0) \geqslant 0$ in the viscosity sense.

The fact $\Delta_\infty u(x_0) \leqslant 0$ in the viscosity sense can be similarly proved.

As the local uniform limit of $\{u_k\}$ in Ω, $\bar{u}$ is clearly in $C(\Omega)$. In order to prove (ii) and (iii), we will apply the comparison with standard functions properties of the viscosity sub- and super-solutions of the equation $\Delta_\infty v = 1$.

In fact,

$$\Delta_\infty \frac{u_k}{\sqrt[3]{\delta_k}} = 1$$

in the viscosity sense in Ω. Fix $x_0 \in \partial\Omega$. For any $b > 0$, the comparison with standard functions by above property states that

$$\frac{u_k(x)}{\sqrt[3]{\delta_k}} - c_0\big(|x - x_0| + b\big)^{\frac{4}{3}} \leqslant \max_{y\in\partial\Omega}\left(\frac{u_k(y)}{\sqrt[3]{\delta_k}} - c_0\big(|y - x_0| + b\big)^{\frac{4}{3}}\right)$$

for all $x \in \Omega$, or equivalently

$$u_k(x) - c_0\sqrt[3]{\delta_k}\big(|x - x_0| + b\big)^{\frac{4}{3}} \leqslant \max_{y\in\partial\Omega}\big(g(y) - c_0\sqrt[3]{\delta_k}\big(|y - x_0| + b\big)^{\frac{4}{3}}\big).$$

For large $b > 0$

$$\begin{aligned}
&c_0\sqrt[3]{\delta_k}\big(|y - x_0| + b\big)^{\frac{4}{3}} \\
&\quad = c_0\sqrt[3]{\delta_k}b^{\frac{4}{3}}\left(1 + \frac{|y - x_0|}{b}\right)^{\frac{4}{3}} \\
&\quad = c_0\sqrt[3]{\delta_k}b^{\frac{4}{3}}\left(1 + \frac{4}{3}\frac{|y - x_0|}{b} + \circ\left(\frac{|y - x_0|}{b}\right)\right) \\
&\quad = c_0\sqrt[3]{\delta_k}b^{\frac{4}{3}} + \frac{4}{3}c_0\sqrt[3]{\delta_k}b^{\frac{1}{3}}|y - x_0| + c_0\sqrt[3]{\delta_k}b^{\frac{4}{3}}\circ\left(\frac{|y - x_0|}{b}\right).
\end{aligned}$$

Take $b = b_k$ large enough so that $\frac{4}{3}c_0\sqrt[3]{\delta_k}b^{\frac{1}{3}} = CL_g(\partial\Omega)$ for some universal constant $C \gg 1$. So, for $y \in \partial\Omega$,

$$\begin{aligned}
&g(y) - c_0\sqrt[3]{\delta_k}\big(|y-x_0|+b\big)^{\frac{4}{3}} \\
&\quad \leqslant g(y) - c_0\sqrt[3]{\delta_k}b^{\frac{4}{3}} - CL_g(\partial\Omega)|y-x_0| \\
&\quad \leqslant g(x_0) - c_0\sqrt[3]{\delta_k}b^{\frac{4}{3}},
\end{aligned}$$

i.e.

$$\max_{y\in\partial\Omega}\Big(g(y) - c_0\sqrt[3]{\delta_k}\big(|y-x_0|+b\big)^{\frac{4}{3}}\Big) = g(x_0) - c_0\sqrt[3]{\delta_k}b^{\frac{4}{3}}.$$

As a result, for $x\in\Omega$ near x_0,

$$\begin{aligned}
u_k(x) &\leqslant g(x_0) + c_0\sqrt[3]{\delta_k}\big(|x-x_0|+b\big)^{\frac{4}{3}} - c_0\sqrt[3]{\delta_k}b^{\frac{4}{3}} \\
&= g(x_0) + \frac{4}{3}c_0\sqrt[3]{\delta_k}b^{\frac{1}{3}}|x-x_0| + \sqrt[3]{\delta_k}b^{\frac{4}{3}}\circ\left(\frac{|x-x_0|}{b}\right) \\
&\leqslant g(x_0) + CL_g(\partial\Omega)|x-x_0|.
\end{aligned}$$

On the other hand, the comparison with standard functions from below property states that, for sufficiently large $b>0$ and all x in Ω,

$$\frac{u_k(x)}{\sqrt[3]{\delta_k}} - c_0\big(b-|x-x_0|\big)^{\frac{4}{3}} \leqslant \max_{y\in\partial\Omega}\left(\frac{u_k(y)}{\sqrt[3]{\delta_k}} - c_0\big(b-|y-x_0|\big)^{\frac{4}{3}}\right),$$

or equivalently

$$u_k(x) - c_0\sqrt[3]{\delta_k}\big(b-|x-x_0|\big)^{\frac{4}{3}} \leqslant \max_{y\in\partial\Omega}\Big(g(y) - c_0\sqrt[3]{\delta_k}\big(b-|y-x_0|\big)^{\frac{4}{3}}\Big).$$

For large $b>0$,

$$\begin{aligned}
&c_0\sqrt[3]{\delta_k}\big(b-|x-x_0|\big)^{\frac{4}{3}} \\
&\quad = c_0\sqrt[3]{\delta_k}b^{\frac{4}{3}}\left(1-\frac{|x-x_0|}{b}\right)^{\frac{4}{3}} \\
&\quad = c_0\sqrt[3]{\delta_k}b^{\frac{4}{3}}\left(1-\frac{4}{3}\frac{|x-x_0|}{b}+\circ\left(\frac{|x-x_0|}{b}\right)\right) \\
&\quad = c_0\sqrt[3]{\delta_k}b^{\frac{4}{3}} - \frac{4}{3}c_0\sqrt[3]{\delta_k}b^{\frac{1}{3}}|x-x_0| + c_0\sqrt[3]{\delta_k}b^{\frac{4}{3}}\circ\left(\frac{|x-x_0|}{b}\right).
\end{aligned}$$

Take $b>0$ large enough so that

$$\frac{4}{3}c_0\sqrt[3]{\delta_k}b^{\frac{1}{3}} = CL_g(\partial\Omega)$$

for some $C\gg 1$ (so that $c_0\sqrt[3]{\delta_k}b^{\frac{4}{3}}\circ(\frac{|x-x_0|}{b}) \ll \frac{4}{3}c_0\sqrt[3]{\delta_k}b^{\frac{1}{3}}|x-x_0|$).

As a result, for $y\in\partial\Omega$,

$$\begin{aligned}
&g(y) - c_0\sqrt[3]{\delta_k}\big(b - |y - x_0|\big)^{\frac{4}{3}} \\
&\quad \geqslant g(y) - c_0\sqrt[3]{\delta_k}b^{\frac{4}{3}} + CL_g(\partial\Omega)|y - x_0| \\
&\quad \geqslant g(x_0) - c_0\sqrt[3]{\delta_k}b^{\frac{4}{3}},
\end{aligned}$$

which means

$$\min_{\partial\Omega}\big(g(x) - c_0\sqrt[3]{\delta_k}\big(b - |x - x_0|\big)^{\frac{4}{3}}\big) = g(x_0) - c_0\sqrt[3]{\delta_k}b^{\frac{4}{3}}.$$

So, for $x \in \Omega$ near x_0,

$$\begin{aligned}
u_k(x) &\geqslant g(x_0) + c_0\sqrt[3]{\delta_k}\big(b - |x - x_0|\big)^{\frac{4}{3}} - c_0\sqrt[3]{\delta_k}b^{\frac{4}{3}} \\
&= g(x_0) - \frac{4}{3}c_0\sqrt[3]{\delta_k}b^{\frac{1}{3}}|x - x_0| + \sqrt[3]{\delta_k}b^{\frac{4}{3}} \circ \left(\frac{|x - x_0|}{b}\right) \\
&\geqslant g(x_0) - CL_g(\partial\Omega)|x - x_0|.
\end{aligned}$$

Therefore, for some $C \gg 1$ independent of k,

$$g(x_0) - CL_g(\partial\Omega)|x - x_0| \leqslant u_k(x) \leqslant g(x_0) + CL_g(\partial\Omega)|x - x_0|,$$

for all k and all $x \in \Omega$ near x_0.

Sending k to ∞, we have

$$g(x_0) - CL_g(\partial\Omega)|x - x_0| \leqslant \bar{u}(x) \leqslant g(x_0) + CL_g(\partial\Omega)|x - x_0|,$$

for all k and all $x \in \Omega$ near x_0.

Now it is clear that (ii) and (iii) hold.

The uniqueness of a solution in $C(\bar{\Omega})$ of the Dirichlet problem for homogeneous equation $\Delta_\infty u = 0$ in Ω under $u|_{\partial\Omega} = g$ implies that $\bar{u} = u$ on $\bar{\Omega}$. As a result, $\{u_k\}$ converges to u locally uniformly in Ω.

Recall that $\sup_\Omega(u - u_k) > \varepsilon_0$. There exists, for each k, an $x_k \in \Omega$ such that

$$u(x_k) > u_k(x_k) + \varepsilon_0$$

and x_k approaches the boundary $\partial\Omega$, since $\{u_k\}$ converges to u locally uniformly in Ω. Without loss of generality, we assume $x_k \to x_0 \in \partial\Omega$.

Then we will have the following contradiction by previous estimate on $u_k(x)$ for $x \in \Omega$ near x_0:

$$\begin{aligned}
g(x_0) = \lim_k u(x_k) &\geqslant \limsup_k u_k(x_k) + \varepsilon_0 \\
&\geqslant \limsup_k\big(g(x_0) - CL_g(\partial\Omega)|x_k - x_0| + \varepsilon_0\big) \\
&= g(x_0) + \varepsilon_0.
\end{aligned}$$

This completes the proof. $\square$

We may also perturb the boundary data and still have the uniform convergence desired. This is the content of the following theorem.

Theorem 12. *Ω is a bounded open subset of $\mathbf{R}^n$. Suppose $\{g_k\}$ is a sequence of functions in $\mathrm{Lip}_\partial(\Omega)$ which converges to $g \in \mathrm{Lip}_\partial(\Omega)$ uniformly on $\partial\Omega$, and $\{f_k\}$ is a sequence of continuous functions on Ω which converges uniformly to 0 in Ω. If $u_k \in C(\bar{\Omega})$ is a viscosity solution of the Dirichlet problem*

$$\begin{cases} \Delta_\infty u_k = f_k & \text{in } \Omega, \\ u_k = g_k & \text{on } \partial\Omega \end{cases}$$

and $u \in C(\bar{\Omega})$ is the unique viscosity solution of the Dirichlet problem

$$\begin{cases} \Delta_\infty u = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases}$$

then u_k converges to u uniformly on $\bar{\Omega}$, i.e.

$$\sup_{\Omega} |u_k - u| \to 0$$

as $k \to \infty$.

Proof. Let $c_k = \| f_k \|_{L^\infty(\Omega)}$ and $\{\varepsilon_k\}$ denotes a sequence of positive numbers that converges to 0.

Proceeding as in the proof of the previous theorem, we let u_k^1 and $u_k^2 \in C(\bar{\Omega})$ be the respective viscosity solutions of the following Dirichlet problems

$$\begin{cases} \Delta_\infty u_k^1 = -c_k - \varepsilon_k & \text{in } \Omega, \\ u_k^1 = g_k & \text{on } \partial\Omega \end{cases}$$

and

$$\begin{cases} \Delta_\infty u_k^2 = c_k + \varepsilon_k & \text{in } \Omega, \\ u_k^2 = g_k & \text{on } \partial\Omega. \end{cases}$$

By Lemma 1, we know that

$$u_k^2 \leqslant u_k \leqslant u_k^1$$

on $\bar{\Omega}$.

So it suffices to show that $\sup_\Omega (u_k^j - u) \to 0$ as $k \to \infty$, for both $j = 1$ and 2.

We introduce v_k^1 and $v_k^2 \in C(\bar{\Omega})$ as the viscosity solutions of the following Dirichlet problems respectively

$$\begin{cases} \Delta_\infty v_k^1 = -c_k - \varepsilon_k & \text{in } \Omega, \\ v_k^1 = g & \text{on } \partial\Omega \end{cases}$$

and

$$\begin{cases} \Delta_\infty v_k^2 = c_k + \varepsilon_k & \text{in } \Omega, \\ v_k^2 = g & \text{on } \partial\Omega. \end{cases}$$

The maximum principle, Theorem 4, implies that

$$\sup_{\bar{\Omega}} \left| u_k^j - v_k^j \right| \leqslant \max_{\partial\Omega} |g_k - g| \to 0, \quad \text{as } k \to \infty,$$

for $j = 1, 2$.

The previous Theorem 11 implies that

$$\sup_{\bar{\Omega}} \left| v_k^j - u \right| \to 0, \quad \text{as } k \to \infty,$$

for $j = 1, 2$. Therefore we have

$$\sup_{\bar{\Omega}} \left| u_k^j - u \right| \to 0, \quad \text{as } k \to \infty,$$

for $j = 1, 2$, as expected. $\square$

Acknowledgments

The authors would like to thank Professor Robert Jensen for bringing recently the preprint [19] to their attention, and they also greatly appreciate the referee's suggestions. The counter-example in Appendix A was added mainly because of the referee's insistence on the justification of the condition $\inf_\Omega f > 0$. This condition exposes a significant feature of the inhomogeneous infinity Laplacian.

Appendix A. A counter-example

In the appendix, we would like to provide a counter-example of the uniqueness theorem without the sign-assumption $\inf_\Omega f > 0$. This example is modified from a counter-example provided in [19]. We include this example for the completeness of this work.

A viscosity solution of the infinity Laplace equation $\Delta_\infty u = 0$ in $\mathbf{R}^2 \backslash \{0\}$ is given by

$$G = \left[\frac{\cos\theta (1 - \tan^{\frac{4}{3}} \frac{\theta}{2})^2}{1 + \tan^{\frac{4}{3}} \frac{\theta}{2} + \tan^{\frac{8}{3}} \frac{\theta}{2}} \right]^{\frac{1}{3}} r^{-\frac{1}{3}}$$

provided by Aronsson (see [4]).

One may write $G = \varphi(\theta) r^{-\frac{1}{3}}$, where φ is real-analytic except at $k\pi$ for $k \in \mathbf{Z}$, and φ is differentiable at $k\pi$, as long as $r > 0$. Furthermore, $|\nabla G|$ is comparable to $r^{-\frac{4}{3}}$. So, for any $L > 0$, if one defines the set Z_L to be the set

$$Z_L = \left\{ x \in \mathbf{R}^2 \;\middle|\; \left| \nabla G(x) \right| > L \right\},$$

then Z_L is bounded and contains an open neighborhood of 0 but not 0, and the outer radius of Z_L shrinks to 0 as $L \to \infty$. Let $R_L = \sup\{|x|\colon x \in Z_L\}$ and $r_L = \inf\{|x|\colon x \notin Z_L \cup \{0\}\}$ be the outer and inner radii of Z_L.

We take L so large that $Z_L \Subset B_1$. Denote

$$A_L = B_1 \backslash B_{\frac{r_L}{2}}.$$

We also use $AM(\Omega)$ to denote the set of viscosity solutions of the homogeneous infinity Laplace equation in Ω.

Lemma A. *Suppose $u \in AM(A_L)$ with $u = G$ on $\partial B_{\frac{r_L}{2}}$ and $u = c$ on ∂B_1 for a fixed constant c. If L is sufficient large, then $u = G$ in Z_L.*

Proof. Let w be the unique absolute minimizer in $AM(B_1 \backslash Z_L)$ such that $w = G$ on Z_L and $w = u$ on ∂B_1.

We claim that $w \in AM(A_L)$. If the claim is true, then $w = u$ on ∂A_L implies $w = u$ in A_L and thus $G = u$ in Z_L.

To prove the claim, we first note that $\mathrm{Lip}_x w \leqslant L$, $\forall x \in B_1 \backslash Z_L$, as $\mathrm{Lip}_{\partial B_1 \cup \partial Z_L} w \leqslant L$.

We complete the proof of the lemma by showing $\mathrm{Lip}_V w \leqslant \mathrm{Lip}_{\partial V} w$ for any $V \Subset A_L$. Without loss of generality, we assume V is connected. If $V \cap Z_L = \Phi$, nothing to prove. Assume $V \cap Z_L \neq \Phi$. Then $L_1 := \mathrm{Lip}_V G > L$. $\exists x_j \neq y_j \in \partial V$ and a path $\gamma_j \subset V \cup \partial V$ connecting x_j and y_j such that $\lim_j \frac{|G(x_j) - G(y_j)|}{l(\gamma_j)} = L_1$, where $l(\gamma_j)$ denotes the length of γ_j. For large j, $|G(x_j) - G(y_j)| > Ll(\gamma_j)$. So $\gamma_j \cap Z_L \neq \Phi$. Let $x_j' = \gamma_j(t_1)$ where $t_1 = \inf\{s\colon (-\infty, s) \cap \gamma^{-1}(Z_L) \neq \Phi\}$ and $y_j' = \gamma(t_2)$ where $t_2 = \sup\{s : \gamma^{-1}(Z_L) \cap (s, \infty) \neq \Phi\}$. It is obvious that

$$\frac{d(x_j', x_j)}{l(\gamma_j)}, \frac{d(y_j', y_j)}{l(\gamma_j)} \to 0,$$

as $j \to \infty$. Since $\sup_{x \notin Z_L} \mathrm{Lip}_x G \leqslant L$, $\sup_{x \notin Z_L} \mathrm{Lip}_x w \leqslant L$ and $w = G$ in Z_L, the inequalities $|G(x_j) - w(x_j)| \leqslant 2Ld(x_j, x_j')$ and $|G(y_j) - w(y_j)| \leqslant 2Ld(y_j, y_j')$ hold. Thus

$$\frac{|w(x_j) - w(y_j)|}{l(\gamma_j)} - \frac{|G(x_j) - G(y_j)|}{l(\gamma_j)} \to 0$$

as $j \to \infty$. So $\mathrm{Lip}_{\partial V} w \geqslant \mathrm{Lip}_V w = L_1$. So $w \in AM(A_L)$. $\square$

Next, for any $r_0 > 0$, one can construct a viscosity solution $v : \mathbf{R}^2 \to \mathbf{R}$ of $\Delta_\infty v = g$ in $\mathbf{R}^2$ and $v = G$ in $\mathbf{R}^2 \backslash B_{r_0}$, where g is a continuous function. Take C^∞ increasing functions $a(r)$ and $\lambda(r)$ so that $a(r) = r^{-\frac{1}{3}}$ and $\lambda(r) = 1$ for all $r \geqslant \frac{r_0}{2}$ and $\lambda = a = 0$ in a small neighborhood of 0. Define $v(r, \theta) = a(r)(\lambda(r)\varphi(\theta) + (1 - \lambda(r))\cos\theta)$, where $\varphi(\theta) = [\frac{\cos\theta(1 - \tan^{\frac{4}{3}}\frac{\theta}{2})^2}{1 + \tan^{\frac{4}{3}}\frac{\theta}{2} + \tan^{\frac{8}{3}}\frac{\theta}{2}}]^{\frac{1}{3}}$ as in the definition of the Aronsson's solution G. It can be checked that $\varphi(\theta) = 1 - \frac{1}{2\sqrt[3]{2}}\theta^{\frac{4}{3}} - \frac{\theta^2}{6} + O(\theta^3)$ for θ near 0. Clearly, if $\theta \neq k\pi$, $\Delta_\infty v$ is C^∞. For θ very close to 0, computation shows that

$$\Delta_\infty v = (a')^2 a'' - \frac{1}{\sqrt[3]{2}}(a'\lambda + a\lambda')a''\theta^{\frac{4}{3}} - \frac{4\lambda^3 a^3}{81r^4} + \frac{4\lambda^3 a^3}{81\sqrt[3]{2}r^4(a')^2}(a'\lambda + a\lambda')\theta^{\frac{4}{3}} + \frac{P(r,\lambda,\lambda',a,a')}{r^6(a')^2}\theta^{\frac{2}{3}} + O(\theta^2),$$

where P is a polynomial. So $\lim_{\theta\to 0}\Delta_\infty v$ exists. So such a continuous function g exists.

In the end, let $u_j \in AM(A_L)$ such that $u_j = G$ on $\partial B_{\frac{r_L}{2}}$ and $u_j = j$ on ∂B_1, for $j = 1, 2$. The preceding lemma implies $u_1 = G = u_2$ in $Z_L \cap A_L$. As $Z_L \cap A_L \supset B_{r_L} \backslash B_{\frac{r_L}{2}}$, one may 'glue' u_j to v with overlapping on $Z_L \backslash B_{\frac{r_L}{2}}$ to obtain a viscosity solution of the inhomogeneous equation in B_1 if r_0 is taken small enough. Now take $v_j = u_j - j$ in A_L and $v_j = v - j$ for $|x| < \frac{r_L}{2}$. One can see that $v_j = 0$ on ∂B_1 and $\Delta_\infty v_j = g$ in the viscosity sense in B_1, but $v_1 \neq v_2$ in Z_L. This is the end of the construction of the counter-example.

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