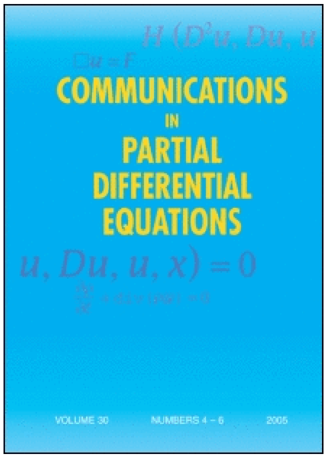

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A note on a poincaré type inequality for solutions to subelliptic equations

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A NOTE ON A POINCARÉ TYPE INEQUALITY FOR SOLUTIONS TO SUBELLIPTIC EQUATIONS

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Abstract

We prove Poincaré type inequalities for solutions to certain classes of quasilinear subelliptic equations, including the well-known p-Sublaplacian. A notable feature in these inequalities is to replace the usual f_B, the average of f over a metric ball B, by $f(x_0)$ for $x_0 \in B$. Result of this kind was considered earlier by Ziemer [18] in the classic case. We mention that our endpoint result, even in the classic case, is not obtainable through the compactness argument.

1 Introduction

In R^n, given vector fields $X_i = \frac{\partial}{\partial x_i}, i = 1, 2, \cdots, n$, then for any Euclidean ball $B = B(x, r) \subset R^n$ and any $1 \leq p < \infty$ we have the Poincaré-Sobolev type

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inequalities as stated below provided that $1 \le q \le q(p)$:

$$\left(\frac{1}{|B|}\int_B |f - f_B|^q\right)^{1/q} \le Cr\left(\frac{1}{|B|}\int_B \sum_{i=1}^n |X_i f|^p\right)^{1/p},$$

where $|E|$ stands for the Lebesgue measure for any measurable set $E \subset R^n$, and f_B is the average of the function f over the ball B. We refer the reader to the book of W. Ziemer [19] for a fairly thorough study of classic Poincaré type inequlities. Weighted inequalities also hold when the Lebesgue measure is replaced by certain pairs of two different weight functions satisfying a balanced condition (see for example, Chanillo-Wheeden [3] and the references therein).

The above inequality is also known to be true when the vector fields $\{X_i\}_{i=1}^m$ are of Hörmander type (see [5], [8], [10], 11]). In section 2 of this note, we consider solutions f to a certain class of quasilinear subelliptic differential equations and prove such a type of Poincaré inequality with f_B replaced by $f(x_0)$ at any distinguished interior point x_0 when the vector fields $\{X_i\}_{i=1}^m$ are degenerate and satisfy Hörmander's condition. We will also establish inequalities on certain extension domains with respect to the vector fields and remark that such an inequality also holds on domains satisfying a certain chain condition. Weighted Poincaré inequalities of such type for solutions to degenerate subelliptic equations will be derived in Section 3.

Let Ω be a bounded, open and pathconnected domain in R^n, and let $X_1, \cdots, X_m$ be a collection of C^∞ real vector fields defined in a neighbourhood of the closure $\overline{\Omega}$ of Ω. For a multi-index $\alpha = (i_1, \cdots, i_k)$, denote by X_α the commutator $[X_{i_1}, [X_{i_2}, \cdots, [X_{i_{k-1}}, X_{i_k}]] \cdots,]$ of length $k = |\alpha|$. Throughout this paper we assume that the vector fields satisfy Hörmander's condition: there exists some positive integer s such that $\{X_\alpha\}_{|\alpha| \le s}$ span the tangent space of R^d at each point of Ω. There is a metric associated with these vector fields and the Lebesgue measure is doubling with respect to the metric balls (see [14]). We also define $Q = \sum_{j=1}^s j m_j$ where m_j is the number of linearly independent free commutators of length j. This number Q is called the homogeneous dimension.

We now define the Sobolev space $W^{1,p}(\Omega)$ to be the completion of Lipschitz (or smooth) functions under the norm

$$||f||_{W^{1,p}(\Omega)} = \left(\int_\Omega |f|^p\right)^{1/p} + \left(\int_\Omega |Xf|^p\right)^{1/p},$$

where $|Xf|$ expresses $(\sum_{i=1}^m |X_i f|^2)^{\frac{1}{2}}$.

We also define $W_0^{1,p}(\Omega)$ as the completion of Lipschitz functions with compact support (or functions in the class $C_0^\infty(\Omega)$) under the above norm $||\cdot||_{W^{1,p}(\Omega)}$.

Let Ω be an open and bounded domain in R^n. We say Ω is a $W^{1,p}$-**extension domain associated with the vector fields** if there is a domain $\Omega' \subset R^n$ containing $\overline{\Omega}$ and an operator $T : W^{1,p}(\Omega) \to W_0^{1,p}(\Omega')$ such that $Tf(x) = f(x)$ a.e. in Ω and $||Tf||_{W^{1,p}(\Omega')} \le C||f||_{W^{1,p}(\Omega)}$ with C independent of f.

The following Poincaré inequality for Hörmander vector fields has been established in ([11], [13]):

Theorem If $E \subset\subset \Omega$, and $1 < p < \infty$, then there exist constants $q(p) > p$, $r_0 > 0$, $C > 0$, such that for all $1 \le q \le q(p)$ and for any metric ball $B = B(x,r) \subset \Omega$, $x \in E$, and any $f \in C^\infty(\overline{B})$, the following inequality holds:

$$\left(\frac{1}{|B|}\int_B |f - f_B|^q\right)^{1/q} \le Cr\left(\frac{1}{|B|}\int_B \sum_{i=1}^m |X_i f|^p\right)^{1/p},$$

provided $0 < r < r_0$, where C, r_0 depend only on E and Ω, and f_B may be taken to be $\frac{1}{|B|}\int_B f$. Here $q(p)$ can be taken to be $\frac{pQ}{Q-p}$ when $1 < p < Q$ and any number less than ∞ when $p = Q$. If f has compact support, then one can replace f_B by 0.

We mention that for $p = 1$ and $q = \frac{Q}{Q-1}$, the inequality has been established in [5] together with an application to a relative isoperimetric inequality. We also like to mention that the Poincare type inequality for the Grushin operator has been obtained in [4].

In this note, we will establish similar inequalities for functions defined on reasonably nice sets Ω with f_Ω replaced by the value of f at any distinguished point in Ω. We also require that the functions f are the solutions to certain classes of quasilinear subelliptic equations.

This note is self-contained. The only things we need here are certain properties of the solutions. We have established in [13] certain Harnack inequalities for weak solutions, subsolutions, and supersolutions of quasilinear second order subelliptic partial differential equations of the form

$$\sum_{j=1}^{m} X_j^* A_j(x, u, X_1u, X_2u, \cdots, X_mu) + B(x, u, X_1u, X_2u, \cdots, X_mu) = 0 \quad (0.1)$$

under certain structural assumptions on the equation (1.1).

We now let $x = (x_1, \cdots, x_n)$, $\eta = (\eta_1, \cdots, \eta_m)$ denote vectors in R^n and R^m respectively and $Xu = (X_1u, \cdots, X_mu)$. $A(x, u, \eta) = (A_1(x, u, \eta), \cdots, A_m(x, u, \eta))$ and $B(x, u, \eta)$ are, respectively, vector and scalar measurable functions defined on $\Omega \times R \times R^m$, where Ω is a domain in R^n on which the vector fields are defined.

The structure of the equation (1.1) throughout this paper will be assumed to satisfy the following: For all $M < \infty$ and for all $(x, u, \eta) \in \Omega \times (-M, M) \times R^n$,

$$\begin{aligned} |A(x,u,\eta)| &\leq a_0|\eta|^{p-1} + (a_1(x)|u|)^{p-1} + (a_3(x))^{p-1}, \\ \eta \cdot A(x,u,\eta) &\geq |\eta|^p - (a_2(x)|u|)^p - (a_4(x))^p, \\ |B(x,u,\eta)| &\leq b_0|\eta|^p + b_1(x)|\eta|^{p-1} + (b_2(x))^p\,|u|^{p-1} + (b_3(x))^p \end{aligned} \quad (0.2)$$

where $p > 1$, a_0, b_0 are constants, $a_i(x), b_i(x)$ are nonnegative measurable functions; $p, a_0, b_0, a_i(x), b_i(x)$ may possibly depend on M. Such equations in Euclidean spaces have been studied in [9], [15], [16] and [17]-[19].

We will assume $p > 1$ and allow $a_i(x), b_i(x)$ to be in certain subspaces of $L^t_{loc}(\Omega)$, where $t = \max(p, Q)$ (see [13]). More precisely, let $\epsilon(\rho)$ be a smooth function defined for $\rho > 0$ and such that $\epsilon(\rho) \to 0$ as $\rho \to 0$. We then define the space $L^{Q,\epsilon(\rho)}$ by

$$L^{Q,\epsilon(\rho)} = \left\{u(x) \in L^Q(\Omega) : ||u||_{Q,\epsilon(\rho);\Omega} < \infty\right\},$$

where

$$||u||_{Q,\epsilon(\rho);\Omega} = \sup_{x_0 \in \Omega, \rho > 0} \epsilon(\rho)^{-1} ||u||_{Q;B_\rho(x_0) \bigcap \Omega}. \tag{0.3}$$

We assume the functions $a_i(x), b_j(x)$ in the structure condition (1.2) are in such spaces with certain $\epsilon(\rho)$. More precisely, we will assume when $p < Q$ that

$$a_i(x), b_j(x) \in L^{Q,\rho^\alpha}(\Omega) \ \textit{for some} \ \alpha > 0, i = 2, 4; j = 1, 2, 3$$

and

$$a_i(x) \in L^Q(\Omega), i = 1, 3,$$

and we in this case set $B = B_{3\rho}(x_0)$ and

$$\lambda = \rho^{-1} ||a_1||_{Q;B \bigcap \Omega} + \rho^{\alpha - 1} ||a_2 + b_1 + b_2||_{Q,\rho^\alpha;B \bigcap \Omega},$$
$$m(\rho) = ||a_3||_{Q,B \bigcap \Omega} + \rho^\alpha ||a_4||_{Q,\rho^\alpha;B \bigcap \Omega} + \left(\rho^\alpha ||b_3||_{Q,\rho^\alpha;B \bigcap \Omega}\right)^{\frac{p}{p-1}}. \tag{0.4}$$

When $p = Q$, we also assume $a_1(x), a_3(x) \in L^{Q,\rho^\alpha}(\Omega)$ and set for $B = B_{3\rho}(x_0)$

$$\lambda = \rho^{\alpha - 1} ||a_1 + a_2 + b_1 + b_2||_{Q,\rho^\alpha;B \bigcap \Omega},$$
$$m(\rho) = \rho^\alpha ||a_3 + a_4||_{Q,\rho^\alpha;B \bigcap \Omega} + \left(\rho^\alpha ||b_3||_{Q,\rho^\alpha;B \bigcap \Omega}\right)^{\frac{p}{p-1}}. \tag{0.5}$$

If $p > Q$ we assume that all a_i, b_j are in $L^p(\Omega)$ and set

$$\lambda = \rho^{-Q/p} ||a_1 + a_2 + b_1 + b_2||_{Q,\rho^\alpha;B \bigcap \Omega},$$
$$m(\rho) = \rho^{1-Q/p} ||a_3 + a_4||_{Q,\rho^\alpha;B \bigcap \Omega} + \left(\rho^{1-Q/p} ||b_3||_{p,\rho^\alpha;B \bigcap \Omega}\right)^{\frac{p}{p-1}}. \tag{0.6}$$

Remark: If we only assume $\epsilon(\rho) > 0$ satisfies a certain Dini condition, i.e., $\int_0^1 \frac{\epsilon(\rho)}{\rho} d\rho < \infty$, then the proofs of all the Theorems proved in [13] still hold with minimal modifications.

The main theorems of this note are the following:

Theorem 0.7 *Let $1 < p < Q$ and $1 \le q < \frac{Qp}{Q-p}$. Assume that $a_3(x) = a_4(x) = b_3(x) = 0$ in (1.2). Suppose that Ω is a $W^{1,p}$-extension domain and let $u \in W^{1,p}(\Omega)$ be a weak solution of (1.1). Then for any point $x_0 \in \Omega$ there is a constant C depending on x_0, the structure condition (1.2), p, q, $||u||_{q,\Omega}$, and Ω such that*

$$||u - u(x_0)||_{q,\Omega} \le C||Xu||_{p,\Omega}.$$

Since we do not know so far if a metric ball is an extension domain with respect to the vector fields, the following theorem becomes interesting and nontrivial. Moreover, Theorem (1.8) below holds for the endpoint $q = \frac{Qp}{Q-p}$.

Theorem 0.8 *Let $1 < p < Q$ and $1 \le q \le \frac{Qp}{Q-p}$. Assume that $a_3(x) = a_4(x) = b_3(x) = 0$ in (1.2). Suppose that $E \subset \Omega$ is any metric ball. Let $u \in W^{1,p}(\Omega)$ be a weak solution of (1.1). Then there is a constant C depending on the structure condition (1.2), p, q, $||u||_{q,\Omega}$, and Ω such that for any point $x_0 \in \frac{1}{2}E$*

$$\left(\frac{1}{|E|}\int_E |u - u(x_0)|^q\right)^{\frac{1}{q}} \le C\rho(E)\left(\frac{1}{|E|}\int_E |Xu|^p\right)^{\frac{1}{p}},$$

where $\rho(E)$ is the radius of E.

The following remarks are in order:

Remark 1: In theorem (1.8), we assume $x_0 \in \frac{1}{2}E$, where $\frac{1}{2}E$ stands for the ball with the same center as E but with half the radius of E. However, $\frac{1}{2}$ is not essential and can be replaced by any number less than 1.

Remark 2: The main feature of Theorem (1.8) is that the integral on the right side is over the ball E. It is easy to see that one can replace the L^q norm on the left hand side by the L^∞ norm when we replace the domain of the integration on the right side by αE for any $\alpha > 1$. However, the constant $C = C(\alpha)$ there will blow up as α goes to 1 (see Lemma (2.1)). Theorem (1.8) says that α can be taken to be 1 if $q \le \frac{pQ}{Q-p}$ with bounded constant C in the inequality. We also note that Theorem (1.8) actually holds for all subsolutions too since the proof only involves the mean value property.

Remark 3: The proof of Theorem (1.7) adapts a well-known compactness argument (see for example Ziemer's book [19] and also [18]) while the proof of Theorem (1.8) needs a covering lemma argument. We need the Rellich-Kondrachov compact embedding theorem in the subelliptic context derived in [12] to prove Theorem (1.7). The above theorems also hold for all $Q \leq p < \infty$ and for any $1 < q < \infty$. We shall not state the results.

Remark 4: The dependence of the constant C in the Poincare inequality on the structure condition (1.2) in Theorems (1.7) and (1.8) above, and also in Theorem (1.9) below, is as follows: C depends on $\lambda\rho$ which is only dependent on the appropriate norms of coefficients of the differential equations (see (1.4), (1.5) and (1.6)), and C is uniformly bounded in ρ; note that λ is defined as in (1.4), (1.5) and (1.6).

We also state the following theorem when the domain satisfies a certain chain condition (see the definition in Section 2). The proof will be similar to that of the case over the metric ball.

Theorem 0.9 *Let $1 < p < Q$ and $1 \leq q \leq \frac{Qp}{Q-p}$. Suppose that Ω is a chain domain and let $u \in W^{1,p}(\Omega)$ be a weak solution of (1.1) below. Then there is a constant C depending on the structure condition (1.2), p, q, $||u||_{q,\Omega}$, and Ω such that for any point $x_0 \in B_0$, where B_0 is the central ball in the definition of chain domain (see Section 2),*

$$||u - u(x_0)||_{q,\Omega} \leq C||Xu||_{p,\Omega}.$$

We now define the notion of solutions, subsolutions and supersolutions of the equations (1.1). A function $u(x)$ is said to be a weak solution of (1.1) in Ω if $u(x) \in W^{1,p}_{loc}$ and

$$\int_\Omega \{X\phi \cdot A(x,u,Xu) - \phi B(x,u,Xu)\}\, dx = 0$$

for all bounded $\phi(x) \in W_0^{1,p}(\Omega)$.

A function $u(x)$ is said to be a weak subsolution (or supersolution) of (1.1) in Ω if $u(x) \in W_{loc}^{1,p}$ and

$$\int_\Omega \{X\phi \cdot A(x,u,Xu) - \phi B(x,u,Xu)\}\,dx \le 0 (or \ge 0)$$

for all bounded $\phi(x) \ge 0, \phi(x) \in W_0^{1,p}(\Omega)$.

We note here that if the above expressions hold for all $\phi(x) \ge 0, \phi(x) \in C_0^1(\Omega)$ and $a_i(x), b_i(x) \in L_{loc}^Q(\Omega), u(x) \in L_{loc}^\infty$, then a standard argument of approximation will show that it still holds for all $\phi(x)$ given in the definition. We need the following two theorems which have been proved in [13]:

Mean value inequality Suppose that $u(x)$ is a weak subsolution of (1.1) in a metric ball $B_{3\rho} \subset \Omega$ with $|u| < M$ in $B_{3\rho}$. Then for any $1 < \alpha < 3$,

$$\max_{B_\rho} |u(x)| \le C\left(|B|^{\frac{-1}{\gamma}} ||u(x)||_{\gamma, B_{\alpha\rho}} + m(\rho)\right) \tag{0.10}$$

for any $\gamma > p-1$, where $C = C(p, Q, a_0, b_0 M, \lambda\rho)$, and $m(\rho)$ and λ are numbers defined as in (1.4), (1.5) and (1.6) ($m(\rho) = 0$ when $a_3(x) = a_4(x) = b_3(x) = 0$).

A Harnack inequality for nonnegative solutions was proved in [13] and one application of it is the Hőlder continuity of the weak solutions of (1.1).

Holder continuity Suppose that $u(x)$ is a weak solution of (1.1) in Ω which is also locally bounded (assuming $|u| < M$). Then $u(x)$ is Holder continuous in Ω and if $B_{\rho_0} \subset \Omega$ then

$$\sup_{x,y \in B_\rho} |u(x) - u(y)| \le C\left(\frac{\rho}{\rho_0}\right)^\beta \left\{\sup_{B_{\rho_0}} |u(x)| + m(\rho_0)\right\}, \tag{0.11}$$

for all $\overline{B}_\rho \subset B_{\rho_0}$ and some $\beta > 0$, and $C = C(p, Q, a_0, b_0 M)$.

1 Proof of Theorems (1.7) and (1.8)

We first prove Theorem (1.7) by adapting the compactness argument together with the mean-value inequality and Holder's continuity of the solutions (see

Ziemer [18]-[19]). The proof of Theorem (1.8) will differ from this and uses a covering lemma argument (and then keeps the endpoint result $q = \frac{pQ}{Q-p}$). We mention some related results in this line, see for example, [1], [2], [4], [7], [8], [11], etc.

Proof of Theorem (1.7): Let $q < \frac{Qp}{Q-p}$. Suppose the theorem is false. Then for any positive integer j there is a weak solution u_j such that

$$||u_j - u_j(x_0)||_{q,\Omega} > j||Xu_j||_{p,\Omega}.$$

If we set $\overline{u_j}(x) = u_j(x) - u_j(x_0)$, then $\overline{u}(x) = \overline{u_j}(x)$ is a weak solution of an equation of the form (1.1):

$$\sum_{i=1}^{m} X_i^* \overline{A_i}(x, \overline{u}, X_1\overline{u}, X_2\overline{u}, \cdots, X_m\overline{u}) + \overline{B}(x, \overline{u}, X_1\overline{u}, X_2\overline{u}, \cdots, X_m\overline{u}) = 0$$

where $\overline{A}(x, \overline{u}, \eta) = A(x, \overline{u} + u_j(x_0), \eta)$ and $\overline{B}(x, \overline{u}, \eta) = B(x, \overline{u} + u_j(x_0), \eta)$.

Thus the equation above for $\overline{u}_j$ has the same structure as those satisfied by u_j except the coefficients can depend on the constant $u_j(x_0)$ which is bounded by $||u_j||_{q,\Omega}$ by the mean value inequality (1.10) because $m(\rho) = 0$ under the assumption that $a_3(x) = a_4(x) = b_3(x) = 0$. For simplicity we drop the "bar" from $\overline{A}, \overline{B}, \overline{u(x)}$ and simply write $A, B, u(x)$. Therefore, we may assume $u_j(x_0) = 0$ by replacing $u_j(x_0)$ by $u_j - u_j(x_0)$. We may also assume that $||u_j||_{q,\Omega} = 1$ by replacing u_j by $\frac{u_j}{||u_j||_{q,\Omega}}$. Thus we have

$$||u_j||_{q,\Omega} > j||Xu_j||_{p,\Omega}$$

with $||u_j||_{q,\Omega} = 1$. Since we have assumed that Ω is an extension domain, then we can extend each u_j to be defined on some Ω' containing $\overline{\Omega}$ with

$$||u_j||_{W^{1,p}(\Omega')} \leq C||u_j||_{W^{1,p}(\Omega)}.$$

Since $||Xu_j||_{p,\Omega} + ||u_j||_{p,\Omega}$ is bounded, $||Xu_j||_{p,\Omega'} + ||u_j||_{p,\Omega'}$ is also bounded by a constant, i.e., u_j have bounded Sobolev norm in $W^{1,p}(\Omega')$. Then we can pick a subsequence (still called u_j) such that u_j converges weakly to some $u \in W^{1,p}(\Omega')$.

Therefore by the Rellich-Kondrachov compactness theorem for vector fields proved by the author in [13] one can get, since $\overline{\Omega} \subset \Omega'$,

$$||u_j - u||_{q,\Omega} \to 0.$$

We note $||Xu_j||_{p,\Omega} \to 0$ as $j \to \infty$ and $||u_j||_{q,\Omega} = 1$ by assumption. Therefore we get $||u||_{q,\Omega} = 1$ and $||Xu||_{p,\Omega} = 0$.

Since each u_j is Holder continuous on any compact subset of Ω by (1.11), we then conclude that $\{u_j\}$ are uniformly bounded. By Ascoli's theorem, there is a subsequence of u_j converging to u uniformly on each compact subset of Ω. Therefore, $u(x_0) = 0$. But $||Xu||_{p,\Omega} = 0$ so $u = constant$ a.e. and then $u(x) = 0$ for all $x \in \Omega$ since $u(x_0) = 0$, which is a contradiction to $||u||_{q,\Omega} = 1$.

Q.E.D.

Before we prove Theorem (1.8), we need the following lemma:

Lemma 1.1 *Let K be any compact subset of Ω. Assume $a_3(x) = a_4(x) = b_3(x) = 0$. Let $1 \leq p < \infty$. Suppose that $B = B(x,r) \subset \Omega$ with $x \in K \subset \Omega$ is any metric ball. Let $u \in W^{1,p}(\Omega)$ be a weak solution of (1.1). Let α be a constant with $\alpha > 1$. Then there is a constant C depending on α, the structure condition (1.2), p, q and $||u||_{q,\Omega}$, and Ω such that for any point $x_0 \in B$ and for all $1 \leq q \leq \infty$,*

$$\left(\frac{1}{|B|}\int_B |u - u(x_0)|^q\right)^{\frac{1}{q}} \leq C\rho(B)\left(\frac{1}{|B|}\int_{\alpha B} |Xu|^p\right)^{\frac{1}{p}}, \qquad (1.2)$$

where $\rho(B)$ is radius of B and αB stands for the ball concentric with B but with radius $\alpha\rho(B)$.

Remark: The proof provided below actually shows that (2.2) holds by replacing p on the right-hand side by any $t > p - 1$.

Proof: We note that in this case $m(\rho) = 0$. By the mean value inequality (1.10) of the solution u, we have for any $x_0 \in B$ and $\alpha > 1$,

$$\sup_{x\in B} |u(x) - u(x_0)| \leq C(\alpha) \left(\frac{1}{|B|} \int_{\alpha B} |u|^t \right)^{\frac{1}{t}}, for\ all\ t > p - 1 \tag{1.3}$$

where $C(\alpha)$ is a constant and usually blows up as $\alpha \to 1$. If we set $\overline{u}(x) = u(x) - u_{\alpha B}$, where $u_{\alpha B} = \frac{1}{|\alpha B|}\int_{\alpha B} u$, then $\overline{u}(x)$ is a weak solution of an equation of the form (1.1):

$$\sum_{j=1}^{m} X_j^* \overline{A}(x, \overline{u}, X_1\overline{u}, X_2\overline{u}, \cdots, X_m\overline{u}) + \overline{B}(x, \overline{u}, X_1\overline{u}, X_2\overline{u}, \cdots, X_m\overline{u}) = 0$$

where $\overline{A}(x, \overline{u}, \eta) = A(x, \overline{u} + u_{\alpha B}, \eta)$ and $\overline{B}(x, \overline{u}, \eta) = B(x, \overline{u} + u_{\alpha B}, \eta)$.

Thus the equation above for $\overline{u}$ has the same structure as that satisfied by u except the coefficients can depend on the constant $u_{\alpha B}$ which is bounded by $||u||_{q,\Omega}$ by the mean value inequality (1.10). For simplicity we again drop the "bar" from $\overline{A}, \overline{B}, \overline{u(x)}$ and simply write $A, B, u(x)$.

Thus we have for any $t > p - 1$ by replacing u by $u - u_{\alpha B}$ in (2.3),

$$\sup_{x\in B} |u(x) - u(x_0)| \leq C(\alpha) \left(\frac{1}{|B|} \int_{\alpha B} |u(x) - u_{\alpha B}|^t \right)^{\frac{1}{t}}.$$

The right-hand side of the above is bounded by the Poincare inequality by

$$C(\alpha)\rho(B) \left(\frac{1}{|B|} \int_{\alpha B} |Xf|^t \right)^{\frac{1}{t}}.$$

Therefore

$$||u(x) - u(x_0)||_{L^\infty(B)} \leq C(\alpha)\rho(B) \left(\frac{1}{|B|} \int_{\alpha B} |Xf|^p \right)^{\frac{1}{p}},$$

by taking $t = p$. The case for any $1 \leq q < \infty$ then follows immediately.

Q.E.D

We give now a defintion of Chain domain:

Defintion: A domain $\Omega \subset R^n$ is called a Chain domain if there exist constants $M > 0$, $\mu \geq 1$ and a family $\mathcal{F}$ of disjoint metric balls B such that

(i) $\Omega = \bigcup_{B\in\mathcal{F}} 2B$

(iii) $\sum_{B\in\mathcal{F}} \chi_{10B}(x) \leq M\chi_{\Omega}(x)$ for all $x \in X$.

(iii) There is a so-called "central ball" $B_0 \in \mathcal{F}$ such that each ball $B \in \mathcal{F}$ can be connected to B_0 by a finite chain of balls $B_0, \cdots, B_{k(B)} = B$ in such a way that $2B_j \bigcap 2B_{j+1} \neq \emptyset$ and $4B_j \bigcap 4B_{j+1}$ contains a metric ball D_j whose volume is comparable to those of both B_j and B_{j+1}

(iv) Moreover, $B \subset \mu B_j$ for all $j = 0, 1, \cdots, k(B)$.

The explicit numbers 2, 4 and 10 are not essential here and are chosen just for simplicity.

Lemma 1.4 *Let $E = E(\xi_1, r_1) \subset \Omega$ be a metric ball. Then E is a Boman chain domain.*

This lemma has been verified in [11].

Let E be a metric ball in $\Omega \subset R^n$. Let $B \in \mathcal{F}$, where $\mathcal{F}$ is the decomposition of E as in the definition. A Lipschitz curve γ connecting two points $x, y \in \Omega$ is called admissible if

$$\gamma : [0, b] \to \Omega,\ \gamma(0) = x,\ \gamma(b) = y,\ \textit{and}\ \gamma'(t) = \sum_{i=1}^{m} a_i(t) X_i(\gamma(t))$$

with $\sum_{i=1}^{m} a_i^2(t) \leq 1$. Then

$$\varrho(x, y) = \inf\{b :\ \exists\ \textit{an admissible curve}\ \gamma :\ [0, b] \to \Omega\ \textit{connecting}\ x\ \textit{and}\ y\}.$$

We now define γ_B as an admissible path from the center η_B of B to ξ_1 (the center of E) of length $\leq r_1$. Denote the subset of E defined by the image of γ_B by γ_B as well. This path may not be unique, but will be fixed throughout this paper. Denote $\mathcal{F}(B) = \{A \in \mathcal{F} : 2A \bigcap \gamma_B \neq \emptyset\}$.

We will need two technical lemmas.

Lemma 1.5 *Given $1 \leq p < \infty$. Let $\{B_\alpha\}$ be an arbitrary family of open metric balls in (Ω, ϱ) with $\mu B_\alpha \subset \Omega$ and $\{a_\alpha\}_{\alpha\in I}$ be nonnegative numbers,* *where $\mu \geq 1$ is a constant. Then*

$$||\sum_\alpha a_\alpha \chi_{\mu B_\alpha}||_{L^p(\Omega)} \leq C||\sum_\alpha a_\alpha \chi_{B_\alpha}||_{L^p(\Omega)},$$

where C is independent of $\{a_\alpha\}$ and $\{B_\alpha\}$.

The proof is standard. We omit it here.

Lemma 1.6 *If $p \geq 1$, then for any metric balls I and B with $I \subset B \subset \Omega$ we have*

$$\left(\frac{\rho(I)}{\rho(B)}\right) \cdot \left(\frac{|I|}{|B|}\right)^{1/q-1/p} \leq C$$

provided that $1 \leq q \leq \frac{Qp}{Q-p}$ and $\rho(B) \leq r_0$ for some $r_0 > 0$.

This lemma is proved in [10]. It is lemma (6.12) in [10].

Proof of theorem (1.8): We set here $f(x) = u(x)$ as the solution to the differential equation (1.1). Fix the central ball B_0 as in the lemma (2.4). We also denote the center of the ball B as x_B. It is clear that we only need to show the theorem for $x_0 = x_{B_0}$. For any other $x_0 \in B_0$ the theorem follows by the mean value inequality and the Poincare inequality by considering the difference $f(x_0) - f(x_{B_0})$. We then have

$$\begin{aligned} &||f - f(x_{B_0})||^q_{L^q(E)} \\ &\leq 2^{q-1} \sum_{B\in\mathcal{F}} ||f - f(x_B)||^q_{L^q(B)} + 2^{q-1} \sum_{B\in\mathcal{F}} ||f(x_B) - f(x_{B_0})||^q_{L^q(B)} \\ &= I + II. \end{aligned} \tag{1.7}$$

We note by Lemma (2.1) (taking $\alpha = 2$),

$$\left(\frac{1}{|B|}\int_B |f(x) - f(x_B)|^q dx\right)^{1/q} \leq c\rho(B)\left(\frac{1}{|B|}\int_{2B}\left(\sum_{i=1}^m |X_i f|\right)^p\right)^{1/p} \tag{1.8}$$

for any given $B \in \mathcal{F}$. Now fix temporarily $B \in \mathcal{F}$ and consider the chain $\mathcal{F}(B) = \{A_1, \cdots, A_{k(B)}\}$ constructed in lemma (2.4). Thus

$$||f(x_B) - f(x_{B_0})||_{L^q(B)} \leq C \sum_{j=1}^{k(B)-1} ||f(x_{A_j}) - f(x_{A_{j+1}})||_{L^q(B)}$$

$$\leq \quad C\sum_{j=1}^{k(B)-1}\left(\frac{|B|}{|4A_j\bigcap 4A_{j+1}|}\right)^{1/q}||f(x_{A_j})-f(x_{A_{j+1}})||_{L^q(4A_j\bigcap 4A_{j+1})}$$

$$\leq \quad C\sum_{j=1}^{k(B)-1}\left(\frac{|B|}{|A_j|}\right)^{1/q}||f-f(x_{A_j})||_{L^q(4A_j)}+C\sum_{j=1}^{k(B)}\left(\frac{|B|}{|A_{j+1}|}\right)^{1/q}||f-f(x_{A_{j+1}})||_{L^q(4A_{j+})}$$

$$\leq \quad 2C\sum_{j=1}^{k(B)-1}\left(\frac{|B|}{|A_j|}\right)^{1/q}||f-f(x_{A_j})||_{L^q(4A_j)}.$$

We note that

$$||f-f(x_{A_j})||_{L^q(4A_j)}\leq ||f-f(x_{4A_j})||_{L^q(4A_j)}+||f(x_{4A_j})-f(x_{A_j})||_{L^q(4A_j)},$$

and

$$||f(x_{4A_j})-f(x_{A_j})||_{L^q(4A_j)}\leq ||f-f(x_{4A_j})||_{L^q(4A_j)}.$$

Thus

$$||f(x_B)-f(x_{B_0})||_{L^q(B)}\leq C\sum_{j=1}^{k(B)-1}\left(\frac{|B|}{|A_j|}\right)^{1/q}||f-f(x_{4A_j})||_{L^q(4A_j)}.$$

Since by the chain condition $B\subset \mu A_j$ for each $A_j\in\mathcal{F}(B)$, we then have

$$\begin{aligned}&||f(x_B)-f(x_{B_0})||_{L^q(B)}\frac{\chi_B(\xi)}{|B|^{1/q}}\\ &\leq C\sum_{A\in\mathcal{F}}\left(\frac{1}{|A|}\right)^{1/q}||f-f_{4A}||_{L^q(4A)}\chi_{4\mu A}(\xi)\\ &= C\sum_{A\in\mathcal{F}}a_A\chi_{4\mu A}(\xi).\end{aligned}$$

In the above expression, a_A is notationally defined in an obvious way. For the term II in (2.7), we have

$$II\leq C\sum_{B\in\mathcal{F}}\int_\Omega ||f(x_B)-f(x_{B_0})||^q_{L^q(B)}\frac{\chi_B(\xi)}{|B|}.$$

Since $\sum_{B\in\mathcal{F}}\chi_B(\xi)\leq C$, we obtain

$$II\leq C\int_\Omega|\sum_{A\in\mathcal{F}}a_A\chi_{4\mu A}|^q.$$

By lemma (2.5), we then get

$$II\leq C\int_\Omega|\sum_{A\in\mathcal{F}}a_A\chi_A|^q.$$

Since $\sum_{A\in\mathcal{F}}\chi_A(\xi)\le C$, we have

$$II \le C\sum_{A\in\mathcal{F}} a_A^q \int_\Omega \chi_A(\xi) \le C\sum_{A\in\mathcal{F}} ||f-f(x_{4A})||^q_{L^q(4A)}.$$

Therefore, by Lemma (1.9),

$$\begin{aligned} II &\le C\sum_{A\in\mathcal{F}} |A|^{1-q/p}\rho(A)^q\left(\int_{8A}\left(\sum_{i=1}^m |X_i f|\right)^p\right)^{q/p} \\ &\le C|E|^{1-q/p}\rho(E)^q \sum_{A\in\mathcal{F}}\left(\int_{8A}\left(\sum_{i=1}^m |X_i f|\right)^p\right)^{q/p} \\ &\le C|E|^{1-q/p}\rho(E)^q \left(\int_{E}\left(\sum_{i=1}^m |X_i f|\right)^p\right)^{q/p}. \end{aligned}$$

In the last inequality we used the fact $q\ge p$, $8A\subset E$ and $\sum_{A\in\mathcal{F}}\chi_{8A}(\xi)\le C$, and in the one next to the last we used lemma (2.6).

For the term I in (2.7), the estimate is the same by replacing $4A$ by $2A$ in the estimate of II. Indeed,

$$\begin{aligned} I &= \sum_{B\in\mathcal{F}} ||f-f(x_B)||^q_{L^q(B)} \\ &\le C\sum_{B\in\mathcal{F}} |B|^{1-q/p}\rho(B)^q\left(\int_{2B}\left(\sum_{i=1}^m |X_i f|\right)^p\right)^{q/p} \\ &\le C|E|^{1-q/p}\rho(E)^q \sum_{B\in\mathcal{F}}\left(\int_{8B}\left(\sum_{i=1}^m |X_i f|\right)^p\right)^{q/p} \\ &\le C|E|^{1-q/p}\rho(E)^q \left(\int_{E}\left(\sum_{i=1}^m |X_i f|\right)^p\right)^{q/p}. \end{aligned}$$

2 Remarks on Poincaré type inequalities for solutions of degenerate subelliptic equations

We now define the weighted Sobolev space $W^{1,p}_w(\Omega)$ to be the completion of all Lipschitz (or smooth) functions f under the norm

$$||f||_{W^{1,p}_w(\Omega)} = \left(\int_\Omega |f|^p w\right)^{1/p} + \left(\int_\Omega |Xf|^p w\right)^{1/p},$$

We also define $W^{1,p}_{w,0}(\Omega)$ as the completion of Lipschitz functions with compact support (or functions in the class $C^\infty_0(\Omega)$) under the above norm $||\cdot||_{W^{1,p}_w(\Omega)}$.

Let Ω be an open and bounded domain in R^n. We say Ω is a $W^{1,p}_w$-**extension domain** if there is a domain $\Omega' \subset R^n$ containing $\overline{\Omega}$ and an operator $T : W^{1,p}_w(\Omega) \to W^{1,p}_{w,0}(\Omega')$ such that $Tf(x) = f(x)$ a.e. in Ω and $||Tf||_{W^{1,p}_w(\Omega')} \leq C||f||_{W^{1,p}_w(\Omega)}$ with C independent of f.

Let X_i^* be the adjoint of X_i. We will consider the differential operators

$$L = \sum_{i,j=1}^{m} X_i^*(x)(a_{ij}(x)X_j(x)),$$

and

$$\mathcal{L} = -\sum_{i,j=1}^{m} X_i(x)(a_{ij}(x)X_j(x))$$

where the coefficients a_{ij} are measurable, real-valued functions whose coefficient matrix $A = (a_{ij})$ is symmetric and satisfies

$$c^{-1}w(x)|\xi|^2 \leq < A\xi, \xi > \leq cw(x)|\xi|^2, \tag{2.1}$$

where $< \cdot, \cdot >$ denotes the usual dot product, and $w \in A_2(\Omega)$ is a Muckenhoupt A_2 weight in the metric space (Ω, ϱ).

We then have the following theorems:

Theorem 2.2 *Suppose that Ω is a $W^{1,2}_w$-extension domain and let $u \in W^{1,2}_w(\Omega)$ be a weak solution of $Lu = 0$ (or $\mathcal{L}u = 0$). Let $1 \leq q < \frac{2Q}{Q-1} + \delta$ for some $\delta > 0$ derived in Theorem B of [13]. Then for any point $x_0 \in \Omega$ there is a constant C depending on the A_2 constant of w, q, $||u||_{w,q,\Omega}$, and Ω such that*

$$||u - u(x_0)||_{w,q,\Omega} \leq C||Xu||_{w,2,\Omega}.$$

Theorem 2.3 *Suppose that $E \subset \Omega$ is any metric ball. Let $u \in W^{1,2}_w(\Omega)$ be a weak solution of $Lu = 0$ (or $\mathcal{L}u = 0$). Let $1 \leq q < \frac{2Q}{Q-1} + \delta$ for some $\delta > 0$ (see*

Theorem B in [13]). Then there is a constant C depending on the A_2 constant of w, q, $||u||_{w,q,\Omega}$, and Ω such that for any point $x_0 \in \frac{1}{2}E$ we have

$$\left(\frac{1}{w(B)}\int_B |u-u(x_0)|^q w\right)^{1/q} \leq Cr\left(\frac{1}{w(B)}\int_B \sum_{i=1}^m |X_i u|^2 w\right)^{1/2}$$

Remark: The weighted L^2 norm on the right-hand side of the above inequalities can be replaced by weighted L^p norms for any $p \geq 1$ with appropriately selected q provided $u \in W_w^{1,p}(\Omega) \bigcap W_w^{1,2}(\Omega)$ because the mean value inequality for the subsolutions holds for any $0 < p < \infty$ (see Theorem (7.5) in [L1]).

The proofs of the above two theorems will follow the pattern of those in Section 2 and adapt the weighted version of the Rellich-Kondrachov compact embedding theorem (see [12]) and the mean-value inequality and Holder continuity derived in [10]. One also needs an adaptation of the covering lemma argument. We omit the details here. A theorem similar to Theorem (1.9) also holds but we shall not state it.

Added in Proof: After this paper was submitted for publication, we learned that Ziemer's result for $X_i = \frac{\partial}{\partial x_i}$ can be extended to the case $p < 1$ (see the work of S. Buckley and P. Koskela, Indiana Journal, 1994). The main result of our present paper for Hormander's vector fields has also been shown to hold for $p < 1$ in the forthcoming joint work of us.

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