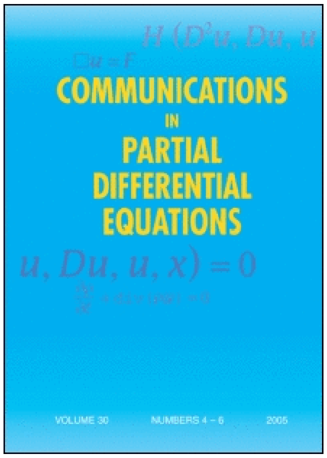

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Guozhen Lu [a]; Guozhen Lu [a]
[a] Department of Mathematics, California Institute of Technology, Pasadena, CA

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EXISTENCE AND SIZE ESTIMATES FOR THE GREEN'S FUNCTIONS OF DIFFERENTIAL OPERATORS CONSTRUCTED FROM DEGENERATE VECTOR FIELDS

Guozhen Lu

Department of Mathematics, 253-37

California Institute of Technology
Pasadena, CA 91125

1 Introduction

Let $\Omega \subset R^d$ ($d \geq 3$) be a bounded, open and connected domain and $X_1, \cdots, X_m$ be C^∞ vector fields satisfying Hőrmander's condition on a neighbourhood of $\overline{\Omega}$, i.e., there is a positive integer s such that all the commutators of $X_1, \cdots, X_m$ up to order s span the tangent space of R^d at every point of Ω (see [H], [RS] or [NSW]).

There has been much important work for the existence and estimates of the fundamental solutions for subelliptic operators formed by vector fields.

In [Fo] Folland obtained the fundamental solutions for the homogeneous operators on the nilpotent groups. Since the Rothschild-Stein lifting theorem for vector fields was proved in [RS], the fundamental solutions for differential operators formed by the general vector fields satisfying Hőrmander's condition have been studied in [NSW] and [Sa]. In particular, Sánchez-Calle [Sa], and Nagel, Stein and Wainger [NSW], Fefferman and Sanchez-Calle [FeS] proved the estimates for the Green kernel for sums of squares of vector fields and certain subelliptic operators. Later on, Jerison and Sánchez-Calle [JS], and Kusuoka and Strook [KS] obtained the size estimates for the heat kernel for certain classes of subelliptic differential operators. We refer the interested reader to the above papers and references therein. The results cited above were in principle for differential operators with smooth and "elliptic" coefficients for the leading terms. Thus it seems interesting to study the differential operators formed by vector fields with nonsmooth and even unbounded coefficients.

The purpose of this article is to deal with the existence and bounds estimate for Green's function of the degenerate differential operators

$$Lu = \sum_{i,j=1}^{m} X_i^* \left(a_{ij}(x) X_j u\right)$$

where X_i^* denotes the adjoint of X_i and the coefficient matrix $A = (a_{ij})$ satisfies

$$c^{-1} w(x)|\xi|^2 \leq < A\xi, \xi > \leq c w(x)|\xi|^2, \xi \in R^m. \tag{1.1}$$

Here, $< \cdot, \cdot >$ denotes the dot product in R^m and w is a nonnegative function which will be specified later. The representation of the solutions to L is proved. We also show a Rellich compact embedding lemma in the weighted Sobolev space for the vector fields which is of special interest. We also remark out that when the coefficient matrix satisfies even stronger degeneracy assumption, say,

$$c^{-1}w_1(x)|\xi|^2 \leq < A\xi, \xi > \leq cw_2(x)|\xi|^2, \xi \in R^m,$$

where w_1, w_2 satisfy certain conditions such that the Poincare-Sobolev inequality holds (see [L1]), then we can also obtain the existence and bounds estimate for the fundamental solutions for the operator $\mathbf{L}$ by mimicing the proof of the present work. We do not intend to do so here.

Precisely speaking, we shall show that a Green function for the operator L exists and will also derive the local interior estimation for its size. By "Green's function for Ω with pole y" we mean a function $G(x,y) = G^y(x), x, y \in \Omega$, which solves $\mathbf{L}G^y = \delta_y$ in the weak sense, i.e.,

$$\int_\Omega < AXG^y, X\phi > dx = \phi(y), for\ \phi \in Lip_0(\Omega),$$

where $Lip_0(\Omega)$ denotes the class of Lipschitz continuous functions supported in Ω and $Xf = (X_1 f, \cdots, X_m f)$. Moreover, G^y vanishes on the boundary $\partial\Omega$ in the sense that it is the limit, in appropriate norm, of functions supported in Ω. We shall also obtain the representation of the solution u to

$$\mathrm{L}u = f\ in\ \Omega,\ with\ u = 0\ on\ \partial\Omega,$$

in terms of a potential of f which has G as its kernel.

Before we state our main theorems in this paper, we like to introduce some notations and definitions. Throughout this paper we will always assume that $w \in A_2(\Omega, \varrho)$, for the metric ϱ defined by vector fields $X_1, \cdots, X_m$ (see [L1]) i.e.,

$$\left(\frac{1}{|B|}\int_B w\right)\left(\frac{1}{|B|}\int_B w^{-1}\right) \leq c\ for\ all\ metric\ balls\ B \subset \Omega,$$

For an example of A_2 weight on the Heisenberg group, we refer the reader to [L1]. We will denote cB the ball with the same center as B and c times as large. We will also denote $B(x,r), B_I(x,r)$ or $B_r(x)$ the metric balls defined in [NSW] with center x and radius r. For the definitions of these metric balls, we refer the reader to [NSW]. We will work on different metric balls whenever necessary. As we proved in [L1], $w \in A_2$ implies that for any balls B=$B_h(x) \subset \Omega$,

$$(1.2) \qquad \frac{s}{h}\left[\frac{w(B_s(x))}{w(B_h(x))}\right]^{1/q} \le c\left[\frac{w(B_s(x))}{w(B_h(x))}\right]^{1/2}, 0 < s < h,$$

for some $q > 2$. We shall also denote $\sigma = \frac{q}{2}$ and $s_0 = \frac{2\sigma}{\sigma+1}$, then $\sigma > 1, 1 < s_0 < 2$.

For $1 \le p < \infty$, we use the notation

$$L^p_w = \left\{ f : \|f\|_{L^p_w} = \left(\int_\Omega |f(x)|^p w(x)dx\right)^{1/p} < \infty \right\}$$

and we write L^p for $w = 1$.

We shall also adopt the notation $X = X_{t,s}$ for the Banach space which is the closure of $Lip_0(\Omega)$ with respect to the norm

$$\|f\|_{L^t_w} + \|Xf\|_{L^s_w},$$

where $|Xf| = (\sum_{i=1}^m |X_i f|)^{1/2}$. For $1 < s < \infty$, define s' by $1/s + 1/s' = 1$.

Since we only consider the local interior estimates for Green's function, we will derive the existence and estimates of Green's function for a small ball B inside Ω when the pole lies in the middle half of B. The important point here is that all the constants below will be independent of balls B inside Ω, but only dependent on the domain Ω, the geometry of the metric defined by the vector fields, the degeneracy constant for the matrix $A = (a_{ij})$ and the A_2 constant for the weight w.

We now are ready to state the main result in this paper.

Theorem 1.3 *Suppose that $w \in A_2(\Omega, \varrho)$, and A is a symmetric matrix which satisfies (1.1). Let $E \subset\subset \Omega$ and let $B = B_R(x_0)$ be a ball with*

$x_0 \in E$ and $B \subset \Omega$. Then for almost every $y \in \frac{1}{2}B$, there is a nonnegative function $G(x,y), x \in B$, which satisfies

(i) $G \in X_{t,s}$ for $t < \sigma$ and $s < s_0$, and the size of the norms are uniform in y; that is,

$$ess \sup_{y\in\frac{1}{2}B} \left[\int_B G(x,y)^t w(x)dx + \int_B |XG(x,y)|^s w(x)dx\right] < \infty$$

for such t and s;

(ii) $\int_B < AX_x G(x,y), X\phi > dx = \phi(y), \phi \in Lip_0(B)$;

For $0 < r < R/4$ and with c independent of B, y, r, we have

(iii) $ess \sup_{x:r/2<\varrho(x,y)<r} G(x,y) \leq c \int_r^R \frac{t^2}{w(B_t(y))}\frac{dt}{t}$

(iv) $ess \inf_{x:r/2<\varrho(x,y)<r} G(x,y) \geq c \int_r^R \frac{t^2}{w(B_t(y))}\frac{dt}{t}$

There are two Hilbert spaces H_0 and H associated with the operator **L**. The definitions and properties are discussed in [L1]. For completeness, we will introduce them briefly in section (2). We recall that H_0 consists of elements of H which vanish at ∂B in an appropriate sense, and that the inner product $a_0(u,\phi)$ on H_0 satisfies

$$a_0(u,\phi) = \int_B < AXu, X\phi >$$

if u, $\phi \in Lip_0(B)$. Moreover, $a_0(u,\phi)$ can be defined for u, $\phi \in H$, and there are associated functions $\tilde{u}$, $\tilde{\phi} \in L^2_w(B)$ (even $L^{2\sigma}_w(B)$) such that $X\tilde{u}$, $X\tilde{\phi} \in L^2_w(B)$ and

$$a_0(u,\phi) = \int_B < AX\tilde{u}, X\tilde{\phi} > .$$

An argument based on the Lax-Milgram theorem shows that if $f/w \in L^{(2\sigma)'}_w(B)$ and the assumption of Theorem A holds, then we can solve the problem

$$\mathbf{L}u = f \ in \ B, with \ u = 0 \ on \ \partial B, \tag{1.4}$$

in the sense that there exixts $u \in \dot{H}_0$ with

$$a_0(u,\phi) = \int_B f\phi, \quad \phi \in H_0$$

We will refer to u as the Lax-Milgram solution of (1.4).

Likewise, if F is a vector with $|F|/w \in L^2_w(B)$, it is possible to solve

$$\mathbf{L}u = X^*F \text{ in } B, \text{with } u = 0 \text{ on } \partial B, \tag{1.5}$$

in the sense that there exists $u \in H_0$ with

$$a_0(u,\phi) = \int_B < F, X\phi >, \quad \phi \in H_0,$$

where $X^*F = \sum_{i=1}^m X_i^* F_i$, $F = (F_1, \cdots, F_m)$. We shall refer to u as the Lax-Milgram solution to (1.5).

The following is the represention theorem of solutions to operator **L** in terms of G.

Theorem 1.6 *Let $w \in A_2$ and assume (1.1) holds. If $f/w \in L^{t'}_w(B)$ for some $t < \sigma$ and u is the Lax-Milgram solution to (1.4), then*

$$\tilde{u}(y) = \int_B f(x)G(x,y)dx \quad for\ a.e.\ \ y \in \frac{1}{2}B.$$

Furthermore, if $\frac{|F|}{w} \in L^{s'}_w(B)$ for some $s < \frac{2\sigma}{\sigma+1}$ and u is the Lax-Milgram solution of (1.5), then

$$\tilde{u}(y) = \int_B < F(x), X_x G(x,y) > dx \quad for\ a.e.\ \ y \in \frac{1}{2}B.$$

The proofs of theorems (1.3) and (1.6) need the mean-value and Harnack inequalities and also the following Sobolev inequality

$$\left(\frac{1}{w(B)}\int_B |f|^q w\right)^{1/q} \le cr\left(\frac{1}{w(B)}\int_B |Xf|^2 w\right)^{1/2}, \ f \in Lip_0(B) \tag{1.7}$$

for $B = B(x,r)$ or $B = B_I(x,r), x \in E \subset\subset \Omega, r \le r_0$, with c independent of f, B. The number q is the same as in (1.2).

This paper is the continuation of our previous work [L1]. In [L1], we proved the Poincaré and Sobolev inequalities with two weights w_1, w_2 satisfying certain condition. A uniform Harnack's inequality and mean value inequalities are also derived in [L1] for the following two types of differential operators:

$$L = \sum_{i,j=1}^{m} X_i^*(a_{ij}(x)X_j), \quad \mathcal{L} = \sum_{i,j=1}^{m} X_i(a_{ij}(x)X_j).$$

The proof of Theorem (1.3) and Theorem (1.6) relies on adapting the methods of finding out first the approximate Green's function developed in [CW] to our case. In [CW], the existence and size estimates for green's function are proved for the operator $L = \sum_{i,j=1}^{m} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial}{\partial x_j})$ in the case of unequal weights. The proof of the same problem in the setting of equal weights was given in [FJK] (for uniformly elliptic case, see [GW] and [LSW]). Thus, our result here extends the one in [FJK] in the sense by letting $X_i = \frac{\partial}{\partial x_i}$, $m = d$. As an application of this paper, we have shown in [L2] the Harnack inequality for a class of strongly degenerate Schrődinger's operators formed by the general vector fields satisfying Hőrmander's condition:

$$L = \sum_{i,j=1}^{m} X_i^*(a_{ij}(x)X_j) + V(x),$$

for the potential V in the so-called Kato-Stummel class. We also like to point out that the **Relich compact embedding** lemma (2.6) for vector fields is of independent importance.

2 Weak maximum principle, and Rellich's embedding lemma for vector fields

Throughout this section, we assume that Ω' is a subdomain of Ω and so small that the Sobolev inequality holds on a ball B_0 containing Ω'. The results obtained in this section will apply to any small ball inside Ω. The constants appearing in all the inequalities below will be independent of the particular ball but only dependent on w, the vector fields, and Ω.

As in [L1], let

$$a_0(u,\phi) = \int_{\Omega'} < AXu, X\phi >, \quad u,\phi \in Lip(\overline{\Omega'}),$$

where $Lip(\overline{\Omega'})$ as usual denotes the class of Lipschitz continuous functions on the closure of Ω'. Then by the degeneracy condition (1.1),

$$\int_{\Omega'} |Xu|^2 w \le a_0(u,u) \le \int_{\Omega'} |Xu|^2 w.$$

It is easy to see that $a_0(u,\phi)$ is an inner product on $Lip_0(\Omega')$ by the assumption that $X_1, \cdots, X_m$ satisfies Hőrmander's condition (see the proof in [L1]) Hence $a_0(u,u)$ is a norm on $Lip_0(\Omega')$. We note that

$$a_0(u,\phi) \le a_0(u,u)^{1/2} a_0(\phi,\phi)^{1/2}$$

and also note that

$$| < Ax, y > | \le < Ax, x >^{1/2} < Ay, y >^{1/2}$$

since A is symmetric.

We define $H_0 = H_0(\Omega')$ to be the completion of $Lip_0(\Omega')$ with respect to the norm $a_0(u,u)$. Thus an element of H_0 is an equivalence class of the Cauchy sequences $\{u_k\}, u_k \in Lip_0(\Omega')$.

If $u, \phi \in H_0$ with $u = \{u_k\}$ and $\phi = \{\phi_k\}, u_k, \phi_k \in Lip_0(\Omega')$, it is easy to check that $a_0(u_k, \phi_k)$ converges, and we define

$$a_0(u,\phi) = \lim_{k\to\infty} a_0(u_k,\phi_k).$$

Thus we see that $\|u\|_0 = a_0(u,u)^{1/2}$ is a norm on H_0.

As noted in [L1], we can associate to each $u \in H_0$ a unique pair $(\tilde{u}, X\tilde{u})$ so that if $u = \{u_k\}$, then $u_k \to \tilde{u}$ in L^2_w (even in $L^{2\sigma}_w$) and $Xu_k \to X\tilde{u}$ in L^2_w. This is because the fact that Sobolev inequality holds on a ball B_0 containing Ω'. We can also argue as in [CW] and show that $X\tilde{u}$ is the distributional "gradient" of $\tilde{u}$, that is,

$$\int \tilde{u} X^* \phi = \int X\tilde{u}\phi$$

where $X^*\phi = (X_1^*\phi, \cdots, X_m^*\phi)$.

The Hilbert space $H = H(\Omega')$ is also introduced in [L1] . $H = H(\Omega')$ is defined as the completion of $Lip(\overline{\Omega'})$ with respect to the inner product

$$a(u,\phi) = a_0(u,\phi) + \int_{\Omega'} u\phi w, \quad u,\phi \in Lip(\overline{\Omega'})$$

Several facts about H are given in [L1]. In particular, H_0 can be viewed as the subspace of H, that is, the inclusion map from H_0 to H is continuous by the Sobolev inequality. Furthermore, if $u \in H, u = \{u_k\}, u_k \in Lip(\overline{\Omega'})$, then u_k converges to $\tilde{u}$ in $L^{2\sigma}_w$ (thus in L^2_w) and Xu_k converges in L^2_w to a vector $X\tilde{u}$. It is also easy to see that if $\phi \in H, \phi = \{\phi_k\}$, then the limits $a(u,\phi) = \lim a(u_k,\phi_k)$ and $a_0(u,\phi) = \lim a_0(u_k,\phi_k)$ exists and the following holds

$$a(u,\phi) = a_0(u,\phi) + \int_{\Omega'} \tilde{u}\tilde{\phi} w,$$

and $a(u,\phi)$ is an inner product on H.

Now we state the following

Lemma 2.1 *Let $u,\phi \in H$ and let $X\tilde{u}$ and $X\tilde{\phi}$ be the associated "gradient" to u and ϕ, respectively. If $u = \{u_k\}$ and $\phi = \{\phi_k\}$, then*

$$< AXu_k, X\phi_k > \to < AX\tilde{u}, X\tilde{\phi} > \ \text{in } L^1(\Omega').$$

In particular,

$$a_0(u,\phi) = \int_{\Omega'} < AX\tilde{u}, X\tilde{\phi} >$$

and

$$a(u,\phi) = \int_{\Omega'} < AX\tilde{u}, X\tilde{\phi} > + \int_{\Omega'} \tilde{u}\tilde{\phi}w.$$

The proof of the above lemma can be found in [CW] in the case $X_i = \frac{\partial}{\partial x_i}$. In our case, the proof is almost the same.

We recall $u \in H$ is nonnegative, or $u \geq 0$, in Ω' if u can be represented by a sequence $\{u_k\}, u_k \in Lip(\overline{\Omega'}), u_k \geq 0$ in Ω'. It is easy to see that if $u \geq 0$, then $\tilde{u} \geq 0$ a.e. in Ω'. We now give the definitions for solutions, supersolutions and subsolutions. An element $u \in H$ is called a solution of Lu=0 if $a_0(u,\phi) = 0$ for all $\phi \in H_0$; u is called a subsolution if $a_0(u,\phi) \leq 0$, for all $\phi \in Lip_0(\Omega'), \phi \geq 0$; u is called a supersolution if -u is a subsolution. We also review two results proved in [L1]. First, if u is a solution in H(2B), then the following mean value inequality holds

$$\text{(2.2)} \qquad ess\sup_B |\tilde{u}| \leq c_p \left(\frac{1}{w(B)} \int_{2B} |\tilde{u}|^p w \right)^{1/p}, 0 < p < \infty.$$

Furthermore, if $u \geq 0$, then we have the following Harnack inequality

$$\text{(2.3)} \qquad ess\sup_B \tilde{u} \leq c \; ess\inf_B \tilde{u}.$$

The next three lemmas will be essential throughout this paper.

Lemma 2.4 (Weak Maximum Principle) *Let u be a supersolution in $H(\Omega')$ to* Lu=0*. Let $u = \{u_k\}, u_k \in Lip(\overline{\Omega'})$, and assume $u_k \geq 0$ in some neighbourhood of $\partial\Omega'$ (depending on k). Then $\tilde{u} \geq 0$ a.e. in Ω'.*

Proof: Let $u_k^- = -\min\{u_k, 0\}$. Note that $u_k^- \in Lip_0(\Omega')$ since $u_k \geq 0$ near $\partial\Omega'$. It is easy to see that $\|u_k^-\|_{H_0}$ is bounded since $\|u_k\|_H$ is bounded. Thus we can select a subsequence $u_{k_j}^-$ converging weakly in H_0 to some $\Psi \in H_0$. Since u is a supersolution,

$$\lim a_0(u_{k_j}, u_{k_j}^-) = \lim a_0(u, u_{k_j}^-) = a_0(u, \Psi) \geq 0$$

Thus

$$\lim \int < AXu_{k_j}, Xu_{k_j}^- > \geq 0,$$

which implies

$$\lim \int < AXu_{k_j}^-, Xu_{k_j}^- > \leq 0.$$

Therefore, we have that $\|Xu_{k_j}^-\|_{L^2_w} \to 0$. By extending $u_{k_j}^-$ to the ball B_0 containing Ω' and applying Sobolev's inequality, we see $\|u_{k_j}\|_{L^2_w} \to 0$. But we have $u_{k_j}^- \to (\tilde{u})^-$ in L^2_w since $u_{k_j} \to \tilde{u}$ in L^2_w. Thus $(\tilde{u})^- = 0$ a.e. in Ω' and we are done.

Lemma 2.5 *Let B_1, B_2 and B_3 be balls with a common center and radii r_1, r_2, r_3, respectively, and satisfying $r_1 < r_2 < r_3$. If $\phi \in H(B_3)$ and $\tilde{\phi} \leq l$ a.e. in $B_3 \backslash B_1$, then given $L > l$, there exists $\phi_k \in Lip(\overline{B_2})$ such that $\phi_k \to \phi$ in $H(B_2)$ and $\phi_k \leq L$ in some neighbourhood of ∂B_2. Moreover, if u is a solution in $H(B_2), u = \{u_k\}$, and if $u_k \leq \phi_k$ near ∂B_2 for these ϕ_k, then $\tilde{u} \leq L$ a.e. in B_2.*

The proof of lemma (2.5) is similar to the proof of lemma (2.7) in [CW]. In fact, the proof given in [CW] not only works for three balls B_1, B_2 and B_3 but also for any three proper subdomains $\Omega_1 \subset \Omega_2 \subset \Omega_3$.

Next we prove a version of Rellich's imbedding lemma adapted to our situation.

Lemma 2.6 (Rellich) *Let $w \in A_2$. Let $\{f_j\}$ be a sequence of functions supported in Ω having the property that*

$$\|f_j\|_{L^2_w(\Omega)} + \|Xf_j\|_{L^2_w(\Omega)} \leq c < \infty, \forall j\ .$$

Then for any compact subset $K \subset \Omega$, there exists a subsequence $\{f_{k_j}\}$ such that $\{f_{k_j}\}$ converges in $L^{q^*}_w(K)$ for any $2 \leq q^* < q$, where q is the exponent in the Sobolev inequality (1.7)

Proof: As in [RS], we lift X_i to $\tilde{X}_i$ for $i = 1, \cdots, m$ and define $\tilde{f}(\xi) = f(x)$, $\tilde{w}(\xi) = w(x)$ for $\xi = (x,t) \in \tilde{\Omega} = \Omega \times T$, where T is a unit ball in R^d for some $d > 0$ (see [RS] for details about the lifting of vector fields). We also denote $\tilde{K} = K \times T$. Since $\tilde{X}_i \tilde{f} = X_i f$, we have

$$\|\tilde{f}_j\|_{L^2_{\tilde{w}}(\tilde{\Omega})} + \|\tilde{X}\tilde{f}_j\|_{L^2_{\tilde{w}}(\tilde{\Omega})} \leq c, \ \forall j \tag{2.7}$$

Now we drop all the tildes for simplicity. We pick $\phi \in C_0^\infty(G)$, $supp\{\phi\} \subset \{|y| \leq 1\}$,
$\int_G \phi = 1, \ 0 \leq \phi \leq 1$, where G is the graded nilpotent Lie group as in [RS]. For $\xi \in K$ and t small enough, we define

$$f^t(\xi) = \int_\Omega f(\eta) I_t\phi(\Theta(\eta,\xi))\, d\eta$$

where $I_t\phi(y) = t^{-Q}\phi(\delta_{t^{-1}}y)$ and δ_t is the dilation on G, and Q is the homogeneous dimension of G (see [RS]). Then, if let $f(\xi) = f_j(\xi)$ and denote $\varrho(\xi,\eta) = |\Theta(\xi,\eta)|$, by (2.7) we have,

$$|f^t(\xi)| \leq ct^{-Q} \int_{\varrho(\xi,\eta)\leq t} |f(\eta)| d\eta \leq ct^{-Q} \int_{\tilde{\Omega}} |f(\eta)| d\eta$$

$$\leq ct^{-Q} \|f\|_{L^2_w} \left(\int_\Omega w^{-1} \right)^{1/2} \leq Ct^{-Q}$$

where C is independent of f, t.

Let $\alpha = (i_1, \cdots, i_k)$, define $X_\alpha = [X_{i_1}, [\cdots, [X_{i_{k-1}}, X_{i_k}], \cdots]]$ and denote by $|\alpha| = k$ the length of X_α. We note (see [J])

$$t^{|\alpha|} X_\alpha f^t(\xi) = \sum_{i=1}^m \int tX_i f(\eta) \phi_{i\alpha} \left(\delta_t^{-1} \circ \Theta(\eta,\xi) \right) t^{-Q} d\eta + \int f(\eta) F_\alpha^t(\xi,\eta) t^{-Q} d\eta$$

where $|F_\alpha^t| \leq ct$, $supp\ F_\alpha^t \subset \{(\xi,\eta) : \varrho(\xi,\eta) \leq ct\}$, and $\phi_{i\alpha} = D_{i\alpha}\phi$ for some differential operators $D_{i\alpha}$ as defined in [J]. Then by (2.7) again,

$$|t^{|\alpha|} X_\alpha f^t(\xi)| \leq \sum_{i=1}^m \int_{\varrho(\xi,\eta)\leq t} t^{-Q+1} |X_i f(\eta)| d\eta + \int_{\varrho(\xi,\eta)\leq t} |f(\eta)| t^{-Q+1} d\eta \leq Ct^{-Q+1}$$

with C independent of f and t. Therefore

$$|X_\alpha f^t(\xi)| \le Ct^{-Q+1-|\alpha|}.$$

Replace $f(\xi)$ by $h(\xi) = \frac{f(\xi)}{g(\xi)}$, where $g(\xi) = \tilde{g}(\xi, 0)$, and $\tilde{g}(\xi, \delta_t y)$ is defined by

$$\left(\left(\delta_t^{-1} \circ \Theta_\xi\right)^{-1}\right)^* t^{-Q} d\eta = \tilde{g}(\xi, \delta_t y) dy$$

and note that

$$g \in C^\infty \left(B(\xi, ct) \times \{|y| < 1\}\right), and\ g \approx 1\ on\ B(\xi, ct) \times \{|y| < 1\}$$

(see [J] also). Since

$$|h(\xi)| \le c|f(\xi)|\ and\ |X_i h(\xi)| \le c(|X_i f| + |f|),$$

the previous computation shows that

$$|h(\xi)| \le ct^{-Q}\ and\ |X_\alpha h^t(\xi)| \le ct^{-Q+1-|\alpha|}$$

Note now,

$$\begin{aligned}
|h^t(\xi) - f(\xi)| &= \int_0^t \frac{\partial}{\partial s} h^s(\xi) ds \\
&\le \int_0^t \left\{ \sum_{i=1}^m \int |X_i h(\eta) I_s \phi^{(i)} \left(\Theta(\eta, \xi)\right)| d\eta + \int |h(\eta) K_s^1(\xi, \eta)| d\eta \right\} ds \\
&\le c \sum_{i=1}^m \int_0^t |X_i h(\eta)| s^{-Q} d\eta ds + c \int_0^t \int_{\varrho(\xi,\eta) \le s} |h(\eta)| s^{-Q} d\eta ds \\
&\le c \int_{\varrho(\xi,\eta)\le t} \frac{\sum_{i=1}^m |X_i h(\eta)| + |h(\eta)|}{\varrho(\xi, \eta)^{Q-1}} d\eta
\end{aligned}$$

where $|K_s^1(\xi, \eta)| \le cs^{-Q}$, $supp\ K_s^1 \subset \{(\xi, \eta) : \varrho(\xi, \eta) \le cs\}$ as in [J].

Now we estimate $\|h^t(\xi) - f(\xi)\|_{L^2_w(K)}$, for $h(\xi) = h_j(\xi) = \frac{f_j(\xi)}{g(\xi)}, f(\xi) = f_j(\xi)$ and $K \subset \Omega$. Let $F(\eta) = (|Xf(\eta)| + |f(\eta|)\chi_\Omega(\eta))$, then

$$|h^t(\xi) - f(\xi)| \le \int_{\varrho(\xi,\eta)\le t} \frac{F(\eta)}{\varrho(\xi, \eta)^{Q-1}} d\eta$$

But we note that

$$\int_{\varrho(\xi,\eta)\leq t}\frac{F(\eta)}{\varrho(\xi,\eta)^{Q-1}}d\eta=\sum_{k=1}^{\infty}\int_{2^{-k}t\leq\varrho(\xi,\eta)\leq 2^{-k+1}t}\frac{F(\eta)}{\varrho(\xi,\eta)^{Q-1}}d\eta$$
$$\leq\sum_{k=1}^{\infty}\frac{1}{(2^{-k}t)^{Q-1}}\int_{\varrho(\xi,\eta)\leq 2^{-k+1}t}F(\eta)d\eta$$
$$\leq\sum_{k=1}^{\infty}\frac{(2^{-k+1}t)^{Q}}{(2^{-k}t)^{Q-1}}\frac{1}{(2^{-k+1}t)^{Q}}\int_{\varrho(\xi,\eta)\leq 2^{-k+1}t}F(\eta)d\eta$$
$$\leq\sum_{k=1}^{\infty}(2^{-k}t)2^{Q}F^{*}(\xi)\leq 2^{Q}tF^{*}(\xi)$$

where F^* is the Hardy-Littlewood Maximal function with respect to the pseudo-metric defined by $\varrho(\xi,\eta)$. Since $w\in A_2$, by a theorem in [Ca],

$$\|h^t(\xi)-f(\xi)\|_{L^2_w(K)}\leq 2^Qt\|F^*(\xi)\|_{L^2_w(\Omega)}$$
$$\leq ct\|F\|_{L^2_w(\Omega)}\leq ct\|(|Xf|+|f|)\|_{L^2_w(\Omega)}$$
$$\leq ct\left[\|Xf\|_{L^2_w(\Omega)}+\|f\|_{L^2_w(\Omega)}\right]\leq Ct$$

with C independent of f, t. By the above, $\exists C$ independent of j and t such that

$$|h^t_j(\xi)|\leq Ct^{-Q},|X_\alpha h^t_j(\xi)|\leq Ct^{-Q+1-|\alpha|}\ and\ \|h^t_j-f_j\|_{L^2_w(\Omega)}\leq Ct$$

If we note $\{X_\alpha\}_{|\alpha|\leq s}$ spans the tangent space of R^n, then

$$|\nabla h^t_j(\xi)|\leq c\sum_{|\alpha|\leq s}|X_\alpha h^t_j(\xi)|$$

So by Ascoli's theorem, we can obtain a subsequence $\{\tilde{f}_{j_k}\}$ convergent in $L^2_{\tilde{w}}(\tilde{\Omega})$. Integration with respect to the variables t shows that the same subsequence $\{f_{j_k}\}$ converges in $L^2_w(\Omega)$.

Now let $q>2$ be the exponent in the Sobolev inequality. For $2<q^*<q$, there exists some $0<\epsilon<1$ such that $\frac{1}{q^*}=\frac{\epsilon}{2}+\frac{1-\epsilon}{q}$. Thus

$$\|f_{j_k}-f_{j_l}\|_{L^{q^*}_w}\leq C\|f_{j_k}-f_{j_l}\|^{\epsilon}_{L^2_w}\cdot\|f_{j_k}-f_{j_l}\|^{1-\epsilon}_{L^q_w}\tag{2.8}$$

Now,

$$\|f_{j_k} - f_{j_l}\|_{L^2_w} \to 0 \;\; as\; j,\; l \to \infty$$

By Sobolev's inequality and the hypothesis in our lemma, we have

$$\|f_{j_k} - f_{j_l}\|_{L^q_w} \le C\|X(f_{j_k} - f_{j_l})\|_{L^2_w} \le C'$$

Thus in (2.8) above the first term on the right in the inequality goes to zero while the second term on the right is uniformly bounded. This establishes the convergence of f_{j_k} in $L^{q^*}_w$.

Q.E.D.

By adapting the above proof to the case $w \equiv 1$ and also use the Sobolev inequality without weights proved in [L1], we will get the following

Lemma 2.9 *Given $1 < p < Q$. Let $\{f_j\}$ be a sequence of functions supported in Ω having the property that*

$$\|f_j\|_{L^p(\Omega)} + \|Xf_j\|_{L^p(\Omega)} \le c < \infty, \forall j\ .$$

Then for any compact subset $K \subset \Omega$, there exists a subsequence $\{f_{k_j}\}$ such that $\{f_{k_j}\}$ converges in $L^{q^*}(K)$ for any $q^* < q$, where $q = \frac{pQ}{Q-p}$ is the exponent in the Sobolev inequality proved in [L1] and Q is the homogeneous dimension of G.

We record lemma (2.9) here just for the future reference and there is no value to the present article .

Corollary 2.10 *Lemma (2.6) holds as stated if we replace the hypothesis that $\{f_j\}$ have support in Ω by the alternate hypothesis that $\exists f_{j_k}$ supported in Ω such that $f_{j_k} \to f_j$ and $Xf_{j_k} \to Xf_j$ in L^2_w as $k \to \infty$.*

The proof of the corollary is easy. We only need to apply the previous lemma to the sequence $f_j^{k_j}$, where for a given j, we select k_j such that the $L^2_w(\Omega)$ norm of both $f_j - f_j^{k_j}$ and $Xf_j - Xf_j^{k_j}$ go to 0.

3 Estimates for the approximate Green function G^ρ

In this section, we are going to define the approximate Green function for the operator $\mathbf{L}$.

Given $y \in \Omega'$, where Ω' defined as before. Fix $B_\rho(y) = B(y,\rho) = \{x \in \Omega' : \varrho(x,y) < \rho\} \subset \Omega$, ρ small enough, where ϱ is the metric on Ω. Define

$$l: \ \phi \to \frac{1}{w(B_\rho)} \int_{B_\rho} \phi w, \ \phi \in H_0.$$

We claim that l is a continuous linear functional on H_0. In fact,

$$|l(\phi)| = |\frac{1}{w(B_\rho)} \int_{B_\rho} \phi w| \le \left(\frac{1}{w(B_\rho)} \int_{B_\rho} \phi^2 w \right)^{1/2}$$

For a ball B_{R_0} containing Ω', we get for ϕ supported in Ω',

$$\begin{aligned} |l(\phi)| &\le \left[\frac{w(B_{R_0})}{w(B_\rho)}\right]^{1/2} \left(\frac{1}{w(B_0)} \int_{B_{R_0}} \phi^2 w \right)^{1/2} \\ &\le c \left[\frac{w(B_{R_0})}{w(B_\rho)}\right]^{1/2} R_0 \left(\frac{1}{w(B_{R_0})} \int_{\Omega'} |X\phi|^2 w \right)^{1/2} \le C \frac{R_0}{w(B_\rho)^{1/2}} \|\phi\|_{H_0} \end{aligned}$$

Thus the claim follows.

We know

$$a_0(u,\phi) = \int < AXu, X\phi >$$

is bounded and a coercive bilinear form on H_0. By the Riesz representation theorem, there exists $G^\rho = G^{\rho,y} \in H_0$ such that $l(\phi) = \int < AXG^\rho, X\phi >$, that is,

$$\int < AXG^\rho, X\phi > = \frac{1}{w(B_\rho)} \int_{B_\rho} \phi w, \ \forall \phi \in H_0.$$

G^ρ is called the approximate Green function for $\mathbf{L}$. Now we shall study G^ρ.

Lemma 3.1 *$G^\rho \ge 0$ as an element of H_0, i.e., $\exists\ G_k^\rho \in Lip_0(\Omega')$ such that $G_k^\rho \ge 0$ and $G_k^\rho \to G^\rho$ in H_0 (Consequently, G^ρ will be nonnegative as a function).*

Proof: Pick $G^\rho_k \in Lip_0(\Omega')$, $\exists\, G^\rho_k \to G^\rho$ in H_0. Now we show that for some subsequence $\{|G^\rho_{k_j}|\}$, $|G_{k_j}| \to G_\rho$ in H_0. Since $|G^\rho_{k_j}| \geq 0$ for each k_j and lies in $Lip_0(\Omega')$, we will be done. Note that

$$X(|G^\rho_k|) = XG^\rho_k \; sgn\; G^\rho_k; \; if \; G^\rho_k \neq 0$$

and

$$X(|G^\rho_k|) = 0 \;\; if \; G^\rho_k = 0$$

holds almost everywhere. Now,

$$\int_{\Omega'} < AX(|G^\rho_k|), X(|G^\rho_k|) >= \int_{\Omega'} < AXG^\rho_k, XG^\rho_k >,$$

Since $\|G^\rho_k\|_{H_0} \to \|G^\rho\|_{H_0}$, $\{|G^\rho_k|\}$ is bounded in H_0 for all k. Thus $\exists\, |G^\rho_{k_j}| \to h$ in H_0 weakly. Since for any $\phi \in H_0$,

$$u \to \int_{\Omega'} < AX\phi, Xu >$$

is a bounded linear functional on H_0, we have

$$\int_{\Omega'} < AX(|G^\rho_{k_j}|), XG^\rho > \to \int_{\Omega'} < AXh, XG^\rho > \; .$$

Note

$$\begin{aligned} 0 < a_0(G^\rho, G^\rho) &= \frac{1}{w(B_\rho)} \int_{B_\rho} G^\rho w \\ &= \lim \frac{1}{w(B_\rho)} \int_{B_\rho} G^\rho_{k_j} w \leq \lim \frac{1}{w(B_\rho)} \int_{B_\rho} |G^\rho_{k_j}| w \\ &= \frac{1}{w(B_\rho)} \int_{B_\rho} hw = a_0(G^\rho, h) \end{aligned}$$

Thus $a_0(G^\rho, h) > 0$. Following the same proof in [CW], we have

$$\|G^\rho - |G^\rho_{k_j}|\|_{H_0} \to 0$$

and thus $|G^\rho_{k_j}| \to G^\rho$ in H_0.

¿From now on, when we write $G^\rho = \{G_k^\rho\}$, we always mean $G_k^\rho \geq 0$. We shall assume $\Omega' = B_R(x_0)$. We outline the suceeding steps as follows.

(i) Obtain an estimate for $w(\{x \in B_R(x_0) : G^\rho(x) > t\})$ for large t, which is independent of ρ, y.

(ii) Use (i) to estimate the size of $L^p_w(\Omega')$ norm for G^ρ for $1 < p < \sigma$.

(iii) Use (ii) and the mean value inequality for nonnegative solutions to estimate $ess\sup_B G^\rho$ for B away from y.

(iv) Estimate XG^ρ

We start with (i). Define,

$$\Psi_k = \left[\frac{1}{t} - \frac{1}{G_k^\rho}\right]^+ = \left(\frac{1}{t} - \frac{1}{G_k^\rho}\right)\chi_{\{G_k^\rho > t\}}$$

Thus $\Psi_k \in Lip_0(B)$ and

$$X\Psi_k = \frac{XG_k^\rho}{(G_k^\rho)^2}\chi_{\{G_k^\rho > t\}} \ \ a.e. \ for \ t \ fixed.$$

Now we claim that $\|\Psi_k\|_0$ is bounded in k. We first compute

$$\|\Psi_k\|_0^2 = \int < AX\Psi_k, X\Psi_k >= \int_{\{G_k^\rho > t\}} < AXG_k^\rho, XG_k^\rho > \frac{1}{(G_k^\rho)^4} \leq \frac{1}{t^4}\|G_k^\rho\|_0^2$$

It is easy to see that $\|\Psi_k\|_0$ is bounded in k for fixed $t > 0$. Therefore, $\exists\ \Psi_k \rightarrow \Psi \in H_0$ weakly, thus

$$a_0(\Psi_k, G^\rho) \rightarrow a_0(\Psi, G^\rho).$$

Note

$$a_0(\Psi_{k_j}, G_{k_j}^\rho) = \int_{\{G_k^\rho > t\}} < \frac{AXG_{k_j}^\rho}{G_{k_j}^\rho}, \frac{XG_{k_j}^\rho}{G_{k_j}^\rho} >$$

and

$$\lim_{k_j \rightarrow \infty} a_0(\Psi_{k_j}, G^\rho) = \lim_{k_j \rightarrow \infty} a_0(\Psi_{k_j}, G_{k_j}^\rho) = a_0(\Psi, G^\rho).$$

We thus obtain

$$\lim_{k_j \rightarrow \infty} \int_{\{G_{k_j} > t\}} < \frac{AXG_{k_j}^\rho}{G_{k_j}^\rho}, \frac{XG_{k_j}^\rho}{G_{k_j}^\rho} >= a_0(\Psi, G^\rho)$$

and

$$\frac{1}{w(B_\rho)}\int_{B_\rho}\Psi w = \lim \frac{1}{w(B_\rho)}\int_{B_\rho}\Psi_{k_j} w \le \frac{1}{t}$$

By the definition of G^ρ,

$$a_0(\Psi, G^\rho) = \frac{1}{w(B_\rho)}\int_{B_\rho}\Psi w.$$

Thus,

$$\limsup_{k_j}\int_{\{G_{k_j}>t\}}\frac{|XG^\rho_{k_j}|^2}{(G^\rho_{k_j})^2}w \le \frac{c}{t}. \tag{3.2}$$

Define $\phi_{k_j} = \log^+\left(\frac{G^\rho_{k_j}}{t}\right)$. We easily verify

$$\phi_{k_j} \in Lip_0(B) \text{ and } X\phi_{k_j} = \frac{XG^\rho_{k_j}}{G^\rho_{k_j}}\chi_{\{G^\rho_{k_j}>t\}}$$

So by (3.2),

$$\limsup_{k_j\to\infty}\int_B |X\phi_{k_j}|^2 w \le \frac{c}{t}. \tag{3.3}$$

By Sobolev's inequality (1.7) applied to (3.3), we see for $\sigma > 1$,

$$\limsup_{k_j\to\infty}\int_B |\phi_{k_j}|^{2\sigma} w \le c\left(\frac{R^2}{w(B)}\right)^\sigma \frac{1}{t^\sigma}w(B).$$

Inserting the definition of ϕ_k above we get,

$$\limsup_{k_j\to\infty}\int_B |\log^+\left(\frac{G^\rho_{k_j}}{t}\right)|^{2\sigma} w \le c\left[\frac{R^2}{w(B)}\right]^\sigma \frac{w(B)}{t^\sigma}$$

Restricting the integeration to $\{G^\rho_{k_j} > 2t\}$ we thus have,

$$(\log 2)^{2\sigma}\limsup_{k_j\to\infty} w\left(\{x\in B : G^\rho_{k_j}(x) > 2t\}\right) \le c\left(\frac{R^2}{w(B)}\right)^\sigma \frac{w(B)}{t^\sigma}$$

where c is independent of ρ, y and B, t. If we replace t by $t/2$,

$$\limsup_{k_j\to\infty} w(\{x\in B : G^\rho_{k_j}(x) > t\}) \le c\left(\frac{R^2}{w(B)}\right)^\sigma \frac{1}{t^\sigma}w(B)$$

We note that $G^\rho_k \rightharpoonup \tilde{G}^\rho$ in L^2_w, this follows that $G^\rho_{k_j} \to \tilde{G}^\rho$ in L^2_w. Picking a further subsequence (called $G^\rho_{k_j}$ again) such that $G^\rho_{k_j} \to \tilde{G}^\rho$ a.e., then

$$\chi_{\{G^\rho>t\}} \le \liminf_{k_j\to\infty}\chi_{\{\tilde{G}_{k_j}>t\}}$$

So,

$$\int_{\{\tilde{G}^\rho>t\}} w \le \int \liminf \chi_{\{G^\rho_{k_j}>t\}} w \le \lim \inf_{k_j\to\infty} \int_{\{G^\rho_{k_j}>t\}} w \le c\left(\frac{R^2}{w(B)}\right)^\sigma \frac{1}{t^\sigma} w(B)$$

Hence

$$w(\{x \in B : \tilde{G}^\rho(x) > t\}) \le c\left(\frac{R^2}{w(B)}\right)^\sigma \frac{1}{t^\sigma} w(B), t > 0. \tag{3.4}$$

We now use this distribution function estimate to get control $\|\tilde{G}^\rho\|_{L^p_w(B)}$ for $p < \sigma$.

Consider now the case $B = B_r(y) \subset \Omega$. We argue assuming $r \le r_0$ for some r_0. Let $G^\rho = G^{\rho,y,B}$. We restrict x so that $r/2 < \varrho(x,y) < 3/4r$, and $\rho < r/4$. Note that $B_{r/4}(x) \subset B\backslash B_\rho$ if $r/2 < \varrho(x,y) < 3r/4$. Furthermore, G^ρ is a solution in $B\backslash B_\rho$ since $\forall \phi \in Lip_0(B\backslash B_\rho)$, $\phi \equiv 0$ on B_ρ, and hence,

$$a_0(G^\rho, \phi) = \frac{1}{w(B_\rho)} \int_{B_\rho} \phi w = 0.$$

Recall $G^\rho \ge 0$, thus by the mean value inequality (2.2), and because G^ρ is a solution in $B\backslash B_\rho$,

$$\tilde{G}^\rho(x) \le C_p \left(\frac{1}{w(B(x,r/4))} \int_{B(x,r/4)} (\tilde{G}^\rho)^p w\right)^{1/p} \tag{3.5}$$

for $0 < p < \infty$. By applying (3.4), we can estimate as follows:

$$\begin{aligned}
\int_{B(x,r/4)} (\tilde{G}^\rho)^p w &= p\int_0^\infty w(\{B(x,r/4) : \tilde{G}^\rho > t\}) t^{p-1} dt \\
&\le cp \int_0^\infty \min\left\{\left[\frac{r^2}{w(B)}\right]^\sigma \frac{1}{t^\sigma} w(B), w(B)\right\} t^{p-1} dt \\
&\le cp \int_0^{\frac{r^2}{w(B)}} w(B) dt + \int_{\frac{r^2}{w(B)}}^\infty \left[\frac{r^2}{w(B)}\right]^\sigma w(B) t^{p-\sigma-1} dt \\
&= cw(B)\left[\frac{r^2}{w(B)}\right]^p + c\left[\frac{r^2}{w(B)}\right]^\sigma w(B)\left[\frac{r^2}{w(B)}\right]^{p-\sigma}
\end{aligned}$$

Thus

$$\int_{B(x,r/4)} (G^\rho)^p w \le C\left[\frac{r^2}{w(B)}\right]^p w(B), \ if\ 0 < p < \sigma. \tag{3.6}$$

Thus by (3.5) and (3.6) we have $\tilde{G}^\rho(x) \le C\frac{r^2}{w(B)}$ for $r/2 < \varrho(x,y) < 3r/4$, which is equivalent to

$$\tilde{G}^\rho(x) \le C\frac{\varrho(x,y)^2}{w(B(y,\varrho(x,y)))}$$

To sum up we have shown,

$$(3.7) \qquad \sup_{B_{3r/4}\setminus B_{r/2}} \tilde{G}^\rho(x) \le C\frac{r^2}{w(B_r)} \approx \frac{r^2}{w(B(y,\varrho(x,y)))} \; for \; \rho < r/4,$$

where C is independent of ρ, y, r.

What we really want to show is that

$$(3.8) \qquad ess \sup_{B_r\setminus B_{r/2}} \tilde{G}^\rho(x) \le C\frac{r^2}{w(B_r)}, \; \rho < r/4$$

Now consider $B^* = B_{4r/3}$ and let $(G^\rho)^*$ be the corresponding approximate Green function for B^*, then by (3.7), we have

$$\sup_{B_{3/4(4r/3)}\setminus B_{1/2(4r/3)}} (\tilde{G}^\rho)^* \le C\frac{(4r/3)^2}{w(B_{4/3r})} \le C\frac{r^2}{w(B_r)}$$

i.e.,

$$ess \sup_{B_r\setminus B_{2r/3}} (\tilde{G}^\rho)^* \le C\frac{r^2}{w(B_r)}$$

If we knew that $G^\rho \le (G^\rho)^*$ in B_r, then we would be done. So we need to show if $G^\rho, (G^\rho)^*$ are the approximate Green functions for two domains $\Omega \subset \Omega^*$ repectively, then $G^\rho \le (G^\rho)^*$ a.e. in Ω. We note that $(G^\rho)^* - G^\rho$ is a solution in Ω since for $\phi \in Lip_0(\Omega)$,

$$\begin{aligned} a_0((G^\rho)^* - G^\rho, \phi) &= a_0((G^\rho)^*, \phi) - a_0(G^\rho, \phi) \\ &= \frac{1}{w(B_\rho)}\int_{B_\rho} \phi w - \frac{1}{w(B_\rho)}\int_{B_\rho} \phi w = 0. \end{aligned}$$

We also note that $(G^\rho)^* - G^\rho \in H(\Omega)$ and is represented by a sequence $(G^\rho_k)^* - G^\rho_k$ and obviously $(G^\rho_k)^* - G^\rho_k \ge 0$ in a neighbourhood of $\partial\Omega$ (depending on k), then by the weak maximum principle $(G^\rho)^* - G^\rho \ge 0$ a.e. in Ω. Thus we have proved (3.8).

Lemma 3.9 *Let $B = B_R(x_0), y \in \frac{1}{2}B$, let G^ρ be the approximate Green function for the ball B with pole y. If $x \in B_{R/2}(y), 0 < \rho < \frac{\varrho(x,y)}{4}$, then*

$$\tilde{G}^\rho(x) \leq C \int_{\varrho(x,y)}^{R} \frac{t^2}{w(B_t(y))} \frac{dt}{t} \tag{3.10}$$

where C is independent of R, x_0, y, ρ, x.

The proof just follows the lines of the proof of lemma (3.5) in [CW].

Corollary 3.11 *For a.e. $y \in \frac{1}{2}B, B = B_R(x_0)$, where R is small enough, there exists c independent of ρ, but dependent on y, R, w such that if $\rho < \frac{\varrho(x,y)}{4}$, then*

$$\tilde{G}^\rho(x) \leq c \min\{\varrho(x,y)^{2-\alpha}, \varrho(x,y)^{-\frac{\sigma}{\alpha}}\}$$

where $\sigma = q/2$ for q as given in (1.2) and (1.7), $0 < \alpha = \alpha(y) \leq Q$ is dependent on y, and Q is the homogeneous dimension as in the proof of lemma (2.6).

Proof: First of all we note by the result in [NSW] that there exists $r_0 > 0$ such that for any $x \in \Omega$, there is a multiple $I = (i_1, \cdots, i_d)$ such that

$$B_I(x, \delta) \subset B(x, \delta) \subset B_I(x, c\delta)$$

for some $c > 1$ independent of x and $\delta \leq r_0$. If we select r_0 even smaller, the Sobolev inequality will be valid on such balls (see [L1]). Thus, for $\alpha = d(I)$ (see [NSW] or [L1] for the definition of $d(I)$), we know that

$$\begin{aligned}\tilde{G}^\rho(x) &\leq c \int_{\varrho(x,y)}^{R} \frac{t^2}{w(B_t(y))} \frac{dt}{t} \\ &\leq c \left(\sup_{t<R} \frac{t^\alpha}{w(B_t(y))} \right) \int_{\varrho(x,y)}^{R} t^{2-\alpha} \frac{dt}{t}\end{aligned}$$

c independent of ρ, x, y, R.

Performing the integration since $\alpha \geq d \geq 3$, we get

$$\tilde{G}^\rho(x) \leq C \sup_{t<R} \frac{t^\alpha}{w(B_t(y))} \varrho^{2-\alpha}(x,y)$$

We now use the symbol M_w for the Hardy-Littlewood function with respect to the weight w. Since $w \in A_2$, by a theorem of Calderón (see [Ca]), then

$$\|M_w\left(\frac{1}{w}\chi_{B_R}\right)\|_{L^2_w} \le c\|\frac{1}{w}\|_{L^2_w(B_R)} < \infty.$$

Thus by noting that $t^\alpha = |B_t(y)|$, we have

$$\begin{aligned}&\frac{t^\alpha}{w(B_t(y))} \le c'(y)\frac{1}{w(B_t(y))}\int_{B_t(y)} dx\\ &\le \frac{1}{w(B_t(y))}\int_{B_t(y)} \frac{1}{w}\cdot w dx \le M_w(\frac{1}{w}\chi_{B_R})(y) < \infty \ a.e.\end{aligned}$$

Thus

$$\tilde{G}^\rho(x) \le C(R,y)\varrho(x,y)^{2-\alpha}.$$

For $w \in A_2$ and $t < R$, we also have (by (1.2))

$$\frac{t}{R}\left[\frac{w(B_t(y))}{w(B_R(y))}\right]^{\frac{1}{2\sigma}} \le C\left[\frac{w(B_t(y))}{w(B_R(y))}\right]^{1/2},$$

which implies

$$\frac{t^2}{w(B_t(y))} \le C_R w(B_t(y))^{-\frac{1}{\sigma}}$$

Inserting this to (3.10), we get

$$\begin{aligned}&G^\rho(x) \le C_R\int_{\varrho(x,y)}^R \left[\frac{t^\alpha}{w(B_t(y))}\right]^{\frac{1}{\sigma}} t^{-\frac{\alpha}{\sigma}}\frac{dt}{t}\\ &\le C_R\left[\sup_{t<R}\left(\frac{t^\alpha}{w(B_t(y))}\right)\right]^{\frac{1}{\sigma}}\int_{\varrho(x,y)}^\infty t^{-\frac{\alpha}{\sigma}}\frac{dt}{t} \le C(R,y)\varrho(x,y)^{-\frac{\alpha}{\sigma}}\end{aligned}$$

Thus the proof of the corollary is complete.

4 Estimates for $X\tilde{G}^\rho$

The purpose of this section is to derive an estimate for $\|X\tilde{G}^\rho\|_{L^s_w(B)}$ which is uniform in ρ for $s < s_0$. We will prove the following

Lemma 4.1 *Let $B = B_R(x_0)$, G^ρ be the approximate Green function for the B with pole y. Then there is a number s_0 with $1 < s_0 < 2$ such that $\int_B |X\tilde{G^\rho}|^s w$ is bounded uniformly in ρ for each $s < s_0$ and a.e. $y \in \frac{1}{2}B$. The bounds depend on s, y, B, w.*

Before we prove this lemma, we need the following lemmas. Lemma (4.2) below is a Cacciopolli type of lemma.

Lemma 4.2 (Cacciopolli) *Let B and G^ρ be as above and let $B_r = B_I(y, r)$ for $r \leq R/2$. For $y \in \frac{1}{2}B$ and $\rho < cr$ with $c < 1/2$*

$$| \int_{B \setminus B_r} < AX\tilde{G^\rho}, X\tilde{G^\rho} > | \leq \frac{C}{r^2} \int_{B_r \setminus B_{r/2}} (\tilde{G^\rho})^2 w$$

Proof: By the existence of a cut off function relative to balls B_I (see [L1]), we can pick η such that $\eta \equiv 1$ outside B_r, $\eta \equiv 0$ in $B_{r/2}$ and $|X\eta| \leq \frac{C}{r}$. Thus $\phi_k = G_k^\rho \eta^2 \in Lip_0(B)$. We can easily see that $\{\phi_k\}$ is bounded for all k in H_0. Then $\exists\ \phi_k \rightarrow \phi \in H_0$ weakly, and thus

$$a_0(G_{k_j}^\rho, \phi_{k_j}) \rightarrow a_0(G^\rho, \phi_{k_j}) \rightarrow a_0(G^\rho, \phi)$$

Since $\phi_{k_j} = 0$ on B_ρ for $\rho < cr$ for some small enough c independent of y (notice $B_\rho(y) \subset B_I(y, r/2)$ when $\rho < cr$ for small c), thus

$$\delta_{k_j} = a_0(G_{k_j}^\rho, \phi_{k_j}) = \delta_{k_j} = \int_B < AXG_{k_j}^\rho, X\phi_{k_j} > \rightarrow 0$$

Dropping the subscripts,

$$\delta = \int_B < AXG^\rho, X\phi >= \int_B < AXG^\rho, X(\eta^2 G^\rho) >$$

$$= \int < AXG^\rho, XG^\rho > \eta^2 + \int < AXG^\rho, 2\eta(X\eta)G^\rho > .$$

Thus,

$$\int < AXG^\rho, XG^\rho > \eta^2 = \delta - 2\int < AXG^\rho, X\eta > \eta G^\rho$$

So we have

$$\int < AXG^{\rho}, XG^{\rho} > \eta^2 \leq |\delta| + \int | < AXG^{\rho}, X\eta > |\eta^2$$

$$\leq |\delta| + \epsilon \int | < AXG^{\rho}, XG^{\rho} > |\eta^2 + \frac{1}{\epsilon} \int < AX\eta, X\eta > (G^{\rho})^2$$

Thus

$$\int < AXG^{\rho}, XG^{\rho} > \eta^2 \leq \frac{C}{r^2} \int_{B_r \setminus B_{r/2}} (G^{\rho})^2 w + C|\delta|$$

Letting $k_j \to \infty$, and noticing that $\delta_{k_j} \to 0$,

$$\int_{B \setminus B_I(y,r/2)} |X\tilde{G}^{\rho}|^2 w \leq \frac{C}{r^2} \int_{B_I(y,r) \setminus B_I(y,r/2)} (\tilde{G}^{\rho})^2 w$$

with C independent of y, r, ρ.

Lemma 4.3 *Let B, r, and $\tilde{G}^{\rho}$ be as above, $y \in \frac{1}{2}B, B = B_R(x_0)$, then given any small $c_1 > 0$, there are c_2, c_3 independent of y, r, ρ such that if $\rho < c_2 \varrho(x,y)$, we have*

$$ess \sup_{c_1 r < \varrho(x,y) < r} \tilde{G}^{\rho}(x) \leq c_3 \int_{\varrho(x,y)}^{R} \frac{t^2}{w(B_t(y))} \frac{dt}{t}$$

where $r < R/2$.

The proof of lemma (4.3) is just a modification of that of lemma (3.9).

By lemma (4.3), we also have,

Corollary 4.4 *Let B, r, y, α, and $\tilde{G}^{\rho}$ be as before, then for any $c_1 > 0$ small, $\exists\ c_2 > 0$ such that for $\rho < c_2 \varrho(x,y)$,*

$$(i)\ ess \sup_{c_1 r \leq \varrho(x,y) \leq r} \tilde{G}^{\rho}(x) \leq C \min\{\varrho(x,y)^{2-\alpha}, \varrho(x,y)^{-\frac{\alpha}{\sigma}}\}$$

$$(ii)\ \int_{B \setminus B_I(y,r)} < AX\tilde{G}^{\rho}, X\tilde{G}^{\rho} > \leq Cr^{-\frac{\alpha}{\sigma}}, \quad C = C(B, y, w).$$

Proof: The proof of (i) is just a modification of that of corollary (3.11). To show (ii), we note that there is c_1 small enough which is independent of y, r such that $B_{c_1 r}(y) \subset B_I(y, r/2)$. For this c_1, choose c_2 as in (i). Then we get,

$$\begin{aligned}
&\int_{B\setminus B_I(y,r)} < AX\tilde{G}^\rho, X\tilde{G}^\rho > \le \frac{C}{r^2}\int_{B_I(y,r)\setminus B_I(y,r/2)} (\tilde{G}^\rho)^2 \\
&\le \frac{C}{r^2}\int_{B_I(y,r)\setminus B(y,c_1 r)} (\tilde{G}^\rho)^2 w \le \frac{C}{r^2}\min\left\{\varrho(x,y)^{2-\alpha}, \varrho(x,y)^{-\frac{\alpha}{\sigma}}\right\}^2 w(B_r) \\
&= Cr^{-\frac{\alpha}{\sigma}}\min\left\{r^{2-\alpha+\frac{\alpha}{\sigma}}, r^{-(2-\alpha+\frac{\alpha}{\sigma})}\right\}\frac{w(B_r)}{r^\alpha}.
\end{aligned}$$

Since $w \in A_2$, we have for a.e. y,

$$w(B_r)r^{-\alpha} \le Mw(y) \le C(y, B, w) < \infty.$$

Thus the expression above is at most $r^{-\frac{\alpha}{\sigma}}C(y, B, w)$, which for a.e. y is finite, for the fixed r.

We are now ready to prove lemma (4.1).

Proof of lemma (4.1): For $B = B_R(x_0)$, $t > 0$,

$$\begin{aligned}
&w(\{x \in B : |X\tilde{G}^\rho| > t\}) \le w(\{x \in B\setminus B_I(y,r) : |X\tilde{G}^\rho| > t\}) + w(B_I(y,r)) \\
&\le \frac{1}{t^2}\int_{B\setminus B_I(y,r)} |X\tilde{G}^\rho|^2 w + w(B_r).
\end{aligned}$$

For $r < R/2$, we may use corollary (4.4) (ii) to get,

$$\frac{1}{t^2}\int_{B\setminus B_I(y,r)} |X\tilde{G}^\rho|^2 w \le \frac{c}{t^2} r^{-\frac{\alpha}{\sigma}}$$

Thus for a.e. $y \in \frac{1}{2}B$, $\exists\ c = c_{y,R,w}$ such that

$$w(\{x \in B : |X\tilde{G}^\rho| > t\}) \le c\left(t^{-2}r^{-\frac{\alpha}{\sigma}} + w(B_r)\right), r \le R/2$$

But $w(B_r) \le cr^\alpha$, thus

$$w(\{x \in B : |X\tilde{G}^\rho| > t\}) \le C(t^{-2}r^{-\frac{\alpha}{\sigma}} + r^\alpha),\ r \le R/2$$

Choose $r = t^{-\frac{2\sigma}{(\sigma+1)\alpha}}$, which is $\le R/2$ if $t > (R/2)^{-\frac{\alpha(\sigma+1)}{\sigma}}$. Using the choice of r in the inequality above we get

$$w(\{x \in B : |X\tilde{G}^\rho| > t\}) \le ct^{-\frac{2\sigma}{\sigma+1}} = ct^{-s_0},\ s_0 = \frac{2\sigma}{\sigma+1}$$

Furthermore we also have the trivial estimate,

$$w(\{x \in B : |X\tilde{G}^{\rho}| > t\}) \leq w(B), \forall t > 0.$$

Thus,

$$\int_B |X\tilde{G}^{\rho}|^s w = s\int_0^{\infty} t^{s-1} w(\{|X\tilde{G}^{\rho}| > t\})dt$$
$$= C\int_0^{(R/2)^{-\frac{\alpha(\sigma+1)}{\sigma}}} w(B)t^{s-1}dt + c\int_{(R/2)^{-\frac{\alpha(\sigma+1)}{\sigma}}}^{\infty} t^{-s_0+s-1}w(B)dt < \infty.$$

5 Existence of the Green function G

In this section, we are going to prove (ii) and (iv) of theorem (1.3). In section (4) we showed that $\exists s_0, 1 < s_0 < 2$, such that for a.e. $y \in \frac{1}{2}B$, $X\tilde{G}^{\rho} \in L^s_w$ uniformly in ρ for $s < s_0$. We also showed (see (3.4)),

$$w(x \in B : \tilde{G}^{\rho} > \lambda) \leq c\min\left\{\left[\frac{R^2}{w(B)}\right]^{\sigma}\frac{1}{\lambda^{\sigma}}, 1\right\} w(B)$$

with c independent of ρ, y, and t.

Consequently, $\tilde{G}^{\rho} \in L^t_w$ uniformly in ρ and y for $t < \sigma$. Since G^{ρ}_k is supported in B and $G^{\rho}_k \to G^{\rho}$ in $L^{2\sigma}_w$ and $X\tilde{G}^{\rho}_k \to X\tilde{G}^{\rho}$ in L^2_w, it follows that for a.e. $y \in \frac{1}{2}B$, $\tilde{G}^{\rho} \in X_{t,s}$ uniformly in ρ for $1 < t < \sigma$, $1 < s < s_0$, where we recall that $X = X_{t,s}$ =closure of $Lip_0(B)$ with respect to

$$\|f\|_{L^t_w} + \|Xf\|_{L^s_w}.$$

We want to show that $\exists\ G = G^y \in X_{t,s}$ such that G^y satisfies the properties stated in theorem (1.3). Since $\|G^{\rho}\|_{X_{t,s}}$ is bounded uniformly in ρ, $\exists\ G^{\rho_j} \to G = G^y$ in $X_{t,s}$ weakly. By using a diagonal procedure, we may assume the same sequence G^{ρ_j} works for all $t < \sigma$, $s < s_0$.

By definition,

$$a_0(G^{\rho}, \phi) = \frac{1}{w(B_{\rho}(y))}\int_{B_{\rho}(y)} \phi w,\ for\ \phi \in Lip_0(B),\ y\ fixed.$$

The right side of the above equality goes to $\phi(y)$ as $\rho \to 0$. We claim that the left side goes to $\int < AXG^y, X\phi >$. If so, then we get

$$\int < AXG^y, X\phi >= \phi(y) \ for \ \phi \in Lip_0(B).$$

To show the claim, fix $\phi \in Lip_0(B)$. Let

$$l(h) = \int < AXh, X\phi >, \ h \in X_{t,s}$$

then

$$\begin{aligned}|l(h)| &\leq \int | < AXh, X\phi > | \leq c \int |Xh| \cdot |X\phi| w \\ &\leq c \int_B |Xh| \cdot \|X\phi\|_{L^\infty} \leq c \|X\phi\|_{L^\infty} \left(\int_B |Xh|^s w \right)^{1/s} \cdot w(B)^{1/s'} \\ &\leq c_{w,B} \|X\phi\|_{L^\infty} \cdot \|h\|_{X_{t,s}}\end{aligned}$$

So l is a continuous linear functional on $X_{t,s}$. Since $G^{\rho_j} \to G^y$ weakly in $X_{t,s}$, the claim follows. Thus this shows (ii) of theorem (1.3). Since $\|\tilde{G}^\rho\|_{L^t_w}$ is uniformly bounded in both y and ρ, it shows the uniformity in y of the L^t_w norm of G by using Fatou's lemma. In section (6), we shall prove the uniformity in y for the norm of XG.

Now we want to show $G^{\rho_j} \to G^y$ pointwise a.e. for some subsequence. Let $B_r = B_r(y)$, $r < R/2$. By lemma (4.2) and corollary (4.4), we have shown

$$\|\tilde{G}^\rho\|_{L^2_w(B \backslash B_r)} + \|X\tilde{G}^\rho\|_{L^2_w(B \backslash B_r)} \leq c_r$$

and $G^{\rho_j} \to G$ in $L^2_w(B \backslash B_r)$. Since G^ρ_k supported in B, $G^\rho_k \to \tilde{G}^\rho$ in L^2_w, and $X\tilde{G}^\rho \to X\tilde{G}$ in L^2_w, by Rellich's lemma (2.6), it follows that $\exists$ a subsequence $G^{\rho_{j_k}}$ convergent in $L^2_w(B \backslash B_r)$ (the subsequence depends on r). We now show the limit must be G by using the similar argument in [CW].

Given a bounded function ϕ, let $l(g) = \int g\phi w$. Furthermore,

$$|l(g)| \leq \|\phi\|_{L^\infty} \cdot \|g\|_{L^1_w} \leq \|\phi\|_{L^\infty} \cdot w(B)^{1/t'} \|g\|_{L^t_w} \leq c\|g\|_{X_{t,s}}$$

Thus l is a continuous linear functional on X. Thus

$$\int G^{\rho_j}\phi w \to \int G\phi w$$

Assume that $G^{\#}$ is the limit in $L^2_w(B\backslash B_r)$ of $G^{\rho_{j_k}}$. Then for any bounded ϕ supported in $B\backslash B_r$,

$$\int G^{\rho_{j_k}}\phi w \to \int G^{\#}\phi w$$

Thus, for ϕ picked as above,

$$\int G\phi w = \int G^{\#}\phi w,$$

and it follows $G^{\#} = G$ a.e. in $B\backslash B_r$.

Summarizing, we have shown now that $G^{\rho_{j_k}} \to G$ in $L^2_w(B\backslash B_r)$ for a subsequence $\{\rho_{j_k}\}$ which depends on r, $\rho_{j_k} \to 0$. Hence there is a further subsequence, again denoted by $\{\rho_{j_k}\}$ such that $G^{\rho_{j_k}} \to G$ pointwise a.e. in $B\backslash B_r$. Letting $r \to 0$ through a sequence and using repeated subsequence and a diagonal process, we have a fixed $\rho_{j_k} \to 0$ such that $G^{\rho_{j_k}} \to G$ a.e. in B.

We have proved in section (4) that

$$ess \sup_{r/2 \le \varrho(x,y) \le r} \tilde{G}^{\rho}(x) \le c \int_{\varrho(x,y)}^{R} \frac{t^2}{w(B_t(y))} \frac{dt}{t}.$$

For $\rho < \frac{\varrho(x,y)}{4}$, taking limits $\rho_{j_k} \to 0$, we obtain

$$ess \sup_{r/2 \le \varrho(x,y) \le r} \tilde{G}^{y}(x) \le c \int_{\varrho(x,y)}^{R} \frac{t^2}{w(B_t(y))} \frac{dt}{t}$$

We have also shown that $G^{\rho_{j_k}} \to G$ weakly in X, and strongly in $L^2_w(B\backslash B_r)$ for any $r > 0$, and $G^{\rho_{j_k}} \to G$ pointwise a.e..

We now prove,

$$ess \inf_{r/2 \le \varrho(x,y) \le r} G(x,y) \ge c \int_{r}^{R} \frac{t^2}{w(B_t(y))} \frac{dt}{t}$$

where $r < R/4$ and c independent of B, y, r.

Let G^ρ be the approximate Green function for $B_{c_4 r}(y)$ and $0 < c_0 < c_1 < c_2 < c_3 < c_4$ independent of y, r be chosen such that

$$B_{c_0 r}(y) \subset B_I(y, c_1 r) \subset B_{r/2}(y) \subset B_{3/2r}(y) \subset B_I(y, c_2 r) \subset B_{c_3 r}(y) \subset B_{c_4 r}(y)$$

For the possibility of the above containment of different metric balls, we refer the reader to [NSW]. Then as shown in lemma (4.2) by taking a proper cut off function

$$\int_{B_{c_4 r}\setminus B_I(y,c_2 r)} < AX\tilde{G}^\rho, X\tilde{G}^\rho > \le \frac{c}{r^2}\int_{B_I(y,c_2 r)\setminus B_I(y,c_1 r)} (\tilde{G}^\rho)^2 w$$
$$\le \frac{c}{r^2}\left(ess \sup_{B_I(y,c_2 r)\setminus B_I(y,c_1 r)} \tilde{G}^\rho\right)^2 w(B_r) \le \frac{c}{r^2}\left(ess \sup_{B_{c_3 r}(y)\setminus B_{c_0 r}(y)} \tilde{G}^\rho\right)^2 w(B_r)\ .$$

Since G^ρ is a nonnegative solution in $B_{c_4 r}(y)\setminus B_\rho$ if $\rho < c_5 r$ for some small c_5 independent of y and r, Harnack's inequality holds, that is

$$ess \sup_{B_{c_3 r}(y)\setminus B_{c_0 r}(y)} \tilde{G}^\rho \le c\ ess \inf_{B_{c_3 r}(y)\setminus B_{c_0 r}(y)} \tilde{G}^\rho.$$

Thus if $\rho < c_5 r$,

$$\left(ess \inf_{B_{3/2r}(y)\setminus B_{r/2}(y)} \tilde{G}^\rho\right)^2 \ge \left(ess \inf_{B_I(y,c_2 r)\setminus B_I(y,c_1 r)} \tilde{G}^\rho\right)^2$$
$$\ge \left(ess \inf_{B_{c_3 r}(y)\setminus B_{c_0 r}(y)} \tilde{G}^\rho\right)^2 \ge c\frac{r^2}{w(B_r)}\int_{B_{c_4 r}(y)\setminus B_I(y,c_2 r)} < AX\tilde{G}^\rho, X\tilde{G}^\rho >$$

Now pick ϕ with $\phi \equiv 1$ on $B_I(y, c_2 r)$, $supp\{\phi\} \subset B_{c_4 r}(y), |X\phi| \le \frac{c}{r}, B_\rho \subset B_I(y, c_2 r)$. With this choice of ϕ,

$$1 = \frac{1}{w(B_\rho)}\int_{B_\rho} \phi w = a_0(G^\rho, \phi).$$

Thus,

$$1 = \int < AX\tilde{G}^\rho, X\phi > \le \left(\int_{supp\{X\phi\}} < AX\tilde{G}^\rho, X\tilde{G}^\rho >\right)^{1/2} \cdot \left(\int < AX\phi, X\phi >\right)^{1/2}$$
$$\le c\left(\int_{B_{c_4 r}\setminus B_I(y,c_2 r)} < AX\tilde{G}^\rho, X\tilde{G}^\rho >\right)^{1/2} \cdot \left(\frac{c}{r^2} w(B_r)\right)^{1/2}$$

Thus

$$ess \inf_{B_{3/2r}(y)\backslash B_{r/2}(y)} \tilde{G}^{\rho} \geq c\frac{r^2}{w(B_r)},\ \rho < c_5 r,$$

where $\tilde{G}^{\rho} = \tilde{G}^{\rho}_{c_4 r}$ is the approximate Green function for the ball $B = B_{c_4 r}(y)$ with pole y. By lemma (2.5), $\exists$ sequence $G^{\rho}_{c_4 r,k} \geq c\frac{r^2}{w(B_r)}$ near ∂B_r. Now near ∂B_r, $G^{\rho}_{r,k} = 0$. Thus near ∂B_r,

$$G^{\rho}_{c_4 r,k}(y) - G^{\rho}_{r,k} \geq c\frac{r^2}{w(B_r)}$$

We also note that $G^{\rho}_{c_4 r} - G^{\rho}_{r}$ is a solution in B_r, so by the maximum principle,

$$\tilde{G}^{\rho}_{c_4 r} - \tilde{G}^{\rho}_{r} \geq c\frac{r^2}{w(B_r)} \ a.e.\ in\ B_r$$

Now assume for simplicity that $B = B_R(y),\ if\ r < \frac{R}{c_4}$, then choose positive m such that $rc_4^m \leq R < rc_4^{m+1}$. In $B_{c_4 r}$,

$$\tilde{G}^{\rho}_{R} \geq G^{\tilde{\rho}}_{rc_4^m} = G^{\tilde{\rho}}_{c_4 r} + \sum_{j=1}^{m-1}\left[G^{\tilde{\rho}}_{rc_4^{j+1}} - G^{\tilde{\rho}}_{rc_4^j}\right]$$

Thus in $B_{3r/2}\backslash B_{r/2}$ a.e.

$$\tilde{G}^{\rho}_{R} \geq c\frac{r^2}{w(B_r)} + c\sum_{j=1}^{m-1}\frac{(rc_4^j)^2}{w(B_{c_4^j r})}$$

Since w is a doubling weight, the expression on the right side above is

$$\geq c'\int_r^R \frac{t^2}{w(B_t)}\frac{dt}{t}.$$

Letting $\rho \to 0$,

$$ess \inf_{B_r\backslash B_{r/2}} G(x,y) \geq c\int_r^R \frac{t^2}{w(B_t)}\frac{dt}{t}$$

for $r < \frac{R}{c_4}$. In case B_R is not centered at y, we note that $y \in \frac{1}{2}B$ implies $B \supset B_{R/2}(y) = B'$ and thus we apply the estimate above to B', $r < \frac{R}{2c_4}$ to get,

$$ess \inf_{B_r\backslash B_{r/2}} G(x,y) \geq ess \inf_{B_r\backslash B_{r/2}} G_{B_{r/2}}(x,y) \geq c\int_r^R \frac{t^2}{w(B_t(y))}\frac{dt}{t}.$$

6 Proof of the representation theorem and uniformity of the norms for G and XG

Now we prove Theorem (1.6). The proof is standard (see also [CW]). Define

$$l(\phi) = \int_B f\tilde{\phi}dx, \quad \tilde{\phi} \in H_0,$$

then

$$\begin{aligned} |l(\phi)| &\leq \left(\int_B (f/w)^{(2\sigma)'}w\right)^{\frac{1}{(2\sigma)'}} \left(\int_B \tilde{\phi}^{2\sigma}w\right)^{\frac{1}{2\sigma}} \\ &\leq c_{w,B}\left(\int_B (f/w)^{(2\sigma)'}w\right)^{\frac{1}{(2\sigma)'}} \left(\int_B |X\tilde{\phi}|^2 w\right)^{1/2} \leq c_{w,B}\|\phi\|_{H_0} \end{aligned}$$

Thus l is a continuous linear functional on H_0. By Lax-Milgram's theorem, $\exists!\ u \in H_0$ such that

$$a_0(u,\phi) = \int_B f\tilde{\phi}$$

i.e.

$$Lu = f \qquad and \qquad \|u\|_{H_0} \leq c_{w,B}\|f/w\|_{L_w^{(2\sigma)'}(B)}$$

Now selecting $\phi = G^\rho$ (with pole y), we get

$$a_0(u, G^\rho) = \int_B f\tilde{G}^\rho dx\ .$$

Letting $\rho \to 0$,

$$a_0(u, G^\rho) = \frac{1}{w(B_\rho)}\int_{B_\rho} uw \to \tilde{u}(y)\ a.e.$$

Further,

$$|\int_B fg| \leq \left(\int (f/w)^{t'}w\right)^{1/t'} \left(\int_B |g|^t w\right)^{1/t} \leq \left(\int_B \left(\frac{|f|}{w}\right)^{t'} w\right)^{1/t'} \|g\|_X$$

Thus under the hypothesis of Theorem (1.6),

$$g \to \int_B fg \quad is\ a\ continuous\ linear\ functional\ on\ X_{t,s}$$

for $t < \sigma$. Since $\tilde{G^{\rho_j}} \to G(x,y)$ weakly in X for $y \in \frac{1}{2}B$ a.e. for $t < \sigma,\ s < s_0$,

$$\int_B f\tilde{G^{\rho_j}} \to \int_B f(x)G(x,y)dx \quad for\ y \in \frac{1}{2}B\ a.e.$$

and hence,

$$\tilde{u}(y) = \int_B f(x)G(x,y)dx\ for\ y \in \frac{1}{2}B\ a.e.$$

This proves the first part of theorem (1.6).

The proof of the second part is similar. Indeed,

$$|\int_B < F, X\phi > | \le \left(\int_B \left(\frac{|F|}{w}\right)^2 w\right)^{1/2} \cdot \left(\int_B |X\phi|^2 w\right)^{1/2} \le \left(\int_B \left(\frac{|F|}{w}\right)^2 w\right)^{1/2} \cdot \|\phi\|_{H_0},$$

thus there exists a unique $u \in H_0$ such that

$$a_0(u,\phi) = \int_B < F, X\tilde{\phi} >, \quad \phi \in H_0\ ,$$

and also

$$\|u\|_{H_o} \le \left(\int_B \left(\frac{|F|}{w}\right)^2 w\right)^{1/2}.$$

Chosing $\phi = G^\rho$ and observing that

$$|\int_B < F, Xg > | \le \left(\int_B \left(\frac{|F|}{w}\right)^{s'} w\right)^{1/s'} \cdot \left(\int_B |Xg|^s w\right)^{1/s} \le \left(\int_B \left(\frac{|F|}{w}\right)^{s'} w\right)^{1/s'} \|g\|_X\ ,$$

the result will follow as in the first part if $s < s_0$.

Now we show

$$\|G\|_{L^t_w} + \|XG\|_{L^s_w} \le C < \infty$$

uniformly in $y \in \frac{1}{2}B$ for $t < \sigma,\ s < s_0$, which will prove (i) of theorem (1.3). The uniformity of the first term follows from (3.4) and Fatou's lemma. But we now give another proof which applies to the uniformity for both $\|G\|$ and $\|XG\|$.

Let u be the Lax-Milgram solution to **Lu**=f, $u \in H_0$, $u = \{u_j\}$, $u_j \in Lip_0(B)$. Let $t < \sigma$ and $k = \left(\int_B \left(\frac{|f|}{w}\right)^{t'} w\right)^{1/t'}$. For $\beta \ge 1$, $k < M < \infty$, define

$$H_M(\tau) = \tau^\beta - k^\beta,\ k \le \tau \le M,$$

and

$$H_M(\tau) = \beta M^{\beta-1}(\tau - M) + M^\beta - k^\beta,\ \tau > M.$$

Let $\Psi_j = u_j^+ + k$. For fixed M, define

$$\phi_j(x) = G(\Psi_j(x)) = \int_k^{\Psi_j} H'_M(\tau)^2 d\tau$$

It is easy to see that $\{\phi_j\} \in Lip_0(B)$ and $\|\phi_j\|_0$ is bounded in j. Thus $\exists$ a subsequence, denoted by ϕ_j again, such that $\phi_j \to \phi \in H_0$ weakly. Thus

$$\lim a_0(u_j, \phi_j) = \lim a_0(u, \phi_j) = a_0(u, \phi)$$

Note,

$$a_0(u_j, \phi_j) = \int < AXu_j, X\phi_j > \ and\ a_0(u, \phi_j) = \int f\phi_j$$

Thus,

$$\int_B < AXu_j, X\phi_j >= \int_B f\phi_j + \delta_j,\ \delta_j \to 0 \tag{5.6.1}$$

By (5.6.1),

$$\begin{aligned}\int |X\Psi_j|^2 G'(\Psi_j)w &\le C\int_B < AX\Psi_j, X\Psi_j > G'(\Psi_j \\ &\le \int_B |f|\phi_j + |\delta_j| \le \frac{1}{k}\int_B |f|\Psi_j^2 G'(\Psi_j) + |\delta_j|\ .\end{aligned}$$

Thus,

$$\int |XH_M(\Psi_j)|^2 w \le \frac{1}{k}\int |f| \cdot |H'_M(\Psi_j)\Psi_j|^2 + |\delta_j|$$

By Sobolev's inequality (1.7) (noticing $H_M(\Psi_j) \in Lip_0(B)$)

$$\begin{aligned}\left(\int_B |H_M(\Psi_j)|^{2\sigma} w\right)^{\frac{1}{2\sigma}} &\le \frac{1}{k}\int_B |f| \cdot |H'_M(\Psi_j)\Psi_j|^2 + |\delta_j| \\ &\le c/k\left(\int_B \left(\frac{|f|}{w}\right)^{t'} w\right)^{1/t'} \left(\int_B |H'_M(\Psi_j)\Psi_j|^{2t}\right)^{1/t} + c|\delta_j|\end{aligned}$$

Note $\Psi_j = u_j + k \to \tilde{u}^+ + k$ a.e. for a subsequence. Dropping subscripts and setting $\Psi = \tilde{u}^+ + k$, we have

$$\left(\int_B |H_M(\Psi)|^{2\sigma} w\right)^{\frac{1}{2\sigma}} \le c\left(\int_B |H'_M(\Psi)\Psi|^{2t}\right)^{\frac{1}{2t}} \tag{5.6.2}$$

Next we observe,

$$(\tau^\beta - k^\beta)\chi_{(k,M)}(\tau) \le H_M(\tau) \text{ and } H'_M(\tau) \le \beta\tau^{\beta-1},$$

where $\chi_{(k,M)}$ is the characteristic function.

Inserting these inequalities into (5.6.2) and letting $M \to \infty$ we get,

$$\left(\int_B (\Psi^\beta - k^\beta)^{2\sigma} w\right)^{\frac{1}{2\sigma}} \le c\beta \left(\int_B \Psi^{2t\beta} w\right)^{\frac{1}{2t}} \tag{5.6.3}$$

Since $\Psi \ge k$ and $\beta \ge 1$,

$$k^\beta \le c\left(\int_B \Psi^{2t\beta} w\right)^{\frac{1}{2t}} \le c\beta \left(\int_B \Psi^{2t\beta} w\right)^{\frac{1}{2t}} \tag{5.6.4}$$

Hence by Minkowski inequality we get by combining (5.6.3) and (5.6.4),

$$\left(\int_B \Psi^{2\sigma\beta} w\right)^{\frac{1}{2\sigma\beta}} \le (c\beta)^{\frac{1}{\beta}} \left(\int_B \Psi^{2t\beta}\right)^{\frac{1}{2t\beta}}. \tag{5.6.5}$$

For $t < \sigma$,

$$\left(\int_B \Psi^{2t} w\right)^{\frac{1}{2t}} \le c\left(\int_B \Psi^{2\sigma} w\right)^{\frac{1}{2\sigma}} \le c\left(\int_B |\tilde{u}|^{2\sigma} w\right)^{\frac{1}{2\sigma}} + ck.$$

Since $\tilde{u}$ has compact support, by Sobolev's inequality (1.7), the expression on the right above is at most,

$$c\left(\int_B |X\tilde{u}|^2 w\right)^{1/2} + ck \le c\|u\|_0 + ck \le ck$$

We now apply an iteration argument to the inequality above with the starting choice $\beta = 1$ to get $\|\Psi\|_{L^\infty(B)} \le ck$. Thus $\|\tilde{u}^+\|_{L^\infty(B)} \le ck$. A similar argument also works for $\tilde{u}^-$. Applying

$$\tilde{u} = \int_B f(x)G(x,y)dy, \ a.e\ y \in \frac{1}{2}B,$$

and

$$\left(\int_B \left(\frac{|f|}{w}\right)^{t'} w\right)^{1/t'} \le k$$

we obtain

$$\int_B G(x,y)^t w dx \leq C, \ \textit{with } C \textit{ independent of } y.$$

The argument for the uniformity in y of the size of the norm $\|X_x G(x,y)\|_{L_w^s}$ is similar. In this case, u is the Lax-Milgram solution to $Lu = X^*F$, i.e., $u \in H_0$ and

$$a_0(u,\phi) = \int < F, X\tilde{\phi} > \quad for\ \phi \in H_0.$$

We assume now that $s < s_0$ and $\frac{|F|}{w} \in L_w^{s'}(B)$. Then by theorem (1.6),

$$\tilde{u} = \int_B < F(x), XG(x,y) > dx, \ \ for\ a.e.\ y \in \frac{1}{2}B$$

Use the same test function ϕ_j as before except that now

$$k = \left(\int_B \left(\frac{|F|}{w}\right)^{s'} w\right)^{1/s'}.$$

The analogue of (5.6.1) is

$$\int_B < AXu_j, X\phi_j > = \int_B < F, X\phi_j > +\delta_j = \int_B < F, G'(\Psi_j)X\Psi_j > +\delta_j.$$

Thus, for $\epsilon > 0$,

$$\int_B |X\Psi_j|^2 G'(\Psi_j)w \leq \epsilon \int_B |X\Psi_j|^2 G'(\Phi_j)w + \frac{1}{\epsilon}\int_B \left(\frac{|F|}{w}\right)^2 G'(\Psi_j)w + |\delta_j|.$$

Selecting $\epsilon = 1/2$ and using the fact that $\Psi_j \geq k$, we get

$$\int_B |X(H_M(\Psi_j))|^2 w \leq 4/k^2 \int_B \left(\frac{|F|}{w}\right)^2 (H'_M(\Psi_j)\Psi_j)^2 w + 2|\delta_j|$$

By Hőlder's inequality with exponent $s'/2$ and $s'/(s'-2)$, the expression on the right above is bounded by

$$4\left[\int_B (H'_M(\Psi_j)\Psi_j)^{\frac{2s'}{s'-2}} w\right]^{\frac{s'-2}{s'}} + 2\delta_j.$$

We now use Sobolev's inequality and let $j \to \infty$ and $M \to \infty$ as before to get

$$\left(\int_B (\Psi^\beta - k^\beta)^{2\sigma} w\right)^{1/2\sigma} \le c\beta \left(\int_B \Psi^{\frac{2s'\beta}{s'-2}} w\right)^{(s'-2)/2s'}.$$

Thus,

$$\left(\int_B \Psi^{2\sigma\beta} w\right)^{\frac{1}{2\sigma\beta}} \le (c\beta)^{1/\beta} \left(\int_B \Psi^{\frac{2s'\beta}{s'-2}} w\right)^{\frac{s'-2}{2s'\beta}}.$$

When $\beta = 1$ and $s'/(s'-2) < \sigma$, the expression on the right above is bounded by ck. Thus, an iteration argument applied to the inequality above starting with $\beta = 1$ shows for $s < s_0$, i.e., $s'/(s' - 2') < \sigma$

$$\int_B |X_x G(x,y)|^s w(x)dx \le c \quad \textit{independent of } y.$$

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