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Some equivalent definitions of high order Sobolev spaces on stratified groups and generalizations to metric spaces

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Abstract. Recently, in the article [LW], the authors use the notion of polynomials in metric spaces $(\mathcal{S}, \rho, \mu)$ of homogeneous type (in the sense of Coifman-Weiss) to prove a relationship between high order Poincaré inequalities and representation formulas involving fractional integrals of high order, assuming only that μ is a doubling measure and that geodesics exist. Motivated by this and by recent work in [H], [FHK], [KS] and [FLW] about first order Sobolev spaces in metric spaces, we define Sobolev spaces of high order in such metric spaces $(\mathcal{S}, \rho, \mu)$. We prove that several definitions are equivalent if functions of polynomial type exist. In the case of stratified groups, where polynomials do exist, we show that our spaces are equivalent to the Sobolev spaces defined by Folland and Stein in [FS]. Our results also give some alternate definitions of Sobolev spaces in the classical Euclidean case.

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1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be an open set and m be a positive integer. The classical Sobolev space $W^{m,p}(\Omega)$ is defined to be the collection of functions $f \in L^p(\Omega)$ whose distributional derivatives $\nabla^k f$ are in $L^p(\Omega)$ for all integers k with $1 \le k \le m$.

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This space is equipped with the norm

$$||f||_{W^{m,p}(\Omega)} = ||f||_{L^p(\Omega)} + \sum_{1\le k\le m} ||\nabla^k f||_{L^p(\Omega)}.$$

Some of the most interesting generalizations of the classical Sobolev spaces are the Folland–Stein spaces associated with the left-invariant vector fields $X_1, \cdots,$ X_l over a stratified group $\mathbb{G}$. Let $\Omega \subset \mathbb{G}$ be an open set. The space $W^{m,p}(\Omega)$ is defined to be the set of functions $f \in L^p(\Omega)$ with distributional derivatives $X^\alpha f \in L^p(\Omega)$ for each $|\alpha| \le m$, where $X^\alpha = X_{i_1}^{\alpha_1} \cdots X_{i_l}^{\alpha_l}$, $1 \le i_j \le l$, $1 \le j \le l$, the $\alpha_i' s$ are nonnegative integers, and $|\alpha| = \alpha_1 + \cdots + \alpha_l$. Here, we say that the distributional derivative $X^\alpha f$ exists and equals a locally integrable function g_α in Ω if for every $\phi \in C_0^\infty(\Omega)$,

$$\int_\Omega f X_{i_l}^{\alpha_l} \cdots X_{i_1}^{\alpha_1} \phi \, dx = (-1)^{|\alpha|} \int_\Omega g_\alpha \phi \, dx.$$

The norm on $W^{m,p}(\Omega)$ is defined to be

$$||f||_{W^{m,p}(\Omega)} = ||f||_{L^p(\Omega)} + \sum_{1\le|\alpha|\le m} ||X^\alpha f||_{L^p(\Omega)}.$$

In [LW], the authors studied the relationship between L^1 to L^1 Poincaré inequalities (i.e., (2.1) when $q = 1$) and integral representation formulas involving polynomials and high order vector field derivatives. Motivated by this and by interesting recent work in [H] and [FHK] on Sobolev spaces of first order in metric spaces, we study notions of high order Sobolev spaces in metric spaces, including as special examples the Folland–Stein spaces on stratified groups. Our results extend some of the notions in [H], [FHK] and [FLW] to high order Sobolev spaces.

The plan of the paper is as follows. In Sect. 2, we list the required properties of polynomials in metric spaces $(\mathcal{S}, \rho, d\mu)$ and recall some results proved in [LW] which will be needed. Section 3 contains the statements and proofs of the equivalence of several definitions of Sobolev classes on metric spaces $(\mathcal{S}, \rho, d\mu)$ under the assumption that μ is merely doubling. In Sect. 4, we prove that on stratified groups the Sobolev classes defined in Sect. 3 are equivalent to the Folland–Stein spaces.

2. Polynomials in metric spaces

Let $(\mathcal{S}, \rho, \mu)$ be a metric space with a metric ρ and a doubling measure μ, namely, for all $x, y, z \in \mathcal{S}$, ρ satisfies

$$\rho(x, y) \le \rho(x, z) + \rho(z, y),$$

and the measure μ satisfies the condition

$$\mu(B(x,2r)) \le A_\mu \mu(B(x,r)), \qquad x \in \mathcal{S},\ r > 0,$$

for an absolute constant A_μ, where by definition $B(x,r) = \{y \in \mathcal{S} : \rho(x,y) < r\}$, and $\mu(B(x,r))$ denotes the μ-measure of $B(x,r)$. Such a metric space is usually called as a metric space of homogeneous type in the sense of Coifman and Weiss. As usual, we refer to $B(x,r)$ as the ball with center x and radius r, and, if B is a ball, we write x_B for its center, $r(B)$ for its radius and cB for the ball of radius $cr(B)$ having the same center as B. We always assume that $(\mathcal{S},\rho)$ is locally compact and μ is doubling.

We now define what we will mean by polynomial functions on $\mathcal{S}$. The properties that we list here are similar to those used in [LW].

Let $(\mathcal{S},\rho,\mu)$ be a metric space of homogeneous type, and let Ω be an open set in $\mathcal{S}$. Our results rely on the existence of a linear class of functions $P(x)$ (called polynomial functions) which satisfy both

(P1) For every metric ball $D \subset \Omega$,

$$\text{ess sup}_{x\in D}|P(x)| \le \frac{C_1(\mu)}{\mu(D)} \int_D |P(y)|d\mu(y),$$

where the essential supremum is taken with respect to μ;

(P2) If D is any metric ball in Ω and E is a subball of D with $\mu(E) > \gamma\mu(D)$, $\gamma > 0$, then

$$\| P \|_{L^\infty_\mu(E)} \ge C_2(\gamma,\mu) \| P \|_{L^\infty_\mu(B)} .$$

In stratified groups, including ordinary Euclidean space, (P1) and (P2) are known to hold for polynomials (see [FS] and §4 below for the definition of polynomials in stratified groups), with constants $C_1(\mu)$, $C_2(\gamma,\mu)$ which depend additionally only on the degree of the polynomial. See [LPR] and the comments at the end of this paper for examples of functions which satisfy (P1) and (P2) in more general situations.

In our case, the role of degree is replaced by an exponent which measures the order of smoothness of a given locally integrable function f in one of several ways, such as in assuming that the following Poincaré estimate holds for f and a positive integer k: there exist $q \ge 1$ and a function g such that for every ball $B \subset \Omega$ and some function $P_k(B,f)$,

$$\text{(2.1)} \qquad \frac{1}{\mu(B)} \int_B |f - P_k(B,f)|\, d\mu \le Cr(B)^k \left(\frac{1}{\mu(B)} \int_B |g|^q d\mu \right)^{1/q},$$

with C independent of B. Typically, (2.1) may be thought of as a way to express the fact that $P_k(B,f)$ is an approximation to f. In (2.1), g and q are allowed to depend on f but not on B. The function $P_k(B,f)$ may also depend on g, q and μ, and we sometimes write $P_k(B,f) = P_k(B,f,g,q,\mu)$. The important assumption for us will be that (P1) and (P2) hold for a linear class that contains $P_k(B, f)$ for all $B \subset \Omega$, with constants $C_1(\mu)$, $C_2(\gamma, \mu)$ depending additionally only on k. Note that if (2.1) holds for one value of q, then it also holds for all larger values by Hölder's inequality.

When (2.1) holds for a given f, we will say that f satisfies an L^q to L^1 Poincaré inequality of order k for every ball $B \subset \Omega$. For a stratified group, every smooth function f satisfies (2.1) with $q = 1$ and $g = |X^k f|$ for several choices of polynomials of degree $k-1$ by the results in [L1], [L2] (see also the discussion in [LW]), and in this case, the polynomials satisfy (P1) and (P2) as we already noted.

Instead of (2.1), we will sometimes consider alternate notions of smoothness of f which involve functions $P_k(B, f)$, but in any case, we always require (P1) and (P2) to hold for a linear class that contains $P_k(B, f)$ for all $B \subset \Omega$, with constants depending only on k, μ, γ. When we consider Sobolev spaces of order m, the functions will be assumed to satisfy such conditions for every integer k with $1 \le k \le m$.

Another basic assumption that we make throughout this paper is that the following geodesic (or segment) property holds in $\mathcal{S}$:

(S) $(\mathcal{S}, \rho)$ has the *segment property*, i.e., for each pair of points $x, y \in \mathcal{S}$, there exists a continuous curve $\gamma = \gamma(t)$ connecting x and y such that $\rho(\gamma(t), \gamma(s)) = |t - s|$ for all t, s.

It is easy to see that, if the segment property holds, then for every ball B with center x_B the following is true: if $x \in B$, there is a continuous curve $\gamma = \gamma_{x_B,x}(t)$, $0 \le t \le 1$, in B with $\gamma(0) = x_B$, $\gamma(1) = x$ and $\rho(x_B, z) = \rho(x_B, y) + \rho(y, z)$ for all $y, z \in \gamma$ with $y = \gamma(s)$, $z = \gamma(t)$, $0 \le s \le t \le 1$.

We also note that the fact that the curve $\gamma = \gamma_{x_B,x}$ lies in B if $x \in B$ is a corollary of the assumed additivity of ρ along γ. To see this, we choose $z = x$ in the additivity statement above, obtaining that for any $y \in \gamma$,

$$r(B) > \rho(x_B, x) = \rho(x_B, y) + \rho(y, x) \ge \rho(x_B, y),$$

and therefore $y \in B$.

A special case of a result in [LW] that will be useful is given in the next theorem.

Theorem A. *Let $(\mathcal{S}, \rho, \mu)$ be a metric space of homogeneous type in which the segment property (S) holds. Let k be a positive integer, B_0 be a fixed ball and f be a function for which the Poincaré inequality (2.1) holds with $q = 1$ for each ball $B \subset B_0$ and with polynomials $P_k(B, f)$ which belong to a linear class satisfying (P1) and (P2) with constants depending only on k, γ, μ. Then for μ-a.e. $x \in B_0$,*

$$\begin{aligned} &|f(x) - P_k(B_0, f)(x)| \\ &\quad \le C \int_{B_0} |g(y)| \frac{\rho(x, y)^k}{\mu(B(x, \rho(x, y)))}\, d\mu(y) + C \frac{r(B_0)^k}{\mu(B_0)} \int_{B_0} |g(y)|\, d\mu(y), \end{aligned}$$

where C depends only on μ and the constants in (2.1), (P1) and (P2). In particular, C is independent of x and B_0.

We will also use a special chain of balls constructed in [LW]. In fact, let B_0 be a ball in which the segment property (S) holds. Then, given $x \in B_0$, there is a sequence $\{B_j\}_{j\geq 1}$ of subballs of B_0 satisfying

(1) $B_j \subset B_0$ and $\rho(B_j, x) \to 0$ as $j \to \infty$.
(2) $r(B_j) \approx 2^{-j} r(B_0)$, so that $r(B_j) \to 0$ as $j \to \infty$.
(3) If $y \in B_j$, then $\rho(y, x) \approx r(B_j)$.
(4) $B_j \cap B_{j-1}$ contains a ball S_j with $r(S_j) \approx r(B_j) \approx r(B_{j-1})$.
(5) If $\ell < j$, then $B_j \subset cB_\ell$.
(6) $\{B_j\}$ has bounded overlaps, i.e., $\sum_j \chi_{B_j}(y) \leq c$ for all y.

The constants of equivalence in properties (2), (3), (4) and the constants c in (5) and (6) are independent of x, j, ℓ and B_0, but the chain $\{B_j\}$ depends on x.

A similar chain without the property that all the balls are subballs of B_0 was constructed earlier in [FW].

3. High order Sobolev spaces on a metric space

The main goal of this section is to show the equivalence of several definitions of high order Sobolev classes on a domain Ω in a metric space $(\mathcal{S}, \rho, \mu)$ with a doubling measure μ. We will use $⨍_E f(x)d\mu(x)$ to denote $\frac{1}{\mu(E)} \int_E f(x)d\mu(x)$, and $||f||_{L^p_\mu(E)}$ to denote the L^p norm of f on E with respect to μ. We now list two definitions of Sobolev classes.

Definition 3.1. *Given a positive integer m and $1 < p < \infty$, we define the Sobolev class $A^{m,p}(\Omega)$ to be the set of functions $f \in L^p(\Omega)$ so that for each $k = 1, \cdots, m$, there exist q_k with $1 \leq q_k < p$, functions $g_k(x)$ with $0 \leq g_k \in L^p(\Omega)$, and polynomials $P_k(B, f)$ with*

$$⨍_B |f(x) - P_k(B, f)(x)| d\mu(x) \leq r(B)^k \left(⨍_B g_k^{q_k}(x) d\mu(x) \right)^{1/q_k} \tag{3.1}$$

for every ball $B \subset \Omega$. The polynomials $P_k(B, f)$ are assumed to belong to a linear class which satisfies (P1) and (P2) with constants depending only on k, γ, μ. If $f \in A^{m,p}(\Omega)$, we define

$$||f||_{A^{m,p}(\Omega)} = ||f||_{L^p(\Omega)} + \inf_{\{g_k\}} \sum_{k=1}^{m} ||g_k||_{L^p(\Omega)},$$

where the infimum is taken over all sequences such that (3.1) holds for f for $k = 1, \ldots, m$.

It is easy to see that $A^{m,p}(\Omega)$ is a linear space; moreover, $||\cdot||_{A^{m,p}(\Omega)}$ is a norm if all $q_k = 1$.

Definition 3.2. *Given a positive integer m and $1 < p < \infty$, we define the Sobolev class $B^{m,p}(\Omega)$ to be the set of functions $f \in L^p(\Omega)$ so that for each $k = 1, \cdots, m$ there exist functions $0 \le g_k \in L^p(\Omega)$ and polynomials $P_k(B, f)$ such that*

$$|f(x) - P_k(B, f)(x)| \le r(B)^k g_k(x) \tag{3.2}$$

for $\mu-$a.e. $x \in B$ for every metric ball $B \subset \Omega$. The polynomials $P_k(B, f)$ are assumed to belong to a linear class which satisfies (P1) and (P2) with constants depending only on k, γ, μ. If $f \in B^{m,p}(\Omega)$, let

$$||f||_{B^{m,p}(\Omega)} = ||f||_{L^p(\Omega)} + \inf_{\{g_k\}} \sum_{k=1}^{m} ||g_k||_{L^p(\Omega)}.$$

The class $B^{m,p}(\Omega)$ is a Banach space with norm $||\cdot||_{B^{m,p}}$.

Remark. We could replace the right-hand side of inequality (3.2) by

$$\int_B \frac{\rho(x, y)^k g_k(y)}{\mu(B(x, \rho(x, y)))} d\mu(y) + r(B)^k \fint_B g_k(y) d\mu(y)$$

for $\mu-$a.e. $x \in B$. It can be shown that the resulting space is equivalent to those given in Definitions 3.1 and 3.2; see the comments at the end of Sect. 3.

We denote the Hardy–Litlewood maximal function of a function f by $M(f)$:

$$M(f)(x) = \sup_{B: x \in B} \frac{1}{\mu(B)} \int_B |f(y)|\, d\mu(y).$$

To show that definitions 3.1 and 3.2 are equivalent, we will use the following fact.

Theorem 1. *Let $1 \le q < \infty$, m be a positive integer, $B_0 \subset \Omega$ be a fixed ball, and suppose that the segment property (S) holds for B_0. Let f be a locally integrable function in Ω for which there exist a function $0 \le g \in L^q(\Omega)$ and polynomials $P_m(B, f)$ such that the Poincaré inequality*

$$\fint_B |f(x) - P_m(B, f)(x)|\, d\mu(x) \le c r(B)^m \left(\fint_B |g(x)|^q d\mu(x) \right)^{1/q}$$

holds for every ball $B \subset \Omega$. The polynomials $P_m(B, f)$ are assumed to belong to a linear class which satisfies (P1) and (P2) with constants depending only on m, γ, μ. Then for $\mu-$a.e. $x \in B_0$,

$$|f(x) - P_m(B_0, f)(x)| \le Cr(B_0)^m M(g^q)(x)^{1/q}$$

with C independent of x.

Proof of Theorem 1. Let $x \in B_0$. We will use the chain of subballs $\{B_j\}$ of B_0 from [LW] mentioned earlier. The chain depends on x. We may assume that x is a Lebesgue point of both $|f - P_m(B_0, f)|$ and $|g|^q$ with respect to μ since almost every point of an integrable function is a Lebesgue point of the function. Then by properties (1), (2) and (3) of the chain,

$$\begin{aligned}|f(x) - P_m(B_0, f)(x)| = &\lim_{j\to\infty} \fint_{B_j} |f(y) - P_m(B_0, f)(y)|\, d\mu(y)\\ \le &\limsup_{j\to\infty} \fint_{B_j} |f(y) - P_m(B_j, f)(y)|\, d\mu(y)\\ &+ \limsup_{j\to\infty} \fint_{B_j} |P_m(B_j, f)(y) - P_m(B_0, f)(y)|\, d\mu(y)\\ = &I_1 + I_2,\end{aligned}$$

where I_1 and I_2 are defined by the last equality.

We will first show that $I_1 = 0$ for every Lebesgue point x of $|g|^q$. By the Poincaré inequality,

$$\begin{aligned}I_1 &\le c \limsup_{j\to\infty} r(B_j)^m \left(\fint_{B_j} |g(y)|^q d\mu(y)\right)^{1/q}\\ &= 0 \cdot |g(x)| = 0\end{aligned}$$

by properties (1), (2) and (3) of the chain.

Thus we only need to estimate I_2. We have

$$\begin{aligned}I_2 \le &\limsup_{j\to\infty} \sum_{\ell=0}^{j-1} \fint_{B_j} |P_m(B_{\ell+1}, f)(y) - P_m(B_\ell, f)(y)|\, d\mu(y)\\ \le &\limsup_{j\to\infty} \sum_{\ell=0}^{j-1} ||P_m(B_{\ell+1}, f) - P_m(B_\ell, f)||_{L^\infty_\mu(B_j)}\\ \le &\limsup_{j\to\infty} \sum_{\ell=0}^{j-1} ||P_m(B_{\ell+1}, f) - P_m(B_\ell, f)||_{L^\infty_\mu(cB_\ell)} \text{ by (5)}\\ \le &C \sum_{\ell=0}^{\infty} ||P_m(B_{\ell+1}, f) - P_m(B_\ell, f)||_{L^\infty_\mu(S_\ell)} \text{ by (4) and (P2)}\\ \le &C \sum_{\ell=0}^{\infty} \frac{1}{\mu(S_\ell)} ||P_m(B_{\ell+1}, f) - P_m(B_\ell, f)||_{L^1_\mu(S_\ell)} \text{ by (P1)}\end{aligned}$$

$$
\begin{aligned}
&\leq C \sum_{\ell=0}^{\infty} \fint_{B_\ell} |P_m(B_\ell, f)(y) - f(y)|\, d\mu(y) \\
&\quad + C \sum_{\ell=0}^{\infty} \fint_{B_{\ell+1}} |P_m(B_{\ell+1}, f)(y) - f(y)|\, d\mu(y) \text{ by (4)} \\
&\leq C \sum_{\ell=0}^{\infty} r(B_\ell)^m \left(\fint_{B_\ell} |g(y)|^q d\mu(y) \right)^{1/q} \\
&\leq C \sum_{\ell=0}^{\infty} r(B_\ell)^m M(|g|^q)(x)^{1/q} \text{ by (3)} \\
&= C \sum_{\ell=0}^{\infty} 2^{-\ell m} r(B_0)^m M(|g|^q)(x)^{1/q} \text{ by (2)} \\
&\leq C r(B_0)^m M(|g|^q)(x)^{1/q}.
\end{aligned}
$$

This completes the proof of Theorem 1.

We now prove the main theorem in this section.

Theorem 2. *Suppose that the segment property (S) holds in a metric space $(\mathcal{S}, \rho, \mu)$ with a doubling measure μ, and let Ω be a domain in $\mathcal{S}$. Then the Sobolev classes $A^{m,p}(\Omega)$, and $B^{m,p}(\Omega)$ are equivalent in the sense that they are the same as sets of functions and, for any function f in these classes,*

$$
||f||_{A^{m,p}(\Omega)} \approx ||f||_{B^{m,p}(\Omega)}
$$

with constants of equivalence which are independent of f.

Proof of Theorem 2. Let $f \in A^{m,p}(\Omega)$. Given $\epsilon > 0$, there are functions $0 \leq g_k \in L^p(\Omega)$, polynomials $P_k(B, f)$ and exponents q_k, $1 \leq q_k < p$ $(k = 1, \cdots, m)$, such that

$$
\fint_B |f - P_k(B, f)| d\mu \leq r(B)^k \left(\fint_B |g_k|^{q_k} d\mu \right)^{1/q_k}
$$

for every ball $B \subset \Omega$, and such that

$$
||f||_{L^p(\Omega)} + \sum_{k=1}^{m} ||g_k||_{L^p(\Omega)} < ||f||_{A^{m,p}(\Omega)} + \epsilon.
$$

By Theorem 1, for each $1 \leq k \leq m$ and $\mu-$a.e. $x \in B$,

$$
|f(x) - P_k(B, f)(x)| \leq C r(B)^k M(|g|^{q_k})(x)^{1/q_k}.
$$

The functions $CM(|g|^{q_k})^{1/q_k}$ are in $L^p(\Omega)$ since $p > q_k$ and the maximal function is bounded on $L^{p/q_k}(\Omega)$. Thus, by definition 3.2, $f \in B^{m,p}(\Omega)$ and

$$\begin{aligned} ||f||_{B^{m,p}(\Omega)} &\leq ||f||_{L^p(\Omega)} + C \sum_{k=1}^{m} ||M(|g_k|^{q_k})^{1/q_k}||_{L^p(\Omega)} \\ &\leq ||f||_{L^p(\Omega)} + C \sum_{k=1}^{m} ||g_k||_{L^p(\Omega)} \leq C(||f||_{A^{m,p}(\Omega)} + \epsilon). \end{aligned}$$

Letting $\epsilon \to 0$, we obtain

$$||f||_{B^{m,p}(\Omega)} \leq C||f||_{A^{m,p}(\Omega)}.$$

Suppose next that $f \in B^{m,p}(\Omega)$. Given $\epsilon > 0$, there are functions $0 \leq g_k \in L^p(\Omega)$ and polynomials $P_k(B, f)$ ($k = 1, \cdots, m$) such that

$$|f(x) - P_k(B, f)(x)| \leq r(B)^k g_k(x)$$

for $\mu-$a.e. $x \in B \subset \Omega$ and

$$||f||_{L^p(\Omega)} + \sum_{k=1}^{m} ||g_k||_{L^p(\Omega)} < ||f||_{B^{m,p}(\Omega)} + \epsilon.$$

It follows by integrating over B that the Poincaré inequality

$$\fint_B |f(x) - P_k(B, f)(x)| d\mu(x) \leq r(B)^k \fint_B |g_k(x)| d\mu(x)$$

holds for every such B. By definition 3.1, applied with $q_k = 1$, we conclude that $f \in A^{m,p}(\Omega)$ and

$$||f||_{A^{m,p}(\Omega)} \leq ||f||_{L^p(\Omega)} + \sum_{k=1}^{m} ||g_k||_{L^p(\Omega)} \leq ||f||_{B^{m,p}(\Omega)} + \epsilon.$$

Thus $||f||_{A^{m,p}(\Omega)} \leq ||f||_{B^{m,p}(\Omega)}$, and we have proved that definitions 3.1 and 3.2 are equivalent.

In passing, let us briefly justify the remark we made after Definition 3.2. A simple argument based on dividing the domain of integration into annuli gives

$$\int_B \frac{\rho(x, y)^k g_k(y)}{\mu(B(x, \rho(x, y)))} d\mu(y) \leq Cr(B)^k M(g_k)(x), \quad x \in B,$$

and it then follows that the condition in the remark implies the one in Definition 3.2. Conversely, the condition in Definition 3.2 leads immediately by integration to an L^1 to L^1 Poincaré estimate, and then Theorem A implies the condition given in the remark.

4. Classical and Folland–Stein Sobolev spaces are special examples

Let $\mathcal{G}$ be a finite-dimensional, stratified, nilpotent Lie algebra. Assume that

$$\mathcal{G} = \oplus_{i=1}^{s} V_i \ ,$$

with $[V_i, V_j] \subset V_{i+j}$ for $i + j \leq s$ and $[V_i, V_j] = 0$ for $i + j > s$. Let $X_1, \ldots, X_l$ be a basis for V_1 and suppose that $X_1, \ldots, X_l$ generate $\mathcal{G}$ as a Lie algebra. Then for $2 \leq j \leq s$, we can choose a basis $\{X_{ij}\}$, $1 \leq i \leq k_j$, for V_j consisting of commutators of length j. We set $X_{i1} = X_i,\ i = 1, \ldots, l$ and $k_1 = l$, and we call X_{i1} a commutator of length 1.

If $\mathbb{G}$ is the simply connected Lie group associated with $\mathcal{G}$, then the exponential mapping is a global diffeomorphism from $\mathcal{G}$ to $\mathbb{G}$. Thus, for each $g \in \mathbb{G}$, there is $x = (x_{ij}) \in \mathbb{R}^N$, $1 \leq i \leq k_j,\ 1 \leq j \leq s,\ N = \sum_{j=1}^{s} k_j$, such that

$$g = \exp(\sum x_{ij} X_{ij}) \ .$$

A homogeneous norm function $|\cdot|$ on $\mathbb{G}$ is defined by

$$|g| = (\sum |x_{ij}|^{2s!/j})^{1/2s!} \ ,$$

and $Q = \sum_{j=1}^{s} jk_j$ is said to be the *homogeneous dimension* of $\mathbb{G}$. The dilation δ_r on $\mathbb{G}$ is defined by

$$\delta_r(g) = \exp(\sum r^j x_{ij} X_{ij}) \ \text{ if } \ g = \exp(\sum x_{ij} X_{ij}).$$

The convolution operation on $\mathbb{G}$ is defined by

$$f * h(x) = \int_{\mathbb{G}} f(xy^{-1}) h(y) dy = \int_{\mathbb{G}} f(y) h(y^{-1}x) dy,$$

where y^{-1} is the inverse of y and xy^{-1} denotes group multiplication of x by y^{-1}. It is known that for any left invariant vector field X on $\mathbb{G}$,

$$X(f * h) = f * (Xh).$$

We now recall the definition of the class of polynomials on $\mathbb{G}$ given by Folland and Stein in [FS]. Let $X_1, \cdots, X_l$ in V_1 be the generators of the Lie algebra $\mathcal{G}$, and let $X_1, \cdots, X_l, \cdots, X_N$ be a basis of $\mathcal{G}$. We denote $d(X_j) = d_j$ to be the length of X_j as a commutator, and we arrange the order so that $1 \leq d_1 \leq \cdots \leq d_N$. Then it is easy to see that $d_j = 1$ for $j = 1, \cdots, l$. Let $\xi_1, \cdots, \xi_N$ be the dual basis for $\mathcal{G}^*$, and let $\eta_i = \xi_i \circ \exp^{-1}$. Each η_i is a real-valued function on $\mathbb{G}$, and $\eta_1, \ldots, \eta_N$ gives a system of global coordinates on $\mathbb{G}$. A function P on $\mathbb{G}$ is said to be a polynomial on $\mathbb{G}$ if $P \circ \exp$ is a polynomial on $\mathcal{G}$. Every polynomial on $\mathbb{G}$ can be written uniquely as

$$P(x) = \sum_I a_I \eta^I(x), \quad \eta^I = \eta_1^{i_1} \cdots \eta_N^{i_N}, \quad a_I \in \mathbb{R},$$

where all but finitely many of the coefficients a_I vanish. Clearly η^I is homogeneous of degree $d(I) = \sum_{j=1}^N i_j d_j$, i.e., $\eta^I(\delta_r x) = r^{d(I)}\eta_i(x)$. If $P = \sum_I a_I \eta^I$, then we define the homogeneous degree (or order) of P to be $\max\{d(I) : a_I \neq 0\}$.

We also adopt the following multi-index notation for higher order derivatives. If $I = (i_1, \cdots, i_N) \in \mathbb{N}^N$, we set

$$X^I = X_1^{i_1} \cdot X_2^{i_2} \cdots X_N^{i_N}.$$

By the Poincaré–Birkhoff–Witt theorem (cf. Bourbaki [B], I.2.7), the differential operators X^I form a basis for the algebra of left-invariant differential operators in $\mathbb{G}$. Furthermore, we set

$$|I| = i_1 + i_2 + \cdots + i_N, \quad d(I) = d_1 i_1 + d_2 i_2 + \cdots + d_N i_N.$$

Thus, $|I|$ is the order of the differential operator X^I, and $d(I)$ is its degree of homogenity; $d(I)$ is called the homogeneous degree (order) of X^I. From the Poincaré–Birkhoff–Witt theorem, it follows in particular that any differential operator $X^\alpha = X_{i_1}^{\alpha_1} \cdots X_{i_k}^{\alpha_k}$ with $1 \leq i_j \leq l$ for $1 \leq j \leq k$ (for any k and α) can be expressed as a linear combination of operators of the special form X^I above with $d(I) = |\alpha| = \alpha_1 + \cdots + \alpha_k$. Thus, instead of considering such differential operators X^α, we will often consider only operators of the special form X^I with $d(I) = |\alpha|$. We will also use the notation

$$|X^m f| = \Big(\sum_{I:d(I)=m} |X^I f|^2 \Big)^{1/2}$$

for any positive integer m.

Let m be a positive integer, $1 < p < \infty$, and Ω be an open set in $\mathbb{G}$. The Folland–Stein Sobolev space $W^{m,p}(\Omega)$ associated with the vector fields $X_1, \cdots, X_l$ is defined to consist of all functions $f \in L^p(\Omega)$ with distributional derivatives $X^I f \in L^p(\Omega)$ for every X^I defined above with $d(I) \leq m$. Here, we say that the distributional derivative $X^I f$ exists and equals a locally integrable function g_I if for every $\phi \in C_0^\infty(\Omega)$,

$$\int_\Omega f X^I \phi \, dx = (-1)^{d(I)} \int_\Omega g_I \phi dx.$$

$W^{m,p}(\Omega)$ is equipped with the norm

$$||f||_{W^{m,p}(\Omega)} = ||f||_{L^p(\Omega)} + \sum_{1 \leq d(I) \leq m} ||X^I f||_{L^p(\Omega)}.$$

This definition is equivalent to the one given in the introduction because of our earlier remarks about the Poincaré–Birkhoff–Witt theorem.

Before we state the main theorems of this section, we need to recall some other results that will be needed. It is known that any polynomial on $\mathbb{G}$ satisfies (P1) (see [FS]) and also (P2) (see [L2]) when μ is Lebesgue measure, with constants which depend only on γ and the homogeneous degree of the polynomial. Now, if $\Omega \subset \mathbb{G}$, let $f \in W^{m,p}(\Omega)$ and let $\mathcal{P}_m$ be the class of polynomials on $\mathbb{G}$ of homogeneous degree strictly less than m. It is shown in [L2], and earlier in [N] for the Heisenberg group, that given any ball $B \subset \Omega$, there exists $P_m(B, f) \in \mathcal{P}_m$ such that

$$\int_B X^I(f - P_m(B, f))dx = 0 \tag{4.1}$$

if $d(I) < m$. Clearly, by iteration of the case $m = 1$, this polynomial $P_m(B, f)$ satisfies the Poincaré inequality of order m for Lebesgue measure, i.e.,

$$||f - P_m(B, f)||_{L^1(B,dx)} \leq Cr(B)^m ||X^m f||_{L^1(B,dx)}. \tag{4.2}$$

In fact, a sharp L^p to L^q Poincaré inequality with optimal exponents holds for $1 \leq p < Q$ and $q = \frac{pQ}{Q-p}$ (see [L2]). It is also shown in [L2] that, for every ball $B \subset \Omega$, there is a linear projection operator

$$\pi_m(B, \cdot) : W^{m,p}(\Omega) \to \mathcal{P}_m$$

satisfying $\pi_m(B, P) = P$ for all $P \in \mathcal{P}_m$ and both

$$\sup_{x \in B} |\pi_m(B, f)(x)| \leq \frac{C}{|B|} \int_B |f(y)|dy \tag{4.3}$$

and

$$||X^I \pi_m(B, f)||_{L^q(B,dx)} \leq C(Q, q)||X^{d(I)} f||_{L^q(B,dx)} \tag{4.4}$$

for all $1 \leq q \leq \infty$. It is easy to see that the polynomials $\pi_m(B, f)$ satisfy (4.2) by using the fact that polynomials satisfying (4.1) do (see, e.g., Lemma 2.8 in [LW]). Polynomials satisfying (4.1), or (4.3) and (4.4), have applications to proving extension theorems on high order Sobolev spaces on the Heisenberg group (see [N]) and stratified groups (see [L2], [L3]) and Sobolev interpolation inequalities of any order (see [L1], [L2]).

The main theorem of this section concerns the equivalence of $W^{m,p}(\Omega)$ and the spaces $A^{m,p}(\Omega)$ and $B^{m,p}(\Omega)$ defined earlier. We always choose $d\mu = dx$ in the rest of the paper.

Theorem 3. *Let m be a positive integer and $1 < p < \infty$. The Folland–Stein space $W^{m,p}(\Omega)$ on a stratified group $\mathbb{G}$ is equivalent to the corresponding classes in definitions (3.1)–(3.2), when μ is taken to be Lebesgue measure in those definitions. Moreover,*

$$||f||_{W^{m,p}} \approx ||f||_{A^{m,p}} \approx ||f||_{B^{m,p}}$$

with constants of equivalence which are independent of f.

To show this, we only need to show that $W^{m,p}(\Omega)$ is equivalent to $A^{m,p}(\Omega)$ since the equivalence of $A^{m,p}(\Omega)$ and $B^{m,p}(\Omega)$ was shown in Sect. 3.

Lemma 4. *Let P be a polynomial of order less than k and $1 \le q \le \infty$. Let I be any multi-index. Then*

$$||X^I P||_{L^q(B)} \le Cr(B)^{-d(I)} ||P||_{L^q(B)} \text{ for all balls } \quad B \in \mathbb{G}$$

where C depends only on Q, q, k.

This lemma is essentially proved in [FS] (see [L2] for a proof of the result stated in the above form).

We begin the proof of Theorem 3 by showing that (when $d\mu = dx$) $A^{m,p}(\Omega) \subset W^{m,p}(\Omega)$.

Theorem 5. *Let m be a positive integer, $1 \le p < \infty$, and $f \in A^{m,p}(\Omega)$. For each $k = 1, \cdots, m$, consider functions $0 \le g_k \in L^p(\Omega)$, polynomials $P_k(B, f)$ and $1 \le q_k < p$ such that for every ball $B \subset \Omega$,*

$$\fint_B |f(x) - P_k(B, f)(x)| dx \le Cr(B)^k \left(\fint_B g_k^{q_k} dx \right)^{1/q_k}. \tag{4.5}$$

Then $f \in W^{m,p}(\Omega)$ and $||f||_{W^{m,p}(\Omega)} \le ||f||_{L^p(\Omega)} + C \sum_{k=1}^m ||g_k||_{L^p(\Omega)}$.

Proof of Theorem 5. The proof adapts ideas of Calderón [Ca] and relies on the group structure of $\mathbb{G}$. See also [H] for a similar adaption of Calderón's ideas. Later, we will give a second proof which works in case Ω is the entire space and which involves polynomials for high order Sobolev functions. A similar argument for $m = 1$ has been used in [FHK] to prove the equivalence of several definitions of Sobolev spaces of first order in metric spaces.

Let $f \in A^{m,p}(\Omega)$, and let (4.5) hold for $k = 1, \cdots, m$ and all $B \subset \Omega$. To show that $f \in W^{m,p}(\Omega)$, we need to show that the distributional derivatives $X^I f = X_1^{i_1} \cdots X_N^{i_N} f$ of f in Ω lie in $L^p(\Omega)$ for all I with $d(I) \le m$. By using the Riesz representation theorem, it is enough to prove that there exists a nonnegative function $h \in L^p(\Omega)$ such that for any $\phi \in C_0^\infty(\Omega)$,

$$\left| \int_\Omega f X_N^{i_N} \cdots X_2^{i_2} X_1^{i_1} \phi \right| dx \le \int_\Omega |\phi| h \, dx.$$

Let us now show that for fixed I with $d(I) = k \le m$, this estimate holds with $h(x)$ taken to be $CM(g_k^{q_k})^{1/q_k}(x)$. We set $Y^I = X_N^{i_N} \cdots X_1^{i_1}$. Fix any $\Psi \in C_0^\infty(B(0,1))$ with $\int \Psi = 1$, and let $\Psi_\epsilon(x) = \epsilon^{-Q} \Psi(\delta_{\epsilon^{-1}} x)$. Note that Ψ_ϵ is supported in the ball $B(0,\epsilon)$. If $\phi \in C_0^\infty(\Omega)$, then

$$\begin{aligned}\int_{\mathbb{G}} f\, Y^I \phi\, dx =& \lim_{\epsilon\to 0}\int_{\mathbb{G}} (f \ast \Psi_\epsilon)\, Y^I \phi\, dx\\ =&(-1)^{d(I)} \lim_{\epsilon\to 0}\int_{\mathbb{G}} X^I (f \ast \Psi_\epsilon)\, \phi\, dx\\ =&(-1)^k \lim_{\epsilon\to 0}\int_{\mathbb{G}} (f \ast X^I \Psi_\epsilon)\, \phi\, dx.\end{aligned}$$

Recall that

$$\begin{aligned}P_k(B(x,\epsilon), f) \ast X^I \Psi_\epsilon(x) =& \int_{\mathbb{G}} P_k(B(x,\epsilon), f)(xy^{-1}) X^I \Psi_\epsilon(y) dy\\ =&(-1)^k \int_{\mathbb{G}} Y^I [P_k(B(x,\epsilon), f)(xy^{-1})] \Psi_\epsilon(y) dy.\end{aligned}$$

Note that for any fixed x, $P_k(B(x,\epsilon), f)(xy^{-1})$ is a polynomial in y of degree less than k (see [FS, p. 23]), so that $Y^I[P_k(B(x,\epsilon), f)(xy^{-1})] = 0$. Therefore,

$$P_k(B(x,\epsilon), f) \ast X^I \Psi_\epsilon(x) = 0$$

and

$$(f \ast X^I \Psi_\epsilon)(x) = (f - P_k(B(x,\epsilon), f)) \ast X^I \Psi_\epsilon(x).$$

Consequently, if $1 \le d(I) = k \le m$,

$$\begin{aligned}|(f \ast X^I \Psi_\epsilon)(x)| =& \left|\int_{\mathbb{G}} (f - P_k(B(x,\epsilon), f))(y) (X^I \Psi_\epsilon)(y^{-1}x)\, dy\right|\\ \le& C \int_{B(x,\epsilon)} |f - P_k(B(x,\epsilon), f)| \cdot \epsilon^{-Q-k} dy.\end{aligned}$$

Assuming as we may that $x \in \Omega$, and choosing ϵ so small that $B(x,\epsilon) \subset \Omega$, we can estimate the last expression by using (4.5). Thus,

$$|(f \ast X^I \Psi_\epsilon)(x)| \le C \left(\fint_{B(x,\epsilon)} |g_k|^{q_k} dy \right)^{1/q_k} \le C M(g_k^{q_k})(x)^{1/q_k},$$

and so

$$\left|\int f\, Y^I \phi\, dx\right| \le C \int |\phi|\, M(g_k^{q_k})^{1/q_k} dx.$$

This shows that if $1 \le d(I) = k \le m$, then for a.e. $x \in \Omega$, the distributional derivative $X^I f$ satisfies

$$|X^I f(x)| \le C M(g_k^{q_k})(x)^{1/q_k}.$$

Thus,

$$||X^I f||_{L^p(\Omega)} \le C ||M(g_k^{q_k})^{1/q_k}||_{L^p(\Omega)} \le C ||g_k||_{L^p(\Omega)}.$$

Hence, $f \in W^{m,p}(\Omega)$ and $||f||_{W^{m,p}(\Omega)} \leq ||f||_{L^p(\Omega)} + C\sum_{k=1}^m ||g_k||_{L^p(\Omega)}$. This completes the proof of Theorem 5.

We now give a second proof of Theorem 5 in case Ω is the entire space $\mathbb{G}$. This proof is less dependent on the group structure of $\mathbb{G}$.

Second Proof of Theorem 5 when $\Omega = \mathbb{G}$. Given $\epsilon > 0$, select a sequence of balls B_i with $r(B_i) = \epsilon$ such that $\cup B_i = \mathbb{G}$ and $\sum_i \chi_{2B_i}(x) \leq M$ for all x. It is easy to see that if $2B_i \cap 2B_j \neq \emptyset$, then $4B_i \cap 4B_j$ contains a ball S_{ij} with $r(S_{ij}) \approx \epsilon$. Here, the constants of equivalence as well as M can be chosen to be independent of ϵ, i, j. (See, e.g., the construction of dyadic grids in [SW].)

Let ξ_i be a C^∞ function supported in $2B_i$ such that $0 \leq \xi_i(x) \leq 1$ for all x, $\xi_i(x) = 1$ for $x \in B_i$, and $|X^I \xi_i(x)| \leq Cr(B_i)^{-d(I)}$ for all x and all multi-indices I (see [FS]). Now let

$$\phi_i = \xi_i / \sum_j \xi_j.$$

Then $\{\phi_i\}$ is a smooth partition of unity associated with $\{B_i\}$, namely,

$$\sum_i \phi_i(x) = 1, \quad \operatorname{supp} \phi_i \subset 2B_i, \quad \text{and } |X^I \phi_i(x)| \leq C\epsilon^{-d(I)}$$

for all x and all multi-indices I.

Fix a function $f \in A^{m,p}(\mathbb{G})$, and assume that the estimates (4.5) hold. For $k = 1, \cdots, m$, define

$$f_{\epsilon,k}(x) = \sum_i \phi_i(x) P_k(4B_i, f)(x), \quad x \in \mathbb{G}.$$

We claim that for any fixed index I with $d(I) = k$,

$$|X^I f_{\epsilon,k}(x)| \leq CM(g_k^{q_k})(x)^{1/q_k}, \quad x \in \mathbb{G}.$$

To show this, fix x and choose a ball B_{i_0} with $x \in B_{i_0}$. Clearly, there are only a finite number of i such that $2B_i \cap 2B_{i_0} \neq \emptyset$. We will use the notation $J + K = I$ if $I = (i_1, \cdots, i_N)$, $J = (j_1, \cdots, j_N)$, $K = (k_1, \cdots, k_N)$ and $i_h + j_h = k_h$ for each $1 \leq h \leq N$. Then

$$\begin{aligned}
|X^I f_{\epsilon,k}(x)| &= |X^I \sum_i \big(P_k(4B_i, f)(x) - P_k(4B_{i_0}, f)(x)\big)\phi_i(x)| \\
&= |\sum_i \sum_{d(J)+d(K)=d(I)} X^J \big(P_k(4B_i, f)(x) - P_k(4B_{i_0}, f)(x)\big) X^K \phi_i(x)|
\end{aligned}$$

$$\leq C\epsilon^{-d(K)} \sum_{i:2B_i \cap 2B_{i_0} \neq \emptyset} \sum_{d(J)+d(K)=d(I)} |X^J \big(P_k(4B_i, f)(x) - P_k(4B_{i_0}, f)(x)\big)|$$

since $x \in B_{i_0}$ and ϕ_i has support in $2B_i$. Recall that if $2B_i \cap 2B_{i_0} \neq \emptyset$, then there is a ball $S_{ii_0} \subset 4B_i \cap 4B_{i_0}$ with $r(S_{ii_0}) \approx \epsilon$. We have

$$\begin{aligned}
&|X^J\left(P_k(4B_i, f)(x) - P_k(4B_{i_0}, f)(x)\right)| \\
\leq &||X^J\left(P_k(4B_i, f) - P_k(4B_{i_0}, f)\right)||_{L^\infty(4B_{i_0})} \\
\leq &Cr(B_{i_0})^{-d(J)}||P_k(4B_i, f) - P_k(4B_{i_0}, f)||_{L^\infty(4B_{i_0})} \text{ by Lemma 4} \\
\leq &Cr(B_{i_0})^{-d(J)}||P_k(4B_i, f) - P_k(4B_{i_0}, f)||_{L^\infty(S_{ii_0})} \\
\leq &Cr(B_{i_0})^{-d(J)} \fint_{S_{ii_0}} |P_k(4B_i, f)(y) - P_k(4B_{i_0}, f)(y)|dy \\
\leq &Cr(B_{i_0})^{-d(J)} \fint_{4B_i} |P_k(4B_i, f)(y) - f(y)|dy \\
&+ Cr(B_{i_0})^{-d(J)} \fint_{4B_{i_0}} |f(y) - P_k(4B_{i_0}, f)(y)|dy \\
\leq &Cr(B_{i_0})^{-d(J)} r(B_i)^k \left(\fint_{4B_i} g_k^{q_k} dy\right)^{1/q_k} \\
&+ Cr(B_{i_0})^{-d(J)} r(B_{i_0})^k \left(\fint_{4B_{i_0}} g_k^{q_k} dy\right)^{1/q_k} \\
\leq &C\epsilon^{-d(J)}\epsilon^k M(g_k^{q_k})(x)^{1/q_k}.
\end{aligned}$$

Here we have used (4.5) to obtain the next-to-last inequality. Thus,

$$|X^I f_{\epsilon,k}(x)| \leq CM(g_k^{q_k})(x)^{1/q_k}.$$

For a.e. x,

$$|f(x) - f_{\epsilon,k}(x)| \leq \sum_i |\phi_i(x)|\,|f(x) - P_k(4B_i, f)(x)| \leq C\epsilon^k M(g_k^{q_k})(x)^{1/q_k}$$

by using the fact from Theorem 1 that (4.5) implies that for a.e. $x \in 4B_i$,

$$|f(x) - P_k(4B_i, f)(x)| \leq Cr(B_i)^k M(g_k^{q_k})(x)^{1/q_k},$$

and the fact that only a finite number of the balls $2B_i$ contain x. Thus, $f(x) = \lim_{\epsilon\to 0} f_{\epsilon,k}(x)$ for a.e. x. Moreover, since $g_k \in L^p(\mathbb{G})$ and M is a bounded operator on $L^{p/q_k}(\mathbb{G})$ (recall that $q_k < p$), $f_{\epsilon,k}$ also converges to f in $L^p(\mathbb{G})$. Therefore, for any $\phi \in C_0^\infty(\mathbb{G})$,

$$\int_{\mathbb{G}} f(x) X_N^{i_N} \cdots X_1^{i_1}\phi(x)dx = \lim_{\epsilon\to 0}\int_{\mathbb{G}} f_{\epsilon,k}(x) X_N^{i_N} \cdots X_1^{i_1}\phi(x)dx.$$

Since $|X^I f_{\epsilon,k}(x)| \leq CM(g_k^{q_k})(x)^{1/q_k}$ for all x, we conclude that

$$|\int_{\mathbb{G}} f(x) X_N^{i_N} \cdots X_1^{i_1}\phi(x)dx| \leq C\int_{\mathbb{G}} M(g_k^{q_k})(x)^{1/q_k}|\phi(x)|dx.$$

Thus the distributional derivative $X^I f(x)$ satisfies $|X^I f(x)| \leq CM(g_k^{q_k})(x)^{1/q_k}$, and the rest of the proof is as before.

Proof of Theorem 3. Suppose $f \in A^{m,p}(\Omega)$, and let δ be any positive number. Then there exist functions $0 \leq g_k \in L^p(\Omega)$, polynomials $P_k(B, f)$ on $\mathbb{G}$, and exponents $1 \leq q_k < p$ ($k = 1, \cdots, m$) such that

$$\fint_B |f(x) - P_k(B, f)(x)|dx \leq r(B)^k \left(\fint_B g_k^{q_k} dx \right)^{1/q_k}$$

for every ball $B \subset \Omega$, and $||f||_{L^p(\Omega)} + \sum_{k=1}^m ||g_k||_{L^p(\Omega)} < ||f||_{A^{m,p}(\Omega)} + \delta$. By Theorem 5, $f \in W^{m,p}(\Omega)$ and

$$||f||_{W^{m,p}(\Omega)} \leq ||f||_{L^p(\Omega)} + C \sum_{k=1}^m ||g_k||_{L^p(\Omega)}.$$

By combining estimates and letting $\delta \to 0$, we obtain

$$||f||_{W^{m,p}(\Omega)} \leq C||f||_{A^{m,p}(\Omega)} \ .$$

Now let $f \in W^{m,p}(\Omega)$. Then by the comments which precede Theorem 3, for $k = 1, \cdots, m$, there are polynomials of degree less than k such that

$$\int_B |f(x) - P_k(B, f)(x)|dx \leq Cr(B)^k \int_B |X^k f|dx$$

for every ball $B \subset \Omega$. Thus by definition (3.1), $f \in A^{m,p}(\Omega)$ and

$$||f||_{A^{m,p}(\Omega)} \leq C||f||_{W^{m,p}(\Omega)}.$$

Combining estimates gives $A^{m,p}(\Omega) = W^{m,p}(\Omega)$ and

$$||f||_{W^{m,p}(\Omega)} \approx ||f||_{A^{m,p}(\Omega)} \ .$$

This completes the proof of Theorem 3.

Final remarks. The referee pointed out to us the relevant work of Cheeger [Ch] on differentiability of Lipschitz functions on metric spaces, written around the same time as the current article. Based on the current paper, a number of other papers have been written. For example, it has been shown in [LP] that the L^1 norm can be replaced by the L^q "norm" for $0 < q < 1$ on the left-hand side of (3.1) of Definition (3.1). It has been shown in [R] and [LRP] that the condition $\mu(E) > \gamma\mu(D)$ in (P2) is equivalent to $r(E) > \gamma_1 r(D)$ for some $\gamma_1 > 0$ depending only on γ and the doubling constant of μ, where $r(E)$ denotes the radius of E. In [LPR], we have shown the existence of polynomials in metric spaces (e.g., positive powers of the distance function), defined notions of degrees of polynomials and higher order gradients in metric spaces, and established distribution theory in metric spaces by using the high order Sobolev space theory developed in the current paper.

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