ON A PRIORI $C^{1,\alpha}$ AND $W^{2,p}$ ESTIMATES FOR A PARABOLIC MONGE-AMPÈRE EQUATION IN THE GAUSS CURVATURE FLOWS

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Abstract. This paper establishes Hölder estimates of Du and L^p estimates of D^2u for solutions u to the parabolic Monge-Ampère equation $-Au_t + (\det D^2u)^{1/n} = f$.

1. Introduction. In this paper, we consider the following parabolic Monge-Ampère equation

$$-A(x)u_t + (\det D^2u)^{1/n} = f(x,t), \quad \text{in } Q = \Omega \times (0,T] \tag{1.1}$$

where $u = u(x,t)$ is convex in x for every $0 < t \leq T$, D^2u denotes the Hessian of u with respect to x, and Ω is a bounded convex domain in $\mathbb{R}^n$.

Equation (1.1) arises in connection with geometric evolution problems involving powers of Gauss curvature which have drawn a great deal of attention and undergone a rapid development. For instance, see [Fir], [Tso1], [Cho], [An 1,2], [Ham], [C-E-I], and [Ur]. Let $X_0(s)$, $s \in S^n$ be a smooth strictly convex closed surface in $\mathbb{R}^{n+1}$, and consider the Gauss curvature flow

$$\begin{cases} \frac{\partial X}{\partial t}(s,t) = -sgn(\beta)K^{\beta}(s,t)\nu(s,t) \\ X(s,0) = X_0(s), \end{cases} \tag{1.2}$$

where $\beta \neq 0$, $sgn(\beta)$ is the sign function of β, K the Gauss curvature, and ν is the unit outward normal at $X(s,t)$.

If the strictly convex surface X is parametrized by the inverse of the Gauss map, then the support function $H(\nu,t)$ of X can be written as

$$H(\nu,t) = X(\nu,t)\cdot\nu = \sup_{x\in X} x\cdot\nu,$$

where the unit outward normal at $X(\nu,t)$ is ν. As in [Tso1] and [Cho], (1.2) is

Manuscript received March 29, 2004.

Research of the first author supported in part by NSF grant No. DMS-0201599; research of the second author supported in part by NSF grant No. DMS99-70352.

American Journal of Mathematics 128 (2006), 453–480.

equivalent to the equation

$$\frac{\partial H}{\partial t}(\nu, t) = -sgn(\beta)K^{\beta}. \tag{1.3}$$

From (1.3), $u(x,t) = H(x, -1, t)$, $x \in \mathbb{R}^n$ satisfies

$$-u_t(\det D^2u)^{\beta} = sgn(\beta)(1 + |x|^2)^{[1-\beta(n+2)]/2}. \tag{1.4}$$

When $\beta = -\frac{1}{n}$, (1.4) leads to (1.1) with $f = 0$.

When $\beta = 1$, (1.4) becomes the following equation

$$-u_t \det D^2u = f(x,t). \tag{1.5}$$

Equation (1.5) is nonlinear both in u_t and D^2u and is parabolic if $u(x,t)$ is strictly decreasing in t and convex in x. This operator plays an essential role in the parabolic version of the Alexandrov-Bakelman-Pucci maximum principle proved in [Kry], [Tso2], which is an indispensable tool for fully nonlinear parabolic equations. Tso [Tso1] also used (1.5) to study the Gauss flow (1.2) with $\beta = 1$ and established the existence and some asymptotic estimates. Firey's conjecture about limiting shape of convex surfaces under the Gauss flow (1.2) with $\beta = 1$ was proved in [An2]. Recently, analytic aspect of (1.5) has been also studied. The Schauder $C^{2,\alpha}$ estimates were established in [W-W1] and $W^{2,p}$ estimates were proved in [G-H3]. A Calabi type result about entire solutions for (1.5) was obtained in [G-H1].

Equation (1.1) with $A(x) = 1$ has recently been investigated by some authors. It was first introduced in [Kry]. Ivochikina and Ladyzhenskaya [I-L] considered the initial-boundary value problem. They proved the existence of classical solutions if given data satisfy certain smoothness and compatibility assumptions. In the case of less smooth given data, the Bernstein technique is not enough for obtaining $C^{2,\alpha}$ estimates. In that case, [W-W2] adapted a nonlinear perturbation that Caffarelli had used for the elliptic Monge-Ampère equation to study (1.1) with $A(x) = 1$ and established the Caffarelli-Schauder type estimates if $f(x,t)$ is Lipschitz together with other assumptions.

It is our purpose in this paper to establish Hölder estimates of Du and L^p estimates of D^2u for solutions u to (1.1), when $f(x,t)$ is merely continuous but not necessarily Lipschitz continuous and f_t satisfies some one-sided boundedness condition and integrability condition as well with other assumptions. For the elliptic Monge-Ampère equation, the estimates of these types were established in [Caf1].

It seems that (1.5) is more nonlinear in appearance than (1.1). However, (1.5) has an advantage that it is the Jacobian of the associated normal mapping $\mathcal{N}$ defined by $\mathcal{N}(x,t) = (Du(x,t), xDu(x,t) - u(x,t))$. The lack of being as a

Jacobian operator for (1.1) gives rise to new difficulty. In deriving estimates about eccentricity of sections which can lead to $C^{1,\alpha}$ estimates, since (1.1) has not the same affine invariance as (1.5) when rescaling, we have to deal with a family of equations with different coefficients and need to carefully control the bounds of these coefficients. We will use the Calderón-Zygmund decomposition in terms of sections to establish $W^{2,p}$ estimates. One ingredient of the proof is the estimate of density of good sets where the Hessians of solutions are controlled. The existent approach heavily relies on the structure of Jacobian operators and is not suitable for (1.1). To overcome the difficulty, we use the concavity of the operator in (1.1) and one-sided $W^{2,\delta}$ estimates ($0 < \delta < 1$), together with the property of (1.1) that if the Hessian of a solution is bounded from above then it is also bounded from below. This approach can also be applied to (1.5) and the elliptic Monge-Ampère equation.

Although the notion of viscosity solutions to (1.1) can be introduced (see [W-W2]), for simplicity, we will work with smooth solutions which are convex in x throughout this paper. But the estimates obtained are independent of smoothness and depend only on the structure constants.

Now we state the main results of the paper.

THEOREM A. *Let $u = u(x,t)$ be convex in x and a solution to (1.1) in the cylinder $Q = \Omega \times (0,T]$ with $u = \phi$ on $\partial_p Q = \partial\Omega \times (0,T] \cup \overline{\Omega} \times \{0\}$. Assume that:*

(A1) *$A \in C(\overline{\Omega})$ and $0 < \lambda \le A(x) \le \Lambda$ in Ω;*

(A2) *$f \in C(\overline{Q})$, $f_t \in L^{n+1}(Q)$;*

(A3) *$\phi \in C^{2,1}(\overline{Q})$ is convex in x, $B_{d_0} \subset \Omega \subset B_d$ convex, $\partial\Omega \in C^{1,\alpha}$ with $\alpha > 1 - \frac{2}{n}$; and*

(A4) *there exist $\nu > 0$, $a \ge 0$ such that $f_t^+ \le a$ in Q and*

$$\inf\left\{\inf_{\partial\Omega\times(0,T]}\phi_t,\ \inf_{\Omega\times\{0\}}\frac{(\det D^2\phi)^{\frac{1}{n}} - f(x,0)}{A(x)}\right\} + \inf_Q\frac{f}{A} - \frac{1}{2}ad^2 \ge \nu.$$

Then:

(i) *$\|u_t\|_{L^\infty(Q)} \le M_1$ and $A(x)u_t + f(x,t) \ge \lambda\nu > 0$ in Q, where M_1 depends only on structure constants above.*

(ii) *For any $0 < \beta < 1$, $\varepsilon > 0$, $\Omega' \subset\subset \Omega$*

$$[D_x u]_{C^{\beta,\beta/2}(\Omega'\times[\varepsilon,T])} \le M_2,$$

where M_2 depends only on β, ε, dist$(\Omega', \partial\Omega)$, and structure constants in (A1)–(A4).

(iii) *For any $1 \le p < \infty$, $\varepsilon > 0$, $\Omega' \subset\subset \Omega$*

$$\|D_x^2 u\|_{L^p(\Omega'\times[\varepsilon,T])} \le M_3,$$

where M_3 depends only on p, ε, dist$(\Omega', \partial\Omega)$, and structure constants in (A1)–(A4).

Remark B. All conclusions of Theorem A still hold, if assumption (A4) in Theorem A is replaced by the following assumption:

(A4)*: there exist $\nu > 0$, $a^* \geq 0$ such that $\|f_t^+\|_{L^{n+1}(Q)} \leq a^*$ and

$$\inf\left\{\inf_{\partial\Omega\times(0,T]} \phi_t, \inf_{\Omega\times\{0\}} \frac{(\det D^2\phi)^{\frac{1}{n}} - f(x,0)}{A(x)}\right\} + \inf_Q \frac{f}{A} - C_n^* a^* \left(\frac{d^n}{\lambda}\right)^{\frac{1}{n+1}} \geq \nu,$$

where $C_n^* = n\left(\dfrac{2^n}{\sigma_n(n+1)^n}\right)^{1/(n+1)}$ and σ_n is the surface area of unit sphere in $\mathbb{R}^n$.

Remark C. If we assume that $A \in C^{\alpha_0}$ and $f \in C^{\alpha_0,\alpha_0/2}$ in addition to assumptions (A1) to (A3) and either (A4) or (A4)*, then the eccentricity of sections can be shown to be uniformly bounded and $C^{2+\alpha_0,1+\frac{\alpha_0}{2}}$ estimates can be established. See Theorem 3.2.

The organization of the paper is as follows. In §2, the bounds of time derivative of solutions are discussed. In §3, estimates on the eccentricity of sections and Hölder estimates of Du and D^2u are given. Finally, §4 contains the proof of L^p estimates of D^2u.

2. Bounds for time derivatives. The purpose of this section is to establish bounds for time derivatives of solutions to (1.1). Noting that if u is a solution to (1.1) then u_t satisfies the linearized equation, we will use the Alexandrov-Bakelman-Pucci maximum principle and an auxilliary function to show that u_t is bounded and $A(x)u_t + f(x,t)$ is strictly positive.

Let us define the linearized operator L_u of (1.1) by

$$L_u v = -A(x)v_t + \frac{1}{n}(\det D^2u)^{1/n}\mathrm{tr}((D^2u)^{-1}D^2v).$$

Given a smooth function v, the parabolic Alexandrov-Bakelman-Pucci maximum principle yields

$$\max_{\overline{Q}} v \leq \max_{\partial_p Q} v + C_n(\mathrm{diam}\Omega)^{\frac{n}{n+1}}\left[\iint_{\Gamma_v} |v_t \det(D^2v)|\, dxdt\right]^{\frac{1}{n+1}},$$

where

$$\Gamma_v = \{(x,t) \in Q\colon\ v_t \geq 0 \,\text{and}\, D^2v \leq 0\} \quad \text{and} \quad C_n = [n(n+1)/\sigma_n]^{\frac{1}{n+1}}.$$

Obviously, on the contact set Γ_v

$$\begin{aligned} v_t \cdot \det(D^2(-v)) &= \frac{1}{A(x)} A(x) v_t \cdot \det\left[(\det D^2 u)^{\frac{1}{n}} (D^2 u)^{-1}(-D^2 v)\right] \\ &\leq \frac{1}{nA(x)} \left[\frac{nA(x)v_t + \mathrm{tr}((\det D^2 u)^{\frac{1}{n}} \cdot (D^2 u)^{-1}(-D^2 v))}{n+1}\right]^{n+1}. \end{aligned}$$

It follows that

$$\max_{\overline{Q}} v \leq \max_{\partial_p Q} v + C\left[\iint_{\Gamma_v} (-L_u v)^{+(n+1)}\, dxdt\right]^{\frac{1}{n+1}}, \tag{2.1}$$

where $C = n[(n+1)^n \sigma_n \lambda]^{-1/(n+1)} (\mathrm{diam}\Omega)^{n/(n+1)}$.

Let u be a solution to (1.1). By differentiating (1.1), we obtain

$$L_u(u_t) = f_t.$$

It is clear that $u_t = \phi_t$ on $\partial\Omega \times (0,T]$ and

$$A(x) u_t = (\det D^2 \phi)^{1/n} - f(x,0) \quad \text{on } \Omega \times \{t = 0\}.$$

Thus, applying (2.1) to u_t yields

$$\begin{aligned} \max_{\overline{Q}} u_t &\leq \max_{\partial_p Q} u_t + C\left[\iint_{\Gamma_{u_t}} (-f_t)^{+(n+1)} dxdt\right]^{\frac{1}{n+1}} \\ &\leq \max\{\max_{\partial\Omega\times(0,T]} \phi_t, \max_{\Omega\times\{0\}} \frac{(\det D^2\phi)^{\frac{1}{n}} - f(x,0)}{A(x)}\} + C\|f_t^-\|_{L^{n+1}(Q)}, \end{aligned} \tag{2.2}$$

which gives an upper bound for u_t.

Now we show that $u_t + f$ is strictly positive and hence u_t is also bounded from below.

In the case that assumption (A4) holds, by observing that

$$\frac{1}{n} (\det D^2 u)^{\frac{1}{n}} \cdot \mathrm{tr}((D^2 u)^{-1}) \geq 1,$$

we obtain that $w = u_t - \frac{1}{2} a |x - x_0|^2$ satisfies

$$L_u(w) \leq f_t - a \leq 0 \quad \text{in } Q.$$

Without loss of generality, one may assume $\Omega \subset B_d(x_0)$ in assumption (A3). By the maximum principle

$$\inf_Q u_t \geq \inf_Q w = \inf_{\partial_p Q} w \geq \inf_{\partial_p Q} u_t - \frac{1}{2}ad^2.$$

Therefore by assumption (A4)

$$\begin{aligned} A(x)u_t + f(x,t) &\geq A(x)\left[\inf_{\partial_p Q} u_t - \frac{1}{2}ad^2 + \inf_Q \frac{f}{A}\right] \\ &\geq A(x)\left[\inf\left\{\inf_{\partial\Omega\times(0,T]} \phi_t, \inf_{\Omega\times\{0\}} \frac{(\det D^2\phi)^{\frac{1}{n}} - f(x,0)}{A(x)}\right\}\right. \\ &\qquad \left. + \inf_Q \frac{f}{A} - \frac{1}{2}ad^2\right] \geq \lambda\nu. \end{aligned} \tag{2.3}$$

If assumption (A4)* holds, by applying (2.1) to $-u_t$, we obtain

$$\begin{aligned} A(x)u_t + f(x,t) &\geq A(x)\left[\inf_{\partial_p Q} u_t - C\|f_t^+\|_{L^{n+1}(Q)} + \inf_Q \frac{f}{A}\right] \\ &\geq A(x)\left[\inf_{\partial_p Q} u_t + \inf_Q \frac{f}{A} - C_n^* a^* \left(\frac{d^n}{\lambda}\right)^{\frac{1}{n+1}}\right] \\ &\geq \lambda\nu. \end{aligned} \tag{2.4}$$

By (2.2), (2.3), and (2.4), we get the following result.

THEOREM 2.1. *Let u be the solution of (1.1) in Q with $u = \phi$ on $\partial_p Q$. Suppose that assumptions (A1) to (A3) and either (A4) or (A4)* hold. Then $u_t \in L^\infty(Q)$ and $A(x)u_t + f(x,t) \geq \lambda\nu > 0$ in Q.*

We make some remarks about assumption (A4). When ϕ satisfies

$$-A(x)\phi_t + (\det D^2\phi)^{\frac{1}{n}} = f \quad \text{on } \overline{\Omega} \times \{t = 0\},$$

assumption (A4) is reduced to

$$\inf_{\partial_p Q} \phi_t + \inf_Q \frac{f}{A} - \frac{1}{2}ad^2 \geq \nu.$$

In the case of the homogeneous equation (i.e., $f = 0$), if ϕ is strictly convex in x and strictly increasing in t, then (A4) holds.

3. Eccentricity of sections and Hölder estimates for Du**.** In this section, we will use a perturbation argument to investigate the eccentricity of sections and derive Hölder estimates for the gradient and Hessian of solutions to (1.1).

Let $u(x,t)$ be a solution of (1.1) with initial-boundary value ϕ. By Theorem 2.1, $u_t \in L^\infty(Q)$ and $A(x)u_t + f(x,t) \geq \lambda\nu$. Clearly, $v(x,t) = u(x,t) - Mt$ has the same regularity as $u(x,t)$, and is strictly decreasing in t and convex in x for large M. Without loss of generality (Otherwise replace $u(x,t)$ by $v(x,t)$), we can assume from now onwards that $u(x,t)$ satisfies

$$0 < m_1 \leq -u_t \leq m_2, \tag{3.1}$$

and

$$0 < \lambda_1 \leq -u_t \det D^2 u \leq \lambda_2, \quad \text{in } Q. \tag{3.2}$$

For $z_0 = (x_0,t_0) \in Q$, recall the section $Q_h(z_0) = Q_h(u,z_0)$ is defined by

$$Q_h(z_0) = \{(x,t) \in Q\colon\ u(x,t) \leq \ell_{z_0}(x) + h \text{ and } t \leq t_0\},$$

where $\ell_{z_0}(x) = u(x_0,t_0) + Du(x_0,t_0)(x - x_0)$, and the elliptic section $S_h(u;x_0|t_0) = S_h(x_0|t_0)$ is defined by

$$S_h(x_0|t_0) = \{x \in \Omega\colon\ u(x,t_0) \leq \ell_{z_0}(x) + h\}.$$

The parabolic boundary $\partial_p Q_h(z_0)$ of $Q_h(z_0)$ is given by

$$\partial_p Q_h(z_0) = \{(x,t) \in Q\colon\ u(x,t) = \ell_{z_0}(x) + h \text{ and } t \leq t_0\}.$$

Let us recall some facts about sections. It follows from (3.1) that there exists c_1, $c_2 > 0$ such that for $0 < \theta < 1$

$$S_{\theta h}(x_0|t_0) \times (t_0 - c_1(1-\theta)h, t_0] \subset Q_h(z_0) \subset S_h(x_0|t_0) \times (t_0 - c_2 h, t_0].$$

By [G-H3, Theorem 6.3], for $(x_0,t_0) \in \Omega' \times (\varepsilon_0, T]$ with $\Omega' \subset\subset \Omega$, there exists h_0 such that

$$Q_h(z_0) \subset\subset Q \qquad \text{for } h \leq h_0,$$

and by [G-H2, Theorem 2.3], $\operatorname{diam}(Q_h(z_0)) \to 0$ as $h \to 0$.

Since A and f are continuous, $f \geq \lambda(\nu + m_1)$, u_t is bounded and $\operatorname{diam} Q_h(x_0,t_0) \longrightarrow 0$, for $\varepsilon > 0$, there exists $h_0 > 0$ such that

$$(1-\varepsilon)f(x_0,t_0) \leq -A(x_0)u_t + (\det D^2 u)^{1/n} \leq (1+\varepsilon)f(x_0,t_0), \quad \text{in } Q_{h_0}(x_0,t_0). \tag{3.3}$$

We now normalize (or rescale) u and Q_{h_0}. From Fritz John's Lemma, there exists an ellipsoid E such that

$$\gamma_0 E \subset S_{h_0}(x_0|t_0) \subset E.$$

Furthermore, by the theory of the elliptic Monge-Ampère equation, E can be chosen such that E is centered at (x_0, t_0) and $\gamma_0 E \subset S_{\theta_0 h_0}(x_0|t_0)$ for some $0 < \theta_0 < 1$. Let T be the invertible affine transformation satisfying $TE = B_1$ and $Tx_0 = 0$, where B_1 is the unit ball in $\mathbb{R}^n$ centered at the origin. Define the transformation $\mathcal{T}$ by

$$(y, s) = \mathcal{T}(x, t) = \left(Tx, \frac{t - t_0}{Kh_0}\right)$$

and set

$$\begin{aligned} u^*(y, s) &= \frac{1}{C}\left[(u - \ell_{z_0})(\mathcal{T}^{-1}(y, s)) - h_0\right] \\ &= \frac{1}{C}\left[u(T^{-1}y, t_0 + Kh_0 s) - \ell_{z_0}(T^{-1}y) - h_0\right], \end{aligned}$$

where $z_0 = (x_0, t_0)$ and $\ell_{z_0}(x)$ is the supporting affine function or hyperplane to $u(\cdot, t_0)$ at $x = x_0$. From (3.3), one obtains

$$(3.4)\quad (1 - \varepsilon)f(x_0, t_0) \leq -\frac{CA(x_0)}{Kh_0}u^*_s + \frac{C(\det D^2u^*)^{1/n}}{|\det T^{-1}|^{2/n}} \leq (1 + \varepsilon)f(x_0, t_0) \quad \text{in } Q^*_{h_0},$$

where $Q^*_{h_0} = \mathcal{T}Q_{h_0}(x_0, t_0)$.

Choose $C = |\det T^{-1}|^{2/n}$. By (3.1), (3.2), and $\det D^2u^* = \det D^2u$, it follows that

$$\lambda_1/m_2 \leq \det D^2u^* \leq \lambda_2/m_1, \quad \text{in } TS_{h_0}(x_0|t_0).$$

By the theory of the elliptic Monge-Ampère equation

$$h_0^n \approx |S_{h_0}(x_0|t_0)|^2 \approx |E|^2 \approx |\det T^{-1}|^2.$$

Here and throughout the paper, we use the symbol $a \approx b$ to denote that the quantity a/b is bounded by two positive universal constants from above and below. Thus, $C/h_0 \approx 1$. We now choose $K = C/h_0 \approx 1$ and get

$$(3.5)\qquad B_{\gamma_0} \times (-c_3, 0] \subset Q^*_{h_0} = \mathcal{T}Q_{h_0}(x_0, t_0) \subset B_1 \times (-c_4, 0].$$

Note that $Q^*_{h_0}$ is not a cylindrical domain but a bowl-shaped domain defined by u^*. Let $\partial_p Q^*_{h_0} = \mathcal{T}\partial_p Q_{h_0}(x_0, t_0)$. From the above argument, we conclude that u^*

satisfies

$$(3.6)\quad (1-\varepsilon)f(x_0,t_0) \le -A(x_0)u_s^* + (\det D^2u^*)^{1/n} \le (1+\varepsilon)f(x_0,t_0) \quad \text{in } Q_{h_0}^*,$$
$$u^* = 0, \quad \text{on } \partial_p Q_{h_0}^*, \quad 0 < m_1 \le -u_s^* \le m_2.$$

Let $\mathcal{P}$ be the standard paraboloid defined by

$$(3.7)\qquad \mathcal{P} = \{(x,t)\colon\ t \le 0 \text{ and } \tfrac{1}{2}|x|^2 - t \le 1\},$$

and denote the parabolic μ-dilation of set S by $\mu S = \{(\mu x, \mu^2 t)\colon\ (x,t) \in S\}$.

To study the regularity of u is reduced to investigate the properties of u^*. Now we prove the following lemma for the normalized solution u^*. To simplify the notation, we still use u instead of u^*.

LEMMA 3.1. *Let u be a strictly parabolically convex function in Q and satisfy*

$$(3.8)\quad (1-\varepsilon)f(z_0) \le -A(x_0)u_t + (\det D^2u)^{1/n} \le (1+\varepsilon)f(z_0) \quad \text{in } Q,$$
$$u = 0 \qquad \text{on } \partial_p Q,$$
$$(3.9)\qquad m_1 \le -u_t \le m_2 \quad \text{in } Q,$$
$$(3.10)\qquad A(x_0)u_t + f(z_0) \ge \lambda\nu/2 \quad \text{in } Q,$$

where $Q = \{(x,t)\colon\ u < 0 \text{ and } t \le 0\}$ is a bowl-shaped domain and $\partial_p Q = \{(x,t)\colon\ u = 0 \text{ and } t \le 0\}$ is the parabolic boundary of Q.

Assume that $u(0,0) = \min_Q u$ and that

$$(3.11)\qquad B_{\gamma_0} \times (-c_3, 0] \subset Q \subset B_1 \times (-c_4, 0].$$

Then there exists a linear transformation $\mathcal{T}\colon\ (x,t) \longrightarrow (Tx, a\,t)$ such that

$$(3.12)\qquad m_1/2 \le a \le 2m_2$$
$$(3.13)\qquad C^{-1} \le \|Tx\| \le C, \quad \text{for } \|x\| = 1$$
$$(3.14)\qquad -aA(x_0) + f(z_0) = (\det T)^{2/n} \ge \lambda\nu/4$$

and for small $\mu > 0$ with $\varepsilon \le \mu^2$

$$(3.15)\qquad [1 - C(\sqrt{\varepsilon/\mu} + \sqrt{\mu})]\mathcal{P} \subset \sqrt{\mu}^{\,-1}\mathcal{T}Q_\mu(u,(0,0))$$
$$\subset [1 + C(\sqrt{\varepsilon/\mu} + \sqrt{\mu})]\mathcal{P},$$

where $\sqrt{\mu}^{\,-1}\mathcal{T}(x,t) = (\sqrt{\mu}^{\,-1}Tx, \mu^{-1}a\,t)$.

Proof. Let $w(x,t)$, convex in x, be the smooth solution to the equation

$$-A(x_0)w_t + (\det D^2 w)^{1/n} = f(z_0) \quad \text{in } Q \tag{3.16}$$

with the boundary value $w = 0$ on $\partial_p Q$. For the existence of w, see [W-W2].

We first establish some estimates for w. Similar to [I-L, Lemma 2.2], one can easily show that the comparison principle holds for (3.8). Therefore, we obtain

$$(1+\varepsilon)w \le u \le (1-\varepsilon)w \quad \text{in } Q.$$

Since $u = w = 0$ on $\partial_p Q$, it implies that

$$(1+\varepsilon)w_t \le u_t \le (1-\varepsilon)w_t \quad \text{on } \partial_p Q.$$

For small $\varepsilon > 0$, on $\partial_p Q$ we obtain

$$\tfrac{1}{2}m_1 \le \frac{-u_t}{1+\varepsilon} \le -w_t \le \frac{-u_t}{1-\varepsilon} \le 2m_2,$$
$$A(x_0)w_t + f(z_0) \ge \frac{A(x_0)u_t}{1-\varepsilon} + f(z_0) \ge \frac{1}{4}\lambda\nu.$$

On the other hand, both w_t and $A(x_0)w_t + f(z_0)$ satisfy the linearized equation

$$-A(x_0)v_t + \frac{1}{n}(\det D^2 w)^{1/n}\operatorname{tr}((D^2 w)^{-1} D^2 v) = 0.$$

By the maximum principle again

$$m_1/2 \le -w_t \le 2m_2 \qquad \text{in } Q,$$
$$(\det D^2 w)^{1/n} = A(x_0)w_t + f(z_0) \ge \tfrac{1}{4}\lambda\nu \qquad \text{in } Q.$$

Together with the estimate of Pogorelov type for (3.16) in [W-W2], interior $C^{2,1}$ estimate for w follows, and hence by the theory of fully nonlinear equations, one obtains estimates for higher order derivatives of w.

Now compare $Q_\mu(u,(0,0))$ with $Q_\mu(w,(0,0))$ for small $\mu > 0$. We use C to denote universal constants.

Recall $u(0,0) = \min_Q u \approx \mathit{const}$ and let $\min_Q w = w(x_1, 0)$. Since $|u - w| \le C\varepsilon$, $|w(x_1,0) - w(0,0)| \le C\varepsilon$ and $0 \in S_{C\varepsilon}(w; x_1|0) \subset B_{C\sqrt{\varepsilon}}(x_1)$. Then $|x_1| \le C\sqrt{\varepsilon}$ and

$$|Dw(0,0)| = |Dw(x_1,0) - Dw(0,0)| \le C|x_1| \le C\sqrt{\varepsilon}.$$

We note that

$$w(x,t) - w(0,0) - Dw(0,0)x = u(x,t) - u(0,0) + [O(\varepsilon) - Dw(0,0)x].$$

Therefore, if $\varepsilon \leq \mu^2$ and μ is small, we obtain

$$Q_{\mu - C\sqrt{\varepsilon}\sqrt{\mu}}(w,(0,0)) \subset Q_\mu(u,(0,0)) \subset Q_{\mu + C\sqrt{\varepsilon}\sqrt{\mu}}(w,(0,0)). \tag{3.17}$$

We now claim that for $\delta << \mu$ (i.e., δ/μ is small)

$$\partial_p Q_{\mu+\delta}(w,(0,0)) \subset N_{C\delta/\sqrt{\mu}}\left(\partial_p Q_\mu(w,(0,0))\right), \tag{3.18}$$

where N_δ is the δ-neighborhood with respect to the distance

$$d_\mu\big((x,t),(y,s)\big) = |x-y| + \sqrt{\mu}^{\,-1}|t-s|.$$

To prove (3.18), let $(x,t) \in \partial_p Q_{\mu+\delta}(w,(0,0))$ and distinguish two cases to discuss. Without loss of generality, one can assume that $Dw(0,0) = 0$. In the first case that $x \in S_\mu(w;0|0)$, there exists $t < t_1 < 0$ such that $(x,t_1) \in \partial_p Q_\mu(w,(0,0))$. Then $w(x,t) - w(x,t_1) = \delta$ and $|t_1 - t| = C\delta$. In the second case that $x \notin S_\mu(w;0|0)$, $w(x,t) - w(x,0) \leq \delta$ and $|t| \leq C\delta$. Let x_2 be the intersecting point of $\partial S_\mu(w;0|0)$ and the segment between 0 and x. Because $S_\mu(w;0|0) \approx B_{C\sqrt{\mu}}(0)$, one obtains

$$\delta \geq |w(x,0) - w(x_2,0)| = |Dw| \cdot |x - x_2| \approx c\sqrt{\mu}|x - x_2|.$$

It yields that $|x_2 - x| \leq C\delta/\sqrt{\mu}$. Thus, we complete the proof of (3.18).

From (3.17) and (3.18), $\partial_p Q_\mu(u;(0,0))$ is in $C\sqrt{\varepsilon}$-neighborhood of $\partial_p Q_\mu(w,(0,0))$. We next compare $Q_\mu(w;(0,0))$ with the paraboloids.

Let $\mathcal{P}_w$ be the paraboloid associated with w given by

$$\mathcal{P}_w = \left\{(x,t)\colon\ t \leq 0 \text{ and } \tfrac{1}{2}D_{ij}w(0,0)x_ix_j + w_t(0,0)t \leq 1\right\}$$

and recall the parabolic dilation $\sqrt{\mu}P_w = \left\{(\sqrt{\mu}x, \mu t)\colon\ (x,t) \in \mathcal{P}_w\right\}$. We now claim that

$$\partial_p Q_\mu(w,(0,0)) \subset N_{C\mu}\left(\partial_p \sqrt{\mu}\mathcal{P}_w\right) \tag{3.19}$$

where N_δ is the δ-neighborhood with respect to the distance d_μ.

To prove (3.19), it is equivalent to show

$$\partial_p Q_\mu(w,(0,0)) \subset (1 + C\sqrt{\mu})\sqrt{\mu}\mathcal{P}_w - (1 - C\sqrt{\mu})\sqrt{\mu}\mathcal{P}_w.$$

If $(x,t) \in \partial_p\left((1+C\sqrt{\mu})\sqrt{\mu}\mathcal{P}_w\right)$, then by the Taylor formula

$$
\begin{aligned}
(3.20)\ w(x,t)-(w(0,0)+Dw(0,0)x) &= \frac{1}{2}D_{ij}w(0,0)x_ix_j + w_t(0,0)t \\
&\quad + O(|D^3w||x|^3+|Dw_t||x||t|+|w_{tt}|t^2) \\
&\geq \left[(1+C\sqrt{\mu})\sqrt{\mu}\right]^2 - K(|x|^3+|x||t|+t^2),
\end{aligned}
$$

where K is a universal constant depending on the bounds of D_x^3w, Dw_t and w_{tt}. For $(x,t) \in \partial_p\left((1+C\sqrt{\mu})\sqrt{\mu}\mathcal{P}_w\right)$, we have that $|x| \leq K\sqrt{\mu}$ and $|t| \leq K\mu$. If $\mu \ll 1$ and $C \gg K$, then

$$w(x,t)-(w(0,0)+Dw(0,0)x) \geq (1+2C\sqrt{\mu})\mu - K(\mu^{3/2}+\mu^2) > \mu.$$

It can be shown similarly that $(1-C\sqrt{\mu})\sqrt{\mu}\mathcal{P}_w$ is contained inside $Q_\mu(w,(0,0))$. Thus we complete the proof of (3.19).

By (3.17), (3.18), and (3.19), we get

$$\partial_p Q_\mu(u,(0,0)) \subset N_{C(\mu+\sqrt{\varepsilon})}\left(\partial_p\sqrt{\mu}\mathcal{P}_w\right). \tag{3.21}$$

Now find the affine transformation $\mathcal{T}$. Since D^2w is positively definite, we can write $D^2w(0,0) = T^t \cdot T$, where T is the composition of rotation and dilation. Set $a = -w_t(0,0)$ and

$$(y,s) = \mathcal{T}(x,t) = (Tx, a\,t).$$

It is easy to verify that (3.12)–(3.14) hold. (3.15) follows from (3.21). The proof of Lemma 3.1 is completed. □

If the shape of Q is close to that of the standard paraboloid $\mathcal{P}$, then one can get better estimates for $\mathcal{T}$ and $Q_\mu(u,(0,0))$. For our purpose, we state the result in the following fashion.

LEMMA 3.2. *Let u be a strictly parabolically convex function in Q satisfying*

$$(1-\varepsilon)f(z_0) \leq -aA(x_0)u_t + B(\det D^2u)^{1/n} \leq (1+\varepsilon)f(z_0) \quad \text{in } Q, \tag{3.22}$$

$$u = 0 \qquad \text{on } \partial_p Q,$$

$$m_1 \leq -au_t \leq m_2 \quad \text{in } Q, \tag{3.23}$$

$$aA(x_0)u_t + f(z_0) \geq \lambda\nu/2 \quad \text{in } Q. \tag{3.24}$$

Assume that $u(0,0) = \min_Q u$ *and that*

$$(1-\sigma)\mathcal{P} \subset Q \subset (1+\sigma)\mathcal{P} \tag{3.25}$$

$$m_1/2 \le a \le 2m_2 \tag{3.26}$$

$$-aA(x_0) + f(z_0) = B \ge \lambda\nu/4. \tag{3.27}$$

Then there exists a linear transformation $\mathcal{T}\colon\ (x,t) \longrightarrow (Tx, a^* t)$ *such that*

$$|a^* - 1| \le C\sigma \tag{3.28}$$

$$1 - C\sigma \le \|Tx\| \le 1 + C\sigma, \quad \textit{for } \|x\| = 1 \tag{3.29}$$

$$m_1/2 \le aa^* \le 2m_2 \tag{3.30}$$

$$-aA(x_0)a^* + f(z_0) = B(\det T)^{2/n} \ge \lambda\nu/4 \tag{3.31}$$

and for small $\mu, \sigma > 0$ *with* $\varepsilon \le \mu^2$

$$\begin{aligned}[1 - C(\sqrt{\varepsilon/\mu} + \sigma\sqrt{\mu})]\mathcal{P} \ &\subset\ \sqrt{\mu}^{\,-1}\mathcal{T}Q_\mu(u,(0,0)) \\ &\subset\ [1 + C(\sqrt{\varepsilon/\mu} + \sigma\sqrt{\mu})]\mathcal{P}.\end{aligned} \tag{3.32}$$

Proof. Let w be convex in x and satisfy

$$-\,aA(x_0)w_t + B(\det D^2 w)^{1/n} = f(z_0), \quad \text{in } Q \tag{3.33}$$

and $w = 0$ on $\partial_p Q$. As in the proof of Lemma 3.1, by the comparison principle

$$(1+\varepsilon)w \le u \le (1-\varepsilon)w, \quad \text{in } Q.$$

Similar to the proof of Lemma 3.1, one can obtain the estimates

$$\begin{aligned} m_1/2 \le -aw_t \le 2m_2, &\quad \text{in } Q \\ aA(x_0)w_t + f(z_0) \ge \lambda\nu/4, &\quad \text{in } Q.\end{aligned} \tag{3.34}$$

From these estimates and the Pogorelov estimate, interior estimates for higher order derivatives of w follows.

Let $P = \frac{1}{2}|x|^2 - t - 1$. By (3.27), it is easy to check that the functions $P \pm 3\sigma$ are also solutions to (3.33). (3.25) implies that $P - 3\sigma \le 0 \le P + 3\sigma$ on $\partial_p Q$. By the comparison principle

$$-3\sigma \le w - \left(\frac{1}{2}|x|^2 - t - 1\right) \le 3\sigma \quad \text{in } Q.$$

Since both w and P satisfy (3.33), $v = w - P$ satisfies the following linear equation of uniformly parabolic type

$$-aA(x_0)v_t + B\operatorname{tr}(D(x,t)D^2 v) = 0,$$

where $D(x,t) = \int_0^1 \frac{1}{n} \det(\theta D^2 w + (1-\theta)I)^{1/n}(\theta D^2 w + (1-\theta)I)^{-1}\, d\theta$, and I is the $n \times n$ unit matrix. By interior Schauder estimates

$$\|w - P\|_{C^{2,1}_{\mathrm{loc}}} \le C\|w - P\|_{L^\infty} \le C\sigma.$$

In particular, $|D^2 w(0,0) - I| \le C\sigma$ and $|w_t(0,0) + 1| \le C\sigma$. It is also easy to verify that the functions $(w-P)_t$ and $D(w-P)$ satisfy the linearized equation

$$-aA(x_0)v_t + B\frac{1}{n}\det(D^2 w)^{1/n}\operatorname{tr}((D^2 w)^{-1} D^2 v) = 0.$$

Again by interior Schauder estimates

$$\begin{aligned}\|(w-P)_{tt}\|_{L^\infty_{\mathrm{loc}}} + \|D(w-P)_t\|_{L^\infty_{\mathrm{loc}}} + \|D^3(w-P)\|_{L^\infty_{\mathrm{loc}}} \\ \le\; C\|(w-P)_t\|_{L^\infty_{\mathrm{loc}}} + C\|D(w-P)\|_{L^\infty_{\mathrm{loc}}} \le C\sigma.\end{aligned}$$

Therefore

$$\|w_{tt}\|_{L^\infty_{\mathrm{loc}}} + \|Dw_t\|_{L^\infty_{\mathrm{loc}}} + \|D^3 w\|_{L^\infty_{\mathrm{loc}}} \le C\sigma.$$

By (3.20) and noting that in current case K can be chosen as $C\sigma$, we have

$$\partial_p Q_\mu(w,(0,0)) \subset N_{C\sigma\mu}\left(\partial_p \sqrt{\mu}\mathcal{P}_w\right),$$

where N_δ denotes δ-neighborhood with respect to the distance $d_\mu((x,t),(y,s)) = |x-y| + \sqrt{\mu}^{-1}|t-s|$ and

$$\mathcal{P}_w = \left\{(x,t)\colon\ t \le 0 \text{ and } \frac{1}{2}D_{ij}w(0,0)x_i x_j + w_t(0,0)t \le 1\right\}.$$

Similar to (3.21), we obtain

$$\partial_p Q_\mu(u,(0,0)) \subset N_{C(\sigma\mu+\sqrt{\varepsilon})}\left(\partial_p \sqrt{\mu}\mathcal{P}_w\right). \tag{3.35}$$

Let $D^2 w(0,0) = T^t T$ and $a^* = -w_t(0,0)$. Define the linear transformation $\mathcal{T}$ by $\mathcal{T}(x,t) = (Tx, a^* t)$. Obviously, (3.28) holds and (3.29) follows from the following estimate

$$(1 - C\sigma)|y|^2 \le y^t \cdot D^2 w(0,0) \cdot y = \|Ty\|^2 \le (1 + C\sigma)|y|^2.$$

Evidently (3.34) implies (3.30) and (3.31), and it is easy to show that (3.32) holds from (3.35). So Lemma 3.2 is proved. □

We apply Lemma 3.1 and Lemma 3.2 to get estimates about eccentricity of sections.

LEMMA 3.3. *Suppose that the assumptions in Lemma 3.1 hold and further assume that there is a sequence* $\{\varepsilon_k\}_{k=0}^{\infty}$ *with* $0 < \varepsilon_{k+1} \le \varepsilon_k$ *and* $\varepsilon_0 = \varepsilon \le \mu^2$ *such that for* $k \ge 1$

$$(1 - \varepsilon_k) f(z_0) \le -A(x_0)u_t + (\det D^2 u)^{1/n} \le (1 + \varepsilon_k) f(z_0) \quad \textit{in } Q_{\mu^k}(u, (0,0)).$$

Then there exist linear transformations $\mathcal{T}_k$: $(x,t) \longrightarrow (T_k x, a_k t)$ *with* $a_k > 0$ *satisfying*

$$\begin{gathered} C^{-1} \le \|T_1 x\| \le C, \quad \textit{for } \|x\| = 1 \\ |a_k - 1| \le C\delta_{k-1}, \quad \textit{for } k \ge 2 \\ 1 - C\delta_{k-1} \le \|T_k x\| \le 1 + C\delta_{k-1}, \quad \textit{for } k \ge 2,\ \|x\| = 1 \\ m_1/2 \le a_1 \cdots a_k \le 2m_2, \quad \textit{for } k \ge 1 \\ -A(x_0)a_1 \cdots a_k + f(z_0) = \det(T_1 \cdots T_k)^{2/n} \ge \lambda\nu/4 \\ (1 - \delta_k)\mathcal{P} \subset \mu^{-k/2}\mathcal{T}_k \cdots \mathcal{T}_1 Q_{\mu^k}(u, (0,0)) \subset (1 + \delta_k)\mathcal{P}, \end{gathered}$$

where $\delta_0 = 1$ *and* $\delta_k = C(\sqrt{\varepsilon_{k-1}/\mu} + \delta_{k-1}\sqrt{\mu})$ *for* $k \ge 1$.

Proof. By Lemma 3.1, there exists $\mathcal{T}_1(x,t) = (T_1 x, a_1 t)$ such that

$$\begin{gathered} C^{-1} \le \|T_1 x\| \le C, \quad \text{for } \|x\| = 1 \\ m_1/2 \le a_1 \le 2m_2 \\ -A(x_0)a_1 + f(z_0) = (\det T_1)^{2/n} \ge \lambda\nu/4 \\ (1 - \delta_1)\mathcal{P} \subset \mu^{-1/2}\mathcal{T}_1 Q_\mu(u, (0,0)) \subset (1 + \delta_1)\mathcal{P}. \end{gathered}$$

Let

$$u_1(x,t) = \mu^{-1}\left[u(\sqrt{\mu}\, T_1^{-1}x, \mu t/a_1) - (\min_Q u + \mu)\right]$$

and

$$Q_1 = \sqrt{\mu}^{-1}\mathcal{T}_1 Q_\mu(u,(0,0)).$$

Simple computations give

$$(1-\varepsilon_1)f(z_0) \le -A(x_0)a_1(u_1)_t + (\det T_1)^{2/n}(\det D^2u_1)^{1/n} \le (1+\varepsilon_1)f(z_0) \quad \text{in } Q_1.$$

It is easy to verify that the assumptions in Lemma 3.2 hold, and one can apply Lemma 3.2 to u_1 in Q_1. Hence, there exists a linear transformation $\mathcal{T}_2(x,t) = (T_2x, a_2t)$ such that

$$\begin{gathered}
|a_2 - 1| \le C\delta_1 \\
1 - C\delta_1 \le \|T_2x\| \le 1 + C\delta_1, \quad \text{for } \|x\| = 1 \\
m_1/2 \le a_1a_2 \le 2m_2 \\
-A(x_0)a_1a_2 + f(z_0) = \det(T_1T_2)^{2/n} \ge \lambda\nu/4 \\
(1-\delta_2)\mathcal{P} \subset \sqrt{\mu}^{-1}\mathcal{T}_2 Q_\mu(u_1,(0,0)) = \mu^{-1}\mathcal{T}_2\mathcal{T}_1 Q_{\mu^2}(u,(0,0)) \subset (1+\delta_2)\mathcal{P}.
\end{gathered}$$

We now use the induction to proceed. Assume that the conclusions in the lemma are valid for the case k. As above, consider the normalized solution and domain by

$$u_k(x,t) = \mu^{-k}\left[u(\mu^{k/2}(T_k\cdots T_1)^{-1}x, \mu^k t/(a_1\cdots a_k)) - (\min_Q u + \mu^k)\right]$$

and $Q_k = \mu^{-k/2}\mathcal{T}_k\cdots\mathcal{T}_1 Q_{\mu^k}(u,(0,0))$. One can easily check that u_k satisfies

$$(1-\varepsilon_k)f(z_0) \le -A(x_0)a_1\cdots a_k(u_k)_t + (\det T_1\cdots T_k)^{2/n}(\det D^2u_k)^{1/n} \le (1+\varepsilon_k)f(z_0),$$

in Q_k. The induction hypotheses imply that the assumptions in Lemma 3.2 are valid. By applying Lemma 3.2 to u_k in Q_k, there exists a linear transfromation $\mathcal{T}_{k+1}(x,t) = (T_{k+1}x, a_{k+1}t)$ such that

$$\begin{gathered}
|a_{k+1} - 1| \le C\delta_k \\
1 - C\delta_k \le \|T_{k+1}x\| \le 1 + C\delta_k, \quad \text{for } \|x\| = 1 \\
m_1/2 \le a_1\cdots a_k a_{k+1} \le 2m_2 \\
-A(x_0)a_1\cdots a_{k+1} + f(z_0) = \det(T_1\cdots T_{k+1})^{2/n} \ge \lambda\nu/4 \\
(1-\delta_{k+1})\mathcal{P} \subset \sqrt{\mu}^{-1}\mathcal{T}_{k+1}Q_\mu(u_k,(0,0)) \\
= \mu^{-(k+1)/2}\mathcal{T}_{k+1}\cdots\mathcal{T}_1 Q_{\mu^{k+1}}(u,(0,0)) \subset (1+\delta_{k+1})\mathcal{P}.
\end{gathered}$$

The proof of Lemma 3.3 is done. □

We now give $C^{1,\alpha}$ estimates.

THEOREM 3.1. *Let u be a strictly parabolically convex function in Q and satisfy*

$$(1-\varepsilon)f(z_0) \leq -A(x_0)u_t + (\det D^2u)^{1/n} \leq (1+\varepsilon)f(z_0) \quad \text{in } Q,$$
$$u = 0 \qquad \text{on } \partial_p Q,$$
$$m_1 \leq -u_t \leq m_2 \quad \text{in } Q,$$
$$A(x_0)u_t + f(z_0) \geq \lambda\nu/2 \quad \text{in } Q,$$

where $Q = \{(x,t)\colon\ u < 0 \text{ and } t \leq 0\}$ *is a bowl-shaped domain.*

Assume that $u(0,0) = \min_Q u$ *and that* $B_{\gamma_0} \times (-c_3, 0] \subset Q \subset B_1 \times (-c_4, 0]$. *Then for small* $\mu > 0$ *and* $\varepsilon \leq \mu^2$*:*

(i) *there exists* $\delta \leq C\sqrt{\mu}$ *such that for* $k \geq 0$

$$B_{C_1\left(\frac{\sqrt{\mu}}{1+C\delta}\right)^k} \times (-C_1\mu^k, 0] \subset Q_{\mu^k}(u,(0,0)) \subset B_{C_2\left(\frac{\sqrt{\mu}}{1-C\delta}\right)^k} \times (-C_2\mu^k, 0].$$

(ii) $0 \leq u(x,0) - u(0,0) \leq C|x|^{1+\beta}$ *for* $(x,0) \in Q$, *and u is* $C^{1,\beta}$ *at (0,0) with respect to x, where* $\beta = \frac{1-2\tau}{1+2\tau}$ *and* $\tau = -\frac{\ln(1+C\delta)}{\ln\mu}$.

Proof. By Lemma 3.3 with all $\varepsilon_k = \varepsilon$, there exist linear transformations $\mathcal{T}_k$ given by $\mathcal{T}_k(x,t) = (T_k x, a_k t)$ such that

$$C^{-1} \leq \|T_1 x\| \leq C, \quad \text{for } \|x\| = 1$$
$$1 - C\delta \leq \|T_k x\| \leq 1 + C\delta, \quad \text{for } k \geq 2,\ \|x\| = 1$$
$$m_1/2 \leq a_1 \cdots a_k \leq 2m_2$$
$$(1-\delta)\mu^{k/2}\mathcal{T}_1^{-1}\cdots\mathcal{T}_k^{-1}\mathcal{P} \subset Q_{\mu^k}(u,(0,0)) \subset (1+\delta)\mu^{k/2}\mathcal{T}_1^{-1}\cdots\mathcal{T}_k^{-1}\mathcal{P},$$

where $\delta \leq C(\sqrt{\varepsilon/\mu} + \sqrt{\mu}) \leq 2C\sqrt{\mu}$. By the estimates of a_k and T_k, it is easy to obtain conclusion (i).

To prove (ii), let $x \in S_{\mu^k}(u;0|0)\backslash S_{\mu^{k+1}}(u;0|0)$, i.e., $\mu^{k+1} \leq u(x,0) - u(0,0) \leq \mu^k$. From (i)

$$C_1\left(\frac{\sqrt{\mu}}{1+C\delta}\right)^{k+1} \leq |x| \leq C_2\left(\frac{\sqrt{\mu}}{1-C\delta}\right)^k.$$

Obviously $1 + C\delta = \mu^{-\tau}$ and $\mu^{(\frac{1}{2}+\tau)(k+1)} \leq C_1^{-1}|x|$. Therefore

$$|u(x,0) - u(0,0)| \leq C\mu^{-1}|x|^{\frac{2}{1+2\tau}} = C|x|^{1+\beta}.$$

So the proof of Theorem 3.1 is completed. □

Proof of Theorem A(ii). As we did in the beginning of this section, we can assume that (3.1) holds and $A(x)u_t + f(x,t) \geq \lambda\nu$. For $z_0 = (x_0, t_0) \in \Omega' \times [\varepsilon_0, T]$, let E be the Fritz John's ellipsoid of $S_{h_0}(x_0|t_0)$ and T be the affine transformation such that $TE = B_1$ and $Tx_0 = 0$. Set

$$(y,s) = \mathcal{T}(x,t) = \left(Tx, \frac{t-t_0}{Kh_0}\right),$$

$$u^*(y,s) = |\det T|^{2/n}\left[(u - \ell_{z_0})(\mathcal{T}^{-1}(y,s)) - h_0\right],$$

where $\ell_{z_0}(x)$ is the supporting affine function of $u(\cdot, t_0)$ at $x = x_0$. Let h_0 be small. If $K = |\det T|^{-2/n}h_0^{-1}$, then u^* satisfies (3.6) and $A(x_0)u_s^* + f(z_0) \geq \lambda\nu/2$, in $\mathcal{T}Q_{h_0}(x_0, t_0)$. By applying Theorem 3.1 to u^*, we have

$$|u^*(y,0) - u^*(0,0)| \leq C|y|^{1+\beta}, \quad \text{for } (y,0) \in \mathcal{T}Q_{h_0}(x_0,t_0).$$

Note that β can be made arbitrarily close to 1 as μ goes to 0. Since $h_0^n \approx |E|^2 \approx |\det T^{-1}|^2$, one obtains that $C_1(h_0) \leq \|T\| \leq C_2(h_0)$. Therefore

$$|u(x,t_0) - \ell_{z_0}(x)| \leq C|x - x_0|^{1+\beta}, \quad \text{for } (x,t_0) \in Q_{h_0}(x_0,t_0).$$

It follows that Du is C^β with respect to x and $|Du(x_1,t) - Du(x_2,t)| \leq C|x_1 - x_2|^\beta$, for (x_1,t), $(x_2,t) \in \Omega' \times [\varepsilon_0, T]$. Since $u(x,t)$ is Lipschitz in (x,t), by [L-S-U, p.78] we get

$$|Du(x_1,t_1) - Du(x_2,t_2)| \leq C\left(|x_1 - x_2| + |t_1 - t_2|^{\frac{1}{2}}\right)^\beta.$$

This completes the proof of Hölder estimates for Du. □

THEOREM 3.2. *Let $u(x,t)$ be convex in x and the solution to (1.1) in $Q = \Omega \times (0,T]$ with $u = \phi$ on $\partial_p Q$. Suppose that (A1)–(A3), and either (A4) or (A4)* hold. Assume that $A \in C^{\alpha_0}(\overline{\Omega})$ and $f \in C^{\alpha_0, \frac{\alpha_0}{2}}(\overline{Q})$. Then:*

(i) *For $\bar{\varepsilon} > 0$, $\Omega' \subset\subset \Omega$, there exist C_3, C_4, h_0 such that for $(x_0,t_0) \in \Omega' \times (\bar{\varepsilon}, T]$, $0 < h \leq h_0$*

$$B_{C_3\sqrt{h}}(x_0) \times (t_0 - C_3 h, t_0] \subset Q_h(x_0,t_0) \subset B_{C_4\sqrt{h}}(x_0) \times (t_0 - C_4 h, t_0].$$

(ii) $u \in C_{loc}^{2+\alpha_0, 1+\frac{\alpha_0}{2}}(Q)$.

Proof. The proof of Theorem 3.2 is similar to that for $C^{1,\beta}$ estimates. Since $A \in C^{\alpha_0}(\overline{\Omega})$ and $f \in C^{\alpha_0,\alpha_0/2}(\overline{Q})$, when applying Lemma 3.3, we can find a sequence of ε_k which geometrically decays to zero, and therefore we can sharpen the estimates established under the assumption that A and f are only continuous.

As before, we can assume that (3.1) holds and $A(x)u_t + f(x,t) \geq \lambda\nu$. Let $z_0 = (x_0, t_0) \in \Omega' \times (\bar{\varepsilon}, T]$, and let ellipsoid E, transformations T and $\mathcal{T}$, and u^* be as in the proof of Theorem A(ii).

It is easy to verify that $u^*(y,s)$ satisfies

$$-A^*(y)u_s^* + (\det D^2u^*)^{1/n} = f^*(y,s) \quad \text{in } Q^* = \mathcal{T}Q_{h_0}(x_0,t_0), \tag{3.36}$$

where $A^*(y) = A(T^{-1}y)$ and $f^*(y,s) = f(T^{-1}y, t_0 + Kh_0s)$. Moreover, simple calculations yield

$$[A^*]_{C^{\alpha_0}(\overline{Q}^*)} \leq \|T^{-1}\|^{\alpha_0}[A]_{C^{\alpha_0}(Q_{h_0}(z_0))},$$
$$[f^*]_{C^{\alpha_0,\frac{\alpha_0}{2}}(\overline{Q}^*)} \leq \left(\|T^{-1}\|^{\alpha_0} + (Kh_0)^{\frac{\alpha_0}{2}}\right)[f]_{C^{\alpha_0,\frac{\alpha_0}{2}}(Q_{h_0}(z_0))}.$$

Note that $\|T^{-1}\| \longrightarrow 0$, as $h_0 \longrightarrow 0$, by Theorem 3.1 or [G-H3, Lemma 4.5]. Given small $\tau_0 > 0$, we can choose h_0 such that

$$[A^*]_{C^{\alpha_0}(\overline{Q}^*)} \leq \tau_0, \quad [f^*]_{C^{\alpha_0,\alpha_0/2}(\overline{Q}^*)} \leq \tau_0.$$

Let $g^*(y,s) = f^*(y,s) - f^*(0,0) + (A^*(y) - A^*(0))u_t^*$. Noting that $f^*(0,0) = f(z_0)$ and $A^*(0) = A(x_0)$, from (3.36) we have

$$-A(x_0)u_s^* + (\det D^2u^*)^{1/n} = f(z_0) + g^*(y,s) \quad \text{in } Q^*.$$

Obviously, $|g^*(y,s)| \leq C\tau_0$ in Q^*. Set $\varepsilon_0 = C\tau_0$. If τ_0 is small enough, then by Theorem 3.1, $Q_{\mu^k}(u^*,(0,0)) \subset B_{C_2\left(\frac{\sqrt{\mu}}{1-C\delta}\right)^k} \times (-C_2\mu^k, 0]$. It is easy to check that

$$|g^*(y,s)| \leq C\tau_0\theta^k, \quad \text{in } Q_{\mu^k}(u^*,(0,0)),$$

where $\theta = (\sqrt{\mu}/(1-C\delta))^{\alpha_0}$. Let $\varepsilon_k = C\tau_0\theta^k$. By Lemma 3.3, there exist linear transformations $\mathcal{T}_k$: $(x,t) \longrightarrow (T_kx, a_kt)$ with $a_k > 0$ satisfying

$$C^{-1} \leq \|T_1x\| \leq C, \quad \text{for } \|x\| = 1$$
$$1 - C\delta_{k-1} \leq \|T_kx\| \leq 1 + C\delta_{k-1}, \quad \text{for } k \geq 2, \ \|x\| = 1$$
$$m_1/2 \leq a_1 \cdots a_k \leq 2m_2, \quad \text{for } k \geq 1$$
$$(1-\delta_k)\mu^{k/2}\mathcal{T}_1^{-1}\cdots\mathcal{T}_k^{-1}\mathcal{P} \subset Q_{\mu^k}(u^*,(0,0)) \subset (1+\delta_k)\mu^{k/2}\mathcal{T}_1^{-1}\cdots\mathcal{T}_k^{-1}\mathcal{P},$$

where $\delta_0 = 1$ and $\delta_k = C(\sqrt{\varepsilon_{k-1}/\mu} + \delta_{k-1}\sqrt{\mu})$ for $k \geq 1$.

Obviously, $\delta_1 = C(\sqrt{\varepsilon_0/\mu} + \sqrt{\mu}) \leq \sqrt{\theta}\delta_0$ if μ and τ_0 are small. If $\delta_k \leq \sqrt{\theta}\delta_{k-1}$, then $\delta_{k+1} \leq C(\sqrt{\theta\varepsilon_{k-1}/\mu} + \sqrt{\theta}\delta_{k-1}\sqrt{\mu}) = \sqrt{\theta}\delta_k$. By induction, $\delta_{k+1} \leq \sqrt{\theta}\delta_k$ and $\delta_k \leq \sqrt{\theta}^{\,k}$ for $k \geq 0$. This implies that $\prod_{k=1}^{\infty}(1+C\delta_k)$ and $\prod_{k=1}^{\infty}(1-C\delta_k)$ both converge. We then obtain that

$$B_{C^{-1}\sqrt{\mu^k}} \times (-C^{-1}\mu^k, 0] \subset Q_{\mu^k}(u^*, (0,0)) \subset B_{C\sqrt{\mu^k}} \times (-C\mu^k, 0].$$

Since $Q_{h_0\mu^k}(u, (x_0, t_0)) = \mathcal{T}^{-1}Q_{\mu^k/K}(u^*, (0,0))$ with $K = |\det T|^{-2/n}h_0^{-1} \approx 1$, conclusion (i) of Theorem 3.2 follows.

To prove (ii), let $(x_0, t_0) \in \Omega' \times (\bar{\varepsilon}, T]$ and $(x,t) \in Q_{h_0}(x_0, t_0)$. Then there exists $h \leq h_0$ such that $(x,t) \in \partial_p Q_h(x_0, t_0)$, i.e., $u(x,t) - \ell_{z_0}(x) = h$, where $\ell_{z_0}(x)$ is the supporting affine function of $u(\cdot, t_0)$ at $x = x_0$. By (i) we have

$$0 \leq u(x,t) - \ell_{z_0}(x) \leq C(|x - x_0|^2 + |t - t_0|).$$

This implies that D^2u is bounded from above. Since $A(x)u_t + f(x,t) \geq \lambda\nu$, and by (1.1), $\det D^2u \geq (\lambda\nu)^n$, D^2u is bounded away from zero. Thus, equation (1.1) becomes a uniformly parabolic fully nonlinear equation. By [Wan2], we conclude that $u \in C_{loc}^{2+\alpha_0, 1+\frac{\alpha_0}{2}}$. □

4. L^p estimates of D^2u. Our goal in this section is to establish L^p estimates for the Hessian of solutions to (1.1). Since the structure of (1.1) is different from that of equation (1.5) and the elliptic Monge-Ampère equation which can be viewed as Jacobian equations, the difficulty is to estimate the density of good sets where the Hessian is bounded. We use one-sided $W^{2,\delta}$ estimates together with properties of equation (1.1) to tackle it.

As in the beginning of §3, assume that (3.1), (3.2) hold, and $A(x)u_t + f(x,t) \geq \lambda\nu > 0$. We first give the estimate of the density of good sets for the normalized problem.

LEMMA 4.1. *Let u be a strictly parabolically convex function in Q and satisfy*

$$(1 - \varepsilon)f(z_0) \leq -A(x_0)u_t + (\det D^2u)^{1/n} \leq (1 + \varepsilon)f(z_0) \quad \text{in } Q, \tag{4.1}$$

$$u = 0 \qquad \text{on } \partial_p Q,$$

$$m_1 \leq -u_t \leq m_2 \quad \text{in } Q, \tag{4.2}$$

$$A(x_0)u_t + f(z_0) \geq \lambda\nu/2 \quad \text{in } Q, \tag{4.3}$$

where $Q = \{(x,t)\colon u < 0 \text{ and } t \leq 0\}$ *is a bowl-shaped domain and satisfies*

$$B_{\gamma_0} \times (-c_3, 0] \subset Q \subset B_1 \times (-c_4, 0].$$

For $0 < \alpha < 1$*, let* $Q_\alpha = \{(x,t) \in Q\colon\ u(x,t) < (1-\alpha)\min_Q u\}$*. Then there exist* $0 < \delta_0 < 1$ *and* $\sigma > 0$ *such that*

$$|Q_\alpha \backslash A_\sigma| \le \varepsilon^{\delta_0}|Q_\alpha|, \tag{4.4}$$

where

$$A_\sigma(u) = A_\sigma = \{(x_0,t_0) \in Q\colon\ u = P_\sigma \text{ at } (x_0,t_0) \text{ and } u \ge P_\sigma \text{ in } Q \cap \{t \le t_0\}\}$$

and $P_\sigma = \sigma(|x-x_0|^2 - (t-t_0)) + \ell(x)$ *and* $\ell(x)$ *is an affine function.*

Proof. Let w, convex in x, be the solution to

$$-A(x_0)w_t + (\det D^2 w)^{1/n} = f(z_0), \quad \text{in } Q, \tag{4.5}$$

with $w = 0$ on $\partial_p Q$. By the comparison principle

$$(1+\varepsilon)w \le u \le (1-\varepsilon)w, \quad \text{in } Q.$$

By (4.2), $|\min_Q u| \approx$ *const*, and hence $\|u - w\|_{L^\infty(Q)} \le C\varepsilon$. By the concavity of the functional $F(M) = (\det M)^{1/n}$, one obtains

$$\begin{aligned}
&[-A(x_0)u_t + (\det D^2 u)^{1/n}] - [-A(x_0)w_t + (\det D^2 w)^{1/n}] \\
&\qquad \le\ -A(x_0)(u-w)_t + \frac{1}{n}(\det D^2 w)^{1/n} \cdot \operatorname{tr}((D^2 w)^{-1} D^2(u-w)).
\end{aligned}$$

From (4.1), (4.5), the function $v = u - w$ satisfies

$$-A(x_0)v_t + \frac{1}{n}(\det D^2 w)^{1/n} \cdot \operatorname{tr}((D^2 w)^{-1} D^2 v) \ge -\varepsilon f(z_0), \quad \text{in } Q.$$

By the interior C^∞ estimates for (4.5), w is smooth in Q and the above linear operator is uniformly parabolic in Q_α with $0 < \alpha < 1$. By one-sided $W^{2,\delta}$ estimates in [Wan1], there exists $0 < \delta_0 < 1$ such that

$$\begin{aligned}
&|Q_\alpha \backslash \{(x_0,t_0) \in Q_\alpha\colon\ v = P_{M_0} \text{ at } (x_0,t_0) \text{ and } v \le P_{M_0} \text{ in } Q \cap \{t \le t_0\}\}| \\
&\qquad \le\ C(\|v\|_{L^\infty(Q)} + |\varepsilon f(z_0)|)^{\delta_0} M_0^{-\delta_0} \le C\varepsilon^{\delta_0} M_0^{-\delta_0} \\
&\qquad \le\ \varepsilon^{\delta_0}|Q_\alpha|, \qquad \text{if } M_0 \text{ is large,}
\end{aligned}$$

where $P_{M_0}(x,t) = M_0(|x-x_0|^2 - (t-t_0)) + \ell(x)$ and $\ell(x)$ is an affine function. Since $w \in C^2$ can be touched from above by some quadratic polynomial, there exists $M_1 > 0$ such that

$$|Q_\alpha \backslash \{(x_0,t_0) \in Q_\alpha\colon\ u = P_{M_1} \text{ at } (x_0,t_0) \text{ and } u \le P_{M_1} \text{ in } Q \cap \{t \le t_0\}\}| \le \varepsilon^{\delta_0}|Q_\alpha|.$$

To finish the proof of Lemma 4.1, we need to show that if u can be touched from above by P_{M_1} at $(x_0, t_0) \in Q_\alpha$, then u can be touched from below by some P_σ at (x_0, t_0).

Since $u \leq P_{M_1}$ in $Q \cap \{t \leq t_0\}$ and $u = P_{M_1}$ at (x_0, t_0), we have

$$u(x, t_0) \leq M_1|x - x_0|^2 + \ell_{z_0}(x), \qquad \text{for } (x, t_0) \in Q, \tag{4.6}$$

where $\ell_{z_0}(x) = u(x_0, t_0) + Du(x_0, t_0)(x - x_0)$. By [G-H3, Lemma 6.1], for $0 < \alpha \leq \alpha_0 < 1$, there exists $\eta_\alpha > 0$ such that $Q_h(x_0, t_0) \subset Q_{\frac{\alpha_0+1}{2}}$, for $h \leq \eta_\alpha$. From (4.6)

$$B_{\sqrt{\frac{h}{M_1}}}(x_0) \subset S_h(x_0|t_0). \tag{4.7}$$

By (4.1)–(4.3), $u(x, t_0)$ satisfies that $C^{-1} \leq \det D^2u \leq C$. As in the beginning of §3, the theory of the elliptic Monge-Ampère equation yields

$$|S_h(x_0|t_0)| \approx h^{\frac{n}{2}}, \quad \text{for } h \leq \eta_\alpha.$$

Together with (4.7), it implies that $\operatorname{diam}(S_h(x_0|t_0)) \leq C\sqrt{M_1^{n-1}h}$ and

$$S_h(x_0|t_0) \subset B_{C\sqrt{M_1^{n-1}h}}(x_0), \quad \text{for } h \leq \eta_\alpha.$$

It follows that

$$u(x, t_0) \geq \frac{C}{M_1^{n-1}}|x - x_0|^2 + \ell_{z_0}(x), \quad \text{if } x \in S_{\eta_\alpha}(x_0|t_0).$$

By choosing $\sigma \leq \min\{\frac{C}{M_1^{n-1}}, \frac{\eta_\alpha}{\operatorname{diam}(Q)^2}\}$, we have

$$u(x, t_0) \geq \sigma|x - x_0|^2 + \ell_{z_0}(x), \quad \text{if } (x, t_0) \in Q.$$

It is easy to check that if $\sigma \leq m_1$, then $u \geq P_\sigma$ in $Q \cap \{t \leq t_0\}$ and $u = P_\sigma$ at (x_0, t_0), where $P_\sigma = \sigma(|x - x_0|^2 - (t - t_0)) + \ell_{z_0}(x)$. Thus, $(x_0, t_0) \in A_\sigma$ and the proof of Lemma 4.1 is completed. □

Recall that η_{α_0} is the constant in [G-H3, Lemma 6.1] such that $Q_h(z_0) \subset Q_{\frac{\alpha_0+1}{2}}$ for $z_0 \in Q_{\alpha_0}$ and $h \leq \eta_{\alpha_0}$. The following is a rescaled result of Lemma 4.1.

LEMMA 4.2. *Suppose that the assumptions in Lemma 4.1 hold. Then there exists a constant $C_0 > 0$ such that for $z_0 = (x_0, t_0) \in Q_{\alpha_0}$ and $(C_0\lambda)^{-1} \leq h \leq \frac{1}{2}\eta_{\alpha_0}$*

$$|Q_h(z_0) \setminus A_{\lambda^{-1}}| \leq \varepsilon^{\delta_0}|Q_h(z_0)|,$$

where δ_0 is the constant in Lemma 4.1.

Proof. The proof is similar to that of Proposition 6.1 in [G-H3]. Let $h \le \eta_{\alpha_0}$. As in the beginning of §3, let T be an affine transformation such that $B_{\gamma_0} \subset TS_h(x_0|t_0) \subset B_1$. Set

$$(y, s) = \mathcal{T}(x, t) = \left(Tx, \frac{t - t_0}{Kh}\right),$$
$$u^*(y, s) = |\det T|^{2/n}\left[(u - \ell_{z_0})(\mathcal{T}^{-1}(y, s)) - h\right].$$

If $K = |\det T|^{-2/n}h^{-1}$, then $u^*_s = u_t$ and $\det D^2_y u^* = \det D^2_x u$. We have

$$(1 - \varepsilon)f(z_0) \le -A(x_0)u^*_s + (\det D^2 u^*)^{1/n} \le (1 + \varepsilon)f(z_0), \quad \text{in } \mathcal{T}Q_h(z_0) = Q^*,$$
$$u^* = 0, \quad \text{on } \partial_p Q^*.$$

Applying Lemma 4.1 to u^* in Q^* with $\alpha = \frac{1}{2}$, we obtain

$$|Q^*_{1/2} \setminus A_\sigma(u^*)| \le \varepsilon^{\delta_0}|Q^*_{1/2}|, \tag{4.8}$$

where $Q^*_{1/2} = \{(y, s) \in Q^*\colon\ u^* < \frac{1}{2}\min_{Q^*} u^*\} = \mathcal{T}(Q_{h/2}(z_0))$.

Now we show that there exists a constant $C_0 > 0$ such that

$$\mathcal{T}^{-1}(Q^*_{1/2} \cap A_\sigma(u^*)) \subset Q_{h/2}(z_0) \cap A_{C_0 h}(u). \tag{4.9}$$

Let $z^*_1 = (y_1, s_1) \in Q^*_{1/2} \cap A_\sigma(u^*)$ and $z_1 = (x_1, t_1) = \mathcal{T}^{-1}z^*_1 \in Q_{h/2}(z_0)$. Then

$$u^*(y, s_1) - \ell^*(y) \ge \sigma|y - y_1|^2, \quad \text{for } (y, s_1) \in Q^*,$$

where ℓ^* is the supporting affine function of $u^*(\cdot, s_1)$ at $y = y_1$. It follows that

$$u(x, t_1) - \ell_{z_1}(x) \ge \sigma Kh|Tx - Tx_1|^2, \quad \text{for } (x, t_1) \in Q_h(z_0).$$

Since $\mathcal{T}Q_h(z_0) = Q^*$, T is a dilation and $|Tx - Tx_1|^2 \ge C|x - x_1|^2$. We obtain

$$u(x, t_1) - \ell_{z_1}(x) \ge C_0 h|x - x_1|^2, \quad \text{for } (x, t_1) \in Q_h(z_0).$$

By [G-H3, Lemma 6.1], there exists $\eta > 0$ such that $Q_\eta(u^*, z^*_1) \subset Q^*$. Obviously

$$Q_{K\eta h}(u, z_1) = \mathcal{T}^{-1}Q_\eta(u^*, z^*_1) \subset \mathcal{T}^{-1}Q^* = Q_h(z_0).$$

For $(x, t_1) \notin Q_h(z_0)$

$$u(x, t_1) - \ell_{z_1}(x) \ge K\eta h \ge C_0 h|x - x_1|^2.$$

Therefore, $z_1 \in A_{C_0 h} \subset A_{\lambda^{-1}}$ for $C_0 h \leq m_1$ and (4.9) is proved. Lemma 4.2 follows immediately from (4.8) and (4.9). □

Let D^α_λ denote the set

$$D^\alpha_\lambda = \{(x_0, t_0) \in Q_\alpha\colon\ S_h(x_0|t_0) \subset B_{\lambda\sqrt{h}}(x_0),\ \text{for } h \leq \eta_{\alpha_0}\}.$$

By Lemma 6.2 in [G-H3]

$$D^\alpha_\lambda = Q_\alpha \cap A_{\lambda^{-2}}, \quad \text{for } \lambda \geq C \text{ and } 0 < \alpha \leq \alpha_0 < 1.$$

The following theorem gives rise to the power decay of distribution function of the Hessian.

THEOREM 4.1. *Suppose that the assumptions in Lemma 4.1 hold. For* $0 < \tau < \alpha \leq \alpha_0 < 1$, *there exist constants* M, p_0, C_1 *such that*

$$|Q_\tau \setminus D^\tau_{M\lambda}| \leq 2\varepsilon^{\delta_0/2} |Q_\alpha \setminus D^\alpha_\lambda|$$

for $\lambda \geq C_1$ *and* $\alpha - \tau = (M\lambda)^{-p_0}$.

Proof. The proof is similar to Proposition 6.2 in [G-H3]. It is easy to see that the set $\mathcal{O} = Q_\tau \backslash D^\tau_{M\lambda}$ is open and $\mathcal{O} = Q_\tau \setminus A_{(M\lambda)^{-2}}$, for $M\lambda \geq C$ and $\tau < \alpha_0$. Consequently

$$Q_h(z_0) \cap \mathcal{O} \subset Q_h(z_0) \setminus A_{(M\lambda)^{-2}}.$$

By Lemma 4.2, we obtain

$$\frac{|Q_h(z_0) \cap \mathcal{O}|}{|Q_h(z_0)|} \leq \frac{|Q_h(z_0) \setminus A_{(M\lambda)^{-2}}|}{|Q_h(z_0)|} \leq \varepsilon^{\delta_0}, \tag{4.10}$$

for $C_0^{-1}(M\lambda)^{-2} \leq h \leq \frac{1}{2}\eta_{\alpha_0}$ and $z_0 \in Q_{\alpha_0}$. Now recall the section $Q^*_h(x_0, t_0)$ defined by (4.2) in [G-H3]. Let δ be a small positive number depending on m_2. Let $(x_0)^h_{min}$ be the minimum point of $u(x, t^h_0) - \ell_{z_0}(x)$, where $t^h_0 = \min\{t_0 + \delta h, 0\}$. Set

$$Q^*_h(z_0) = Q_h((x_0)^h_{min}, t^h_0).$$

Since $\mathcal{O}$ is open, we have

$$\lim_{h \to 0} \frac{|Q^*_h(z_0) \cap \mathcal{O}|}{|Q^*_h(z_0)|} = 1, \quad \text{for } z_0 \in \mathcal{O}.$$

On the other hand, since $((x_0)^h_{min}, t^h_0) \in Q_{\frac{\alpha_0+1}{2}}$, from (4.10)

$$\frac{|Q^*_h(x_0,t_0)\cap\mathcal{O}|}{|Q^*_h(x_0,t_0)|} \le \varepsilon^{\delta_0},$$

for $C^{-1}(M\lambda)^{-2} \le h \le \frac{1}{2}\eta_{(\alpha_0+1)/2}$. For $z_0 \in \mathcal{O}$ we choose h_{z_0}, the largest $h \le C^{-1}(M\lambda)^{-2}$ such that

$$\frac{|Q^*_h(z_0)\cap\mathcal{O}|}{|Q^*_h(z_0)|} = 4\varepsilon^{\delta_0}.$$

Applying the Calderón-Zygmund decomposition ([G-H3, Theorem 4.1]), we obtain a family of sections $\{Q^*_{h_k}(z_k)\}_{k=1}^{\infty}$, $z_k = (x_k, t_k) \in \mathcal{O}$ with $h_k \le C^{-1}(M\lambda)^{-2}$ such that $\mathcal{O} \subset \cup_{k=1}^{\infty} Q^*_{h_k}(z_k)$, $|\mathcal{O}| \le 2\varepsilon^{\delta_0/2}|\cup_{k=1}^{\infty} Q^*_{h_k}(z_k)|$, and

$$\frac{|Q^*_{h_k}(z_k)\cap\mathcal{O}|}{|Q^*_{h_k}(z_k)|} = 4\varepsilon^{\delta_0}. \tag{4.11}$$

To finish the proof of Theorem 4.1, it suffices to show that

$$Q^*_{h_k}(z_k) \subset Q_\alpha \setminus D^\alpha_\lambda. \tag{4.12}$$

Since $(x_k, t_k) \in Q_\tau$, by Lemma 4.6 and Remark 4.1 in [G-H3], there exists $p_0 > 0$ such that $((x_k)^{h_k}_{min}, t_k) \in Q_{m_2\delta h_k}(x_k, t_k) \subset Q_{\tau+\frac{1}{2}(M\lambda)^{-p_0}}$. If $\tau \le \alpha - (M\lambda)^{-p_0}$, then

$$Q^*_{h_k}(x_k, t_k) \subset Q_{\tau+(M\lambda)^{-p_0}} \subset Q_\alpha.$$

We now use an argument of contradicition to prove (4.12). Suppose there exists $z_0 = (x_0, t_0) \in Q^*_{h_k}(z_k) \cap D^\alpha_\lambda$. By the engulfing property at different times ([G-H3, Lemma 4.2])

$$S_{2h_k}((x_k)^{h_k}_{min}|t^{h_k}_k) \subset S_{\theta 2h_k}(x_0|t_0) \subset B_{\lambda\sqrt{\theta 2h_k}}(x_0),$$

where $t^{h_k}_k = \min\{t_k + \delta h_k, 0\}$. Let T be an affine transformation such that $B_{\gamma_0} \subset T(S_{2h_k}((x_k)^{h_k}_{min}|t^{h_k}_k)) \subset B_1$. Set

$$\begin{aligned} (y,s) &= \mathcal{T}(x,t) = \big(Tx, \frac{t - t^{h_k}_k}{K2h_k}\big), \\ \hat{u}(y,s) &= |\det T|^{2/n}\big[(u - \ell_{z^{h_k}_k})(\mathcal{T}^{-1}(y,s)) - 2h_k\big], \end{aligned}$$

where $z^{h_k}_k = ((x_k)^{h_k}_{min}, t^{h_k}_k)$. Choose $K = |\det T|^{-2/n}(2h_k)^{-1}$. Then $\hat{u}$ satisfies (4.1)–

(4.3) in $\hat{Q}_k = \mathcal{T}(Q_{2h_k}(z_k^{h_k}))$. Applying Lemma 4.1 to $\hat{u}$ in $\hat{Q}_k$, we obtain

$$|(\hat{Q}_k)_{\frac{1}{2}} \setminus A_\sigma(\hat{u})| \leq \varepsilon^{\delta_0} |(\hat{Q}_k)_{\frac{1}{2}}|,$$

where $(\hat{Q}_k)_{\frac{1}{2}} = \mathcal{T}(Q_{h_k}(z_k^{h_k}))$. We claim that for large M

$$\mathcal{T}^{-1}\big((\hat{Q}_k)_{\frac{1}{2}} \cap A_\sigma(\hat{u})\big) \subset D^\alpha_{M\lambda}. \tag{4.13}$$

Let $\hat{z}_1 = (\hat{x}_1, \hat{t}_1) \in (\hat{Q}_k)_{\frac{1}{2}} \cap A_\sigma(\hat{u})$ and $z_1 = \mathcal{T}^{-1}\hat{z}_1 = (x_1, t_1) \in Q_{h_k}(z_k^{h_k})$. Since $S_h(\hat{u}; \hat{x}_1|\hat{t}_1) \subset B_{\sqrt{h/\sigma}}(\hat{x}_1)$, we have for $h \leq C_0$

$$T^{-1}(S_h(\hat{u}; \hat{x}_1|\hat{t}_1)) \subset T^{-1}(B_{\sqrt{h/\sigma}}(\hat{x}_1)).$$

Because T dilates by at least $(C\lambda\sqrt{h_k})^{-1}$ and T^{-1} contracts by at least $C\lambda\sqrt{h_k}$, it follows that for $h \leq C_0$

$$S_{2Kh_kh}(u; x_1|t_1) \subset T^{-1}(B_{\sqrt{h/\sigma}}(\hat{x}_1)) \subset B_{C\lambda\sqrt{h_k}\sqrt{h/\sigma}}(x_1).$$

Therefore, $S_h(u; x_1|t_1) \subset B_{C\lambda\sqrt{h}}(x_1)$ for $h \leq C_0 h_k$.

If $h_k \leq h \leq \eta_{\alpha_0}$, then (x_0, t_0), $(x_1, t_1) \in Q_h((x_k)^{h_k}_{min}, t_k^{h_k})$. By the engulfing property at different times

$$S_h(x_1, t_1) \subset S_{\theta h}(x_0|t_0) \subset B_{\lambda\sqrt{\theta h}}(x_0).$$

Thus $(x_1, t_1) \in D^\alpha_{M\lambda}$ for large M and (4.13) follows.

It is easy to see from (4.13) that

$$Q^*_{h_k}(x_k, t_k) \cap \mathcal{O} \subset Q^*_{h_k}(x_k, t_k) \setminus D^\alpha_{M\lambda} \subset \mathcal{T}^{-1}((\hat{Q}_k)_{\frac{1}{2}} \setminus A_\sigma(\hat{u})).$$

It implies that

$$\frac{|Q^*_{h_k}(x_k, t_k) \cap \mathcal{O}|}{|Q^*_{h_k}(x_k, t_k)|} \leq \frac{|(\hat{Q}_k)_{\frac{1}{2}} \setminus A_\sigma(\hat{u})|}{|(\hat{Q}_k)_{\frac{1}{2}}|} \leq \varepsilon^{\delta_0},$$

which contradicts (4.11). Thus, the proof of Theorem 4.1 is completed. □

Proof of Theorem A(iii). The proof is similar to that in [G-H3]. We apply Theorem 4.1 repeatedly to obtain the power decay of the distribution function of D^2u and hence L^p estimates of D^2u in the interior of a normalized domain. Then use the covering argument to obtain interior L^p estimates of D^2u in $Q = \Omega \times (0, T]$. For details, see the proofs of Theorem 6.1 and Theorem 2.1(B) in [G-H3]. □

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