DISCRETE LITTLEWOOD-PALEY-STEIN THEORY
AND MULTI-PARAMETER HARDY SPACES
ASSOCIATED WITH FLAG SINGULAR INTEGRALS

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ABSTRACT.

   The main purpose of this paper is to develop a unified approach of multi-parameter Hardy space theory using the discrete Littlewood-Paley-Stein analysis in the setting of implicit multi-parameter structure. It is motivated by the goal to establish and develop the Hardy space theory for the flag singular integral operators studied by Muller-Ricci-Stein [MRS] and Nagel-Ricci-Stein [NRS]. This approach enables us to avoid the use of transference method of Coifman-Weiss [CW] as often used in the L^p theory for $p > 1$ and establish the Hardy spaces H_F^p and its dual spaces associated with the flag singular integral operators for all $0 < p \leq 1$. We also prove the boundedness of flag singular integral operators on BMO_F and H_F^p, and from H_F^p to L^p for all $0 < p \leq 1$ without using the deep atomic decomposition. As a result, it bypasses the use of Journe's type covering lemma in this implicit multi-parameter structure. The method used here provides alternate approaches of those developed by Chang-R. Fefferman [CF1-3], Chang [Ch], R. Fefferman [F], Journe [J1-2], Pipher [P] in their important work in pure product setting. A Calderón-Zygmund decomposition and interpolation theorem are also proved for the implicit multi-parameter Hardy spaces.

Typeset by $\mathcal{AMS}$-TEX

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1. Introduction and statement of results

The multi-parameter structures play a significant role in Fourier analysis. On the one hand, the classical Calderón-Zygmund theory can be regarded as centering around the Hardy-Littlewood maximal operator and certain singular integrals which commute with the usual dilations on R^n, given by $\delta \cdot x = (\delta x_1, ..., \delta x_n)$ for $\delta > 0$. On the other hand, if we consider the multi-parameter dilations on R^n, given by $\delta \cdot x = (\delta_1 x_1, ..., \delta_n x_n)$, where $\delta = (\delta_1, ..., \delta_n) \in R_+^n = (R_+)^n$, then these n-parameter dilations are naturally associated with the strong maximal function ([JMZ]), given by

$$(1.1) \qquad M_s(f)(x) = \sup_{x \in R} \frac{1}{|R|} \int_R |f(y)| dy,$$

where the supremum is taken over the family of all rectangles with sides parallel to the axes.

This multi-parameter pure product theory has been developed by many authors over the past thirty years or so. For Calderón-Zygmund theory in this setting, one considers operators of the form $Tf = K * f$, where K is homogeneous, that is, $\delta_1...\delta_n K(\delta \cdot x) = K(x)$, or, more generally, $K(x)$ satisfies the certain differential inequalities and cancellation conditions such that $\delta_1...\delta_n K(\delta \cdot x)$ also satisfy the same bounds. This type of operators has been the subject of extensive investigations in the literature, see for instances the fundamental works of Gundy-Stein ([GS]), R. Fefferman and Stein [FS1], R. Fefferman ([F]), Chang and R. Fefferman ([CF1], [CF2], [CF3]), Journe ([J1], [J2]), Pipher [P], etc.

It is well-known that there is a basic obstacle to the pure product Hardy space theory and pure product BMO space. Indeed, the role of cubes in the classical atomic decomposition of $H^p(R^n)$ was replaced by arbitrary open sets of finite measures in the product $H^p(R^n \times R^m)$. Suggested by a counterexample constructed by L. Carleson [Car], the very deep product $BMO(R^n \times R^m)$ and Hardy space $H^p(R^n \times R^m)$ theory was developed by Chang and R. Fefferman ([Ch],[CF3]). Because of the complicated nature of atoms in product space, a certain geometric lemma, namely

Journe's covering lemma([J1], [J2] and [P]), played an important role in the study of the boundedness of product singular integrals on $H^p(R^n \times R^m)$ and $BMO(R^n \times R^m)$.

While great progress has been made in the case of pure product structure for both L^p and H^p theory, multi-parameter analysis has only been developed in recent years for the L^p theory when the underlying multi -parameter structure is not explicit, but implicit, such as the flag multi-parameter structure studied in [MRS] and [NRS]. The main goal of this paper is to develop a theory of Hardy space in this setting. One of the main ideas of our program is to develop a discrete version of Calderón reproducing formula associated with the given multiparameter structure, and thus prove a Plancherel-Pôlya type inequality in this setting. This discrete scheme of Littlewood-Paley-Stein analysis is particularly useful in dealing with the Hardy spaces H^p for $0 < p \le 1$.

We now recall two instances of implicit multiparameter structures which are of interest to us in this paper. We begin with reviewing one of these cases first. In the work of Muller-Ricci-Stein [MRS], by considering an implicit multi-parameter structure on Heisenberg(-type) groups, the Marcinkiewicz multipliers on the Heisenberg groups yield a new class of flag singular integrals. To be more precise, let $m(\mathcal{L}, iT)$ be the Marcinkiewicz multiplier operator, where $\mathcal{L}$ is the sub-Laplacian, T is the central element of the Lie algebra on the Heisenberg group $\mathbb{H}^n$, and m satisfies the Marcinkiewicz conditions. It was proved in [MRS] that the kernel of $m(\mathcal{L}, iT)$ satisfies the standard one-parameter Calderón-Zygmund type estimates associated with automorphic dilations in the region where $|t| < |z|^2$, and the multi-parameter product kernel in the region where $|t| \ge |z|^2$ on the space $\mathbb{C}^n \times R$. The proof of the $L^p, 1 < p < \infty$, boundedness of $m(\mathcal{L}, iT)$ given in [MRS] requires lifting the operator to a larger group, $\mathbb{H}^n \times R$. This lifts K, the kernel of $m(\mathcal{L}, iT)$ on $\mathbb{H}^n$, to a product kernel $\widetilde{K}$ on $\mathbb{H}^n \times R$. The lifted kernel $\widetilde{K}$ is constructed so that it projects to K by

$$K(z,t) = \int\limits_{-\infty}^{\infty} \widetilde{K}(z, t-u, u)du$$

taken in the sense of distributions.

The operator $\widetilde{T}$ corresponding to product kernel $\widetilde{K}$ can be dealt with in terms of tensor products of operators, and one can obtain their $L^p, 1 < p < \infty$, boundedness by the known pure product theory. Finally, the $L^p, 1 < p < \infty$, boundedness of operator with kernel K follows from transference method of Coifman and Weiss ([CW]), using the projection $\pi : \mathbb{H}^n \times R \to \mathbb{H}^n$ by $\pi((z,t), u) = (z, t+u)$.

Another example of implicit multi-parameter structure is the flag singular integrals on $R^n \times R^m$ studied by Nagel-Ricci-Stein [NRS]. The simplest form of flag singular integral kernel $K(x, y)$ on $R^n \times R^m$ is defined through a projection of a product kernel $\widetilde{K}(x, y, z)$ defined on $R^{n+m} \times R^m$ given by

(1.2) $$K(x,y) = \int\limits_{R^m} \widetilde{K}(x, y-z, z)dz.$$

A more general definition of flag singular kernel was introduced in [NRS], see more details of definitions and applications of flag singular integrals there. We will also briefly recall them later in the introduction. Note that convolution with a flag singular kernel is a special case of product singular kernel. As a consequence, the $L^p, 1 < p < \infty$, boundedness of flag singular integral follows directly from the product theory on $R^n \times R^m$. We note the regularity satisfied by flag singular

kernels is better than that of the product singular kernels. More precisely, the singularity of the standard pure product kernel on $R^n \times R^m$, is sets $\{(x,0)\} \cup \{(0,y)\}$ while the singularity of $K(x,y)$, the flag singular kernel on $R^n \times R^m$ defined by (1.2), is a flag set given by $\{(0,0)\} \subseteq \{(0,y)\}$. For example, $K_1(x,y) = \frac{1}{xy}$ is a product kernel on R^2 and $K_2(x,y) = \frac{1}{x(x+iy)}$ is a flag kernel on R^2.

The work of [NRS] suggests that a satisfactory Hardy space theory should be developed and boundedness of flag singular integrals on such spaces should be established. Thus some natural questions arise. From now on, we will use the subscript "F" to express function spaces or functions associated with the multi-parameter flag structure without further explanation.

Question 1: What is the analogous estimate when $p = 1$? Namely, do we have a satisfactory flag Hardy space $H_F^1(R^n \times R^m)$ theory associated with the flag singular integral operators? More generally, can we develop the flag Hardy space $H_F^p(R^n \times R^m)$ theory for all $0 < p \leq 1$ such that the flag singular integral operators are bounded on such spaces?

Question 2: Do we have a boundedness result on a certain type of $BMO_F(R^n \times R^m)$ space for flag singular integral operators considered in [NRS]? Namely, does an endpoint estimate of the result by Nagel-Ricci-Stein hold when $p = \infty$?

Question 3: What is the duality theory of so defined flag Hardy space? More precisely, do we have an analogue of BMO and Lipchitz type function spaces which are dual spaces of the flag Hardy spaces.

Question 4: Is there a Calderón-Zygmund decomposition in terms of functions in flag Hardy spaces $H_F^p(R^n \times R^m)$? Furthermore, is there a satisfactory theory of interpolation on such spaces?

Question 5: What is the difference and relationship between the Hardy space $H^p(R^n \times R^m)$ in the pure product setting and $H_F^p(R^n \times R^m)$ in flag multiparameter setting?

The original goal of our work is to address these questions. As in the L^p theory for $p > 1$ considered in [MRS], one is naturally tempted to establish the Hardy space theory under the implicit multi-parameter structure associated with the flag singular kernel by lifting method to the pure product setting together with the transference method in [CW]. However, this direct lifting method is not adaptable directly to the case of $p \leq 1$ because the transference method is not known to be valid when $p \leq 1$. This suggests that a different approach in dealing with the Hardy $H_F^p(R^n \times R^m)$ space associated with this implicit multi-parameter structure is necessary. This motivated our work in this paper. In fact, we will develop a unified approach to study multi-parameter Hardy space theory. Our approach will be carried out in the order of the following steps.

(1) We first establish the theory of Littlewood-Paley-Stein square function g_F associated with the implicit multi-parameter structure and the L^p estimates of g_F ($1 < p < \infty$). We then develop a discrete Calderón reproducing formula and a Plancherel-Polya type inequality in a test function space associated to this structure. As in the classical case of pure product setting, these L^p estimates can be used to provide a new proof of Nagel-Ricci-Stein's $L^p(1 < p < \infty)$ boundedenss of flag singular integral operators.

(2) We next develop the theory of Hardy spaces H_F^p associated to the multi-parameter flag structures and the boundedness of flag singular integrals on these spaces; We then establish the boundedness of flag singular integrals from H_F^p to L^p. We refer to the reader the work of product multi-parameter Hardy space theory by Chang-R. Fefferman [CF1-3], R. Fefferman [F1-3], Journe [J1-2] and Pipher [P].

(3) We then establish the duality theory of the flag Hardy space H_F^p and introduce the dual space CMO_F^p, in particular, the duality of H_F^1 and the space BMO_F. We then establish the boundedness of flag singular integrals on BMO_F. It is worthwhile to point out that in the classical one-parameter or pure product case, $BMO(R^n)$ or $BMO(R^n \times R^m)$ is related to the Carleson measure. The space CMO_F^p for all $0 < p \leq 1$, as the dual space of H_F^p introduced in this paper, is then defined by a generalized Carleson measure.

(4) We further establish a Calderón-Zygmund decomposition lemma for any $H_F^p(R^n \times R^m)$ function $(0 < p < \infty)$ in terms of functions in $H_F^{p_1}(R^n \times R^m)$ and $H_F^{p_2}(R^n \times R^m)$ with $0 < p_1 < p < p_2 < \infty$. Then an interpolation theorem is established between $H_F^{p_1}(R^n \times R^m)$ and $H_F^{p_2}(R^n \times R^m)$ for any $0 < p_2 < p_1 < \infty$ (it is noted that $H_F^p(R^n \times R^m) = L^p(R^{n+m})$ for $1 < p < \infty$).

In the present paper, we will use the above approach to study the Hardy space theory associated with the implicit multi-parameter structures induced by the flag singular integrals. We now describe our approach and results in more details.

We first introduce the continuous version of the Littlewood-Paley-Stein square function g_F. Inspired by the idea of lifting method of proving the $L^p(R^n \times R^m)$ boundedness given in [MRS], we will use a lifting method to construct a test function defined on $R^n \times R^m$, given by the non-standard convolution $*_2$ on the second variable only:

$$(1.3) \qquad \psi(x,y) = \psi^{(1)} *_2 \psi^{(2)}(x,y) = \int_{R^m} \psi^{(1)}(x, y-z)\psi^{(2)}(z)dz,$$

where $\psi^{(1)} \in \mathcal{S}(R^{n+m}), \psi^{(2)} \in \mathcal{S}(R^m)$, and satisfy

$$\sum_j |\widehat{\psi^{(1)}}(2^{-j}\xi_1, 2^{-j}\xi_2)|^2 = 1$$

for all $(\xi_1, \xi_2) \in R^n \times R^m \backslash \{(0,0)\}$, and

$$\sum_k |\widehat{\psi^{(2)}}(2^{-k}\eta)|^2 = 1$$

for all $\eta \in R^m \backslash \{0\}$, and the moment conditions

$$\int_{R^{n+m}} x^\alpha y^\beta \psi^{(1)}(x,y)dxdy = \int_{R^m} z^\gamma \psi^{(2)}(z)dz = 0$$

for all multi-indices α, β, and γ. We remark here that it is this subtle convolution $*_2$ which provides a rich theory for the implicit multi-parameter analysis.

For $f \in L^p, 1 < p < \infty, g_F(f)$, the Littlewood-Paley-Stein square function of f, is defined by

$$(1.4) \qquad g_F(f)(x,y) = \left\{ \sum_j \sum_k |\psi_{j,k} * f(x,y)|^2 \right\}^{\frac{1}{2}}$$

where functions

$$\psi_{j,k}(x,y) = \psi_j^{(1)} *_2 \psi_k^{(2)}(x,y),$$

$$\psi_j^{(1)}(x,y) = 2^{(n+m)j}\psi^{(1)}(2^j x, 2^j y) \text{ and } \psi_k^{(2)}(z) = 2^{mk}\psi^{(2)}(2^k z).$$

We remark here that the terminology "implicit multi-parameter structure" is clear from the fact that the dilation $\psi_{j,k}(x,y)$ is not induced from $\psi(x,y)$ explicitly.

By taking the Fourier transform, it is easy to see the following continuous version of the Calderón reproducing formula holds on $L^2(R^{n+m})$,

$$(1.5) \qquad\qquad f(x,y) = \sum_j \sum_k \psi_{j,k} * \psi_{j,k} * f(x,y).$$

Note that if one considers the summation on the right hand side of (1.5) as an operator then, by the construction of function ψ, it is a flag singular integral and has the implicit multi-parameter structure as mentioned before. Using iteration and the vector-valued Littlewood-Paley-Stein estimate together with the Calderón reproducing formula on L^2 allows us to obtain the $L^p, 1 < p < \infty$, estimates of g_F.

Theorem 1.1:. *Let $1 < p < \infty$. Then there exist constants C_1 and C_2 depending on p such that for*

$$C_1\|f\|_p \leq \|g_F(f)\|_p \leq C_2\|f\|_p.$$

In order to state our results for flag singular integrals, we need to recall some definitions given in [NRS]. Following closely from [NRS], we begin with the definitions of a class of distributions on an Euclidean space R^N. A $k-normalized$ bump function on a space R^N is a C^k-function supported on the unit ball with C^k-norm bounded by 1. As pointed out in [NRS], the definitions given below are independent of the choices of k, and thus we will simply refer to "normalized bump function" without specifying k.

For the sake of simplicity of presentations, we will restrict our considerations to the case $R^N = R^{n+m} \times R^m$. We will rephrase Definition 2.1.1 in [NRS] of product kernel in this case as follows:

Definition 1.2:. *A product kernel on $R^{n+m} \times R^m$ is a distribution K on R^{n+m+m} which coincides with a C^∞ function away from the coordinate subspaces $(0,0,z)$ and $(x,y,0)$, where $(0,0) \in R^{n+m}$ and $(x,y) \in R^{n+m}$, and satisfies*

(1) (Differential Inequalities) For any multi-indices $\alpha = (\alpha_1, \cdots, \alpha_n)$, $\beta = (\beta_1, \cdots, \beta_m)$ and $\gamma_m = (\gamma_1, \cdots, \gamma_m)$

$$|\partial_x^\alpha \partial_y^\beta \partial_z^\gamma K(x,y,z)| \leq C_{\alpha,\beta,\gamma}(|x|+|y|)^{-n-m-|\alpha|-|\beta|} \cdot |z|^{-m-|\gamma|}$$

for all $(x,y,z) \in R^n \times R^m \times R^m$ with $|x|+|y| \neq 0$ and $|z| \neq 0$.

(2) (Cancellation Condition)

$$\left| \int_{R^m} \partial_x^\alpha \partial_y^\beta K(x,y,z)\phi_1(\delta z)dz \right| \leq C_{\alpha,\beta}(|x|+|y|)^{-n-m-|\alpha|-|\beta|}$$

for all multi-indices α, β and every normalized bump function ϕ_1 on R^m and every $\delta > 0$;

$$\left| \int_{R^m} \partial_z^\gamma K(x,y,z)\phi_2(\delta x, \delta y)dxdy \right| \leq C_\gamma |z|^{-m-|\gamma|}$$

for every multi-index γ and every normalized bump function ϕ_2 on R^{n+m} and every $\delta > 0$; and

$$|\int_{R^{n+m+m}} K(x,y,z)\phi_3(\delta_1 x, \delta_1 y, \delta_2 z)dxdydz| \leq C$$

for every normalized bump function ϕ_3 on R^{n+m+m} and every $\delta_1 > 0$ and $\delta_2 > 0$.

Definition 1.3:. *A flag kernel on $R^n \times R^m$ is a distribution on R^{n+m} which coincides with a C^∞ function away from the coordinate subspace $\{(0,y)\} \subset R^{n+m}$, where $0 \in R^n$ and $y \in R^m$ and satisfies*

(1) (Differential Inequalities) For any multi-indices $\alpha = (\alpha_1, \cdots, \alpha_n)$, $\beta = (\beta_1, \cdots, \beta_m)$

$$|\partial_x^\alpha \partial_y^\beta K(x,y)| \leq C_{\alpha,\beta}|x|^{-n-|\alpha|} \cdot (|x| + |y|)^{-m-|\beta|}$$

for all $(x,y) \in R^n \times R^m$ with $|x| \neq 0$.

(2) (Cancellation Condition)

$$|\int_{R^m} \partial_x^\alpha K(x,y)\phi_1(\delta y)dy| \leq C_\alpha |x|^{-n-|\alpha|}$$

for every multi-index α and every normalized bump function ϕ_1 on R^m and every $\delta > 0$;

$$|\int_{R^n} \partial_y^\beta K(x,y)\phi_2(\delta x)dx| \leq C_\gamma |y|^{-m-|\beta|}$$

for every multi-index β and every normalized bump function ϕ_2 on R^n and every $\delta > 0$; and

$$|\int_{R^{n+m}} K(x,y)\phi_3(\delta_1 x, \delta_2 y)dxdy| \leq C$$

for every normalized bump function ϕ_3 on R^{n+m} and every $\delta_1 > 0$ and $\delta_2 > 0$.

By a result in [MRS], we may assume first that a flag kernel K lies in $L^1(R^{n+m})$. Thus, there exists a product kernel $K^\sharp$ on $R^{n+m} \times R^m$ such that

$$K(x,y) = \int_{R^m} K^\sharp(x, y-z, z)dz.$$

Conversely, if a product kernel $K^\sharp$ lies in $L^1(R^{n+m} \times R^m)$, then $K(x,y)$ defined as above is a flag kernel on $R^n \times R^m$. As pointed out in [MRS], we may always assume that $K(x,y)$, a flag kernel, is integrable on $R^n \times R^m$ by using a smooth truncation argument.

As a consequence of Theorem 1.1, we give a new proof of the $L^p, 1 < p < \infty$, boundedness of flag singular integrals due to Nagel, Ricci and Stein in [NRS]. More precisely, let $T(f)(x,y) = K * f(x,y)$ be a flag singular integral on $R^n \times R^m$. Then K is a projection of a product kernel $K^\sharp$ on $R^{n+m} \times R^m$.

Theorem 1.4:. *Suppose that T is a flag singular integral defined on $R^n \times R^m$ with the flag kernel $K(x,y) = \int\limits_{R^m} K^\sharp(x, y - z, z)dz$, where the product kernel $K^\sharp$ satisfies the conditions of Definition 1.2 above. Then T is bounded on L^p for $1 < p < \infty$. Moreover, there exists a constant C depending on p such that for $f \in L^p, 1 < p < \infty$,*

$$\|T(f)\|_p \leq C\|f\|_p.$$

In order to use the Littlewood-Paley-Stein square function g_F to define the Hardy space, one needs to extend the Littlewood-Paley-Stein square function to be defined on a suitable distribution space. For this purpose, we first introduce the product test function space on $R^{n+m} \times R^m$.

Definition 1.5:. *A Schwartz test function $f(x, y, z)$ defined on $R^n \times R^m \times R^m$ is said to be a product test function on $R^{n+m} \times R^m$ if*

$$(1.6) \qquad \int f(x, y, z)x^\alpha y^\beta dxdy = \int f(x, y, z)z^\gamma dz = 0$$

for all multi-indices α, β, γ of nonnegative integers.

If f is a product test function on $R^{n+m} \times R^m$ we denote $f \in \mathcal{S}_\infty(R^{n+m} \times R^m)$ and the norm of f is defined by the norm of Schwartz test function.

We now define the test function space $\mathcal{S}_F$ on $R^n \times R^m$ associated with the flag structure.

Definition 1.6:. *A function $f(x, y)$ defined on $R^n \times R^m$ is said to be a test function in $\mathcal{S}_F(R^n \times R^m)$ if there exists a function $f^\sharp \in \mathcal{S}_\infty(R^{n+m} \times R^m)$ such that*

$$(1.7) \qquad f(x, y) = \int\limits_{R^m} f^\sharp(x, y - z, z)dz.$$

If $f \in \mathcal{S}_F(R^n \times R^m)$, then the norm of f is defined by

$$\|f\|_{\mathcal{S}_F(R^n \times R^m)} = \inf\{\|f^\sharp\|_{\mathcal{S}_\infty(R^{n+m} \times R^m)} : \text{for all representations of } f \text{ in (1.7)}\}.$$

We denote by $(\mathcal{S}_F)'$ the dual space of $\mathcal{S}_F$.

We would like to point out that the implicit multi-parameter structure is involved in $\mathcal{S}_F$. Since the functions $\psi_{j,k}$ constructed above belong to $\mathcal{S}_F(R^n \times R^m)$, so the Littlewood-Paley-Stein square function g_F can be defined for all distributions in $(\mathcal{S}_F)'$. Formally, we can define the flag Hardy space as follows.

Definition 1.7:. *Let $0 < p \leq 1$. $H_F^p(R^n \times R^m) = \{f \in (\mathcal{S}_F)' : g_F(f) \in L^p(R^n \times R^m)\}$.*

If $f \in H_F^p(R^n \times R^m)$, the norm of f is defined by

$$(1.8) \qquad \|f\|_{H_F^p} = \|g_F(f)\|_p.$$

A natural question arises whether this definition is independent of the choice of functions $\psi_{j,k}$. Moreover, to study the H_F^p-boundedness of flag singular integrals and establish the duality result of H_F^p, this formal definition is not sufficiently good. We need to discretize the norm of H_F^p. In order to obtain such a discrete H_F^p norm we will prove the Plancherel-Pôlya-type inequalities. The main tool to provide such inequalities is the Calderón reproducing formula (1.5). To be more specific, we will prove that the formula (1.5) still holds on test function space $\mathcal{S}_F(R^n \times R^m)$ and its dual space $(\mathcal{S}_F)'$ (see Theorem 3.6 below). Furthermore, using an approximation procedure and the almost orthogonality argument, we prove the following discrete Calderón reproducing formula.

Theorem 1.8:. *Suppose that $\psi_{j,k}$ are the same as in (1.4). Then*

$$(1.9) \qquad f(x,y) = \sum_j \sum_k \sum_J \sum_I |I||J|\widetilde{\psi}_{j,k}(x,y,x_I,y_J)\psi_{j,k} * f(x_I,y_J)$$

where $\widetilde{\psi}_{j,k}(x,y,x_I,y_J) \in \mathcal{S}_F(R^n \times R^m), I \subset R^n, J \subset R^m$, are dyadic cubes with side-length $\ell(I) = 2^{-j-N}$ and $\ell(J) = 2^{-k-N} + 2^{-j-N}$ for a fixed large integer N, x_I, y_J are any fixed points in I, J, respectively, and the series in (1.9) converges in the norm of $\mathcal{S}_F(R^n \times R^m)$ and in the dual space $(\mathcal{S}_F)'$.

The discrete Calderón reproducing formula (1.9) provides the following Plancherel-Pôlya-type inequalities. We use the notation $A \approx B$ to denote that two quantities A and B are comparable independent of other substantial quantities involved in the context.

Theorem 1.9:. *Suppose $\psi^{(1)}, \phi^{(1)} \in \mathcal{S}(R^{n+m}), \psi^{(2)}, \phi^{(2)} \in \mathcal{S}(R^m)$ and*

$$\psi(x,y) = \int_{R^m} \psi^{(1)}(x,y-z)\psi^{(2)}(z)dz,$$

$$\phi(x,y) = \int_{R^m} \phi^{(1)}(x,y-z)\psi^{(2)}(z)dz,$$

and ψ_{jk}, ϕ_{jk} satisfy the conditions in (1.4). Then for $f \in (\mathcal{S}_F)'$ and $0 < p < \infty$,

$$\|\{\sum_j \sum_k \sum_J \sum_I \sup_{u \in I, v \in J} |\psi_{j,k} * f(u,v)|^2 \chi_I(x)\chi_J(y)\}^{\frac{1}{2}}\|_p$$

$$(1.10) \qquad \approx \|\{\sum_j \sum_k \sum_J \sum_I \inf_{u \in I, v \in J} |\phi_{j,k} * f(u,v)|^2 \chi_I(x)\chi_J(y)\}^{\frac{1}{2}}\|_p$$

where $\psi_{j,k}(x,y)$ and $\phi_{j,k}(x,y)$ are defined as in (1.4), $I \subset R^n, J \subset R^m$, are dyadic cubes with side-length $\ell(I) = 2^{-j-N}$ and $\ell(J) = 2^{-k-N} + 2^{-j-N}$ for a fixed large integer N, χ_I and χ_J are indicator functions of I and J, respectively.

The Plancherel-Pôlya-type inequalities in Theorem 1.9 give the discrete Littlewood-Paley-Stein square function

$$(1.11) \qquad g_F^d(f)(x,y) = \left\{ \sum_j \sum_k \sum_J \sum_I |\psi_{j,k} * f(x_I,y_J)|^2 \chi_I(x)\chi_J(y) \right\}^{\frac{1}{2}}$$

where I, J, x_I, and y_J are the same as in Theorem 1.8 and Theorem 1.9.

From this it is easy to see that the Hardy space H_F^p in (1.8) is well defined and the H_F^p norm of f is equivalent to the L^p norm of g_F^d. By use of the Plancherel-Pôlya-type inequalities, we will prove the boundedness of flag singular integrals on H_F^p.

Theorem 1.10:. *Suppose that T is a flag singular integral with the kernel $K(x,y)$ satisfying the same conditions as in Theorem 1.4. Then T is bounded on H_F^p, for $0 < p \leq 1$. Namely, for all $0 < p \leq 1$ there exists a constant C_p such that*

$$\|T(f)\|_{H_F^p} \leq C_p \|f\|_{H_F^p}.$$

To obtain the $H_F^p \to L^p$ boundedness of flag singular integrals, we prove the following general result:

Theorem 1.11. *Let $0 < p \leq 1$. If T is a linear operator which is bounded on $L^2(R^{n+m})$ and $H_F^p(R^n \times R^m)$, then T can be extended to a bounded operator from $H_F^p(R^n \times R^m)$ to $L^p(R^{n+m})$.*

From the proof, we can see that this general result holds in a very broad setting, which includes the classical one-parameter and product Hardy spaces and the Hardy spaces on spaces of homogeneous type.

In particular, for flag singular integral we can deduce from this general result the following

Corollary 1.12:. *Let T be a flag singular integral as in Theorem 1.4. Then T is bounded from $H_F^p(R^n \times R^m)$ to $L^p(R^{n+m})$ for $0 < p \leq 1$.*

To study the duality of H_F^p, we introduce the space CMO_F^p.

Definition 1.13:. *Let $\psi_{j,k}$ be the same as in (1.4). We say that $f \in CMO_F^p$ if $f \in (\mathcal{S}_F)'$ and it has the finite norm $\|f\|_{CMO_F^p}$ defined by*

$$(1.12) \qquad \sup_\Omega \left\{ \frac{1}{|\Omega|^{\frac{2}{p}-1}} \sum_{j,k} \int_\Omega \sum_{I,J:I\times J\subseteq\Omega} |\psi_{j,k} * f(x,y)|^2 \chi_I(x)\chi_J(y) dxdy \right\}^{\frac{1}{2}}$$

for all open sets Ω in $R^n \times R^m$ with finite measures, and $I \subset R^n, J \subset R^m$, are dyadic cubes with side-length $\ell(I) = 2^{-j}$ and $\ell(J) = 2^{-k} + 2^{-j}$ respectively.

Note that the Carleson measure condition is used and the implicit multi-parameter structure is involved in CMO_F^p space. When $p = 1$, as usual, we denote by BMO_F the space CMO_F^1. To see the space CMO_F^p is well defined, one needs to show the definition of CMO_F^p is independent of the choice of the functions $\psi_{j,k}$. This can be proved, again as in the Hardy space H_F^p, by the following Plancherel-Pôlya-type inequality.

Theorem 1.14:. *Suppose ψ, ϕ satisfy the same conditions as in Theorem 1.9. Then for $f \in (\mathcal{S}_F)'$,*

$$\sup_\Omega \left\{ \frac{1}{|\Omega|^{\frac{2}{p}-1}} \sum_j \sum_k \sum_{I\times J\subseteq\Omega} \sup_{u\in I, v\in J} |\psi_{j,k} * f(u,v)|^2 |I||J| \right\}^{\frac{1}{2}} \approx$$

$$(1.13) \qquad \sup_\Omega \left\{ \frac{1}{|\Omega|^{\frac{2}{p}-1}} \sum_j \sum_k \sum_{I\times J\subseteq\Omega} \inf_{u\in I, v\in J} |\phi_{j,k} * f(u,v)|^2 |I||J| \right\}^{\frac{1}{2}}$$

where $I \subset R^n, J \subset R^m$, are dyadic cubes with side-length $\ell(I) = 2^{-j-N}$ and $\ell(J) = 2^{-k-N} + 2^{-j-N}$ for a fixed large integer N respectively, and Ω are all open sets in $R^n \times R^m$ with finite measures.

To show that space CMO_F^p is the dual space of H_F^p, we also need to introduce the sequence spaces.

Definition 1.15:. *Let s^p be the collection of all sequences $s = \{s_{I \times J}\}$ such that*

$$\|s\|_{s^p} = \left\| \left\{ \sum_{j,k} \sum_{I,J} |s_{I \times J}|^2 |I|^{-1} |J|^{-1} \chi_I(x) \chi_J(y) \right\}^{\frac{1}{2}} \right\|_{L^p} < \infty,$$

where the sum runs over all dyadic cubes $I \subset R^n, J \subset R^m$ with side-length $\ell(I) = 2^{-j-N}$ and $\ell(J) = 2^{-k-N} + 2^{-j-N}$ for a fixed large integer N, and χ_I, and χ_J are indicator functions of I and J respectively.

Let c^p be the collection of all sequences $s = \{s_{I \times J}\}$ such that

$$(1.14) \qquad \|s\|_{c^p} = \sup_{\Omega} \left\{ \frac{1}{|\Omega|^{\frac{2}{p}-1}} \sum_{j,k} \sum_{I,J:I \times J \subseteq \Omega} |s_{I \times J}|^2 \right\}^{\frac{1}{2}} < \infty,$$

where Ω are all open sets in $R^n \times R^m$ with finite measures and the sum runs over all dyadic cubes $I \subset R^n, J \subset R^m$, with side-length $l(I) = 2^{-j-N}$ and $l(J) = 2^{-k-N} + 2^{-j-N}$ for a fixed large integer N.

We would like to point out again that certain dyadic rectangles used in s^p and c^p reflect the implicit multi-parameter structure. Moreover, the Carleson measure condition is used in the definition of c^p. Next, we obtain the following duality theorem.

Theorem 1.16:. *Let $0 < p \leq 1$. Then we have $(s^p)^* = c^p$. More precisely, the map which maps $s = \{s_{I \times J}\}$ to $<s, t> \equiv \sum_{I \times J} s_{I \times J} \bar{t}_{I \times J}$ defines a continuous linear functional on s^p with operator norm $\|t\|_{(s^p)^*} \approx \|t\|_{c^p}$, and moreover, every $\ell \in (s^p)^*$ is of this form for some $t \in c^p$.*

When $p = 1$, this theorem in the one-parameter setting on R^n was proved in [FJ]. The proof given in [FJ] depends on estimates of certain distribution functions, which seems to be difficult to apply to the multi-parameter case. For all $0 < p \leq 1$ we give a simple and more constructive proof of Theorem 1.16, which uses the stopping time argument for sequence spaces. Theorem 1.16 together with the discrete Calderón reproducing formula and the Plancherel-Pôlya-type inequalities yields the duality of H_F^p.

Theorem 1.17:. *Let $0 < p \leq 1$. Then $(H_F^p)^* = CMO_F^p$. More precisely, if $g \in CMO_F^p$, the map ℓ_g given by $\ell_g(f) = <f, g>$, defined initially for $f \in \mathcal{S}_F$, extends to a continuous linear functional on H_F^p with $\|\ell_g\| \approx \|g\|_{CMO_F^p}$. Conversely, for every $\ell \in (H_F^p)^*$ there exists some $g \in CMO_F^p$ so that $\ell = \ell_g$. In particular, $(H_F^1)^* = BMO_F$.*

As a consequence of the duality of H_F^1 and the H_F^1-boundedness of flag singular integrals, we obtain the BMO_F-boundedness of flag singular integrals. Furthermore, we will see that $L^\infty \subseteq BMO_F$ and, hence, the $L^\infty \to BMO_F$ boundedness of flag singular integrals is also obtained. These provide the endpoint results of those in [MRS] and [NRS]. These can be summarized as follows:

Theorem 1.18:. *Suppose that T is a flag singular integral as in Theorem 1.4. Then T is bounded on BMO_F. Moreover, there exists a constant C such that*

$$\|T(f)\|_{BMO_F} \leq C\|f\|_{BMO_F}.$$

Next we prove the Calderón-Zygmund decomposition and interpolation theorems on the flag Hardy spaces. We note that $H_F^p(R^n \times R^m) = L^p(R^{n+m})$ for $1 < p < \infty$.

Theorem 1.19. *(Calderón-Zygmund decomposition for flag Hardy spaces) Let $0 < p_2 \leq 1, p_2 < p < p_1 < \infty$ and let $\alpha > 0$ be given and $f \in H_F^p(R^n \times R^m)$. Then we may write $f = g + b$ where $g \in H_F^{p_1}(R^n \times R^m)$ with $p < p_1 < \infty$ and $b \in H_F^{p_2}(R^n \times R^m)$ with $0 < p_2 < p$ such that $\|g\|_{H_F^{p_1}}^{p_1} \leq C\alpha^{p_1-p}\|f\|_{H_F^p}^p$ and $\|b\|_{H_F^{p_2}}^{p_2} \leq C\alpha^{p_2-p}\|f\|_{H_F^p}^p$, where C is an absolute constant.*

Theorem 1.20. *(Interpolation theorem on flag Hardy spaces) Let $0 < p_2 < p_1 < \infty$ and T be a linear operator which is bounded from $H_F^{p_2}$ to L^{p_2} and bounded from $H_F^{p_1}$ to L^{p_1}, then T is bounded from H_F^p to L^p for all $p_2 < p < p_1$. Similarly, if T is bounded on $H_F^{p_2}$ and $H_F^{p_1}$, then T is bounded on H_F^p for all $p_2 < p < p_1$.*

We point out that the Calderón-Zygmund decomposition in pure product domains was established for all L^p functions ($1 < p < 2$) into H^1 and L^2 functions by using atomic decomposition on H^1 space (see for more precise statement in Section 6).

We end the introduction of this paper with the following remarks. First of all, our approach in this paper will enable us to revisit the pure product multi-parameter theory using the corresponding discrete Littlewood-Paley-Stein theory. This will provide alternative proofs of some of the known results of Chang-R. Fefferman, R. Fefferman, Journe, Pipher and establish some new results in the pure product setting. We will clarify all these in the future. Second, as we can see from the definition of flag kernels, the regularity satisfied by flag singular kernels is better than that of the product singular kernels. It is thus natural to conjecture that the Hardy space associated with flag singular integrals should be larger than the classical pure product Hardy space. This is indeed the case. In fact, if we define the flag kernel on $R^n \times R^m$ by

$$K(x,y) = \int\limits_{R^n} \widetilde{\widetilde{K}}(x - z, z, y)dz,$$

where $\widetilde{\widetilde{K}}(x, z, y)$ is a pure product kernel on $R^n \times R^{n+m}$, and let $\widetilde{H_F^p}$ be the flag Hardy space associated with this structure, thus we have shown in a forthcoming paper that $H^p(R^n \times R^m) = H_F^p(R^n \times R^m) \cap \widetilde{H_F^p}(R^n \times R^m)$. Results in [MRS] and [NRS] together with those in this paper demonstrate that the implicit multi-parameter structure, the geometric property of sets of singularities and regularities of singular kernels and multipliers are closely related. Third, the authors have carried out in [HL] the discrete Littlewood-Paley-Stein analysis and Hardy space theory in the multi-parameter structure induced by the Zygmund dilation and proved the endpoint estimates such as boundedness of singular integral operators considered by Ricci-Stein [RS] on H_Z^p ($0 < p \leq 1$) and BMO_Z, the Hardy and BMO spaces associated with the Zygmund dilation, e.g., on R^3, given by $\delta(x, y, z) = (\delta_1 x, \delta_2 y, \delta_1\delta_2 z), \delta_1, \delta_2 > 0$, where the L^p ($1 < p < \infty$) boundedness has been established (see [RS] and [FP]).

This paper is organized as follows. In Section 2, we establish the L^p estimates for the multi-parameter Littlewood-Paley-Stein $g-$ function for $1 < p < \infty$ and prove Theorems 1.1 and 1.4.

In Section 3 we first introduce the test function spaces associated with the multi-parameter flag structure and show the Calderón reproducing formula in (1.5) still holds on test function space $\mathcal{S}_F(R^n \times R^m)$ and its dual space $(\mathcal{S}_F)'$, and then prove the almost orthogonality estimates and establish the discrete Calderón reproducing formula on the test function spaces, i.e., Theorem 1.8. Some crucial strong maximal function estimates are given (e.g. Lemma 3.7) and together with the discrete Calderón reproducing formula we derive the Plancherel-Pôlya-type inequalities,i.e., Theorem 1.9. Section 4 deals with numerous properties of Hardy space H_F^p and a general result of bounding the L^p norm of the function by its H_F^p norm (see Theorem 4.3), and then prove the H_F^p boundedness of flag singular integrals for all $0 < p < 1$, i.e., Theorem 1.10. The boundedness from H_F^p to L^p for all $0 < p \leq 1$ for the flag singular integral operators, i.e., Theorem 1.11, is thus a consequence of Theorem 1.10 and Theorem 4.3. The duality of the Hardy space H_F^p is then established in Section 5. The boundedness of flag singular integral operators on BMO_F space is also proved in Section 5. Thus the proofs of Theorem 1.14, 1.16, 1.17 and 1.18 will be all given in the Section 5. In Section 6, we prove a Calderón-Zygmund decomposition in flag multi-parameter setting and then derive an interpolation theorem.

2. L^p estimates for Littlewood-Paley-Stein square function: Proofs of Theorems 1.1 and 1.4

The main purpose of this section is to show that the L^p $(p > 1)$ norm of f is equivalent to the L^p norm of $g_F(f)$, and thus use this to provide a new proof of the L^p boundedness of flag singular integral operators given in [MRS]. Our proof here is quite different from those in [MRS] in the sense that we do not need to apply the lifting procedure used in [MRS] directly. We first prove the L^p estimate of the Littlewood-Paley-Stein square function g_F.

Proof of Theorem 1.1: The proof is similar to that in the pure product case given in [FS] and follows from iteration and standard vector-valued Littlewood-Paley-Stein inequalities. To see this, define $F : R^{n+m} \to H = \ell^2$ by $F(x,y) = \{\psi_j^{(1)} * f(x,y)\}$ with the norm

$$\|F\|_H = \{\sum_j |\psi_j^{(1)} * f(x,y)|^2\}^{\frac{1}{2}}.$$

When x is fixed, set

$$\widetilde{g}(F)(x,y) = \{\sum_k \|\psi_k^{(2)} *_2 F(x,\cdot)(y)\|_H^2\}^{\frac{1}{2}}.$$

It is then easy to see that $\widetilde{g}(F)(x,y) = g_F(f)(x,y)$. If x is fixed, by the vector-valued Littlewood-Paley-Stein inequality,

$$\int_{R^m} \widetilde{g}(F)^p(x,y)dy \leq C \int_{R^m} \|F\|_H^p dy.$$

However, $\|F\|_H^p = \{\sum_j |\psi_j^{(1)} * f(x,y)|^2\}^{\frac{p}{2}}$, so integrating with respect to x together with the

standard Littlewood-Paley-Stein inequality yields

$$\int_{R^n}\int_{R^m} g_F(f)^p(x,y)dydx \leq C\int_{R^n}\int_{R^m}\{\sum_j |\psi_j^{(1)} * f(x,y)|^2\}^{\frac{p}{2}}dydx \leq C\|f\|_p^p,$$

which shows that $\|g_F(f)\|_p \leq C\|f\|_p$.

The proof of the estimate $\|f\|_p \leq C\|g_F(f)\|_p$ is routine and it follows from the Calderon reproducing formula (1.5) on $L^2(R^{n+m})$, for all $f \in L^2 \cap L^p$, $g \in L^2 \cap L^{p'}$ and $\frac{1}{p} + \frac{1}{p'} = 1$, and the inequality $\|g_F(f)\|_p \leq C\|f\|_p$, which was just proved. This completes the proof of Theorem 1.1.   **Q.E.D.**

Remark 2.1: Let $\psi^{(1)} \in \mathcal{S}(R^{n+m})$ be supported in the unit ball in R^{n+m} and $\psi^{(2)} \in \mathcal{S}(R^m)$ be supported in the unit ball of R^m and satisfy

$$\int_0^\infty |\widehat{\psi^{(1)}}(t\xi_1, t\xi_2)|^4 \frac{dt}{t} = 1$$

for all $(\xi_1, \xi_2) \in R^n \times R^m \backslash \{(0,0)\}$, and

$$\int_0^\infty |\widehat{\psi^{(2)}}(s\eta)|^4 \frac{ds}{s} = 1$$

for all $\eta \in R^m \backslash \{0\}$. We define $\psi^\sharp(x,y,z) = \psi^{(1)}(x, y-z)\psi^{(2)}(z)$. Set $\psi_t^{(1)}(x,y) = t^{-n-m}\psi^{(1)}(\frac{x}{t}, \frac{y}{t})$ and $\psi_s^{(2)}(z) = s^{-m}\psi(\frac{z}{s})$ and

$$\psi_{t,s}(x,y) = \int_{R^m} \psi_t^{(1)}(x, y-z)\psi_s^{(2)}(z)dz.$$

Repeating the same proof as that of Theorem 1.1, we can get for $1 < p < \infty$

$$(2.1) \qquad \|\{\int_0^\infty \int_0^\infty |\psi_{t,s} * f(x,y)|^2 \frac{dt}{t}\frac{ds}{s}\}^{\frac{1}{2}}\|_p \leq C\|f\|_p,$$

and

$$(2.2) \qquad \|f\|_p \approx \|\{\int_0^\infty \int_0^\infty |\psi_{t,s} * \psi_{t,s} * f(x,y)|^2 \frac{dt}{t}\frac{ds}{s}\}^{\frac{1}{2}}\|_p.$$

The L^p boundedness of flag singular integrals is then a consequence of Theorem 1.1 and Remark 2.1. We give a detailed proof of this below.

Proof of Theorem 1.4: We may first assume that K is integrable function and shall prove the $L^p, 1 < p < \infty$, boundedness of T is independent of the L^1 norm of K. The conclusion for general K then follows by the argument used in [MRS]. For all $f \in L^p, 1 < p < \infty$, by (2.2)

$$(2.3) \qquad \|T(f)\|_p \leq C\|\{\int_0^\infty \int_0^\infty |\psi_{t,s} * \psi_{t,s} * K * f(x,y)|^2 \frac{dt}{t}\frac{ds}{s}\}^{\frac{1}{2}}\|_p.$$

Now we claim the following estimate: for $f \in L^p, 1 < p < \infty$,

$$(2.4) \qquad |\psi_{t,s} * K * f(x,y)| \leq C M_s(f)(x,y),$$

where C is a constant which is independent of the L^1 norm of K and $M_s(f)$ is the maximal function of f defined in the first section.

Assuming (2.4) for the moment, we obtain from (2.3)

$$(2.5) \qquad \|Tf\|_p \leq C\|\{\int_0^\infty \int_0^\infty (M_s(\psi_{t,s} * f)(x,y))^2 \frac{dt\,ds}{t\,s}\}^{\frac{1}{2}}\|_p \leq C\|f\|_p,$$

where the last inequality follows from the Fefferman-Stein vector-valued maximal function and Remark 2.1.

We now prove the claim (2.4). Note that $\psi_{t,s} * K(x,y) = \int \psi_{t,s}^{\sharp} * K^{\sharp}(x, y-z, z)dz$, where $\psi_{t,s}^{\sharp}(x,y,z)$ is given in Remark 2.1 and $K(x,y) = \int K^{\sharp}(x, y-z, z)dz$, where $K^{\sharp}(x,y,z)$ is a product kernel satisfying the conditions of definition 2.1.1 in [NRS] (or Definition 1.2 in our paper). The estimate in (2.4) will follow by integrating with respect to z variable from the following estimate:

$$(2.6) \qquad |\psi_{t,s}^{\sharp} * K^{\sharp}(x,y,z)| \leq C \frac{t}{(t + |x| + |y|)^{n+m+1}} \frac{s}{(s + |z|)^{m+1}},$$

where the constant C is independent of the L^1 norm of K. The estimate (2.6) follows from that in the pure product setting $R^{n+m} \times R^m$ given by R. Fefferman and Stein [FS]. **Q.E.D.**

3. Test function spaces, almost orthogonality estimates and discrete Calderón reproducing formula: Proofs of Theorems 1.8 and 1.9

In this section, we develop the discrete Calderón reproducing formula and the Plancherel-Pôlya-type inequalities on test function spaces. These are crucial tools in establishing the theory of Hardy spaces associated with the flag type multi-parameter dilation structure. The key ideas to provide the discrete Calderón reproducing formula and the Plancherel-Pôlya-type inequalities are the continuous version of the Calderón reproducing formula on test function spaces and the almost orthogonality estimates.

To be more precise, we say that a function $a(x,y)$ for $(x,y) \in R^n \times R^n$ belongs to the class $\mathcal{S}^\infty(R^n \times R^n)$ if $a(x,y)$ is smooth and satisfies the differential inequalities

$$(3.1) \qquad |\partial_x^\alpha \partial_y^\beta a(x,y)| \leq A_{N,\alpha,\beta}(1 + |x-y|)^{-N}$$

and the cancellation conditions

$$(3.2) \qquad \int a(x,y)x^\alpha dx = \int a(x,y)y^\beta dy = 0$$

for all positive integers N and multi-indices α, β of nonnegative integers.

The following almost orthogonality estimate is the simplest one and its proof can be adapted to the more complicated orthogonal estimates in subsequent steps.

Lemma 3.1. *If ψ and ϕ are in the class $\mathcal{S}^\infty(R^n \times R^n)$, then for any given positive integers L and M, there exists a constant $C = C(L, M)$ depending only on L, M and the constants $A_{N,\alpha,\beta}$ in (3.1) such that for all $t, s > 0$*

$$(3.3) \qquad |\int_{R^n} \psi_t(x, z)\phi_s(z, y)dz| \le C(\frac{t}{s} \wedge \frac{s}{t})^M \frac{(t \vee s)^L}{(t \vee s + |x - y|)^{(n+L)}},$$

where $\psi_t(x, z) = t^{-n}\psi(\frac{x}{t}, \frac{z}{t})$ and $\phi_s(z, y) = s^{-n}\phi(\frac{z}{s})$, and $t \wedge s = min(t, s), t \vee s = max(t, s)$.

Proof: We only consider the case $M = L = 1$ and $t \ge s$. Then

$$\int_{R^m} \psi_t(x, z)\phi_s(z, y)dz$$

$$= \int_{R^m} [\psi_t(x, z) - \psi_t(x, y)] \phi_s(z, y)dz$$

$$= \int_{|z-y| \le \frac{1}{2}(t+|x-y|)} + \int_{|z-y| \ge \frac{1}{2}(t+|x-y|)} = I + II$$

We use the smoothness condition for ψ_t and size condition for ϕ_s to estimate term I. To estimate term II, we use the size condition for both ψ_t and ϕ_s.

For the case $M > 1$ and $L > 1$, we only need to use the Taylor expansion of $\psi_t(x, \cdot)$ at y and use the moment condition of ψ_t. We shall not give the details.   **Q.E.D.**

Similarly, if $\psi^\sharp(x, y, z, u, v, w)$ for $(x, y, z), (u, v, w) \in R^n \times R^m \times R^m$ is a smooth function and satisfies the differential inequalities

$$|\partial_x^{\alpha_1}\partial_y^{\beta_1}\partial_z^{\gamma_1}\partial_u^{\alpha_2}\partial_v^{\beta_2}\partial_w^{\gamma_2}\psi^\sharp(x, y, z, u, v, w)|$$

$$(3.4) \qquad \le A_{N,M,\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma_1,\gamma_2}(1 + |x - u| + |y - v|)^{-N}(1 + |z - w|)^{-M}$$

and the cancellation conditions

$$\int \psi^\sharp(x, y, z, u, v, w)x^{\alpha_1}y^{\beta_1}dxdy = \int \psi^\sharp(x, y, z, u, v, w)z^{\gamma_1}dz$$

$$(3.5) \qquad = \int \psi^\sharp(x, y, z, u, v, w)u^{\alpha_2}v^{\beta_2}dudv = \int \psi^\sharp(x, y, z, u, v, w)w^{\gamma_2}dw = 0,$$

and for fixed $x_0 \in R^n, y_0 \in R^m, \phi^\sharp(x, y, z, x_0, y_0) \in \mathcal{S}_\infty(R^{n+m} \times R^m)$ and satisfies

$$|\partial_x^{\alpha_1}\partial_y^{\beta_1}\partial_z^{\gamma_1}\phi^\sharp(x, y, z, x_0, y_0)|$$

$$(3.6) \qquad \le B_{N,M,\alpha_1,\beta_1,\gamma_1,}(1 + |x - x_0| + |y - y_0|)^{-N}(1 + |z|)^{-M},$$

for all positive integers N, M and multi-indices $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2$ of nonnegative integers. Then we have the following almost orthogonality estimate:

Lemma 3.2. *For any given positive integers L_1, L_2 and K_1, K_2, there exists a constant $C = C(L_1, L_2, K_1, K_2)$ depending only on L_1, L_2, K_1, K_2 and the constants in (3.4) and (3.6) such that for all positive t, s, t', s' we have*

$$\left| \int_{R^{n+m+m}} \psi^{\sharp}_{t,s}(x, y, z, u, v, w) \phi^{\sharp}_{t',s'}(u, v, w, x_0, y_0) \, du \, dv \, dw \right|$$

(3.7)

$$\leq C \left(\frac{t}{t'} \wedge \frac{t'}{t} \right)^{L_1} \left(\frac{s}{s'} \wedge \frac{s'}{s} \right)^{L_2} \frac{(t \vee t')^{K_1}}{(t \vee t' + |x - x_0| + |y - y_0|)^{(n+m+K_1)}} \frac{(s \vee s')^{K_2}}{(s \vee s' + |z|)^{(m+K_2)}},$$

where $\psi^{\sharp}_{t,s}(x, y, z, u, v, w) = t^{-n-m} s^{-m} \psi^{\sharp}(\frac{x}{t}, \frac{y}{t}, \frac{z}{s}, \frac{u}{t}, \frac{v}{t}, \frac{w}{s})$ and

$$\phi^{\sharp}_{t,s}(x, y, z, x_0, y_0) = t^{-n-m} s^{-m} \phi^{\sharp}(\frac{x}{t}, \frac{y}{t}, \frac{z}{s}, \frac{x_0}{t}, \frac{y_0}{t},).$$

The proofs of the almost orthogonality estimate in (3.7) is similar to that in (3.3). We will only provide a brief proof.

Proof of Lemma 3.2: We only consider the case $L_1 = L_2 = K_1 = K_2 = 1$, $t \geq t'$ and $s \leq s'$. Thus

$$\left| \int_{R^n \times R^m \times R^m} \psi^{\sharp}_{t,s}(x, y, z, u, v, w) \phi^{\sharp}_{t',s'}(u, v, w, x_0, y_0) \, du \, dv \, dw \right|$$

$$= \left| \int_{R^n \times R^m \times R^m} A \cdot B \, du \, dv \, dw \right|$$

where

$$A = \psi^{\sharp}_{t,s}(x, y, z, u, v, w) - \psi^{\sharp}_{t,s}(x, y, z, x_0, y_0, w)$$

and

$$B = \phi^{\sharp}_{t',s'}(u, v, w, x_0, y_0) - \phi^{\sharp}_{t',s'}(u, v, z, x_0, y_0).$$

In the above, we have used the cancelation properties

$$\int_{R^n \times R^m} \phi^{\sharp}_{t',s'}(u, v, w, x_0, y_0) u^\alpha v^\beta \, du \, dv = 0, \quad \int_{R^m} \psi^{\sharp}_{t,s}(x, y, z, u, v, w) w^\gamma = 0$$

for all multi-indices α, β and γ.

Next,

$$\left| \int_{R^n \times R^m \times R^m} A \cdot B \, du \, dv \, dw \right.$$

$$= \int_{|u-x_0|+|v-y_0| \leq \frac{1}{2}(t+|x-x_0|+|y-y_0|),\ |w-z| \leq \frac{1}{2}(s'+|z|)} A \cdot B \, du \, dv \, dw$$

$$+ \int_{|u-x_0|+|v-y_0| \leq \frac{1}{2}(t+|x-x_0|+|y-y_0|),\ |w-z| \geq \frac{1}{2}(s'+|z|)} A \cdot B \, du \, dv \, dw$$

$$+ \int_{|u-x_0|+|v-y_0| \geq \frac{1}{2}(t+|x-x_0|+|y-y_0|),\ |w-z| \leq \frac{1}{2}(s'+|z|)} A \cdot B \, du \, dv \, dw$$

$$+ \int_{|u-x_0|+|v-y_0| \geq \frac{1}{2}(t+|x-x_0|+|y-y_0|),\ |w-z| \geq \frac{1}{2}(s'+|z|)} A \cdot B \, du \, dv \, dw$$

$$= I + II + III + IV$$

For term I, we use the smoothness conditions on $\psi_{t,s}^{\sharp}$ and $\phi_{t',s'}^{\sharp}$; For term II, we use the smoothness conditions on $\psi_{t,s}^{\sharp}$ and size conditions on $\phi_{t',s'}^{\sharp}$; For term III, we use the size conditions on $\psi_{t,s}^{\sharp}$ and smoothness conditions on $\phi_{t',s'}^{\sharp}$; For term IV, use the size conditions on both $\psi_{t,s}^{\sharp}$ and $\phi_{t',s'}^{\sharp}$. We shall not provide the details here. **Q.E.D.**

The crucial feature, however, is that the almost orthogonality estimate still holds for functions in $\mathcal{S}_F(R^n \times R^m)$. To see this, we first derive the relationship of convolutions on $R^n \times R^m$ and $R^{n+m} \times R^m$. We will use this relationship frequently in this paper.

Lemma 3.3. *Let $\psi, \phi \in \mathcal{S}_F(R^n \times R^m)$, and $\psi^{\sharp}, \phi^{\sharp} \in \mathcal{S}_{\infty}(R^{n+m} \times R^m)$ such that*

$$\psi(x,y) = \int_{R^m} \psi^{\sharp}(x, y-z, z)dz, \ \phi(x,y) = \int_{R^m} \phi^{\sharp}(x, y-z, z)dz.$$

Then

$$(\psi * \phi)(x,y) = \int_{R^m} \left(\psi^{\sharp} * \phi^{\sharp}\right)(x, y-z, z)dz.$$

Lemma 3.3 can be proved very easily. Using this lemma and the almost orthogonality estimates on $R^{n+m} \times R^m$, we can get the following

Lemma 3.4. *For any given positive integers L_1, L_2 and K_1, K_2, there exists a constant $C = C(L_1, L_2, K_1, K_2)$ depending only on L_1, L_2, K_1, K_2 such that if $t \vee t' \leq s \vee s'$, then*

$$|\psi_{t,s} * \phi_{t',s'}(x,y)|$$
$$\leq C(\frac{t}{t'} \wedge \frac{t'}{t})^{L_1}(\frac{s}{s'} \wedge \frac{s'}{s})^{L_2} \cdot \frac{(t \vee t')^{K_1}}{(t \vee t' + |x|)^{(n+K_1)}} \frac{(s \vee s')^{K_2}}{(s \vee s' + |y|)^{(m+K_2)}},$$

and if $t \vee t' \geq s \vee s'$, then

$$|\psi_{t,s} * \phi_{t',s'}(x,y)|$$
$$\leq C(\frac{t}{t'} \wedge \frac{t'}{t})^{L_1}(\frac{s}{s'} \wedge \frac{s'}{s})^{L_2} \cdot \frac{(t \vee t')^{K_1}}{(t \vee t' + |x|)^{(n+K_1)}} \frac{(t \vee t')^{K_2}}{(t \vee t' + |y|)^{(m+K_2)}}.$$

Proof of Lemma 3.4: We first remark that we will prove this lemma with K_1, K_2, L_1, L_2 replaced by K_1', K_2', L_1', L_2'. Thus, we are given any fixed K_1', K_2', L_1', L_2'.

Note that

$$\psi_{t,s} * \phi_{t',s'}(x,y) = \int_{R^m} \psi_{t,s}^{\sharp} * \phi_{t',s'}^{\sharp}(x, y-z, z)dz,$$

where $\psi^{\sharp}, \phi^{\sharp} \in \mathcal{S}_{\infty}(R^{n+m} \times R^m)$, and

$$\psi_{t,s}^{\sharp} * \phi_{t',s'}^{\sharp}(x,y,z) = \int_{R^m \times R^m \times R^n} \psi_{t,s}^{\sharp}(x-u, y-v, z-w)\phi_{t',s'}^{\sharp}(u,v,w)dudvdw,$$

Then by the estimate in (3.7), for any given positive integers L_1, L_2 and K_1, K_2, there exists a constant $C = C(L_1, L_2, K_1, K_2)$ depending only on L_1, L_2, K_1, K_2 such that

$$|\psi_{t,s} * \phi_{t',s'}(x,y)|$$

(3.8)

$$\leq C(\frac{t}{t'} \wedge \frac{t'}{t})^{L_1}(\frac{s}{s'} \wedge \frac{s'}{s})^{L_2} \int_{R^m} \frac{(t \vee t')^{K_1}}{(t \vee t' + |x| + |y - z|)^{(n+m+K_1)}} \frac{(s \vee s')^{K_2}}{(s \vee s' + |z|)^{(m+K_2)}} dz.$$

Case 1: If $t \vee t' \leq s \vee s'$ and $|y| \geq s \vee s'$, write

$$\int_{R^m} \frac{(t \vee t')^{K_1}}{(t \vee t' + |x| + |y - z|)^{(n+m+K_1)}} \frac{(s \vee s')^{K_2}}{(s \vee s' + |z|)^{(m+K_2)}} dz$$

$$= \int_{|z| \leq \frac{1}{2}|y|, \text{ or } |z| \geq 2|y|} + \int_{\frac{1}{2}|y| \leq |z| \leq 2|y|} = I + II$$

It is easy to see that

$$|I| \leq C \frac{(t \vee t')^{K_1}}{(t \vee t' + |x| + |y|)^{(n+m+K_1)}}$$

$$\leq C \frac{(t \vee t')^{K'_1}}{(t \vee t' + |x|)^{(n+K'_1)}} \frac{(t \vee t')^{K_2'}}{|y|^{K'_2}}$$

$$\leq C \frac{(t \vee t')^{K'_1}}{(t \vee t' + |x|)^{(n+K'_1)}} \cdot \frac{(s \vee s')^{m+K'_2}}{(s \vee s' + |y|)^{m+K'_2}}$$

where we have taken $K_1 = K'_1 + K'_2$ and used the fact that $t \vee t' \leq s \vee s'$ and $|y| \geq s \vee s'$.

Next, we estimate

$$|II| \leq \frac{(s \vee s')^{K_2}}{(s \vee s' + |y|)^{(m+K_2)}} \int_{R^m} \frac{(t \vee t')^{K_1}}{(t \vee t' + |x| + |y - z|)^{(n+m+K_1)}} dz$$

$$\leq C \frac{(s \vee s')^{K_2}}{(s \vee s' + |y|)^{(m+K_2)}} \cdot \frac{(t \vee t')^{K_1}}{(t \vee t' + |x|)^{(n+K_1)}}$$

$$\leq C \frac{(s \vee s')^{K'_2}}{(s \vee s' + |y|)^{(m+K'_2)}} \cdot \frac{(t \vee t')^{K'_1}}{(t \vee t' + |x|)^{(n+K'_1)}}$$

where we have used $K_2 \geq K'_2$ and $K_1 = K'_1 + K'_2 > K'_1$.

Case 2: If $t \vee t' \leq s \vee s'$ and $|y| \leq s \vee s'$, then

$$\int_{R^m} \frac{(t \vee t')^{K_1}}{(t \vee t' + |x| + |y - z|)^{(n+m+K_1)}} \frac{(s \vee s')^{K_2}}{(s \vee s' + |z|)^{(m+K_2)}} dz$$

$$\leq \frac{1}{(s \vee s')^m} \int_{R^m} \frac{(t \vee t')^{K_1}}{(t \vee t' + |x| + |y - z|)^{(n+m+K_1)}} dz$$

$$\leq C \frac{(s \vee s')^{K_2}}{(s \vee s' + |y|)^{m+K_2}} \frac{(t \vee t')^{K_1}}{(t \vee t' + |x|)^{(n+K_1)}}.$$

Case 3: We now consider the case $t \vee t' \geq s \vee s'$ and $|y| \leq t \vee t'$. Then

$$\int_{R^m} \frac{(t \vee t')^{K_1}}{(t \vee t' + |x| + |y - z|)^{(n+m+K_1)}} \frac{(s \vee s')^{K_2}}{(s \vee s' + |z|)^{(m+K_2)}} dz$$

$$\leq C \frac{(t \vee t')^{K_1}}{(t \vee t' + |x|)^{(n+m+K_1)}}$$

$$\leq \frac{(t \vee t')^{K_1'}}{(t \vee t' + |x|)^{(n+K_1')}} \cdot \frac{(t \vee t')^{K_2'}}{(t \vee t' + |x| + |y|)^{(m+K_2')}}.$$

by noticing that $K_1 = K_1' + K_2'$.

Case 4: If we assume $t \vee t' \geq s \vee s'$ and $|y| \geq t \vee t'$, then we divide the integral into two parts I and II as in the case $t \vee t' \leq s \vee s'$. Thus we have

$$|I| \leq \frac{(t \vee t')^{K_1}}{(t \vee t' + |x| + |y|)^{(n+m+K_1)}}$$

$$\leq C \frac{(t \vee t')^{K_1'}}{(t \vee t' + |x|)^{(n+m+K_1')}} \cdot \frac{(t \vee t')^{K_2'}}{(t \vee t' + |y|)^{(m+K_2')}}$$

where we have taken $K_1 = K_1' + K_2'$.

To estimate II, we have

$$|II| \leq C \frac{(t \vee t')^{K_1}}{(t \vee t' + |x|)^{(n+K_1)}} \cdot \frac{(s \vee s')^{K_2}}{(s \vee s' + |y|)^{(m+K_2)}}$$

$$\leq C \frac{(t \vee t')^{K_1}}{(t \vee t' + |x|)^{(n+K_1)}} \cdot \frac{(t \vee t')^{K_2}}{(t \vee t' + |y|)^{(m+K_2)}}$$

Q.E.D.

Roughly speaking, $\psi_{t,s} * \phi_{t',s'}(x, y)$ satisfies the one-parameter almost orthogonality when $t \vee t' \geq s \vee s'$ and the product multi-parameter almost orthogonality when $t \vee t' \leq s \vee s'$. More precisely, we have the following

Corollary 3.5. *Given any positive integers L_1, L_2, K_1 and K_2, there exists a constant $C = C(L_1, L_2, K_1, K_2) > 0$ such that*

(i) If $t \geq s$ we obtain the one-parameter almost orthogonality

$$(3.8) \qquad |\psi_{t,s} * \phi_{t',s'}(x, y)| \leq C (\frac{t}{t'} \wedge \frac{t'}{t})^{L_1} (\frac{s}{s'} \wedge \frac{s'}{s})^{L_2} \frac{t^{K_1}}{(t + |x|)^{(n+K_1)}} \frac{t^{K_2}}{(t + |y|)^{(m+K_2)}}$$

and, if $t \leq s$ the product multi-parameter almost orthogonality is given by

$$(3.9) \qquad |\psi_{t,s} * \phi_{t',s'}(x, y)| \leq C (\frac{t}{t'} \wedge \frac{t'}{t})^{L_1} (\frac{s}{s'} \wedge \frac{s'}{s})^{L_2} \frac{t^{K_1}}{(t + |x|)^{(n+K_1)}} \frac{s^{K_2}}{(s + |y|)^{(m+K_2)}}.$$

(ii) Similarly, if $t' \geq s'$,

$$(3.10) \qquad |\psi_{t,s} * \phi_{t',s'}(x, y)| \leq C (\frac{t}{t'} \wedge \frac{t'}{t})^{L_1} (\frac{s}{s'} \wedge \frac{s'}{s})^{L_2} \frac{t'^{K_1}}{(t' + |x|)^{(n+K_1)}} \frac{t'^{K_2}}{(t' + |y|)^{(m+K_2)}},$$

and if $t' \leq s'$,

$$(3.11) \qquad |\psi_{t,s} * \phi_{t',s'}(x,y)| \leq C(\frac{t}{t'} \wedge \frac{t'}{t})^{L_1}(\frac{s}{s'} \wedge \frac{s'}{s})^{L_2} \frac{t'^{K_1}}{(t'+|x|)^{(n+K_1)}} \frac{s'^{K_2}}{(s'+|y|)^{(m+K_2)}}.$$

Corollary 3.5 is actually what we will use frequently in the subsequent parts of the paper. The proof of Corollary 3.5 is a case by case study and can be checked with patience. We shall omit the details of the proof here. All these estimates will be used to prove the following continuous version of the Calderón reproducing formula on test function space $\mathcal{S}_F(R^n \times R^m)$ and its dual space $(\mathcal{S}_F)'$.

Theorem 3.6. *Suppose that $\psi_{j,k}$ are the same as in (1.4). Then*

$$(3.12) \qquad f(x,y) = \sum_j \sum_k \psi_{j,k} * \psi_{j,k} * f(x,y),$$

where the series converges in the norm of $\mathcal{S}_F$ and in dual space $(\mathcal{S}_F)'$.

Proof: Suppose $f \in \mathcal{S}_F$ and $f(x,y) = \int_{R^m} f^\sharp(x, y-z, z)dz$, where $f^\sharp \in \mathcal{S}_\infty(R^{n+m} \times R^m)$. Then, by the classical Calderón reproducing formula as mentioned in the first section, for all $f^\sharp \in L^2$,

$$(3.13) \qquad f^\sharp(x,y,z) = \sum_j \sum_k \psi_{j,k}^\sharp * \psi_{j,k}^\sharp * f^\sharp(x,y,z),$$

where $\psi_{j,k}^\sharp(x,y,z) = \psi_j^{(1)}(x,y)\psi_k^{(2)}(z)$.

We **claim** that the above series in (3.13) converges in $\mathcal{S}_\infty(R^{n+m} \times R^m)$. This claim yields

$$\left\| f(x,y) - \sum_{-N \leq j \leq N} \sum_{-M \leq k \leq M} \psi_{j,k} * \psi_{j,k} * f(x,y) \right\|_{\mathcal{S}_F}$$

$$= \left\| \int_{R^m} [f^\sharp(x,y-z,z) - \sum_{-N \leq j \leq N} \sum_{-M \leq k \leq M} \psi_{j,k}^\sharp * \psi_{j,k}^\sharp * f^\sharp(x,y-z,z)]dz \right\|_{\mathcal{S}_F}$$

$$\leq \left\| f^\sharp(x,y,z) - \sum_{-N \leq j \leq N} \sum_{-M \leq k \leq M} \psi_{j,k}^\sharp * \psi_{j,k}^\sharp * f^\sharp(x,y,z) \right\|_{\mathcal{S}_\infty}$$

where the last term above goes to zero as N and M tend to infinity by the above claim.

To show the claim, it suffices to prove that all the following three summations

$$\sum_{|j|>N} \sum_{|k| \leq M} \psi_{j,k}^\sharp * \psi_{j,k}^\sharp * f^\sharp, \quad \sum_{|j| \leq N} \sum_{|k|>M} \psi_{j,k}^\sharp * \psi_{j,k}^\sharp * f^\sharp, \quad \sum_{|j|>N} \sum_{|k|>M} \psi_{j,k}^\sharp * \psi_{j,k}^\sharp * f^\sharp$$

tend to zero in $\mathcal{S}_\infty(R^{n+m} \times R^m)$ as N and M tend to infinity. Since all proofs are similar for each of the summations, we only prove the assertion for the first summation which we denote by $f_{N,M}^\sharp$. Note that

$$f_{N,M}^\sharp(x,y,z) = \sum_{|j|>N} \sum_{|k| \leq M} \int_{R^n \times R^m \times R^m} \psi_{j,k}^\sharp * \psi_{j,k}^\sharp(x-u, y-v, z-w)f^\sharp(u,v,w)dudvdw$$

where $\psi^\sharp * \psi^\sharp(x-u, y-v, z-w)$ satisfies the conditions (3.4) and (3.5), and $f^\sharp(u,v,w)$ satisfies the conditions (3.6) with $x_0 = y_0 = 0$. The almost orthogonality estimate in (3.7) with $t = 2^{-j}, s = 2^{-k}, t' = s' = 1$ and $x_0 = y_0 = 0$, implies

$$|\int_{R^n \times R^m \times R^m} \psi^\sharp_{j,k} * \psi^\sharp_{j,k}(x-u, y-v, z-w) f^\sharp(u,v,w) dudvdw|$$

$$\leq C 2^{-|j|L_1} 2^{-|k|L_2} \frac{(2^{-j} \vee 1)^{K_1}}{(2^{-j} \vee 1 + |x| + |y|)^{(n+m+K_1)}} \frac{(2^{-k} \vee 1)^{K_2}}{(2^{-k} \vee 1 + |z|)^{(m+K_2)}}.$$

This, by taking $L_1 > K_1$ and $L_2 > K_2$, gives us

$$\lim_{N,M \to \infty} \sup_{x \in R^n, y \in R^m, z \in R^m} (1 + |x| + |y|)^{n+m+K_1} (1 + |z|)^{m+K_2} |f^\sharp_{N,M}(x,y,z)| = 0.$$

Since $\partial_x^\alpha \partial_y^\beta \partial_z^\gamma (f^\sharp_{N,M})(x,y,z) = (\partial_x^\alpha \partial_y^\beta \partial_z^\gamma f^\sharp)_{N,M}(x,y,z)$ and applying the above estimate to $\partial_x^\alpha \partial_y^\beta \partial_z^\gamma f^\sharp$ which also satisfies the conditions in (3.6) with $x_0 = y_0 = 0$, we obtain

$$\lim_{N,M \to \infty} \sup_{x \in R^n, y \in R^m, z \in R^m} (1 + |x| + |y|)^{n+m+K_1} (1 + |z|)^{m+K_2} |(\partial_x^\alpha \partial_y^\beta \partial_z^\gamma f^\sharp)_{N,M}(x,y,z)| = 0,$$

which shows the claim.

The convergence in dual space follows from the duality argument. The proof of Theorem 3.6 is complete.   **Q.E.D.**

Using Theorem 3.6, we prove the discrete Calderón reproducing formula.

Proof of Theorem 1.8: We first discretize (3.12) as follows. For $f \in S_F$, by (3.12) and using an idea similar to that of decomposition of the identity operator due to Coifman, we can rewrite

$$f(x,y) = \sum_{j,k} \sum_{I,J} \int_J \int_I \psi_{j,k}(x-u, y-w) (\psi_{j,k} * f)(u,w) dudw$$

(3.14) $$= \sum_{j,k} \sum_{I,J} \left[\int_J \int_I \psi_{j,k}(x-u, y-w) dudw \right] (\psi_{j,k} * f)(x_I, y_J) + \mathcal{R}(f)(x,y).$$

We shall show that $\mathcal{R}$ is bounded on S_F with the small norm when I and J are dyadic cubes in R^n and R^m with side length 2^{-j-N} and $2^{-k-N} + 2^{-j-N}$ for a large given integer N, and x_I, y_J are any fixed points in I, J, respectively.

To do this, assuming $f(x,y) = \int_{R^m} f^\sharp(x, y-z, z) dz$, where $f^\sharp \in S_\infty(R^{n+m} \times R^m)$. When $k \leq j$, we write

$$(\psi_{j,k} * f)(u,w) - (\psi_{j,k} * f)(x_I, y_J)$$

$$= \int_{R^m} \left(\psi^\sharp_{jk} * f^\sharp\right)(u,v,w-v) dv - \int_{R^m} \left(\psi^\sharp_{jk} * f^\sharp\right)(x_I, v, y_J - v) dv$$

$$= \int \left[\psi^\sharp_{jk}(u-u', v-v', w-v-w') - \psi^\sharp_{jk}(x_I - u', v-v', y_J - v - w') \right] f^\sharp(u',v',w') du' dv' dw' dv$$

where the last integral above is over $R^n \times R^m \times R^m \times R^m$.

When $k > j$, we write

$$(\psi_{j,k} * f)(u, w) - (\psi_{j,k} * f)(x_I, y_J)$$
$$= \int_{R^m} \left(\psi_{jk}^{\sharp} * f^{\sharp} \right)(u, w - v, v)dv - \int_{R^m} \left(\psi_{jk}^{\sharp} * f^{\sharp} \right)(x_I, y_J - v, v)dv$$
$$= \int \left[\psi_{jk}^{\sharp}(u - u', w - v - v', v - w') - \psi_{jk}^{\sharp}(x_I - u', y_J - v - v', v - w') \right] f^{\sharp}(u', v', w')du'dv'dw'dv$$

We now set

$$\mathcal{R}(f)(x, y)$$
$$= \sum_{j,k} \sum_{I,J} \int_J \int_I \psi_{j,k}(x - u, y - w)\left[(\psi_{j,k} * f)(u, w) - (\psi_{j,k} * f)(x_I, y_J) \right] dudw$$
$$= \int \int \int \int \mathcal{R}^{\sharp}(x, y - z, z, u', v', w')f^{\sharp}(u', v', w')du'dv'dw'dz$$
$$= \int_{R^m} \mathcal{R}^{\sharp}(f^{\sharp})(x, y - z, z)dz,$$

where $\mathcal{R}^{\sharp}(x, y, z, u', v', w')$ is the kernel of $\mathcal{R}^{\sharp}$ and

$$\mathcal{R}^{\sharp}(x, y - z, z, u', v', w')$$
$$= \sum_{k \leq j} \sum_{I,J} \int_J \int_I \int \psi_j^{(1)}(x - u, y - z - w)\psi_k^{(2)}(z)$$
$$\times \left[\psi_j^{(1)}(u - u', v - v')\psi_k^{(2)}(w - v - w') - \psi_j^{(1)}(x_I - u', v - v')\psi_k^{(2)}(y_J - w' - v) \right] dudwdv$$
$$+ \sum_j \sum_{k > j} \sum_J \sum_I \int_J \int_I \int \psi_j^{(1)}(x - u, y - z - w)\psi_k^{(2)}(z)$$
$$\times \left[\psi_j^{(1)}(u - u', w - v - v') - \psi_j^{(1)}(x_I - u', y_J - v - v') \right] \psi_k^{(2)}(v - w')dudwdv.$$

Using the change of variables from z to $z + v - w$ in the term of $k \leq j$, and z to $z - v$ in the term of $k > j$, we can rewrite

$$\mathcal{R}^{\sharp}(x, y, z, u', v', w')$$
$$= \sum_{k \leq j} \sum_{I,J} \int_J \int_I \int \psi_j^{(1)}(x - u, y - v)\psi_k^{(2)}(z + v - w)$$
$$\times \left[\psi_j^{(1)}(u - u', v - v')\psi_k^{(2)}(w - w' - v) - \psi_j^{(1)}(x_I - u', v - v')\psi_k^{(2)}(y_J - w' - v) \right] dudwdv$$
$$+ \sum_j \sum_{k > j} \sum_J \sum_I \int_J \int_I \int \psi_j^{(1)}(x - u, y + v - w)\psi_k^{(2)}(z - v)$$
$$\times \left[\psi_j^{(1)}(u - u', w - v - v') - \psi_j^{(1)}(x_I - u', y_J - v - v') \right] \psi_k^{(2)}(v - w')dudwdv$$
$$= \sum_{k \leq j} A_{jk} + \sum_{k > j} B_{jk}.$$

We **claim** that $\mathcal{R}^\sharp$ is bounded in $\mathcal{S}_\infty(R^{n+m} \times R^m)$.

To see this, write

$$
\begin{aligned}
A_{jk} &= \sum_J \sum_I \int \int_J \int_I \psi_j^{(1)}(x-u, y-v)\psi_k^{(2)}(z+v-w) \\
&\times [\psi_j^{(1)}(u-u', v-v') - \psi_j^{(1)}(x_I - u', v-v')]\psi_k^{(2)}(w-w'-v)dudwdv \\
&+ \sum_J \sum_I \int \int_J \int_I \psi_j^{(1)}(x-u, y-v)\psi_k^{(2)}(z+v-w) \\
&\times \psi_j^{(1)}(x_I - u', v-v')[\psi_k^{(2)}(w-w'-v) - \psi_k^{(2)}(y_J - w'-v)]dudwdv \\
&= A_{jk}^{(1)}(x, y, z, u', v', w') + A_{jk}^{(2)}(x, y, z, u', v', w')
\end{aligned}
$$

It is not difficult to check that $\int_{R^m} \psi_k^{(2)}(z+v-w)\psi_k^{(2)}(w-w'-v)dw$ satisfies all the conditions as $\psi_k^{(2)}(z-w')$ does with the comparable constants of $\mathcal{S}_F(R^n \times R^m)$ norm and that

$$
\sum_J \sum_I \int \int_J \int_I \psi_j^{(1)}(x-u, y-v)[\psi_j^{(1)}(u-u', v-v') - \psi_j^{(1)}(x_I - u', v-v')]dudv
$$

satisfies all conditions as $\psi_j^{(1)}(x-u', y-v')$ but with the constants of $\mathcal{S}_F(R^n \times R^m)$ norm replaced by $C2^{-N}$. This follows from the smoothness condition on $\psi_j^{(1)}$ (say the mean-value theorem) and the fact that $u, x_I \in I$ and $l(I) = 2^{-N-j}$.

If we write

$$
\begin{aligned}
&A_{jk}^{(1)}(x, y, z, u', v', w') \\
&= \sum_J \sum_I \int \int_J \int_I \psi_j^{(1)}(x-u, y-v)[\psi_j^{(1)}(u-u', v-v') - \psi_j^{(1)}(x_I - u', v-v')] \\
&\times \left[\int_{R^m} \psi_k^{(2)}(z+v-w)\psi_k^{(2)}(w-w'-v)dw \right] dudv
\end{aligned}
$$

then the function $A_{jk}^{(1)}(x, y, z, u', v', w')$ satisfies all conditions as

$$
\psi_j^{(1)}(x-u', y-v') \cdot \psi_k^{(2)}(z-w')
$$

does but with the $\mathcal{S}_F(R^n \times R^m)$ norm constant replaced by $C2^{-N}$.

By the proof of Theorem 3.6, we conclude that

$$
\int A_{jk}^{(1)}(x, y, z, u', v', w')f^\sharp(u', v', w')du'dv'dw'
$$

is a test function in $\mathcal{S}_\infty(R^{n+m} \times R^m)$ and its test function norm is bounded by $C2^{-N}$.

Similarly,

$$
\sum_J \int_J \psi_k^{(2)}(z+v-w)[\psi_k^{(2)}(w-w'-v) - \psi_k^{(2)}(y_J - w'-v)]dw
$$

satisfies all conditions as $\psi_k^{(2)}(z - w')$ but with the constant replaced by $C2^{-N}$, which follows from the smoothness condition on $\psi_k^{(2)}$ and the fact $w, y_J \in J$ and $l(J) = 2^{-N-j} + 2^{-N-k}$ and $k \leq j$.

We also note that

$$\sum_J \sum_I \int_J \int_I \int \psi_j^{(1)}(x - u, y - v)\psi_j^{(1)}(x_I - u', v - v')dudv$$

satisfies the same conditions as $\psi_j^{(1)}(x - u', y - v')$ does with comparable $\mathcal{S}_F(R^n \times R^m)$ norm constant. Thus, we conclude that

$$\int A_{jk}^{(2)}(x, y, z, u', v', w')f^\sharp(u', v', w')du'dv'dw'$$

is a test function in $\mathcal{S}_\infty(R^{n+m} \times R^m)$ and its test function norm is bounded by $C2^{-N}$. Therefore,

$$\int A_{jk}(x, y, z, u', v', w')f^\sharp(u', v', w')du'dv'dw'$$

is a test function in $\mathcal{S}_\infty(R^{n+m} \times R^m)$ and its test function norm is bounded by $C2^{-N}$.

Similarly, we can conclude that

$$\int B_{jk}(x, y, z, u', v', w')f^\sharp(u', v', w')du'dv'dw'$$

is also a test function in $\mathcal{S}_\infty(R^{n+m} \times R^m)$ and its test function norm is bounded by $C2^{-N}$.

This shows that $\mathcal{R}^\sharp(f^\sharp)(x, y, z) \in \mathcal{S}_\infty(R^{n+m} \times R^m)$ and

$$||\mathcal{R}^\sharp(f^\sharp)||_{\mathcal{S}_\infty(R^{n+m} \times R^m)} \leq C2^{-N}||f^\sharp||_{\mathcal{S}_\infty(R^{n+m} \times R^m)},$$

which implies that $\mathcal{R}(f) \in \mathcal{S}_F(R^n \times R^m)$ and

$$(3.15) \qquad ||\mathcal{R}(f)||_{\mathcal{S}_F(R^n \times R^m)} \leq C2^{-N}||f||_{\mathcal{S}_F(R^n \times R^m)}.$$

By (3.14) together with the boundedness of $\mathcal{R}$ on $\mathcal{S}_F$ with the norm at most $C2^{-N}$, if N is chosen large enough, then we obtain

$$f(x, y) = \sum_j \sum_k \sum_J \sum_I \left[\sum_{i=0}^\infty \mathcal{R}^i \int_J \int_I \psi_{j,k}(\cdot - u, \cdot - v)dudv \right](x, y)\,(\psi_{j,k} * f)(x_I, y_J).$$

Set

$$\left[\sum_{i=0}^\infty \mathcal{R}^i \int_J \int_I \psi_{j,k}(\cdot - u, \cdot - v)dudv \right](x, y) = |I||J|\widetilde{\psi}_{j,k}(x, y, x_I, y_J).$$

It remains to show $\widetilde{\psi}_{j,k}(x, y, x_I, y_J) \in \mathcal{S}_F$. This, however, follows easily from (3.15).

We next establish a relationship between test function in $f^\sharp \in \mathcal{S}_\infty(R^{n+m} \times R^m)$ and test function $f(x,y) = \int\limits_{R^m} f^\sharp(x, y-z, z)dz$ under the actions of $\mathcal{R}^\sharp$, $\mathcal{R}$ and the implicit multi-parameter dilations.

To be more precise, we define

$$f_{j,k}(x,y) = \int\limits_{R^m} (f^\sharp)_{j,k}(x, y-z, z)dz,$$

where

$$(f^\sharp)_{j,k}(x,y,z) = 2^{(n+m)j}2^{mk}f^\sharp(2^j x, 2^j y, 2^k z).$$

Note that

$$\mathcal{R}^\sharp((f^\sharp)_{j,k})(x,y,z)$$

$$= \int\int\int R^\sharp(x,y,z,u',v',w')(f^\sharp)_{jk}(u',v',w')du'dv'dw'$$

$$= 2^{(n+m)j}2^{mk}\int\int\int \mathcal{R}^\sharp(x,y,z,u',v',w')f^\sharp(2^j u', 2^j v', 2^k w')du'dv'dw'$$

$$= \int\int\int \mathcal{R}^\sharp(x,y,z,2^{-j}u',2^{-j}v',2^{-k}w')f^\sharp(u',v',w')du'dv'dw'$$

and

$$\mathcal{R}^\sharp(2^{-j}x, 2^{-j}y, 2^{-k}z, 2^{-j}u', 2^{-j}v', 2^{-k}w') = 2^{(n+m)j}2^{mk}\mathcal{R}^\sharp(x,y,z,u',v',w').$$

Thus we have

$$\mathcal{R}^\sharp((f^\sharp)_{j,k})(x,y,z) = (\mathcal{R}^\sharp(f^\sharp))_{jk}(x,y,z).$$

This implies

$$\mathcal{R}(f_{j,k})(x,y) = \int_{R^m} \mathcal{R}^\sharp((f^\sharp)_{j,k})(x, y-z, z)dz$$

$$= \int_{R^m} (\mathcal{R}^\sharp(f^\sharp))_{jk}(x, y-z, z)dz = (\mathcal{R}(f))_{jk}(x,y)$$

It is worthwhile to point out that

$$\widetilde{\psi}_{j,k}(x,y,x_I,y_J) = \int\limits_{R^m} \widetilde{\psi}^\sharp_{j,k}(x, y-z, z, x_I, y_J)dz,$$

where

$$\widetilde{\psi}^\sharp_{j,k}(x,y,z,x_I,y_J) = (\widetilde{\psi}^\sharp)_{j,k}(x,y,z,x_I,y_J), \ \widetilde{\psi}^\sharp(x,y,z,x_I,y_J) \in \mathcal{S}_\infty(R^{n+m} \times R^m)$$

and satisfies the condition in (3.6) with $x_0 = x_I, y_0 = y_J$.

Remark 3.1: If we begin with discretizing (3.12) by

$$f(x,y) = \sum_j \sum_k \sum_J \sum_I \psi_{j,k}(x - x_I, y - y_J) \int_J \int_I (\psi_{j,k} * f)(u,v) du dv + \widetilde{\widetilde{\mathcal{R}}}(f)(x,y),$$

and repeating the similar proof, then the discrete Calderón reproducing formula can also be given by the following form

$$f(x,y) = \sum_j \sum_k \sum_J \sum_I |I||J| \psi_{j,k}(x - x_I, y - y_J) \widetilde{\widetilde{\psi}}_{j,k}(f)(x_I, y_J),$$

where $|I||J| \widetilde{\widetilde{\psi}}_{j,k}(f)(x_I, y_J) = \sum\limits_{i=0}^{\infty} \int_J \int_I \psi_{j,k} * (\widetilde{\widetilde{\mathcal{R}}})^i(f)(u,v) du dv$. We leave the details of these proofs to the reader.

Before we prove the Plancherel-Pôlya-type inequality, we first prove the following lemma.

Lemma 3.7. *Let I, I', J, J' be dyadic cubes in R^n and R^m respectively such that $\ell(I) = 2^{-j-N}$, $\ell(J) = 2^{-j-N} + 2^{-k-N}$, $\ell(I') = 2^{-j'-N}$ and $\ell(J') = 2^{-j'-N} + 2^{-k'-N}$. Thus for any $u \in I$ and $v \in J$ we have*

$$\sum_{k' \leq j'} \sum_{I', J'} \frac{2^{-(|j-j'|+|k-k'|)K} 2^{-(j'+k')K} |I'||J'|}{(2^{-j'} + |u - x_{I'}|)^{n+K} (2^{-k'} + |v - y_{J'}|)^{m+K}} |\phi_{j',k'} * f(x_{I'}, y_{J'})|$$

$$\leq C \sum_{k' \leq j'} 2^{-|j-j'|K} \cdot 2^{-|k-k'|K} \left\{ M_s \left(\sum_{J'} \sum_{I'} |\phi_{j',k'} * f(x_{I'}, y_{J'})| \chi_{J'} \chi_{I'} \right)^r \right\}^{\frac{1}{r}} (u,v)$$

and

$$\sum_{k' > j'} \sum_{I', J'} \frac{2^{-(|j-j'|+|k-k'|)K} 2^{-2j'K} |I'||J'|}{(2^{-j'} + |u - x_{I'}|)^{n+K} (2^{-j'} + |v - y_{J'}|)^{m+K}} |\phi_{j',k'} * f(x_{I'}, y_{J'})|$$

$$\leq C \sum_{k' > j'} 2^{-|j-j'|K} 2^{-|k-k'|K} \left\{ M \left(\sum_{J'} \sum_{I'} |\phi_{j',k'} * f(x_{I'}, y_{J'})| \chi_{I'} \chi_{J'} \right)^r \right\}^{\frac{1}{r}} (u,v)$$

where M is the Hardy-Littlewood maximal function on R^{n+m}, M_s is the strong maximal function on $R^n \times R^m$ as defined in (1.1), and $\max\left\{ \frac{n}{n+K}, \frac{m}{m+K} \right\} < r$.

Proof: We set

$$A_0 = \{I' : \ell(I') = 2^{-j'-N}, \ \frac{|u - x_{I'}|}{\ell(I')} \leq 1\}$$

$$B_0 = \{J' : \ell(J') = 2^{-j'-N} + 2^{-k'-N}, \ \frac{|v - y_{J'}|}{\ell(J')} \leq 1\}$$

and for $\ell \geq 1$, $i \leq 1$

$$A_\ell = \{I' : \ell(I') = 2^{-j'-N}, \ 2^{\ell-1} < \frac{|u - x_{I'}|}{\ell(I')} \leq 2^\ell\}.$$

$$B_i = \{J' : \ell(J') = 2^{-j'-N} + 2^{-k'-N}, \ 2^{i-1} < \frac{|v - y_{J'}|}{\ell(J')} \le 2^i\}.$$

Then

$$\frac{2^{-(j'+k')K}|I'||J'|}{(2^{-j'} + |u - x_{I'}|)^{n+K}(2^{-k'} + |v - y_{J'}|)^{m+K}} \cdot |\phi_{j',k'} * f(x_{I'}, y_{J'})|$$

$$\le \sum_{\ell,i \ge 0} 2^{-\ell(n+K)} 2^{-i(m+K)} \sum_{I' \in A_\ell, J' \in B_i} |\phi_{j',k'} * f(x_{I'}, y_{J'})|$$

$$\le \sum_{\ell,i \ge 0} 2^{-\ell(n+K)} 2^{-i(m+K)} \left(\sum_{I' \in A_\ell, J' \in B_i} (|\phi_{j',k'} * f(x_{I'}, y_{J'})|)^r \right)^{\frac{1}{r}}$$

$$\le \sum_{\ell,i \ge 0} 2^{-\ell(n+K)} 2^{-i(m+K)} \left(\int_{R^n \times R^m} |I'|^{-1}|J'|^{-1} \sum_{I' \in A_\ell, J' \in B_i} |\phi_{j',k'} * f(x_{I'}, y_{J'})|^r \chi_{I'} \chi_{J'} \right)^{\frac{1}{r}}$$

$$\le \sum_{\ell,i \ge 0} 2^{-\ell(n+K-\frac{n}{r})} 2^{-i(m+K-\frac{m}{r})} \left(M_s \left(\sum_{I' \in A_\ell, J' \in B_i} |\phi_{j',k'} * f(x_{I'}, y_{J'})|^r \chi_{I'} \chi_{J'} \right)(u, v) \right)^{\frac{1}{r}}$$

$$\le C \left(M_s \left(\sum_{I',J'} |\phi_{j',k'} * f(x_{I'}, y_{J'})|^r \chi_{I'} \chi_{J'} \right)(u, v) \right)^{\frac{1}{r}}$$

The last inequality follows from the assumption that $r > \frac{n}{n+K}$ and $r > \frac{m}{m+K}$ which can be done by choosing K large enough.

Similarly, we can prove the second inequality of the lemma.   **Q.E.D.**

We now are ready to give the

Proof of Theorem 1.9: By Theorem 1.8, $f \in \mathcal{S}_F$ can be represented by

$$f(x, y) = \sum_{j'} \sum_{k'} \sum_{J'} \sum_{I'} |J'||I'| \widetilde{\phi}_{j',k'}(x, y, x_{I'}, y_{J'}) (\phi_{j',k'} * f)(x_{I'}, y_{J'}).$$

We write

$$(\psi_{j,k} * f)(u, v)$$
$$= \sum_{j'} \sum_{k'} \sum_{J'} \sum_{I'} |I'||J'| \left(\psi_{j,k} * \widetilde{\phi}_{j',k'}(\cdot, \cdot, x_{I'}, y_{J'}) \right)(u, v) (\phi_{j',k'} * f)(x_{I'}, y_{J'}).$$

By the almost orthogonality estimates in (3.10) and (3.11), for any given positive integer K, we have if $j' \ge k'$,

$$|\psi_{j,k} * \widetilde{\phi}_{j',k'}(\cdot, \cdot)(u, v)|$$

(3.15) $$\le C 2^{-|j-j'|K} \cdot 2^{-|k-k'|K} \cdot \frac{2^{-j'K}}{(2^{-j'} + |u - x_{I'}|)^{n+K}} \cdot \frac{2^{-k'K}}{(2^{-k'} + |v - y_{J'}|)^{m+K}}$$

and if $j' \leq k'$, we have

$$|\psi_{j,k} * \widetilde{\phi}_{j',k'}(\cdot,\cdot)(u,v)|$$

$$(3.16) \qquad \leq C 2^{-|j-j'|K} \cdot 2^{-|k-k'|K} \cdot \frac{2^{-j'K}}{(2^{-j'} + |u - x_{I'}|)^{n+K}} \cdot \frac{2^{-j'K}}{(2^{-j'} + |v - y_{J'}|)^{m+K}}.$$

Using Lemma 3.7, for any $u, x_{I'} \in I, v, y_{J'} \in J$,

$$|\psi_{j,k} * f(u,v)|$$

$$\leq C \sum_{k' \leq j'} \sum_{I'} \sum_{J'} 2^{-|j-j'|K} 2^{-|k-k'|K} |I'||J'| \times$$

$$\frac{2^{-j'K}}{(2^{-j'} + |u - x_{I'}|)^{n+K}} \cdot \frac{2^{-k'K}}{(2^{-k'} + |v - y_{J'}|)^{m+K}} |\phi_{j',k'} * f(x_{I'}, y_{J'})|$$

$$+ C \sum_{k' > j'} \sum_{J'} \sum_{I'} 2^{-|j-j'|K} 2^{-|k-k'|K} |I'||J'| \times$$

$$\frac{2^{-j'K}}{(2^{-j'} + |u - x_{I'}|)^{n+K}} \cdot \frac{2^{-j'K}}{(2^{-j'} + |v - y_{J'}|)^{m+K}} |\phi_{j',k'} * f(x_{I'}, y_{J'})|$$

$$\leq C \sum_{k' \leq j'} 2^{-|j-j'|K} \cdot 2^{-|k-k'|K} \left\{ M_s \left(\sum_{J'} \sum_{I'} |\phi_{j',k'} * f(x_{I'}, y_{J'})| \chi_{J'} \chi_{I'})^r \right) \right\}^{\frac{1}{r}} (u,v)$$

$$+ C \sum_{k' > j'} 2^{-|j-j'|K} 2^{-|k-k'|K} \left\{ M \left(\sum_{J'} \sum_{I'} |\phi_{j',k'} * f(x_{I'}, y_{J'})| \chi_{I'} \chi_{J'} \right)^r \right\}^{\frac{1}{r}} (u,v)$$

where M is the Hardy-Littlewood maximal function on R^{n+m}, M_s is the strong maximal function on $R^n \times R^m$, and $\max\{\frac{n}{n+K}, \frac{m}{m+K}\} < r < p$.

Applying the Holder's inequality and summing over j, k, I, J yields

$$\left\{ \sum_j \sum_k \sum_J \sum_I \sup_{u \in I, v \in J} |\psi_{j,k} * f(u,v)|^2 \chi_I \chi_J \right\}^{\frac{1}{2}}$$

$$\leq C \left\{ \sum_{j'} \sum_{k'} \left\{ M_s (\sum_{J'} \sum_{I'} |\phi_{j',k'} * f(x_{I'}, y_{J'})| \chi_{I'} \chi_{J'})^r \right\}^{\frac{2}{r}} \right\}^{\frac{1}{2}}.$$

Since $x_{I'}$ and $y_{J'}$ are arbitrary points in I' and J', respectively, we have

$$\left\{ \sum_j \sum_k \sum_J \sum_I \sup_{u \in I, v \in J} |\psi_{j,k} * f(u,v)|^2 \chi_I \chi_J \right\}^{\frac{1}{2}}$$

$$\leq C \left\{ \sum_{j'} \sum_{k'} \left\{ M_s (\sum_{J'} \sum_{I'} \inf_{u \in I', v \in J'} |\phi_{j',k'} * f(u,v)| \chi_{I'} \chi_{J'})^r \right\}^{\frac{2}{r}} \right\}^{\frac{1}{2}},$$

and hence, by the Fefferman-Stein vector-valued maximal function inequality [FS] with $r < p$, we get

$$\|\left\{\sum_j\sum_k\sum_J\sum_I \sup_{u\in I, v\in J}|\psi_{j,k}*f(u,v)|^2\chi_I\chi_J\right\}^{\frac{1}{2}}\|_p$$

$$\leq C\|\left\{\sum_{j'}\sum_{k'}\sum_{J'}\sum_{I'} \inf_{u\in I', v\in J'}|\phi_{j',k'}*f(u,v)|^2\chi_{I'}\chi_{J'}\right\}^{\frac{1}{2}}\|_p.$$

This ends the proof of Theorem 1.9.   **Q.E.D.**

4. Discrete Littlewood-Paley-Stein square function, boundedness of flag singular integrals on Hardy spaces H_F^p, from H_F^p to L^p: Proofs of Theorems 1.10 and 1.11

The main purpose of this section is to establish the Hardy space theory associated with the flag multi-parameter structure using the results we have proved in Section 3. As a consequence of Theorem 1.9, it is easy to see that the Hardy space H_F^p is independent of the choice of the functions ψ. Moreover, we have the following characterization of H_F^p using the discrete norm.

Proposition 4.1. *Let $0 < p \leq 1$. Then we have*

(4.1) $$\|f\|_{H_F^p} \approx \|\left\{\sum_j\sum_k\sum_J\sum_I |\psi_{j,k}*f(x_I, y_J)|^2\chi_I(x)\chi_J(y)\right\}^{\frac{1}{2}}\|_p$$

where $j, k, \psi, \chi_I, \chi_J, x_I, y_J$ are same as in Theorem 1.9.

Before we give the proof of the boundedness of flag singular integrals on H_F^p, we show several properties of H_F^p.

Proposition 4.2. *$\mathcal{S}_F(R^n \times R^m)$ is dense in H_F^p.*

Proof: Suppose $f \in H_F^p$, and set $W = \{(j, k, I, J) : |j| \leq L, |k| \leq M, I \times J \subseteq B(0, r)\}$, where I, J are dyadic cubes in R^n, R^m with side length $2^{-j-N}, 2^{-k-N} + 2^{-j-N}$, respectively, and $B(0, r)$ are balls in R^{n+m} centered at the origin with radius r. It is easy to see that

$$\sum_{(j,k,I,J)\in W} |I||J|\widetilde{\psi}_{j,k}(x, y, x_I, y_J)\psi_{j,k}*f(x_I, y_J)$$

is a test function in $\mathcal{S}_F(R^n \times R^m)$ for any fixed L, M, r. To show the proposition, it suffices to prove

$$\sum_{(j,k,I,J)\in W^c} |I||J|\widetilde{\psi}_{j,k}(x, y, x_I, y_J)\psi_{j,k}*f(x_I, y_J)$$

tends to zero in the H_F^p norm as L, M, r tend to infinity. This follows from (4.1) and a similar proof as in the proof of Theorem 1.9. In fact, repeating the same proof as in theorem 1.9 yields

$$\|\sum_{(j,k,I,J)\in W^c} |I||J|\widetilde{\psi}_{j,k}(x, y, x_I, y_J)\psi_{j,k}*f(x_I, y_J)\|_{H_F^p}$$

$$\leq C\|\{\sum_{(j,k,I,J)\in W^c}|\psi_{j,k}*f(x_I,y_J)|^2\chi_I\chi_J\}^{\frac{1}{2}}\|_p,$$

where the last term tends to zero as L, M, r tend to infinity whenever $f \in H_F^p$. **Q.E.D.**

As a consequence of Proposition 4.2, $L^2(R^{n+m})$ is dense in $H_F^p(R^n \times R^m)$. Furthermore, we have

Theorem 4.3. *If $f \in L^2(R^{n+m}) \cap H_F^p(R^n \times R^m), 0 < p \leq 1$, then $f \in L^p(R^{n+m})$ and there is a constant $C_p > 0$ which is independent of the L^2 norm of f such that*

(4.3) $$\|f\|_p \leq C\|f\|_{H_F^p}.$$

To show theorem 4.3, we need a discrete Calderón reproducing formula on $L^2(R^{n+m})$. To be more precise, take $\phi^{(1)} \in C_0^\infty(R^{n+m})$ with

$$\int_{R^{n+m}} \phi^{(1)}(x,y)x^\alpha y^\beta dxdy = 0, \text{ for all } \alpha, \beta \text{ satisfying } 0 \leq |\alpha| \leq M_0, 0 \leq |\beta| \leq M_0,$$

where M_0 is a large positive integer which will be determined later, and

$$\sum_j |\widehat{\phi^{(1)}}(2^{-j}\xi_1, 2^{-j}\xi_2)|^2 = 1, \text{ for all } (\xi_1, \xi_2) \in R^{n+m}\setminus\{(0,0)\},$$

and take $\phi^{(2)} \in C_0^\infty(R^m)$ with

$$\int_{R^m} \phi^{(2)}(z)z^\gamma dz = 0 \text{ for all } 0 \leq |\gamma| \leq M_0,$$

and $\sum_k |\widehat{\phi^{(2)}}(2^{-k}\xi_2)^2 = 1$ for all $\xi_2 \in R^m\setminus\{0\}$.

Furthermore, we may assume that $\phi^{(1)}$ and $\phi^{(2)}$ are radial functions and supported in the unit balls of R^{n+m} and R^m respectively. Set again

$$\phi_{jk}(x,y) = \int_{R^m} \phi_j^{(1)}(x, y-z)\phi_k^{(2)}(z)dz.$$

By taking the Fourier transform, it is easy to see the following continuous version of Calderón reproducing formula on L^2: for $f \in L^2(R^{n+m})$,

$$f(x,y) = \sum_j \sum_k \phi_{jk} * \phi_{jk} * f(x,y).$$

For our purpose, we need the discrete version of the above reproducing formula.

Theorem 4.4. *There exist functions $\widetilde{\phi}_{jk}$ and an operator T_N^{-1} such that*

$$f(x,y) = \sum_j \sum_k \sum_J \sum_I |I||J|\widetilde{\phi}_{j,k}(x-x_I, y-y_J)\phi_{j,k} * \left(T_N^{-1}(f)\right)(x_I, y_J)$$

where functions $\widetilde{\phi}_{jk}(x - x_I, y - y_J)$ *satisfy the conditions in (3.6) with* $\alpha_1, \beta_1, \gamma_1, N, M$ *depending on* M_0, $x_0 = x_I$ *and* $y_0 = y_J$. *Moreover,* T_N^{-1} *is bounded on* $L^2(R^{n+m})$ *and* $H_F^p(R^n \times R^m)$, *and the series converges in* $L^2(R^{n+m})$.

Remark 4.1: The difference between Theorem 4.4 and Theorem 1.8 are that our $\widetilde{\phi}_{jk}$ in Theorem 4.4 has compact support. The price we pay here is that $\widetilde{\phi}_{jk}$ only satisfies the moment condition of finite order, unlike that in Theorem 1.8 where the moment condition of infinite order is satisfied. Moreover, the formula in Theorem 4.4 only holds on $L^2(R^{n+m})$ while the formula in Theorem 1.18 holds in test function space $\mathcal{S}_F$ and its dual space $(\mathcal{S}_F)'$.

Proof of Theorem 4.4: Following the proof of Theorem 1.8, we have

$$(4.4) \qquad f(x, y) = \sum_j \sum_k \sum_J \sum_I [\int_J \int_I \phi_{j,k}(x - u, y - v) du dv] \left(\phi_{j,k} * f\right)(x_I, y_J) + \mathcal{R}(f)(x, y).$$

where I, J, j, k and $\mathcal{R}$ are the same as in Theorem 1.8.

We need the following

Lemma 4.5. *Let* $0 < p \leq 1$. *Then the operator* $\mathcal{R}$ *is bounded on* $L^2(R^{n+m}) \cap H_F^p(R^n \times R^m)$ *whenever* M_0 *is chosen to be a large positive integer. Moreover, there exists a constant* $C > 0$ *such that*

$$||\mathcal{R}(f)||_2 \leq C 2^{-N} ||f||_2$$

and

$$||\mathcal{R}(f)||_{H_F^p(R^n \times R^m)} \leq C 2^{-N} ||f||_{H_F^p(R^n \times R^m)}.$$

Proof of Lemma 4.5: Following the proofs of Theorems 1.8 and 1.9 and using the discrete Calderón reproducing formula for $f \in L^2(R^{n+m})$, we have

$$||g_F(\mathcal{R}(f))||_p$$

$$\leq || \left\{ \sum_j \sum_k \sum_J \sum_I |(\psi_{j,k} * \mathcal{R}(f))|^2 \chi_I \chi_J \right\}^{\frac{1}{2}} ||_p$$

$$= \left|\left| \left\{ \sum_{j,k,J,I} \sum_{j',k',J',I'} |J'||I'| \left(\psi_{j,k} * \mathcal{R} \left(\widetilde{\psi_{j',k'}}(\cdot, x_{I'}, \cdot, y_{J'}) \cdot \psi_{j'k'} * f(x_{I'}, y_{J'}) \right) \right) |^2 \chi_I \chi_J \right\}^{\frac{1}{2}} \right|\right|_p$$

where $j, k, \psi, \chi_I, \chi_J, x_I, y_J$ are the same as in Theorem 1.9.

We **claim:**

$$\left| \left(\psi_{j,k} * \mathcal{R} \left(\widetilde{\psi}_{j',k'}(\cdot, x_{I'}, \cdot, y_{J'}) \right) \right)(x, y) \right|$$

$$\leq C 2^{-N} 2^{-|j-j'|K} 2^{-|k-k'|K} \cdot \int_{R^m} \frac{2^{-(j \wedge j')K}}{(2^{-(j \wedge j')} + |x - x_{I'}| + |y - z - y_{J'})^{n+m+K}} \frac{2^{-(k \wedge k')K}}{(2^{-(k \wedge k')} + |z|)^{m+K}} dz$$

where $K < M_0, max(\frac{n}{n+K}, \frac{m}{m+K}) < p$, and M_0 is chosen to be a lager integer later.

Assuming the claim for the moment, repeating a similar proof in Lemma 3.7 and then Theorem 1.9, we obtain

$$\|g_F(\mathcal{R}f)\|_p \leq C2^{-N}\|\{\sum_{j'}\sum_{k'}\{M_s(\sum_{J'}\sum_{I'}|\psi_{j',k'} * f(x_{I'},y_{J'})|\chi_{J'}\chi_{I'})^r\}^{\frac{2}{r}}\}^{\frac{1}{2}}\|_p$$

$$\leq C2^{-N}\|\{\sum_{j'}\sum_{k'}\sum_{J'}\sum_{I'}|\psi_{j',k'} * f(x_{I'},y_{J'})|^2\chi_{I'}\chi_{J'}\}^{\frac{1}{2}}\|_p \leq C2^{-N}\|f\|_{H_F^p(R^n \times R^m)}.$$

It is clear that the above estimates still hold when p is replaced by 2. These imply the assertion of Lemma 4.5.

We now prove the Claim. Again, by the proof of Theorem 1.8,

$$\mathcal{R}\left(\widetilde{\psi}_{j',k'}(\cdot,x_{I'},\cdot,y_{J'})\right)(x,y) = \int_{R^m} \mathcal{R}^\sharp(x,y-z,z,u',v',w')\widetilde{\psi}_{j',k'}(\cdot,x_{I'},\cdot,y_{J'})du'dv'dw'dz$$

where $\mathcal{R}^\sharp(x,y,z,u',v',w')$ is similar to $\mathcal{R}^\sharp$ as given in the proof of Theorem 1.8 but, as we pointed out in Remark 4.1, that the difference between $\mathcal{R}^\sharp$ here and $\mathcal{R}^\sharp$ given in the proof of Theorem 1.8 is the moment conditions. However, the almost orthogonality estimate still holds if we only require sufficiently high order of moment conditions. More precisely, if we replace the moment conditions in (3.5) "for all $\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2$" by "for all $|\alpha_1|, |\beta_1|, |\gamma_1|, |\alpha_2|, |\beta_2|, |\gamma_2| \leq M_0$ where M_0 is a large integer, then the estimate in (3.7) still holds with L_1, L_2, K_1, K_2 depending on M_0. Thus, the claim follows by applying the same proof as that of Theorem 1.8, and the proof of Lemma 4.5 is complete. **Q. E. D.**

We now return to the proof of Theorem 4.4.

Denote $(T_N)^{-1} = \sum_{i=1}^{\infty} \mathcal{R}^i$, where

$$T_N(f) = \sum_j\sum_k\sum_J\sum_I[\int_J\int_I \phi_{j,k}(x-u,y-v)dudv] (\phi_{j,k} * f)(x_I,y_J).$$

Lemma 4.5 shows that if N is large enough, then both of T_N and $(T_N)^{-1}$ are bounded on $L^2(R^{n+m}) \cap H_F^p(R^n \times R^m)$. Hence, we can get the following reproducing formula

$$f(x,y) = \sum_j\sum_k\sum_J\sum_I |I||J|\widetilde{\phi}_{j,k}(x-x_I,y-y_J)\phi_{j,k} * (T_N^{-1}(f))(x_I,y_J)$$

where $\widetilde{\phi}_{jk}(x-x_I,y-y_J) = \frac{1}{|I|}\frac{1}{|J|}\int_J\int_I \phi_{jk}(x-x_I-(u-x_I),y-y_J-(v-y_J))dudv$ satisfies the estimate in (3.6) and the series converges in $L^2(R^{n+m})$.

This completes the proof of Theorem 4.4. **Q.E.D.**

As a consequence of Theorem 4.4, we obtain the following

Corollary 4.6. *If $f \in L^2(R^{n+m}) \cap H_F^p(R^n \times R^m)$ and $0 < p \leq 1$, then*

$$\|f\|_{H_F^p} \approx \|\{(\sum_j\sum_k\sum_J\sum_I |\phi_{jk} * (T_N^{-1}(f))(x_I,y_J)|^2\chi_I(x)\chi_J(y)\})^{\frac{1}{2}}\|_p$$

where the constants are independent of the L^2 norm of f.

To see the proof of Corollary 4.6, note that if $f \in L^2(R^{n+m})$, one can apply the Calderoń reproducing formulas in Theorem 1.8 and 4.4 and then repeat the same proof as in Theorem 1.9. We leave the details to the reader. We now start the

Proof of Theorem 4.3: We define a square function by

$$\widetilde{g}(f)(x,y) = \{\sum_j \sum_k \sum_J \sum_I |\phi_{j,k} * \left(T_N^{-1}(f)\right)(x_I, y_J)|^2 \chi_I(x)\chi_J(y)\}^{\frac{1}{2}}$$

where ϕ_{jk} are the same as in Theorem 4.4. By Corollary 4.6, for $f \in L^2(R^{n+m}) \cap H_F^p(R^n \times R^m)$ we have,

$$||\widetilde{g}(f)||_{L^p(R^{n+m})} \le C||f||_{H_F^p(R^n \times R^m)}$$

.

To complete the proof of Theorem 4.3, let $f \in L^2(R^{n+m}) \cap H_F^p(R^n \times R^m)$. Set

$$\Omega_i = \{(x,y) \in R^n \times R^m : \widetilde{g}(f)(x,y) > 2^i\}.$$

Denote

$$\mathcal{B}_i = \{(j,k,I,J) : |(I \times J) \cap \Omega_i| > \frac{1}{2}|I \times J|, |(I \times J) \cap \Omega_{i+1}| \le \frac{1}{2}|I \times J|\},$$

where I, J are dyadic cubes in R^n, R^m with side length $2^{-j-N}, 2^{-k-N} + 2^{-j-N}$, respectively. Since $f \in L^2(R^{n+m})$, by the discrete Calderón reproducing formula in Theorem 4.4,

$$f(x,y)$$
$$= \sum_j \sum_k \sum_J \sum_I \widetilde{\phi}_{j,k}(x - x_I, y - y_J)|I||J|\phi_{j,k} * \left(T_N^{-1}(f)\right)(x_I, y_J)$$
$$= \sum_i \sum_{(j,k,I,J)\in\mathcal{B}_i} |I||J|\widetilde{\phi}_{j,k}(x - x_I, y - y_J)\phi_{j,k} * \left(T_N^{-1}(f)\right)(x_I, y_J),$$

where the series converges in L^2 norm, and hence it also converges almost everywhere.

We **claim**

$$||\sum_{(j,k,I,J)\in\mathcal{B}_i} |I||J|\widetilde{\phi}_{j,k}(x - x_I, y - y_J)\phi_{j,k} * \left(T_N^{-1}(f)\right)(x_I, y_J)||_p^p \le C2^{ip}|\Omega_i|,$$

which together with the fact $0 < p \le 1$ yields

$$||f||_p^p \le \sum_i ||\sum_{(j,k,I,J)\in\mathcal{B}_i} |I||J|\widetilde{\phi}_{j,k}(x - x_I, y - y_J)\phi_{j,k} * \left(T_N^{-1}(f)\right)(x_I, y_J)||_p^p$$
$$\le C\sum_i 2^{ip}|\Omega_i|$$
$$\le C||\widetilde{g}(f)||_p^p \le C||f||_{H_F^p}^p.$$

To show the claim, note that $\phi^{(1)}$ and $\psi^{(2)}$ are radial functions supported in unit balls. Hence, if $(j,k,I,J) \in \mathcal{B}_i$ then $\phi_{j,k}(x - x_I, y - y_J)$ are supported in

$$\widetilde{\Omega_i} = \{(x,y) : M_s(\chi_{\Omega_i})(x,y) > \frac{1}{100}\}.$$

Thus, by Hölder's inequality,

$$\|\sum_{(j,k,I,J)\in\mathcal{B}_i}|J||I|\widetilde{\phi}_{j,k}(x-x_I,y-y_J)\phi_{j,k}*\left(T_N^{-1}(f)\right)(x_I,y_J)\|_p^p$$

$$\leq|\widetilde{\Omega_i}|^{1-\frac{p}{2}}\|\sum_{(j,k,I,J)\in\mathcal{B}_i}|J||I|\widetilde{\phi}_{j,k}(x-x_I,y-y_J)\phi_{j,k}*\left(T_N^{-1}(f)\right)(x_I,y_J)\|_2^p.$$

By the duality argument, for all $g\in L^2$ with $\|g\|_2\leq 1$,

$$|<\sum_{(j,k,I,J)\in\mathcal{B}_i}|J||I|\widetilde{\phi}_{j,k}(x-x_I,y-y_J)\phi_{j,k}*\left(T_N^{-1}(f)\right)(x_I,y_J),g>|$$

$$=|\sum_{(j,k,I,J)\in\mathcal{B}_i}|J||I|\widetilde{\phi}_{j,k}*g(x_I,y_J)\phi_{j,k}*\left(T_N^{-1}(f)\right)(x_I,y_J)|$$

$$\leq C\left(\sum_{(j,k,I,J)\in\mathcal{B}_i}|I||J||\phi_{j,k}*\left(T_N^{-1}(f)\right)(x_I,y_J)|^2\right)^{\frac{1}{2}}\cdot\left(\sum_{(j,k,I,J)\in\mathcal{B}_i}|I||J||\widetilde{\phi}_{j,k}*g(x_I,y_J)|^2\right)^{\frac{1}{2}}.$$

Since

$$\left(\sum_{(j,k,I,J)\in\mathcal{B}_i}|I||J||\widetilde{\phi}_{j,k}*g(x_I,y_J)|^2\right)^{\frac{1}{2}}$$

$$\leq\left(\sum_{(j,k,I,J)\in\mathcal{B}_i}|I||J|\left(M_s\left(\widetilde{\phi}_{j,k}*g\right)(x,y)\chi_I(x)\chi_J(y)\right)^2\right)^{\frac{1}{2}}$$

$$\leq C\left(\sum_{j,k}\int_{R^n}\int_{R^m}\left(M_s\left(\widetilde{\phi}_{j,k}*g\right)^2(x,y)dxdy\right)\right)^{\frac{1}{2}}\leq C\|g\|_2$$

thus the claim now follows from the fact that $|\widetilde{\Omega_i}|\leq C|\Omega_i|$ and the following estimate:

$$C2^{2i}|\Omega_i|\geq\int_{\widetilde{\Omega_i}\backslash\Omega_{i+1}}\widetilde{g}^2(f)(x,y)dxdy$$

$$\geq\sum_{(j,k,I,J)\in\mathcal{B}_i}|\phi_{j,k}*\left(T_N^{-1}(f)\right)(x_I,y_J)|^2|(I\times J)\cap\widetilde{\Omega_i}\backslash\Omega_{i+1}|$$

$$\geq\frac{1}{2}\sum_{(j,k,I,J)\in\mathcal{B}_i}|I||J||\phi_{j,k}*\left(T_N^{-1}(f)\right)(x_I,y_J)|^2,$$

where the fact that $|(I\times J)\cap\widetilde{\Omega_i}\backslash\Omega_{i+1}|>\frac{1}{2}|I\times J|$ when $(j,k,I,J)\in\mathcal{B}_i$ is used in the last inequality. This finishes the proof of Theorem 4.3. **Q.E.D.**

As a consequence of Theorem 4.3, we have the following

Corollary 4.7 $H_F^1(R^n\times R^m)$ is a subspace of $L^1(R^n\times R^m)$.

Proof: Given $f \in H_F^1(R^{n+m})$, by Proposition 4.2, there is a sequence $\{f_n\}$ such that $f_n \in L^2(R^{n+m}) \cap H_F^1(R^{n+m})$ and f_n converges to f in the norm of $H_F^1(R^{n+m})$. By Theorem 4.3, f_n converges to g in $L^1(R^{n+m})$ for some $g \in L^1(R^{n+m})$. Therefore, $f = g$ in $(\mathcal{S}_F)'$.   **Q.E.D.**

We now turn to the

Proof of Theorem 1.10: We assume that K is the kernel of T. Applying the discrete Calderón reproducing formula in Theorem 4.4 implies that for $f \in L^2(R^{n+m}) \cap H_F^p(R^n \times R^m)$,

$$\|\{\sum_j \sum_k \sum_J \sum_I |\phi_{j,k} * K * f(x,y)|^2 \chi_I(x)\chi_J(y)\}^{\frac{1}{2}}\|_p =$$

$$\|\{\sum_j \sum_k \sum_J \sum_I |\sum_{j'} \sum_{k'} \sum_{J'} \sum_{I'} |J'||I'|\phi_{j,k} * K * \widetilde{\phi}_{j',k'}(\cdot - x_{I'}, \cdot - y_{J'})(x,y) \times$$

$$\phi_{j',k'} * (T_N^{-1}(f))(x_{I'}, y_{J'})|^2 \chi_I(x)\chi_J(y)\}^{\frac{1}{2}}\|_p,$$

where the discrete Calderón reproducing formula in $L^2(R^{n+m})$ is used.

Note that ϕ_{jk} are dilations of bump functions, by estimates similar to the those in (2.5), one can easily check that

$$|\phi_{j,k} * K * \widetilde{\phi}_{j',k'}(\cdot - x_{I'}, \cdot - y_{J'})(x,y)| \leq C 2^{-|j-j'|K} 2^{-|k-k'|K}$$

$$\int_{R^m} \frac{2^{-(j\wedge j')K}}{(2^{-(j\wedge j')} + |x - x_{I'}| + |y - z - y_{J'}|)^{n+m+K}} \cdot \frac{2^{-(k\wedge k')K}}{(2^{-(k\wedge k')} + |z|)^{m+K}} dz,$$

where K depends on M_0 given in Theorem 4.4 and M_0 is chosen to be large enough. Repeating a similar proof in Theorem 1.9 together with Corollary 4.6, we obtain

$$\|Tf\|_{H_F^p} \leq C\|\{\sum_{j'} \sum_{k'} \{M_s(\sum_{J'} \sum_{I'} |\phi_{j',k'} * (T_N^{-1}(f))(x_{I'}, y_{J'})|\chi_{J'}\chi_{I'})^r\}^{\frac{2}{r}}(x,y)\}^{\frac{1}{2}}\|_p$$

$$\leq C\|\{\sum_{j'} \sum_{k'} \sum_{J'} \sum_{I'} |\phi_{j',k'} * (T_N^{-1}(f))(x_{I'}, y_{J'})|^2 \chi_{J'}(y)\chi_{I'}(x)\}^{\frac{1}{2}}\|_p \leq C\|f\|_{H_F^p},$$

where the last inequality follows from Corollary 4.6.

Since $L^2(R^{n+m})$ is dense in $H_F^p(R^n \times R^m)$, T can extend to a bounded operator on $H_F^p(R^n \times R^m)$. This ends the proof of Theorem 1.10.

Proof of Theorem 1.11 We note that $H_F^p \cap L^2$ is dense in H_F^p, so we only have to show this for $f \in H_F^p \cap L^2$. Thus Theorem 1.11 follows from Theorems 4.3 and 1.10 immediately. **Q.E.D.**

5. Duality of Hardy spaces H_F^p and boundedness of flag singular integrals on BMO_F space: Proofs of Theorems 1.14, 1.16 and 1.18

This section deals with the duality theory of flag Hardy spaces $H_F^p(R^n \times R^m)$ for all $0 < p \leq 1$. We first prove Theorem 1.14, the Plancherel-Pôlya inequalities for CMO_F^p space.

Proof of Theorem 1.14: The idea of the proof of this theorem is, as in the proof of Theorem 1.9, again to use the discrete Calderón reproducing formula and the almost orthogonality estimate.

For the reader's convenience we choose to present the proof of Theorem 1.14 in the case when $n = m = 1$. However, it will be clear from the proof that its extension to general n and m is straightforward. Moreover, to simplify notation, we denote $f_{j,k} = f_R$ when $R = I \times J \subset R^2$ and $|I| = 2^{-j-N}, |J| = 2^{-k-N} + 2^{-j-N}$ are dyadic intervals respectively. Here N is the same as in Theorem 1.8. We also denote by $dist(I, I')$ the distance between intervals I and I',

$$S_R = \sup_{u \in I, v \in J} |\psi_R * f(u,v)|^2, \quad T_R = \inf_{u \in I, v \in J} |\phi_R * f(u,v)|^2.$$

With these notations, we can rewrite the discrete Calderón reproducing formula in (1.9) by

$$f(x,y) = \sum_{R=I \times J} |I||J| \widetilde{\phi}_R(x,y) \phi_R * f(x_I, y_J),$$

where the sum runs over all rectangles $R = I \times J$.

Let $R' = I' \times J', |I'| = 2^{-j'-N}, |J'| = 2^{-j'-N} + 2^{-k'-N}, j' > k'$. Applying the above discrete Calderón reproducing formula and the estimates in Corollary 3.5 yields for all $(u,v) \in R$,

$$|\psi_R * f(u,v)|^2 \leq C \sum_{R'=I' \times J', j' > k'} \left(\frac{|I|}{|I'|} \wedge \frac{|I'|}{|I|}\right)^L \left(\frac{|J|}{|J'|} \wedge \frac{|J'|}{|J|}\right)^L \times$$

$$\frac{|I'|^K}{(|I'| + |u - x_{I'}|)^{(1+K)}} \frac{|J'|^K}{(|J'| + |v - y_{J'}|)^{(1+K)}} |I'||J'||\phi_{R'} * f(x_{I'}, y_{J'})|^2$$

$$+ C \sum_{R'=I' \times J', j' \leq k'} \left(\frac{|I|}{|I'|} \wedge \frac{|I'|}{|I|}\right)^L \left(\frac{|J|}{|J'|} \wedge \frac{|J'|}{|J|}\right)^L \times$$

$$\frac{|I'|^K}{(|I'| + |u - x_{I'}|)^{(1+K)}} \frac{|I'|^K}{(|I'| + |v - y_{J'}|)^{(1+K)}} |I'||J'||\phi_{R'} * f(x_{I'}, y_{J'})|^2,$$

where K, L are any positive integers which can be chosen by $L, K > \frac{2}{p} - 1$ (for general n, m, K can be chosen by $K > (n \vee m)(\frac{2}{p} - 1)$), the constant C depends only on K, L and functions ψ and ϕ, here $x_{I'}$ and $y_{J'}$ are any fixed points in I', J', respectively.

Adding up all the terms with multiplying $|I||J|$ over $R \subseteq \Omega$, we obtain

(5.1) $$\sum_{R \subseteq \Omega} |I||J|S_R \leq C \sum_{R \subseteq \Omega} \sum_{R'} |I'||J'| r(R, R') P(R, R') T_{R'},$$

where

$$r(R, R') = \left(\frac{|I|}{|I'|} \wedge \frac{|I'|}{|I|}\right)^{L-1} \left(\frac{|J|}{|J'|} \wedge \frac{|J'|}{|J|}\right)^{L-1}$$

and

$$P(R, R') = \frac{1}{(1 + \frac{dist(I,I')}{|I'|})^{1+K} (1 + \frac{dist(J,J')}{|J'|})^{1+K}}$$

if $j' > k'$, and

$$P(R, R') = \frac{1}{(1 + \frac{dist(I,I')}{|I'|})^{1+K} (1 + \frac{dist(J,J')}{|I'|})^{1+K}}$$

if $j' \leq k'$.

To estimate the right-hand side in the above inequality (5.1), where we first consider $R' = I' \times J', |I'| = 2^{-j'-N}, |J'| = 2^{-j'-N} + 2^{-k'-N}, j' > k'$.

Define
$$\Omega^{i,\ell} = \bigcup_{I \times J \subset \Omega} 3(2^i I \times 2^\ell J) \text{ for } i, \ell \geq 0.$$

Let $B_{i,\ell}$ be a collection of dyadic rectangles R' so that for $i, \ell \geq 1$

$$B_{i,\ell} = \{R' = I' \times J', 3(2^i I' \times 2^\ell J') \bigcap \Omega^{i,\ell} \neq \emptyset \text{ and } 3(2^{i-1} I' \times 2^{\ell-1} J') \bigcap \Omega^{i,\ell} = \emptyset\},$$

and

$$B_{0,\ell} = \{R' = I' \times J', 3(I' \times 2^\ell J') \bigcap \Omega^{0,\ell} \neq \emptyset \text{ and } 3(I' \times 2^{\ell-1} J') \bigcap \Omega^{0,\ell} = \emptyset\} \text{ for } \ell \geq 1,$$

and

$$B_{i,0} = \{R' = I' \times J', 3(2^i I' \times J') \bigcap \Omega^{i,0} \neq \emptyset \text{ and } 3(2^{i-1} I' \times J') \bigcap \Omega^{i,0} = \emptyset\} \text{ for } i \geq 1,$$

and

$$B_{0,0} = \{R' : R' = I' \times J', 3(I' \times J') \bigcap \Omega^{0,0} \neq \emptyset\}.$$

We write

$$\sum_{R \subseteq \Omega} \sum_{R'} |I'||J'| r(R, R') P(R, R') T_{R'} = \sum_{i \geq 0, \ell \geq 0} \sum_{R' \in B_{i,\ell}} \sum_{R \subseteq \Omega} |I'||J'| r(R, R') P(R, R') T_{R'}.$$

To estimate the right-hand side of the above equality, we first consider the case when $i = \ell = 0$. Note that when $R' \in B_{0,0}$, $3R' \bigcap \Omega^{0,0} \neq \emptyset$. For each integer $h \geq 1$, let $\mathcal{F}_h = \{R' = I' \times J' \in B_{0,0}, |(3I' \times 3J') \bigcap \Omega^{0,0}| \geq (\frac{1}{2^h})|3I' \times 3J'|\}$. Let $\mathcal{D}_h = \mathcal{F}_h \backslash \mathcal{F}_{h-1}$, and $\Omega_h = \bigcup_{R' \in \mathcal{D}_h} R'$. Finally, assume that the right-hand side in (1.12) is finite, that is, for any open set $\Omega \subset R^2$,

$$\sum_{R = I \times J \subseteq \Omega} |I||J| T_R \leq C|\Omega|^{\frac{2}{p} - 1}. \tag{5.2}$$

Since $B_{0,0} = \bigcup_{h \geq 1} \mathcal{D}_h$ and for each $R' \in B_{0,0}, P(R, R') \leq 1$, thus,

$$\sum_{R' \in B_{0,0}} \sum_{R \subseteq \Omega} |I'||J'| r(R, R') P(R, R') T_{R'}$$

$$\leq \sum_{h \geq 1} \sum_{R' \subseteq \Omega_h} \sum_{R \subseteq \Omega} |I'||J'| r(R, R') T_{R'}$$

For each $h \geq 1$ and $R' \subseteq \Omega_h$, we decompose $\{R : R \subseteq \Omega\}$ into

$$A_{0,0}(R') = \{R \subseteq \Omega : \text{ dist}(I, I') \leq |I| \vee |I'|, \text{ dist}(J, J') \leq |J| \vee |J'|\};$$

$$A_{i',0}(R') = \{R \subseteq \Omega : 2^{i'-1}(|I| \vee |I'|) < \text{ dist}(I, I') \leq 2^{i'}(|I| \vee |I'|), \text{ dist}(J, J') \leq |J| \vee |J'|\};$$

$$A_{0,\ell'}(R') = \{R \subseteq \Omega : \mathrm{dist}(I, I') \leq |I| \vee |I'|,\ 2^{\ell'-1}(|J| \vee |J'|) < \mathrm{dist}(J, J') \leq 2^{\ell'}(|J| \vee |J'|)\};$$

$$A_{i',\ell'}(R') = \{R \subseteq \Omega :\ 2^{i'-1}(|I| \vee |I'|) < \mathrm{dist}(I, I') \leq 2^{i'}(|I| \vee |I'|),$$

$$2^{\ell'-1}(|J| \vee |J'|) < \mathrm{dist}(J, J') \leq 2^{\ell'}(|J| \vee |J'|)\},$$

where $i', \ell' \geq 1$.

Now we split $\sum_{h \geq 1} \sum_{R' \subseteq \Omega_h} \sum_{R \subseteq \Omega} |I'||J'| r(R, R') P(R, R') T_{R'}$ into

$$\sum_{h \geq 1} \sum_{R' \in \Omega_h} \left(\sum_{R \in A_{0,0}(R')} + \sum_{i' \geq 1} \sum_{R \in A_{i',0}(R')} + \sum_{\ell' \geq 1} \sum_{R \in A_{0,\ell'}(R')} + \sum_{i',\ell' \geq 1} \sum_{R \in A_{i',\ell'}(R')} \right) |I'||J'|$$

$$\times r(R, R') P(R, R') T_{R'} =: I_1 + I_2 + I_3 + I_4.$$

To estimate the term I_1, we only need to estimate $\sum_{R \in A_{0,0}(R')} r(R, R')$ since $P(R, R') \leq 1$ in this case.

Note that $R \in A_{0,0}(R')$ implies $3R \bigcap 3R' \neq \emptyset$. For such R, there are four cases:

Case 1: $|I'| \geq |I|$, $|J'| \leq |J|$; Case 2: $|I'| \leq |I|$, $|J'| \geq |J|$; Case 3: $|I'| \geq |I|$, $|J'| \geq |J|$; Case 4: $|I'| \leq |I|$, $|J'| \leq |J|$.

In each case, we can show $\sum_{R \in A_{0,0}} r(R, R') \leq C 2^{-hL}$ by using a simple geometric argument similar to that of Chang-R. Fefferman [CF3]. This, together with (5.1), implies that I_1 is bounded by

$$\sum_{h \geq 1} 2^{-hL} |\Omega_h|^{\frac{2}{p}-1} \leq C \sum_{h \geq 1} h^{\frac{2}{p}-1} 2^{-h(L - \frac{2}{p} + 1)} |\Omega^{0,0}|^{\frac{2}{p}-1} \leq C |\Omega|^{\frac{2}{p}-1},$$

since $|\Omega_h| \leq C h 2^h |\Omega^{0,0}|$ and $|\Omega^{0,0}| \leq C|\Omega|$.

Thus it remains to estimate the term I_4, since estimates of I_2 and I_3 can be derived using the same techniques as in I_1 and I_4. This term is more complicated to estimate than term I_1.

For each $i', \ell' \geq 1$, when $R \in A_{i',\ell'}(R')$, we have $P(R, R') \leq 2^{-i'(1+K)} 2^{-\ell'(1+K)}$. Similar to estimating term I_1, we only need to estimate the sum $\sum_{R \in A_{i',\ell'}} r(R, R')$. Note that $R \in A_{i',\ell'}(R')$ implies that $3(2^{i'} I \times 2^{\ell'} J) \cap 3(2^{i'} I' \times 2^{\ell'} J') \neq \emptyset$. We also split into four cases in estimating this sum.

Case 1: $|2^{i'} I'| \geq |2^{i'} I|$, $|2^{\ell'} J'| \leq |2^{\ell'} J|$. Then

$$\frac{|2^{i'} I|}{|3 \cdot 2^{i'} I'|} |3(2^{i'} I' \times 2^{\ell'} J')| \leq |3(2^{i'} I' \times 2^{\ell'} J') \cap 3(2^{i'} I \times 2^{\ell'} J)|$$

$$\leq C 2^{i'} 2^{\ell'} |3R' \cap \Omega^{0,0}| \leq C 2^{i'} 2^{\ell'} \frac{1}{2^{h-1}} |3R'| \leq C \frac{1}{2^{h-1}} |3(2^{i'} I' \times 2^{\ell'} J')|.$$

Thus $|2^{i'} I'| = 2^{h-1+n} |2^{i'} I|$ for some $n \geq 0$. For each fixed n, the number of such $2^{i'} I$'s must be $\leq 2^n \cdot 5$. As for $|2^{\ell'} J| = 2^m |2^{\ell'} J'|$ for some $m \geq 0$, for each fixed m, $3 \cdot 2^{\ell'} J \cap 3 \cdot 2^{\ell'} J' \neq \emptyset$ implies that the number of such $2^{\ell'} J'$ is less than 5. Thus

$$\sum_{R \in case1} r(R, R') \leq \sum_{m,n \geq 0} \left(\frac{1}{2^{n+m+h-1}} \right)^L 2^n \cdot 5^2 \leq C 2^{-hL}.$$

Similarly, we can handle

Case 2: $|2^{i'}I'| \le |2^{i'}I|$, $|2^{\ell'}J'| \ge |2^{\ell'}J|$,

Case 3: $|2^{i'}I'| \ge |2^{i'}I|$, $|2^{\ell'}J'| \ge |2^{\ell'}J|$ and

Case 4: $|2^{i'}I'| \le |2^{i'}I|$, $|2^{\ell'}J'| \le |2^{\ell'}J|$.

Combining the four cases, we have $\sum_{R \in A_{i',\ell'}(R')} r(R, R') \le C 2^{-hL}$, which, together with the estimate of $P(R, R')$, implies that

$$I_4 \le C \sum_{h \ge 1} \sum_{i',\ell' \ge 1} \sum_{R' \subseteq \Omega_h} 2^{-hL} 2^{-i'(1+K)} 2^{-\ell'(1+K)} |I'||J'| T_{R'}.$$

Hence I_4 is bounded by

$$\sum_{h \ge 1} 2^{-hL} |\Omega_h|^{\frac{2}{p}-1} \le C \sum_{h \ge 1} h^{\frac{2}{p}-1} 2^{-h(L-\frac{2}{p}+1)} |\Omega^{0,0}|^{\frac{2}{p}-1} \le C |\Omega|^{\frac{2}{p}-1},$$

since $|\Omega_h| \le C h 2^h |\Omega^{0,0}|$ and $|\Omega^{0,0}| \le C |\Omega|$.

Combining I_1, I_2, I_3 and I_4, we have

$$\frac{1}{|\Omega|^{\frac{2}{p}-1}} \sum_{R' \in B_{0,0}} \sum_{R \subseteq \Omega} |I'||J'| r(R, R') P(R, R') T_{R'} \le C \sup_{\bar{\Omega}} \frac{1}{|\bar{\Omega}|^{\frac{2}{p}-1}} \sum_{R' \subseteq \bar{\Omega}} |I'||J'| T_{R'}.$$

Now we consider

$$\sum_{i,\ell \ge 1} \sum_{R' \in B_{i,\ell}} \sum_{R \subseteq \Omega} |I'||J'| r(R, R') P(R, R') T_{R'}.$$

Note that for $R' \in B_{i,\ell}$, $3(2^i I' \times 2^\ell J') \cap \Omega^{i,l} \ne \emptyset$. Let

$$\mathcal{F}_h^{i,\ell} = \left\{ R' \in B_{i,\ell} : |3(2^i I' \times 2^\ell J') \cap \Omega^{i,l}| \ge \frac{1}{2^h} |3(2^i I' \times 2^\ell J')| \right\},$$

$$\mathcal{D}_h^{i,\ell} = \mathcal{F}_h^{i,\ell} \setminus \mathcal{F}_{h-1}^{i,\ell}$$

and

$$\Omega_h^{i,\ell} = \bigcup_{R' \in \mathcal{D}_h^{i,\ell}} R'.$$

. Since $B_{i,\ell} = \bigcup_{h \ge 1} \mathcal{D}_h^{i,\ell}$, we first estimate

$$\sum_{R' \in \mathcal{D}_h^{i,\ell}} \sum_{R \subseteq \Omega} |I'||J'| r(R, R') P(R, R') T_{R'}$$

for some $i, \ell, h \ge 1$.

Note that for each $R' \in \mathcal{D}_h^{i,\ell}$, $3(2^i I' \times 2^\ell J') \cap \Omega^{i-1,l-1} = \emptyset$. So for any $R \subseteq \Omega$, we have $2^i(|I| \vee |I'|) \le \text{dist}(I, I')$ and $2^\ell(|J| \vee |J'|) \le \text{dist}(J, J')$. We decompose $\{R : R \subseteq \Omega\}$ by

$$A_{i',\ell'}(R') = \{R \subset \Omega : 2^{i'-1} \cdot 2^i(|I| \vee |I'|) \le \text{dist}(I, I') \le 2^{i'} \cdot 2^i(|I| \vee |I'|),$$

$$2^{\ell'-1} \cdot 2^{\ell}(|J| \vee |J'|) \leq \operatorname{dist}(J, J') \leq 2^{\ell'} \cdot 2^{\ell}(|J| \vee |J'|)\},$$

where $i', \ell' \geq 1$. Then we write

$$\sum_{R' \in \mathcal{D}_h^{i,\ell}} \sum_{R \subseteq \Omega} |I'||J'|r(R, R')P(R, R')T_{R'} = \sum_{i',\ell' \geq 1} \sum_{R' \in \mathcal{D}_h^{i,\ell}} \sum_{R \in A_{i',\ell'}(R')} |I'||J'|r(R, R')P(R, R')T_{R'}$$

Since $P(R, R') \leq 2^{-i(1+K)}2^{-\ell(1+K)}2^{-i'(1+K)}2^{-\ell'(1+K)}$ for $R' \in B_{i,\ell}$ and $R \in A_{i',\ell'}(R')$, repeating the same proof with $B_{0,0}$ replaced by $B_{i,\ell}$ and a necessary modification yields

$$\sum_{R' \in \mathcal{D}_h^{i,\ell}} \sum_{R \in A_{i',\ell'}(R')} |I'||J'|r(R, R')P(R, R')T_{R'} \leq C2^{-i(1+K)}2^{-\ell(1+K)}2^{-i'(1+K)}2^{-\ell'(1+K)} \times$$

$$i^{\frac{2}{p}-1}2^{i(\frac{2}{p}-1)}\ell^{\frac{2}{p}-1}2^{\ell(\frac{2}{p}-1)}h^{\frac{2}{p}-1}2^{-h(L-\frac{2}{p}+1)} \sup_{\bar{\Omega}} \frac{1}{|\bar{\Omega}|^{\frac{2}{p}-1}} \sum_{R' \subseteq \bar{\Omega}} |I'||J'|T_{R'}.$$

Adding over all $i, \ell, i', \ell', h \geq 1$, we have

$$\frac{1}{|\Omega|^{\frac{2}{p}-1}} \sum_{i,\ell \geq 1} \sum_{R' \in B_{i,\ell}} \sum_{R \subseteq \Omega} |I'||J'|r(R, R')P(R, R')T_{R'} \leq C \sup_{\bar{\Omega}} \frac{1}{|\bar{\Omega}|^{\frac{2}{p}-1}} \sum_{R' \subseteq \bar{\Omega}} |I'||J'|T_{R'}.$$

Similar estimates hold for

$$\sum_{i \geq 1} \sum_{R' \in B_{i,0}} \sum_{R \subseteq \Omega} |I'||J'|r(R, R')P(R, R')T_{R'}$$

and

$$\sum_{\ell \geq 1} \sum_{R' \in B_{0,\ell}} \sum_{R \subseteq \Omega} |I'||J'|r(R, R')P(R, R')T_{R'},$$

which, after adding over all $i, \ell \geq 0$, shows Theorem 1.21. We leave the details to the reader.
Q.E.D.

As a consequence of Theorem 1.14, it is easy to see that the space CMO_F^p is well defined. Particularly, we have

Corollary 5.1. *We have*

$$\|f\|_{CMO_F^p} \approx \sup_{\Omega} \left\{ \frac{1}{|\Omega|^{\frac{2}{p}-1}} \sum_{j} \sum_{k} \sum_{I \times J \subseteq \Omega} |\psi_{j,k} * f(x_I, y_J)|^2 |I||J| \right\}^{\frac{1}{2}},$$

where I, J are dyadic cubes in R^n, R^m with length $2^{-j-N}, 2^{-j-N} + 2^{-k-N}$, and x_I, y_J are any fixed points in I, J, respectively.

Before we prove Theorem 1.16, we remark again that Theorem 1.16 with $p = 1$ in the one-parameter setting was proved in [FJ] on R^n by use of the distribution inequalities. This method is difficult to apply to multi-parameter case. We will give a simpler and more constructive proof

which also gives a new proof of the result in [FJ]. Moreover, this constructive proof works also for other multi-parameter cases.

Proof of Theorem 1.16: We first prove $c^p \subseteq (s^p)^*$. Applying the proof in Theorem 4.4, set

$$s(x,y) = \{\sum_{I \times J} |s_{I \times J}|^2 |I|^{-1} |J|^{-1} \chi_I(x) \chi_J(y)\}^{\frac{1}{2}}$$

and

$$\Omega_i = \{(x,y) \in R^n \times R^m : s(x,y) > 2^i\}.$$

Denote

$$\mathcal{B}_i = \{(I \times J) : |(I \times J) \cap \Omega_i| > \frac{1}{2}|I \times J|, |(I \times J) \cap \Omega_{i+1}| \leq \frac{1}{2}|I \times J|\},$$

where I, J are dyadic cubes in R^n, R^m, with side length 2^{-j-N}, and $2^{-j-N} + 2^{-k-N}$, respectively. Suppose $t = \{t_{I \times J}\} \in c^p$ and write

$$|\sum_{I \times J} s_{I \times J} \bar{t}_{I \times J}||$$

$$= |\sum_i \sum_{(I \times J) \in \mathcal{B}_i} s_{I \times J} \bar{t}_{I \times J}|$$

$$\leq \{\sum_i \{\sum_{(I \times J) \in \mathcal{B}_i} |s_{I \times J}|^2\}^{\frac{p}{2}} \{\sum_{(I \times J) \in \mathcal{B}_i} |t_{I \times J}|^2\}^{\frac{p}{2}}\}^{\frac{1}{p}}$$

(5.2) $$\leq C\|t\|_{c^p} \{\sum_i |\Omega_i|^{1-\frac{p}{2}} \{\sum_{(I \times J) \in \mathcal{B}_i} |s_{I \times J}|^2\}^{\frac{p}{2}}\}^{\frac{1}{p}}$$

since if $I \times J \in \mathcal{B}_i$, then

$$I \times J \subseteq \widetilde{\Omega_i} = \{(x,y) : M_s(\chi_{\Omega_i})(x,y) > \frac{1}{2}\}, |\widetilde{\Omega_i}| \leq C|\Omega_i|,$$

and $\{t_{I \times J}\} \in c^p$ yields

$$\{\sum_{(I \times J) \in \mathcal{B}_i} |t_{I \times J}|^2\}^{\frac{1}{2}} \leq C\|t\|_{c^p} |\Omega_i|^{\frac{1}{p}-\frac{1}{2}}.$$

The same proof as in the claim of Theorem 4.4 implies

$$\sum_{(I \times J) \in \mathcal{B}_i} |s_{I \times J}|^2 \leq C2^{2i}|\Omega_i|.$$

Substituting the above term back to the last term in (5.2) gives $c^p \subseteq (s^p)^*$.

The proof of the converse is simple and is similar to one given in [FJ] for $p = 1$ in the one-parameter setting on R^n. If $\ell \in (s^p)^*$, then it is clear that $\ell(s) = \sum_{I \times J} s_{I \times J} \bar{t}_{I \times J}$ for some $t = \{t_{I \times J}\}$. Now fix an open set $\Omega \subset R^n \times R^m$ and let S be the sequence space of all $s = \{s_{I \times J}\}$ such that $I \times J \subseteq \Omega$. Finally, let μ be a measure on S so that the $\mu-$measure of the "point" $I \times J$ is $\frac{1}{|\Omega|^{\frac{2}{p}-1}}$. Then,

$$\{\frac{1}{|\Omega|^{\frac{2}{p}-1}} \sum_{I \times J \subseteq \Omega} |t_{I \times J}|^2\}^{\frac{1}{2}} = \|t_{I \times J}\|_{\ell^2(S, d\mu)}$$

$$= \sup_{\|s\|_{\ell^2(S,d\mu)} \leq 1} |\frac{1}{|\Omega|^{\frac{2}{p}-1}} \sum_{I\times J\subseteq\Omega} s_{I\times J}\bar{t}_{I\times J}|$$

$$\leq \|t\|_{(s^p)^*} \sup_{\|s\|_{\ell^2(S,d\mu)} \leq 1} \|s_{I\times J}\frac{1}{|\Omega|^{\frac{2}{p}-1}}\|_{s^p}.$$

By Holder's inequality,

$$\|s_{I\times J}\frac{1}{|\Omega|^{\frac{2}{p}-1}}\|_{s^p}$$

$$=\frac{1}{|\Omega|^{\frac{2}{p}-1}}\{\int_\Omega (\sum_{I\times J\subseteq\Omega} |s_{I\times J}|^2 |I\times J|^{-1}\chi_I(x)\chi_J(y))^{\frac{p}{2}}dxdy\}^{\frac{1}{p}}$$

$$\leq\{\frac{1}{|\Omega|^{\frac{2}{p}-1}}\int_\Omega \sum_{I\times J\subseteq\Omega} |s_{I\times J}|^2 |I\times J|^{-1}\chi_I(x)\chi_J(y)dxdy\}^{\frac{1}{2}}$$

$$=\|s\|_{\ell^2(S,d\mu)} \leq 1,$$

which shows $\|t\|_{c^p} \leq \|t\|_{(s^p)^*}$. **Q.E.D.**

In order to use Theorem 1.16 to show Theorem 1.17, we define a map S which takes $f \in (\mathcal{S}_F)'$ to the sequence of coefficients $\{s_{I\times J}\} = \{|I|^{\frac{1}{2}}|J|^{\frac{1}{2}}\psi_{j,k} * f(x_I, y_J)\}$, where I, J are cubes in R^n, R^m, with side length $2^{-j-N}, 2^{-j-N} + 2^{-k-N}$, and x_I, y_J are any fixed points in I, J, respectively. For any sequence $s = \{s_{I\times J}\}$, we define the map T which takes s to $T(s) = \sum_j\sum_k\sum_J\sum_I |I|^{\frac{1}{2}}|J|^{\frac{1}{2}}\widetilde{\psi}_{j,k}(x,y)s_{I\times J}$, where $\widetilde{\psi}_{j,k}$ are the same as in (1.9).

The following result together with Theorem 1.16 will show theorem 1.17.

Theorem 5.2:. *The maps $S : H_F^p \to s^p$ and $CMO_F^p \to c^p$, and $T : s^p \to H_F^p$ and $c^p \to CMO_F^p$ are bounded, and $T \circ S$ is the identity on H_F^p and CMO_F^p.*

Proof of Theorem 5.2: The boundedness of S on H_F^p and CMO_F^p follows directly from the Plancherel-Pôlya inequalities, Theorem 1.9 and Theorem 1.14. The boundedness of T follows from the same proofs in Theorem 1.9 and 1.14. To be precise, to see T is bounded from s^p to H_F^p, let $s = \{s_{I\times J}\}$. Then, by Proposition 4.1,

$$\|T(s)\|_{H_F^p} \leq C\|\{\sum_j\sum_k\sum_J\sum_I |\psi_{j,k} * T(s)(x,y)|^2\chi_I(x)\chi_J(y)\}^{\frac{1}{2}}\|_p.$$

By adapting a similar the proof in Theorem 1.9, we have for some $0 < r < p$

$$|\psi_{j,k} * T(s)(x,y)\chi_I(x)\chi_J(y)|^2$$

$$=|\sum_{j'}\sum_{k'}\sum_{J'}\sum_{I'} |I'||I'|\psi_{j,k} * \widetilde{\psi}_{j',k'}(\cdot,\cdot)(x,y)s_{I'\times J'}|I'|^{-\frac{1}{2}}|J'|^{-\frac{1}{2}}\chi_I(x)\chi_J(y)|^2$$

$$\leq C\sum_{k'\leq j'} 2^{-|j-j'|K}2^{-|k-k'|K}\{M_s(\sum_{J'}\sum_{I'} |s_{I'\times J'}||I'|^{-1}|J'|^{-1}\chi_{J'\chi_{I'}})^r\}^{\frac{2}{r}}(x,y)\chi_I(x)\chi_J(y)$$

$$+\sum_{k'>j'} 2^{-|j-j'|K}2^{-|k-k'|K}\{M(\sum_{J'}\sum_{I'} |s_{I'\times J'}||I'|^{-1}|J'|^{-1}\chi_{J'\chi_{I'}})^r\}^{\frac{2}{r}}(x,y)\chi_I(x)\chi_J(y).$$

Repeating the proof in Theorem 1.9 gives the boundedness of T from s^p to H_F^p. The similar proof given in the proof of Theorem 1.14 applies to the boundedness of T from c^p to CMO_F^p. We leave the details to the reader. The discrete Calderón reproducing formula, Theorem 1.8, and Theorem 1.14 show that $T \circ S$ is the identity on H_F^p and CMO_F^p.

We are now ready to give the

Proof of Theorem 1.17: If $f \in \mathcal{S}_F$ and $g \in CMO_F^p$, then the discrete Calderón reproducing formula, Theorem 1.16 and Theorem 5.2 imply

$$|\ell_g| = | < f, g > | =$$

$$| \sum_{R = I \times J} |I||J| \psi_R * f(x_I, y_J) \widetilde{\psi}_R(g)(x_I, y_J)| \leq C \|f\|_{H_F^p} \|g\|_{CMO_F^p}.$$

Because $\mathcal{S}_F$ is dense in H_F^p, this shows that the map $\ell_g = < f, g >$, defined initially for $f \in \mathcal{S}_F$ can be extended to a continuous linear functional on H_F^p and $\|\ell_g\| \leq C \|g\|_{CMO_F^p}$.

Conversely, let $\ell \in (H_F^p)^*$ and set $\ell_1 = \ell \circ T$, where T is defined as in Theorem 5.2. Then, by theorem 5.2, $\ell_1 \in (s^p)^*$, so by Theorem 1.16, there exists $t = \{t_{I \times J}\}$ such that $\ell_1(s) = \sum_{I \times J} s_{I \times J} \bar{t}_{I \times J}$ for all $s = \{s_{I \times J}\}$ and $\|t\|_{c^p} \approx \|\ell_1\| \leq C \|\ell\|$ because T is bounded. Again, by Theorem 1.16, $\ell = \ell \circ T \circ S = \ell_1 \circ S$. Hence, with $f \in \mathcal{S}_F$ and $g = \sum_{I \times J} t_{I \times J} \psi_R(x - x_I, y - y_J)$, where, without loss the generality, we may assume ψ is a radial function,

$$\ell(f) = \ell_1(S(f)) = < S(f), t > = < f, g >,$$

This proves $\ell = \ell_g$ and, by Theorem 1.16, $\|g\|_{CMO_F^p} \leq C \|t\|_{c^p} \leq C \|\ell_g\|$.   **Q.E.D.**

Proof of Theorem 1.18 As mentioned in section 4, since H_F^1 is a subspace of L^1, by the duality of H_F^1 and BMO_F, and the boundedness of flag singular integrals on H_F^1, one concludes that L^∞ is a subspace of BMO_F, and flag singular integrals are bounded on BMO_F and from L^∞ to BMO_F. This shows Theorem 1.18.   **Q.E.D.**

6. Calderón-Zygmund decomposition and interpolation on flag Hardy spaces $H_F^p(R^n \times R^m)$: Proofs of Theorems 1.19 and 1.20

The main purpose of this section is to derive a Calderón-Zygmund decomposition using functions in flag Hardy spaces. Furthermore, we will prove an interpolation theorem on $H_F^p(R^n \times R^m)$.

We first recall that Chang and R. Fefferman established the following Calderón-Zygmund decomposition on the pure product domains $R_+^2 \times R_+^2$ ([CF2]).

Calderón-Zygmund Lemma: Let $\alpha > 0$ be given and $f \in L^p(R^2)$, $1 < p < 2$. Then we may write $f = g + b$ where $g \in L^2(R^2)$ and $b \in H^1(R_+^2 \times R_+^2)$ with $\|g\|_2^2 \leq \alpha^{2-p} \|f\|_p^p$ and $\|b\|_{H^1(R_+^2 \times R_+^2)} \leq C \alpha^{1-p} \|f\|_p^p$, where c is an absolute constant.

We now prove the Calderón-Zygmund decomposition in the setting of flag Hardy spaces, namely we give the

Proof of Theorem 1.19 We first assume $f \in L^2(R^{n+m}) \cap H_F^p(R^n \times R^m)$. Let $\alpha > 0$ and $\Omega_\ell = \{(x, y) \in R^n \times R^m : S(f)(x, y) > \alpha 2^\ell\}$, where, as in Corollary 4.6,

$$S(f)(x, y) = \left\{ \sum_{j,k} \sum_{I,J} |\phi_{jk} * \left(T_N^{-1}(f)\right)(x_I, y_J)|^2 \chi_I(x) \chi_J(y) \right\}^{\frac{1}{2}}.$$

It has been shown in Corollary 4.6 that $f \in L^2(R^{n+m}) \cap H_F^p(R^n \times R^m)$ then $||f||_{H_F^p} \approx ||S(f)||_p$.

In the following we take $R = I \times J$ as all dyadic rectangles in $R^n \times R^m$ with $|I| = 2^{-j-N}$, $|J| = 2^{-j-N} + 2^{-k-N}$, where j, k are integers and N is large enough.

Let

$$\mathcal{R}_0 = \left\{ R = I \times J, \text{ such that } |R \cap \Omega_0| < \frac{1}{2}|R| \right\}$$

and for $\ell \geq 1$

$$\mathcal{R}_\ell = \left\{ R = I \times J, \text{ such that } |R \cap \Omega_{\ell-1}| \geq \frac{1}{2}|R| \text{ but } |R \cap \Omega_\ell| < \frac{1}{2}|R| \right\}.$$

By the discrete Calderón reproducing formula in Theorem 4.4,

$$f(x, y) = \sum_{j,k} \sum_{I,J} |I||J|\widetilde{\phi}_{jk}(x - x_I, y - y_J)\phi_{jk} * \left(T_N^{-1}(f)\right)(x_I, y_J)$$

$$= \sum_{\ell \geq 1} \sum_{I \times J \in \mathcal{R}_\ell} |I||J|\widetilde{\phi}_{jk}(x - x_I, y - y_J)\phi_{jk} * \left(T_N^{-1}(f)\right)(x_I, y_J)$$

$$+ \sum_{I \times J \in \mathcal{R}_0} |I||J|\widetilde{\phi}_{jk}(x - x_I, y - y_J)\phi_{jk} * \left(T_N^{-1}(f)\right)(x_I, y_J)$$

$$= b(x, y) + g(x, y)$$

When $p_1 > 1$, using duality argument, it is easy to show

$$||g||_{p_1} \leq C|| \left\{ \sum_{R = I \times J \in \mathcal{R}_0} |\phi_{jk} * \left(T_N^{-1}(f)\right)(x_I, y_J)|^2 \chi_I \chi_J \right\}^{\frac{1}{2}} ||_{p_1}.$$

Next, we estimate $||g||_{H_F^{p_1}}$ when $0 < p_1 \leq 1$. Clearly, the duality argument will not work here. Nevertheless, we can estimate the $H_F^{p_1}$ norm directly by using the discrete Calderón reproducing formula in Theorem 1.8. To this end, we note that

$$||g||_{H_F^{p_1}} \leq || \left\{ \sum_{j',k'} \sum_{I',J'} |(\psi_{j'k'} * g)(x_{I'}, y_{J'})|^2 \chi_{I'}(x) \chi_{J'}(y) \right\}^{\frac{1}{2}} ||_{L^{p_1}}.$$

Since

$$(\psi_{j',k'} * g)(x_{I'}, y_{J'}) = \sum_{I \times J \in \mathcal{R}_0} |I||J| \left(\psi_{j'k'} * \widetilde{\phi}_{jk}\right)(x_{I'} - x_I, y_{J'} - y_J)\phi_{jk} * \left(T_N^{-1}(f)\right)(x_I, y_J)$$

Repeating the same proof of Theorem 1.9, we have

$$
\left\| \left\{ \sum_{j',k'} \sum_{I',J'} |(\psi_{j'k'} * g)(x_{I'}, y_{J'})|^2 \chi_{I'}(x)\chi_{J'}(y) \right\}^{\frac{1}{2}} \right\|_{L^{p_1}}
$$

$$
\leq C \left\| \left\{ \sum_{R=I\times J\in\mathcal{R}_0} |\phi_{jk} * (T_N^{-1}(f))(x_I, y_J)|^2 \chi_I \chi_J \right\}^{\frac{1}{2}} \right\|_{p_1}.
$$

This shows that for all $0 < p_1 < \infty$

$$
\|g\|_{H_F^{p_1}} \leq C \left\| \left\{ \sum_{R=I\times J\in\mathcal{R}_0} |\phi_{jk} * (T_N^{-1}(f))(x_I, y_J)|^2 \chi_I \chi_J \right\}^{\frac{1}{2}} \right\|_{p_1}.
$$

Claim 1:

$$
\int_{S(f)(x,y)\leq\alpha} S^{p_1}(f)(x,y)dxdy \geq C \left\| \left\{ \sum_{R=I\times J\in\mathcal{R}_0} |\phi_{jk} * (T_N^{-1}(f))(x_I, y_J)|^2 \chi_I \chi_J \right\}^{\frac{1}{2}} \right\|_{p_1}.
$$

This claim implies

$$
\|g\|_{p_1} \leq C \int_{S(f)(x,y)\leq\alpha} S^{p_1}(f)(x,y)dxdy
$$

$$
\leq C\alpha^{p_1-p} \int_{S(f)(x,y)\leq\alpha} S^p(f)(x,y)dxdy
$$

$$
\leq C\alpha^{p_1-p} \|f\|_{H_F^p(R^n\times R^m)}^p.
$$

To show Claim 1, we denote $R = I \times J \in \mathcal{R}_0$. We choose $0 < q < p_1$ and note that

$$
\int_{S(f)(x,y)\leq\alpha} S^{p_1}(f)(x,y)dxdy
$$

$$
= \int_{S(f)(x,y)\leq\alpha} \left\{ \sum_{j,k} \sum_{I,J} |\phi_{jk} * (T_N^{-1}(f))(x_I, y_J)|^2 \chi_I(x)\chi_J(y) \right\}^{\frac{p_1}{2}} dxdy
$$

$$
\geq C \int_{\Omega_0^c} \left\{ \sum_{R\in\mathcal{R}_0} |\phi_{jk} * (T_N^{-1}(f))(x_I, y_J)|^2 \chi_I \chi_J \right\}^{\frac{p_1}{2}} dxdy
$$

$$
= C \int_{R^n\times R^m} \left\{ \sum_{R\in\mathcal{R}_0} |\phi_{jk} * (T_N^{-1}(f))(x_I, y_J)|^2 \chi_{R\cap\Omega_0^c}(x,y) \right\}^{\frac{p_1}{2}} dxdy
$$

$$
\geq C \int_{R^n\times R^m} \left\{ \left\{ \sum_{R\in\mathcal{R}_0} \left(M_s \left(|\phi_{jk} * (T_N^{-1}(f))(x_I, y_J)|^q \chi_{R\cap\Omega_0^c} \right)(x,y) \right)^{\frac{2}{q}} \right\}^{\frac{q}{2}} \right\}^{\frac{p_1}{q}} dxdy
$$

$$
\geq C \int_{R^n\times R^m} \left\{ \sum_{R\in\mathcal{R}_0} |\phi_{jk} * (T_N^{-1}(f))(x_I, y_J)|^2 \chi_R(x,y) \right\}^{\frac{p_1}{2}} dxdy
$$

where in the last inequality we have used the fact that $|\Omega_0^c \cap (I \times J)| \geq \frac{1}{2}|I \times J|$ for $I \times J \in \mathcal{R}_0$, and thus

$$\chi_R(x, y) \leq 2^{\frac{1}{q}} M_s(\chi_{R \cap \Omega_0^c})^{\frac{1}{q}}(x, y)$$

and in the second to the last inequality we have used the vector-valued Fefferman-Stein inequality for strong maximal functions

$$\left\| \left(\sum_{k=1}^{\infty} (M_s(f_k))^r \right)^{\frac{1}{r}} \right\|_p \leq C \left\| \left(\sum_{k=1}^{\infty} |f_k|^r \right)^{\frac{1}{r}} \right\|_p$$

with the exponents $r = 2/q > 1$ and $p = p_1/q > 1$. Thus the claim follows.

We now recall $\widetilde{\Omega_\ell} = \{(x, y) \in R^n \times R^m : M_s(\chi_{\Omega_\ell}) > \frac{1}{2}\}$.

Claim 2: For $p_2 \leq 1$,

$$\left\| \sum_{I \times J \in \mathcal{R}_\ell} |I||J|\widetilde{\phi}_{jk}(x - x_I, y - y_J)\phi_{jk} * \left(T_N^{-1}(f)\right)(x_I, y_J) \right\|_{H_F^{p_2}}^{p_2} \leq C(2^\ell \alpha)^{p_2} |\widetilde{\Omega_{\ell-1}}|.$$

Claim 2 implies

$$
\begin{aligned}
\|b\|_{H_F^{p_2}}^{p_2} &\leq \sum_{\ell \geq 1} (2^\ell \alpha)^{p_2} |\widetilde{\Omega_{\ell-1}}| \\
&\leq C \sum_{\ell \geq 1} (2^\ell \alpha)^{p_2} |\Omega_{\ell-1}| \leq C \int_{S(f)(x,y)>\alpha} S^{p_2}(f)(x, y)dxdy \\
&\leq C\alpha^{p_2-p} \int_{S(f)(x,y)>\alpha} S^p(f)(x, y)dxdy \leq C\alpha^{p_2-p} \|f\|_{H_F^p}^p.
\end{aligned}
$$

To show Claim 2, again we have

$$\left\| \sum_{I \times J \in \mathcal{R}_\ell} |I||J|\widetilde{\phi}_{jk}(x - x_I, y - y_J)\phi_{jk} * \left(T_N^{-1}(f)\right)(x_I, y_J) \right\|_{H_F^{p_2}}^{p_2}$$

$$\leq C \left\| \left\{ \sum_{j'k'} \sum_{I',J'} \left| \sum_{I \times J \in \mathcal{R}_\ell} |I||J| \left(\psi_{j'k'} * \widetilde{\phi}_{jk}\right)(x_{I'} - x_I, y_{J'} - y_J)\phi_{jk} * \left(T_N^{-1}(f)\right)(x_I, y_J) \right|^2 \right\}^{\frac{1}{2}} \right\|_{L^{p_2}}$$

$$\leq C \left\| \left\{ \sum_{R=I \times J \in \mathcal{R}_\ell} |\phi_{jk} * \left(T_N^{-1}(f)\right)(x_I, y_J)|^2 \chi_I \chi_J \right\}^{\frac{1}{2}} \right\|_{p_2}$$

where we can use a similar argument in the proof of Theorem 1.19 to prove the last inequality.

However,

$$\sum_{\ell=1}^{\infty}(2^{\ell}\alpha)^{p_2}|\widetilde{\Omega}_{\ell-1}|$$

$$\geq \int_{\widetilde{\Omega}_{\ell-1}\backslash\Omega_{\ell}} S(f)^{p_2}(x,y)dxdy$$

$$= \int_{\widetilde{\Omega}_{\ell-1}\backslash\Omega_{\ell}} \left\{ \sum_{j,k}\sum_{I,J} |\phi_{jk}*(T_N^{-1}(f))(x_I,y_J)|^2 \chi_I(x)\chi_J(y) \right\}^{\frac{p_2}{2}} dxdy$$

$$= \int_{R^n\times R^m} \left\{ \sum_{j,k}\sum_{I,J} |\phi_{jk}*(T_N^{-1}(f))(x_I,y_J)|^2 \chi_{(I\times J)\cap\widetilde{\Omega}_{\ell-1}\backslash\Omega_{\ell})}(x,y) \right\}^{\frac{p_2}{2}} dxdy$$

$$\geq \int_{R^n\times R^m} \left\{ \sum_{I\times J\in\mathcal{R}_{\ell}} |\phi_{jk}*(T_N^{-1}(f))(x_I,y_J)|^2 \chi_{(I\times J)\cap\widetilde{\Omega}_{\ell-1}\backslash\Omega_{\ell})}(x,y) \right\}^{\frac{p_2}{2}} dxdy$$

$$\geq \int_{R^n\times R^m} \left\{ \sum_{I\times J\in\mathcal{R}_{\ell}} |\phi_{jk}*(T_N^{-1}(f))(x_I,y_J)|^2 \chi_I(x)\chi_J(y) \right\}^{\frac{p_2}{2}} dxdy$$

In the above string of inequalities, we have used the fact that for $R\in\mathcal{R}_{\ell}$ we have

$$|R\cap\Omega_{\ell-1}| > \frac{1}{2}|R| \text{ and } |R\cap\Omega_{\ell}| \leq \frac{1}{2}|R|$$

and consequently $R\subset\widetilde{\Omega}_{\ell-1}$. Therefore $|R\cap(\widetilde{\Omega}_{\ell-1}\backslash\Omega_{\ell})| > \frac{1}{2}|R|$. Thus the same argument applies here to conclude the last inequality above. Finally, since $L^2(R^{n+m})$ is dense in $H_F^p(R^n\times R^m)$, Theorem 6.1 is proved.   **Q.E.D.**

We are now ready to prove the interpolation theorem on Hardy spaces H_F^p for all $0<p<\infty$.

Proof of Theorem 1.20: Suppose that T is bounded from $H_F^{p_2}$ to L^{p_2} and from $H_F^{p_1}$ to L^{p_1}. For any given $\lambda>0$ and $f\in H_F^p$, by the Calderón-Zygmund decomposition,

$$f(x,y) = g(x,y) + b(x,y)$$

with

$$||g||_{H_F^{p_1}}^{p_1} \leq C\lambda^{p_1-p}||f||_{H_F^p}^p \text{ and } ||b||_{H_F^{p_2}}^{p_2} \leq C\lambda^{p_2-p}||f||_{H_F^p}^p.$$

Moreover, we have proved the estimates

$$||g||_{H_F^{p_1}}^{p_1} \leq C\int_{S(f)(x,y)\leq\alpha} S(f)^{p_1}(x,y)dxdy$$

and

$$||b||_{H_F^{p_2}}^{p_2} \leq C\int_{S(f)(x,y)>\alpha} S(f)^{p_2}(x,y)dxdy$$

which implies that

$$
\begin{aligned}
||Tf||_p^p = & \, p \int_0^\infty \alpha^{p-1} |\{(x,y) : |Tf(x,y)| > \lambda\}| \, d\alpha \\
\leq & \, p \int_0^\infty \alpha^{p-1} \left|\left\{(x,y) : |Tg(x,y)| > \frac{\lambda}{2}\right\}\right| d\alpha + p \int_0^\infty \alpha^{p-1} \left|\left\{(x,y) : |Tb(x,y)| > \frac{\lambda}{2}\right\}\right| d\alpha \\
\leq & \, p \int_0^\infty \alpha^{p-1} \int_{S(f)(x,y) \leq \alpha} S(f)^{p_1}(x,y) \, dx \, dy \, d\alpha + p \int_0^\infty \alpha^{p-1} \int_{S(f)(x,y) > \alpha} S(f)^{p_2}(x,y) \, dx \, dy \, d\alpha \\
\leq & \, C ||f||_{H_F^p}^p
\end{aligned}
$$

Thus,

$$
||Tf||_p \leq C ||f||_{H_F^p}
$$

for any $p_2 < p < p_1$. Hence, T is bounded from H_F^p to L^p.

To prove the second assertion that T is bounded on H_F^p for $p_2 < p < p_1$, for any given $\lambda > 0$ and $f \in H_F^p$, by the Calderón-Zygmund decomposition again

$$
\begin{aligned}
& |\{(x,y) : |g(Tf)(x,y)| > \alpha\}| \\
\leq & \left|\left\{(x,y) : |g(Tg)(x,y)| > \frac{\alpha}{2}\right\}\right| + \left|\left\{(x,y) : |g(Tb)(x,y)| > \frac{\alpha}{2}\right\}\right| \\
\leq & \, C\alpha^{-p_1} ||Tg||_{H_F^{p_1}}^{p_1} + C\alpha^{-p_2} ||Tb||_{H_F^{p_2}}^{p_2} \\
\leq & \, C\alpha^{-p_1} ||g||_{H_F^{p_1}}^{p_1} + C\alpha^{-p_2} ||b||_{H_F^{p_2}}^{p_2} \\
\leq & \, C\alpha^{-p_1} \int_{S(f)(x,y) \leq \alpha} S(f)^{p_1}(x,y) \, dx \, dy + C\alpha^{-p_2} \int_{S(f)(x,y) > \alpha} S(f)^{p_2}(x,y) \, dx \, dy
\end{aligned}
$$

which, as above, shows that $||Tf||_{H_F^p} \leq C ||g(TF)||_p \leq C ||f||_{H_F^p}$ for any $p_2 < p < p_1$.   **Q.E.D.**

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