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On a relative Alexandrov-Fenchel inequality for convex bodies in Euclidean spaces

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Abstract. In this note we prove a localized form of Alexandrov-Fenchel inequality for convex bodies, i.e. we prove a class of isoperimetric inequalities in a ball involving Federer curvature measures.

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1 Introduction

In the Euclidean space $\mathbb{R}^n$ the (global) isoperimetric inequality states that, for any bounded open set $E \subset \mathbb{R}^n$ with (say) Lipschitz boundary ∂E, we have

$$(1) \qquad |E|^{(n-1)/n} \le \frac{1}{n\omega_n^{1/n}} \mathscr{H}^{n-1}(\partial E),$$

where $|E|$ is the Lebesgue measure of E, $\mathscr{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure, and ω_n is the Lebesgue measure of the unit ball in $\mathbb{R}^n$. Notice that the constant $(n\omega_n^{1/n})^{-1}$ appearing in (1) is sharp. We remark here that the isoperimetric inequality (1.1) with the best constant is equivalent to the L^1 to $L^{n/(n-1)}$ Sobolev inequality for compactly supported functions in $\mathbb{R}^n$ with the same best constant (see [6], [4], [9]). We also refer the reader to [2] and [12] for proofs of such equivalence.

On the other hand, the *relative* isoperimetric inequality provides a much richer information: if E is as above and $U = U(x_0, r)$ is any open Euclidean ball in $\mathbb{R}^n$ of radius $r > 0$ centred at the point x_0, then

$$(2) \qquad \min\{|E \cap U|, |U \backslash E|\}^{(n-1)/n} \le c_n \mathscr{H}^{n-1}(U \cap \partial E),$$

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for a suitable dimensional constant $c_n > 0$. Inequality (2) is stronger than inequality (1) in the sense that (2) is a localized version of (1) and implies (1) with $(n\omega_n^{1/n})^{-1}$ replaced by c_n by taking the ball U sufficiently large. Moreover, the inequality (2) is equivalent to the Poincaré inequality on balls for functions not necessarily compactly supported (see also [2] and [12]), whereas inequality (1) is basically equivalent to Sobolev inequality for compactly supported functions as we pointed out earlier. At this point, it might be somehow surprising to see that we can derive (2) from (1), since (2) contains more geometric information than (1). Nevertheless, we shall see that this is possible since we know the sharp constant $(n\omega_n^{1/n})^{-1}$ in (1). Thus, the geometry of the Euclidean space, which provides the information we need to prove (2), is somehow hidden in this constant.

Let us make our previous argument more precise, by deriving (2) from (1). Indeed, the following result is a simpler version of the more general theorem proved in this paper. The argument given here is of independent interest. We thus state this theorem separately.

Theorem I. *The global isoperimetric inequality* (1) *implies* (2). *Namely, let $E \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary ∂E, and let U be a ball. Then the following relative isoperimetric inequality holds.*

$$\min\{|U\backslash E|^{(n-1)/n}, |U \cap E|^{(n-1)/n}\} \leq \frac{2^{1+1/n}}{n\omega_n^{1/n}(2^{1/n}-1)} \mathscr{H}^{n-1}(U \cap \partial E).$$

Proof of Theorem I. The proof is divided into several steps.

Step 1.1. *Let E be a Lipschitz domain in $\mathbb{R}^n$, and let $U = U(x_0, r)$ be an Euclidean ball. Since $\partial(U \cap E) = (\partial U \cap E) \cup (\partial E \cap U)$, we have*

$$\mathscr{H}^{n-1}(\partial U \cap E) \leq n\frac{|U \cap E|}{r} + \mathscr{H}^{n-1}(U \cap \partial E). \tag{3}$$

Indeed, by Divergence Theorem we get

$$n|U \cap E| = \int_{\partial U(x_0,r)\cap E} \left\langle x - x_0, \frac{x - x_0}{r} \right\rangle d\mathscr{H}^{n-1} + \int_{U\cap\partial E} \langle x - x_0, v\rangle\, d\mathscr{H}^{n-1},$$

where v is the outward normal to ∂E. Then (3) follows since $|x - x_0| \leq r$ in $U \cap \partial E$.

Step 1.2. *Let E and U be as above. Then*

$$|U \cap E|^{(n-1)/n} - \omega_n^{-1/n}\frac{|U \cap E|}{r} \leq \frac{2\mathscr{H}^{n-1}(U \cap \partial E)}{n\omega_n^{1/n}}. \tag{4}$$

Indeed, from the global isoperimetric inequality (1) it follows that

$$(5)\qquad |U \cap E|^{(n-1)/n} \le \frac{1}{n\omega_n^{1/n}}(\mathscr{H}^{n-1}(\partial U \cap E) + \mathscr{H}^{n-1}(U \cap \partial E)).$$

Inserting then (3) in (5) we get (4).

Step 1.3. *Let E and U be as above, and let* $0 < \lambda < 1$. *Assume* $|U \cap E| \le \lambda |U|$. *Then*

$$(6)\qquad |U \cap E|^{(n-1)/n} - \frac{1}{\omega_n^{1/n}}\frac{|U \cap E|}{r} \ge (1 - \lambda^{1/n})|U \cap E|^{(n-1)/n}.$$

Note that (6) is equivalent to

$$\lambda^{1/n}|U \cap E|^{-1/n} \ge \frac{1}{\omega_n^{1/n} r},$$

which follows directly from $|U \cap E| \le \lambda |U|$ and the fact that $|U| = \omega_n r^n$.

Step 1.4. *Let E and U be as above. Then the following relative isoperimetric inequality holds.*

$$(7)\qquad \min\{|U \backslash E|^{(n-1)/n}, |U \cap E|^{(n-1)/n}\} \le \frac{2^{1+1/n}}{n\omega_n^{1/n}(2^{1/n} - 1)}\mathscr{H}^{n-1}(U \cap \partial E).$$

Without loss of generality, we may assume that $|U \cap E| \le \frac{1}{2}|U|$. Namely, we take $\lambda = \frac{1}{2}$. Then by Step 1.3

$$(8)\qquad |U \cap E|^{(n-1)/n} - \frac{1}{\omega_n^{1/n}}\frac{|U \cap E|}{r} \ge \alpha |U \cap E|^{(n-1)/n}$$

where $\alpha = 1 - 2^{-1/n}$. Then, combining this with Step 1.2, the proof of Step 1.4 will be achieved. □

In the present paper we want to apply the same idea in order to localize a class of (generalized) isoperimetric inequalities for convex sets that are known in the literature as isoperimetric inequalities for quermassintegrals (see, e.g., [10] and [1]). In their simplest form, these inequalities state that suitable powers of the volume of a smooth compact convex set E can be bounded (up to a dimensional constant) by the integral over ∂E of the elementary symmetric functions of the principal curvatures of the boundary itself. This result can be extended to nonsmooth convex sets (yielding Alexandrov-Fenchel's inequalities) by means of the so called curvature measures introduced by Federer [3]. We refer to [1], [8] and [7] for an exhaustive introduction to the subject.

For the convenience of comparing our main result with the known classical theorem, we now recall briefly the Alexandrov-Fenchel inequality. For more detailed information, we refer the reader to Section 2.

From now on, we denote by $U = U(x,r)$ the Euclidean open balls in $\mathbb{R}^n$, whereas $B = B(x,r)$ will denote a closed ball.

If E is a convex body (i.e. a non-empty compact convex set), we denote by $\Phi_m(E,\cdot)$, $m = 0,\dots,n$, Federer's curvature measures associated with E using Steiner's formula (see Definition 2.1 in Section 2 for details). The Alexandrov-Fenchel isoperimetric inequality for convex bodies (see, e.g. [1], Chapter IV, 20.2, formula (20)) states: let $0 \le m \le l \le n$ and E be a convex body, then

$$\left(\frac{\Phi_l(E,\mathbb{R}^n)}{\Phi_l(B(0,1),\mathbb{R}^n)}\right)^{m/l} \le \frac{\Phi_m(E,\mathbb{R}^n)}{\Phi_m(B(0,1),\mathbb{R}^n)}. \tag{9}$$

Equivalently, if we let $V_i(E)$ $(i = 0,\dots,n)$ be the cross sectional measures (see [1]), then the Alexandrov-Fenchel inequality also states that

$$\left(\frac{V_l(E)}{\omega_n}\right)^{m/l} \le \frac{V_m(E)}{\omega_n}.$$

As we pointed out earlier, the goal of this paper is to localize inequality (9), roughly speaking by estimating the Lebesgue measure $|U \cap E|$ of the intersection of a ball U with a convex body E in terms of the curvature of ∂E in U, namely $\Phi_m(E,U)$, and of a "singular" term. Indeed the form of the localized Alexandrov-Fenchel type isoperimetric inequality for $m < n-1$ (see Theorem II) cannot involve only the curvature of the boundary in a ball. Geometrically speaking, this is basically because the boundary can be locally flat so that all curvatures vanish even if the volume of the set in the ball is positive. This phenomenon clearly does not occur for the standard relative isoperimetric inequality (2). This is due to the fact that the $(n-1)$-dimensional Hausdorff measure of a portion of plane in a ball is enough to control the volume bounded by the plane itself in the ball. Thus, our local isoperimetric inequalities for curvature measures require new supplementary terms. These new terms involve suitable measures of the intersection of the boundary of the set with the boundary of the ball (i.e. $\partial E \cap \partial U$). They depend basically on the angle between the two surfaces at the intersection points. From the technical point of view, this depends on the following fact. Suppose for the sake of simplicity that we are dealing only with smooth sets. Then unlike in the case of the $(n-1)$-dimensional Hausdorff measure, the curvature measures of the boundary of the intersection is not the sum of the corresponding measures of two smooth pieces. Instead, a new term appears in the sum. Such a term is concentrated on the 'ridge' given by the intersection. For an explicit form of it, see Corollary 2.1.

We now are ready to state the main result of this paper. Let $\Phi_m(E \cap B,\cdot) = \Phi_m^{\mathrm{a}}(E \cap B,\cdot) + \Phi_m^{\mathrm{s}}(E \cap B,\cdot)$ be the Lebesgue decomposition of $\Phi_m(E \cap B,\cdot)$ with respect to $\mathscr{H}^{n-1} \llcorner \partial(E \cap B)$. Then we have

Theorem II. *Let $E \subset \mathbb{R}^n$ be a convex body, let $U = U(x,r)$ be an open Euclidean ball centred at a point $x \in \partial E$, and set $B = B(x,r) = \overline{U}$. Then, if $m = 0,\dots,n-1$, we have*

$$|E \cap U|^{m/n} \le \alpha(m,n)\{\Phi_m(E,U) + \Phi_m^s(E \cap B, \partial E \cap \partial U)\}, \tag{10}$$

where

$$\alpha(m,n) = (1 - 2^{-1+m/n})^{-1}\omega_n^{m/n-1}\omega_{n-m}\binom{n}{m}^{-1}.$$

We point out that the constant $\alpha(m,n)$ is not sharp in general (consider for instance the case $m = n-1$), but the result is sharp, in the sense that, if $m < n-1$, we cannot get rid of the term $\Phi_m^s(E \cap B, \partial E \cap \partial U)$: see Remark 2.5.

The main idea of proving Theorem II is inspired by that used in the proof of Theorem I. However, to adapt this idea to show a relative Alexandrov-Fenchel inequality, extra care must be taken to estimate $\Phi_m(E \cap B, \partial(E \cap U))$. It turns out that this is considerably more difficult when $m < n-1$.

If we replace the assumption $x \in \partial E$ by the more familiar assumption $|E \cap B| \le \frac{1}{2}|B|$, it follows from our proof that a third term appears in the right-hand side of the relative isoperimetric inequality. Such a term takes the form of a dimensional constant times *the negative part* of

$$r^{m-n}\int_{U \cap \partial E} \langle y - x_0, \nu_E(y)\rangle \, d\mathscr{H}^{n-1}(y).$$

Clearly, if $m = n-1$, this term can be absorbed by the term $\Phi_m(E, U)$.

More precisely, we have:

Theorem III. *Let $E \subset \mathbb{R}^n$ be a convex body, let $U = U(x,r)$ be an open Euclidean ball such that*

$$|E \cap U| \le \frac{1}{2}|U|,$$

and set $B = B(x,r) = \overline{U}$. Then, if $m = 0, \ldots, n-1$, we have

$$|E \cap U|^{m/n} \le \alpha(m,n)\{\Phi_m(E,U) + \Phi_m^s(E \cap B, \partial E \cap \partial U)\} - \gamma(m,n)r^{m-n}\int_{U \cap \partial E} \langle y - x_0, \nu_E(y)\rangle \, d\mathscr{H}^{n-1}(y),$$

where

$$\alpha(m,n) = (1 - 2^{-1+m/n})^{-1}\omega_n^{m/n-1}\omega_{n-m}\binom{n}{m}^{-1}$$

and

$$\gamma(m,n) = \frac{\omega_n^{m/n-1}}{n(1 - 2^{-1+m/n})}.$$

2 Proof of the Main Result

To prove Theorem 2, a fair amount of preparation is necessary. We fix preliminarily a few notations: in the sequel, if $E \subset \mathbb{R}^k$, we denote both by $|E|$ and $\mathscr{L}^k(E)$ the k-dimensional Lebesgue measure of E. We begin with the following

Definition 2.1 (Federer, [3], Theorem 5.6). Let $E \subset \mathbb{R}^n$ be a convex body. Then there exist $n+1$ Radon measures $\Phi_0(E,\cdot),\dots,\Phi_n(E,\cdot)$ such that, if β is any Borel set, then the following generalized Steiner formula holds:

$$|\{x \in \mathbb{R}^n : \delta_E(x) \le r, \xi_E(x) \in \beta\}| = \sum_{i=0}^{n} r^{n-i}\omega_{n-i}\Phi_i(E,\beta),$$

where

i) $\delta_E(x)$ denotes the Euclidean distance from x to E;

ii) $\xi_E(x)$ is the unique point of E nearest to x;

iii) ω_j is the j-dimensional Hausdorff measure of the unit ball in $\mathbb{R}^j$.

We notice that, since E is a convex body, following Federer's definition, $\mathrm{reach}(E) = \infty$. In addition, $\Phi_0(E,\cdot),\dots,\Phi_{n-1}(E,\cdot)$ are concentrated on ∂E, i.e. for any Borel set β

$$\Phi_j(E,\beta) = \Phi_j(E,\beta \cap \partial E), \quad j = 0,\dots,n-1 \tag{11}$$

and

$$\Phi_n(E,\beta) = |E \cap \beta|$$

([3], Remark 5.8).

Later on, the following remark will be used.

Remark 2.1. Let D and V be convex bodies. Then

$$\Phi_m(D \cap V, \mathrm{Int}(V) \cap \partial D) = \Phi_m(D, \mathrm{Int}(V)), \quad m = 0,\dots,n-1. \tag{12}$$

Indeed, by (11),

$$\Phi_m(D \cap V, \mathrm{Int}(V) \cap \partial D) = \Phi_m(D \cap V, \mathrm{Int}(V) \cap \partial(V \cap D)) = \Phi_m(D \cap V, \mathrm{Int}(V)) \tag{13}$$

Then (12) follows from Lemma 6.6 in [3].

Remark 2.2. Since

$$|\{x \in \mathbb{R}^n : \delta_E(x) \le r, \xi_E(x) \in \beta\}| = |\{x \in \mathbb{R}^n : 0 < \delta_E(x) \le r, \xi_E(x) \in \beta\}| + |E \cap \beta|,$$

we can write alternatively

$$|\{x \in \mathbb{R}^n : 0 < \delta_E(x) \le r, \xi_E(x) \in \beta\}| = \sum_{i=0}^{n-1} r^{n-i}\omega_{n-i}\Phi_i(E,\beta).$$

This shows that

$$\Phi_i(E,\beta) = \sigma_i C_i(E,\beta), \quad i = 0, \dots, n-1,$$

where $\sigma_i = (n\omega_{n-i})^{-1}\binom{n}{i}$, and hence $C_i(E,\beta)$ is the different normalization of the same curvature measure used in [8] (see Section 4.2, and, e.g., formula (4.2.8)). This remark enables us to use alternatively identities taken from [8].

Remark 2.3. Since $\{x \in \mathbb{R}^n : \delta_{B(0,R)}(x) \le r\} = B(0, R+r)$, then

$$|\{x \in \mathbb{R}^n : \delta_{B(0,R)}(x) \le r\}| = \omega_n \sum_{i=0}^{n} \binom{n}{i} r^{n-i} R^i,$$

so that

$$\Phi_i(B(0,R), \mathbb{R}^n) = \frac{\omega_n}{\omega_{n-i}} \binom{n}{i} R^i.$$

Remark 2.4. If we denote by D the closed unit ball in $\mathbb{R}^n$, we have $\{x \in \mathbb{R}^n : \delta_E(x) \le r\} = E + rD$, so that, by Steiner's decomposition as in [1], Chapter IV, 19.3.6, we obtain the identities

$$\Phi_i(E, \mathbb{R}^n) = \frac{1}{\omega_{n-i}} \binom{n}{i} V_i(E), \quad i = 0, \dots, n,$$

where $V_0(E), \dots, V_n(E)$ are the so-called cross-sectional measures defined for instance in [1], Chapter IV, 19.3.1. In particular

$$V_i(E) = \frac{\omega_n}{\Phi_i(D, \mathbb{R}^n)} \Phi_i(E, \mathbb{R}^n), \quad i = 0, \dots, n. \tag{14}$$

The following result (see e.g. [8], (4.2.19), and [5], Theorem 3.2 and following Remark 3.3) provides an integral representation of $\Phi_i(E, \cdot)$ when ∂E is sufficiently regular.

Proposition 2.1. *Suppose ∂E is a C^2 manifold. Then, if $0 \le i \le n-1$ and β is a Borel set, we have*

$$\Phi_i(E,\beta) = \sigma_i \int_{\partial E \cap \beta} H_{n-i-1}(E,x)\, d\mathscr{H}^{n-1}(x), \tag{15}$$

where $\mathscr{H}^{n-1}$ *is the* $(n-1)$*-dimensional Hausdorff measure, and, if* $1 \le j \le n-1$,

$$H_j(E,x) = \binom{n-1}{j}^{-1} S_j(k_1(E,x),\dots,k_{n-1}(E,x)) \tag{16}$$

is the j-th normalized elementary symmetric function of the principal curvatures $k_1,\dots,k_{n-1}$ *of* ∂E*, in* $x \in \partial E$ *i.e.*

$$S_j(k_1(E,x),\dots,k_{n-1}(E,x)) = \sum_{1 \le i_1 < \dots < i_j \le n-1} k_{i_1}(E,x) \cdots k_{i_j}(E,x).$$

Moreover, we set $H_0 \equiv 1$*. Notice that, by elementary computations, we can write also*

$$\Phi_i(E,\beta) = \frac{1}{(n-i)\omega_{n-i}} \int_{\partial E \cap \beta} S_{n-i-1}(k_1(E,x),\dots,k_{n-1}(E,x))\, d\mathscr{H}^{n-1}(x). \tag{17}$$

The previous result has been improved by Hug in [5]. Indeed, it is possible to generalize the notion of principal curvatures for sets of positive reach, as it is done by Zähle in [11], in such a way that when the set is smooth then generalized and classical notions coincide. By Lebesgue decomposition theorem, if K is a convex body, then for every $i = 0,\dots,n-1$ there exist two measures, the absolutely continuous part (with respect to $\mathscr{H}^{n-1} \llcorner \partial K$) and the singular part, denoted respectively by $\Phi_i^{\mathrm{a}}(K,\cdot)$ and $\Phi_i^{\mathrm{s}}(K,\cdot)$, such that

$$\Phi_i(K,\cdot) = \Phi_i^{\mathrm{a}}(K,\cdot) + \Phi_i^{\mathrm{s}}(K,\cdot),$$

$\Phi_i^{\mathrm{a}}(K,\cdot) \ll \mathscr{H}^{n-1} \llcorner \partial K$, $\Phi_i^{\mathrm{a}}(K,\cdot) \perp \Phi_i^{\mathrm{s}}(K,\cdot)$. Thus Hug, [5] Theorem 3.2, proved that the absolute continuous part of $\Phi_m(K,\cdot)$ can be written in the form

$$\Phi_i^{\mathrm{a}}(K,\beta) = \sigma_i \int_{\partial K \cap \beta} H_{n-i-1}(K,x)\, d\mathscr{H}^{n-1}$$

for every Borel set β of $\mathbb{R}^n$, where $H_{n-i-1}(K,x)$ is defined as in (16), the principal curvatures $k_i(K,\cdot)$, $i = 1,\dots,n-1$, being replaced by suitable generalized curvatures.

The singular part $\Phi_i^{\mathrm{s}}(K,\cdot)$ of $\Phi_i(K,\cdot)$ is represented for any Borel set β by

$$\Phi_i^{\mathrm{s}}(K,\beta) = \sigma_i \int_{\mathscr{N}^{\mathrm{s}}(K)} \chi_\beta(x) \mathbb{H}_{n-i-1}(K,(x,u))\, d\mathscr{H}^{n-1}(x,u),$$

where we refer to [5] for the definition of $\mathbb{H}_{n-i-1}(K,(x,u))$, and $\mathscr{N}^{\mathrm{s}}(K)$.

Moreover Hug proved Remark 3.3 in [5] that $\Phi_i^{\mathrm{a}}(K,\beta)$ is concentrated on the set of normal boundary points of K, denoted by $\mathscr{M}(K)$. We refer to [8], Section 2.5, for the notion of normal point; here we restrict ourselves to stress the fact that $\mathscr{H}^{n-1}(\partial K \backslash \mathscr{M}(K)) = 0$, and that $\mathscr{M}(K)$ coincides with the set of points where the local representing function of ∂K is twice differentiable.

It is well known that, if u, v are Lipschitz continuous functions, $u \equiv v$ on E, then $\nabla u \equiv \nabla v$ a.e. on E. The following lemma is in a sense the second-order counterpart for convex bodies of this result. We provide a full proof since we do not know whether it is stated explicitly in the literature.

Lemma 2.1. *Let E be a convex body and $B = B(x_0, R)$ a closed ball of radius R. Then*

$$\Phi_i^{\mathrm{a}}(E \cap B, \partial E \cap \partial B) = \Phi_i(B, \partial E \cap \partial B), \quad i = 1, \dots, n.$$

Proof. To avoid cumbersome notations, we assume that B is centred at the origin. Arguing locally, without loss of generality, we may assume that both ∂E and ∂U are graphs of Lipschitz functions, so that the normals n_E and n_B exist and coincide $\mathscr{H}^{n-1}$-a.e. on $\partial E \cap \partial B$. Thus, there exists a Borel set $M_0 \subset \partial E \cap \partial B$ with $\mathscr{H}^{n-1}((\partial E \cap \partial B) \backslash M_0) = 0$ such that we have $n_E(x) = n_B(x) = x/R$ for $x \in M_0$. Put

$$\beta_0 := \{\lambda x : \lambda \geq 1, x \in M_0\}.$$

Clearly β_0 is a Borel set. We denote now by $(E \cap B)_\varepsilon := \{x : \delta_{E \cap B}(x) \leq \varepsilon\}$ the ε-enlargement of $E \cap B$. We have

$$\partial(E \cap B)_\varepsilon \cap \beta_0 = \left(1 + \frac{\varepsilon}{R}\right)(\partial(E \cap B) \cap \beta_0) = \left(1 + \frac{\varepsilon}{R}\right) M_0. \tag{18}$$

Indeed, if $x \in \partial(E \cap B)_\varepsilon \cap \beta_0$ then we can write $x = \lambda\eta$, with $\lambda \geq 1$ and $\eta \in M_0$. If we prove that $\lambda = 1 + \frac{\varepsilon}{R}$, then it follows that $x \in \left(1 + \frac{\varepsilon}{R}\right)(\partial(E \cap B) \cap \beta_0)$, since $M_0 \subset \partial(E \cap B)$. Now, by convexity, and keeping into account that $n_E(\eta) = n_B(\eta)$, we get $\eta = \xi_{E \cap B}(\lambda\eta)$, and hence $\varepsilon = |x - \xi_{E \cap B}(x)| = |x - \xi_{E \cap B}(\lambda\eta)| = |\lambda\eta - \eta| = (\lambda - 1)R$ that yields the first inclusion. The same argument applies to prove the reverse inclusion. Thus (18) is proved.

By [3], the boundary of $(E \cap B)_\varepsilon$ is of class $\mathscr{C}^{1,1}$. By (18), if $x \in \partial(E \cap B)_\varepsilon \cap \beta_0$, then $n_{(E \cap B)_\varepsilon}(x) = x/(R + \varepsilon)$. But now the normal $n_{(E \cap B)_\varepsilon}$ is Lipschitz continuous and hence the curvatures of $(E \cap B)_\varepsilon$ coincide $\mathscr{H}^{n-1}$-a.e. on β_0 with those of the ball $\left(1 + \frac{\varepsilon}{R}\right)B$. Thus, keeping into account that (17) still holds for $(E \cap B)_\varepsilon$ (see [3], Theorem 5.5), we get

$$\begin{aligned}
\Phi_i((E \cap B)_\varepsilon, \beta_0) &= \frac{S_{n-i-1}((R+\varepsilon)^{-1}, \dots, (R+\varepsilon)^{-1})}{(n-i)\omega_{n-i}} \mathscr{H}^{n-1}(\partial(E \cap B)_\varepsilon \cap \beta_0) \\
&= \left(\frac{R}{R+\varepsilon}\right)^{n-i-1} \frac{S_{n-i-1}(R^{-1}, \dots, R^{-1})}{(n-i)\omega_{n-i}} \mathscr{H}^{n-1}(\partial(E \cap B)_\varepsilon \cap \beta_0) \\
&= \left(\frac{R}{R+\varepsilon}\right)^{-i} \frac{S_{n-i-1}(R^{-1}, \dots, R^{-1})}{(n-i)\omega_{n-i}} \mathscr{H}^{n-1}(M_0) = \left(\frac{R}{R+\varepsilon}\right)^{-i} \Phi_i(B, M_0).
\end{aligned}$$

Take now $s > 0$; by generalized Steiner formula, we get

$$(19) \qquad |\{x \in \mathbb{R}^n : \delta_{(E\cap B)_\varepsilon}(x) \le s, \xi_{(E\cap B)_\varepsilon}(x) \in \beta_0\}|$$

$$= \sum_{i=0}^{n} s^{n-i}\omega_{n-i}\Phi_i((E\cap B)_\varepsilon, \beta_0) = \sum_{i=0}^{n} s^{n-i}\omega_{n-i}\left(\frac{R}{R+\varepsilon}\right)^{-i}\Phi_i(B, M_0).$$

On the other hand, by [3], Corollary 4.9,

$$\{x \in \mathbb{R}^n : \delta_{(E\cap B)_\varepsilon}(x) \le s\} = \{x \in \mathbb{R}^n : \delta_{(E\cap B)}(x) \le s + \varepsilon\}.$$

Moreover $\xi_{(E\cap B)_\varepsilon}(x) \in \beta_0$ if and only if $\xi_{E\cap B}(x) \in M_0$.

Indeed, by [3], Corollary 4.9, for every $x \in \mathbb{R}^n$ $\xi_{E\cap B}(\xi_{(E\cap B)_\varepsilon}(x)) = \xi_{E\cap B}(x)$, so that we have but to prove that $\xi_{E\cap B}(\beta_0) = M_0$.

On the other hand, $z \in \beta_0$ if and only if $z = \lambda\bar{z}$, $\lambda \ge 1$, $\bar{z} \in M_0 \subset \partial B$. Then, the assertion follows noticing that $|\bar{z} - \lambda\bar{z}| \le |y - \lambda\bar{z}|$ for every $y \in B$, that yields $\xi_{E\cap B}(z) = \bar{z}$.

Replacing in (19) and using again Steiner formula, we get

$$\sum_{i=0}^{n}(s+\varepsilon)^{n-i}\omega_{n-i}\Phi_i(E\cap B, M_0) = \sum_{i=0}^{n} s^{n-i}\omega_{n-i}\left(\frac{R}{R+\varepsilon}\right)^{-i}\Phi_i(B, M_0).$$

Letting $\varepsilon \to 0+$ it follows from the identity of the two polynomials that

$$\Phi_i(E\cap B, M_0) = \Phi_i(B, M_0), \quad i = 0, \ldots, n.$$

Now, by Remark 3.3 in [5],

$$\Phi_i^{\mathrm{a}}(E\cap B, \partial E\cap\partial B) = \Phi_i^{\mathrm{a}}(E\cap B, M_0) = \Phi_i(E\cap B, M_0)$$

$$= \Phi_i(B, M_0) = \Phi_i(B, \partial E\cap\partial B),$$

since B is smooth. Thus the proof is completed. □

The following definition is standard (see, e.g., [8], Section 2.2).

Definition 2.2. Let K be a convex body. The *normal cone* of K at $x \in \mathbb{R}^n$ is the set

$$(20) \qquad \mathrm{Nor}(K, x) = \{\xi \in \mathbb{R}^n\backslash\{0\} : \xi \text{ is an exterior normal vector}$$

$$\text{of a supporting hyperplane to } K \text{ in } x\} \cup \{0\}.$$

The following result is proved in [8], Theorem 4.2.5, keeping in mind Remark 2.2.

Proposition 2.2. *If K is a convex body, and β is a Borel set in $\mathbb{R}^n$, then we have*

$$\Phi_0(K,\beta) = (n\omega_n)^{-1}\mathscr{H}^{n-1} \llcorner \mathbb{S}^{n-1}\Bigg(\bigcup_{x\in K\cap\beta}\mathrm{Nor}(K,x)\Bigg),$$

where $\mathbb{S}^{n-1}$ is the unit sphere in $\mathbb{R}^n$, and, as usual, $\mathscr{H}^{n-1} \llcorner \mathbb{S}^{n-1}$ denotes the restriction of $\mathscr{H}^{n-1}$ to $\mathbb{S}^{n-1}$.

Again by [8], Theorem 4.5.5, the following identity holds.

Theorem 2.1. *Le K be a convex body, and let β be a Borel set in $\mathbb{R}^n$. If $0 \le m < n$, then*

$$\Phi_m(K,\beta) = \alpha_{n,n-m}^{-1}\int_{A(n,n-m)}\Phi_0(K\cap\Pi,\beta\cap\Pi)\,d\mu_{n-m}(\Pi) \tag{21}$$

$$= \alpha_{n,n-m}^{-1} n\omega_n \int_{A(n,n-m)}\mathscr{H}^{n-1} \llcorner \mathbb{S}^{n-1}\Bigg(\bigcup_{x\in K\cap\Pi\cap\beta}\mathrm{Nor}(K\cap\Pi,x)\Bigg)\,d\mu_{n-m}(\Pi),$$

where $A(n,n-m)$ is the set of $(n-m)$-dimensional affine manifolds in $\mathbb{R}^n$, μ_{n-m} is the canonical measure on $A(n,n-m)$ (see, e.g. [8] *Section 4.5), and*

$$\alpha_{n,n-m} := \frac{\Gamma\left(\frac{n-m+1}{2}\right)\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}.$$

Proof of Theorem II. Recall that the Alexandrov-Fenchel isoperimetric inequality for convex bodies (see, e.g. [1], Chapter IV, 20.2, formula (20), keeping in mind (14)), reads as follows: if F is a convex set, then for $0 \le m < l \le n$

$$\left(\frac{\Phi_l(F,\mathbb{R}^n)}{\Phi_l(D,\mathbb{R}^n)}\right)^{m/l} \le \frac{\Phi_m(F,\mathbb{R}^n)}{\Phi_m(D,\mathbb{R}^n)}. \tag{22}$$

If we choose $l = n$ and $F = E\cap B$, then

$$|E\cap U|^{m/n} = |F|^{m/n} \le \omega_n^{m/n}(\Phi_m(D,\mathbb{R}^n))^{-1}\Phi_m(E\cap B,\mathbb{R}^n).$$

Setting $\beta_{m,n} = \omega_n^{m/n-1}\omega_{n-m}\binom{n}{m}^{-1}$, we get

$$|E\cap U|^{m/n} \le \beta_{m,n}\Phi_m(E\cap B,\mathbb{R}^n) = \beta_{m,n}\Phi_m(E\cap B,\partial(E\cap U)) \tag{23}$$

$$= \beta_{m,n}\{\Phi_m(E\cap B, E\cap\partial U) + \Phi_m(E\cap B,\partial E\cap U)\}$$

$$= \beta_{m,n}\{\Phi_m^{\mathrm{a}}(E\cap B, E\cap\partial U) + \Phi_m^{\mathrm{s}}(E\cap B, E\cap\partial U) + \Phi_m(E\cap B,\partial E\cap U)\}.$$

Notice that, recalling the result by Hug, see [5], we get

$$\Phi_m^{\mathrm{a}}(E\cap U, E\cap\partial U) = \sigma_m \int_{\partial(E\cap U)\cap(E\cap\partial U)} H_{n-m-1}(E\cap B, x)\, d\mathscr{H}^{n-1}(x) \tag{24}$$

$$= \sigma_m \int_{E\cap\partial U} H_{n-m-1}(E\cap U, x)\, d\mathscr{H}^{n-1}(x).$$

Now $E\cap\partial U = (\partial E\cap\partial U)\cup(\mathrm{Int}(E)\cap\partial U)$. If $\xi\in\mathrm{Int}(E)\cap\partial U$, then there exists an open ball $\tilde{U} = \tilde{U}(\xi,\rho)\subset\mathrm{Int}(E)$, and hence $\partial(E\cap U)\cap\tilde{U} = \partial U\cap\tilde{U}$, (since $\tilde{U}\cap((\partial E\cap U)\cup(E\cap\partial U)) = \tilde{U}\cap E\cap\partial U = \tilde{U}\cap\partial U$), so that

$$H_{n-m-1}(E\cap U,\cdot)\equiv r^{-(n-m-1)}$$

on $\mathrm{Int}(E)\cap\partial U$. On the other hand, the same identity holds H^{n-1}-a.e. on $\partial E\cap\partial U$, by Lemma 2.1, so that

$$\Phi_m^{\mathrm{a}}(E\cap B, E\cap\partial U) = \sigma_m \int_{(E\cap\partial U)} H_{n-m-1}(E\cap U, x)\, d\mathscr{H}^{n-1}(x) \tag{25}$$

$$= \frac{1}{n\omega_{n-m}}\binom{n}{m}\mathscr{H}^{n-1}(\partial U\cap E)r^{-(n-m-1)}.$$

Thus, if we apply the divergence theorem in $U\cap E$ to the function $V(y) = y - x_0$, we get (remember that $\partial(U\cap E) = (\partial U\cap E)\cup(\partial E\cap U)$ and that $\nu_{E\cap B} = \nu_B$ $\mathscr{H}^{n-1}$-a.e. in $\partial U\cap E = \partial U\cap\partial(B\cap E)$ and $\nu_{E\cap B} = \nu_E$ $\mathscr{H}^{n-1}$-a.e. in $\partial E\cap U\subset\partial E\cap\partial(B\cap E)$)

$$\begin{aligned}
n|U\cap E| &= \int_{\partial(U\cap E)} \langle y - x_0, \nu_{U\cap E}(y)\rangle\, d\mathscr{H}^{n-1}(y)\\
&= \int_{\partial U\cap E} \langle y - x_0, \nu_U(y)\rangle\, d\mathscr{H}^{n-1}(y) + \int_{U\cap\partial E} \langle y - x_0, \nu_E(y)\rangle\, d\mathscr{H}^{n-1}(y)\\
&= r\mathscr{H}^{n-1}(\partial U\cap E) + \int_{U\cap\partial E} \langle y - x_0, \nu_E(y)\rangle\, d\mathscr{H}^{n-1}(y)\\
&= \frac{n\omega_{n-m}}{\binom{n}{m}} r^{n-m}\Phi_m^{\mathrm{a}}(E\cap B, E\cap\partial U) + \int_{U\cap\partial E} \langle y - x_0, \nu_E(y)\rangle\, d\mathscr{H}^{n-1}(y),
\end{aligned}$$

by identity (25). Notice now that

$$\int_{U\cap\partial E} \langle y - x_0, \nu_E(y)\rangle\, d\mathscr{H}^{n-1}(y) \ge 0. \tag{26}$$

Indeed, $\langle x_0 - y, \nu_E(y)\rangle \le 0$ on ∂E, because $x_0 - y\in E - y$, so that $\langle x_0 - y, \nu_E(y)\rangle$

$\le h_{E-y}(v_E(y))$, where $h_{E-y}(v_E(y))$ is the supporting function of $E - y$ evaluated at $v_E(y)$. Thus, combining the last inequalities we obtain

$$\Phi_m^{\mathrm{a}}(E \cap B, E \cap \partial U) \le \frac{1}{\omega_{n-m}} \binom{n}{m} |U \cap E| r^{m-n}. \tag{27}$$

But, keeping in mind that the center of U lies on ∂E and hence that, by convexity, $|E \cap U| \le \frac{1}{2}|U|$,

$$|U \cap E| = |U \cap E|^{m/n} |U \cap E|^{1-m/n} \le |U \cap E|^{m/n} \left(\frac{1}{2}\right)^{1-m/n} |U|^{1-m/n}$$

$$\le 2^{-1+m/n} |U \cap E|^{m/n} (\omega_n r^n)^{1-m/n} = 2^{-1+m/n} \omega_n^{1-m/n} |U \cap E|^{m/n} r^{n-m},$$

so that

$$\Phi_m^{\mathrm{a}}(E \cap B, E \cap \partial U) \le 2^{-1+m/n} \frac{\omega_n^{1-m/n}}{\omega_{n-m}} \binom{n}{m} |U \cap E|^{m/n}. \tag{28}$$

Replacing in (23) we obtain eventually

$$|E \cap U|^{m/n} \le \omega_n^{m/n-1} \omega_{n-m} \binom{n}{m}^{-1} \{\Phi_m(E \cap B, \partial E \cap U) \tag{29}$$

$$+ \Phi_m^{\mathrm{s}}(E \cap B, E \cap \partial U)\} + 2^{-1+m/n} |U \cap E|^{m/n}.$$

As a consequence we get

$$(1 - 2^{-1+m/n}) |E \cap U|^{m/n} \le \beta_{m,n} \{\Phi_m(E \cap B, \partial E \cap U) + \Phi_m^{\mathrm{s}}(E \cap B, E \cap \partial U)\}$$

and in particular (since $\Phi_m^{\mathrm{s}}(E \cap B, \cdot)$ is supported in $\partial(E \cap B)$), we have

$$|E \cap U|^{m/n} \le \alpha(m,n) \{\Phi_m(E \cap B, \partial E \cap U) + \Phi_m^{\mathrm{s}}(E \cap B, E \cap \partial U)\}$$

$$= \alpha(m,n) \{\Phi_m(E \cap B, \partial E \cap U) + \Phi_m^{\mathrm{s}}(E \cap B, \partial E \cap \partial U)\},$$

where

$$\alpha(m,n) = (1 - 2^{-1+m/n})^{-1} \beta_{m,n} = (1 - 2^{-1+m/n})^{-1} \omega_n^{m/n-1} \omega_{n-m} \binom{n}{m}^{-1}.$$

Recalling now (12), the above inequality can be written as

$$|E \cap U|^{m/n} \le \alpha(m,n) \{\Phi_m(E, U) + \Phi_m^{\mathrm{s}}(E \cap B, \partial E \cap \partial U)\}. \qquad \square$$

If $m = n-1$, the singular term $\Phi^s_m(E \cap B, \partial E \cap \partial B)$ vanishes, and we reobtain the classical relative isoperimetric inequality, since $\Phi^s_{n-1}(E \cap B, \cdot) \equiv 0$. If $m < n-1$, an explicit form of the singular term can be found in [5], Theorem 3.2. Moreover, since $\Phi^s_m(E \cap B, \cdot) \leq \Phi_m(E \cap B, \cdot)$, combining Theorem II with the characterization of curvature measures given in Theorem 2.1 and (17) we obtain the following estimate.

Corollary 2.1. *Let $E \subset \mathbb{R}^n$ be a compact convex set, and let $U = U(x,r)$ be an open Euclidean ball centred at a point $x \in \partial E$. Then, if $m = 0, \dots, n-1$, we have*

$$|E \cap U|^{m/n} \leq \alpha(m,n)\Bigg\{\Phi_m(E,U)$$

$$+ \alpha^{-1}_{n,n-m} n\omega_n \int_{A(n,n-m)} \mathscr{H}^{n-1} \llcorner \mathbb{S}^{n-1}\left(\bigcup_{x \in \partial E \cap \partial U \cap \Pi} \mathrm{Nor}(E \cap B \cap \Pi, x)\right) d\mu_{n-m}(\Pi)\Bigg\}.$$

Remark 2.5. We point out that in general we can not take out one of the two terms at the right hand side of (10). Indeed, take first $m = n-1$, so that, by [8], Theorem 4.2.5, $\Phi_{n-1}(E \cap B, \cdot)$ is proportional to the $\mathscr{H}^{n-1} \llcorner \partial(E \cap B)$. Then $\Phi_{n-1}(E \cap B, \partial U \cap \partial E) = 0$, and hence the left hand side is bounded only by $\Phi_{n-1}(E, U)$. On the other hand, if we take for instance $n \geq 3$ and E is a smooth set such that its boundary ∂E is a portion of a 2-dimensional plane near a fixed point x, then, by (17), $\Phi_m(E, U) = 0$ for $m = 0, \dots, n-2$, when $U = U(x,r)$, r sufficiently small. Thus, in this case, the right hand side reduces to $\Phi^s_m(E \cap B, \partial U \cap \partial E)$.

Remark 2.6. Take $n \geq 2$ and let U be any Euclidean ball centred on ∂E. Suppose ∂E is a smooth manifold that is transversal to ∂U (i.e. suppose the outward normals ν_U to ∂U and ν_E to ∂E are linearly independent on $\partial U \cap \partial E$), so that $\partial U \cap \partial E$ is a smooth $(n-2)$-dimensional manifold. Referring to Federer's notations ([3]), if $x \in \partial U \cap \partial E$, then, by Theorem 4.12 in [3]

$$\mathrm{Nor}(E \cap U, x) = \{\lambda \nu_E(x) + \mu \nu_U(x), \lambda, \mu \geq 0\}.$$

Since $\dim \mathrm{Nor}(E \cap U, x) = 2$, then, using again Federer's notations (see Theorem 4.15 (3) in [3]) $\partial U \cap \partial E \subset (E \cap U)^{(n-2)}$, so that, by Remark 6.14 in [3], $\Phi_m(E \cap B, \partial U \cap \partial E) = 0$ for $m = n, n-1$, and $\Phi_{n-2}(E \cap B, \partial U \cap \partial E) \leq \mathscr{H}^{n-2}(\partial U \cap \partial E)$. Thus, in the case $n = 3$, it follows from (10) and (17) that

$$|E \cap U|^{2/3} \leq \alpha(2,3)\Phi_2(E,U) = \frac{\alpha(2,3)}{3\omega_3}\mathscr{H}^2(\partial E \cap U), \tag{30}$$

$$|E \cap U|^{1/3} \leq \alpha(1,3)\{\Phi_1(E,U) + \mathscr{H}^1(\partial U \cap \partial E)\} \tag{31}$$

$$= \alpha(1,3)\left\{\frac{1}{2\omega_2}\int_{\partial E \cap U} S_1(k_1,k_2)\, d\mathscr{H}^2 + \mathscr{H}^1(\partial U \cap \partial E)\right\},$$

and

$$
\begin{aligned}
(32)\qquad 1 &\le \alpha(0,3)\{\Phi_0(E,U) + \Phi_0(E\cap U, \partial U \cap \partial E)\} \\
&= \alpha(0,3)\left\{ \frac{1}{3\omega_3} \int_{\partial E \cap U} S_2(k_1,k_2)\, d\mathscr{H}^2 \right. \\
&\qquad\qquad \left. + 3\omega_3 \mathscr{H}^2 \llcorner \mathbb{S}^2 \left(\bigcup_{x\in \partial U \cap \partial E} \mathrm{Nor}(E\cap U, x) \right) \right\}.
\end{aligned}
$$

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