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Global Poincaré Inequalities on the Heisenberg Group and Applications

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Abstract Let f be in the localized nonisotropic Sobolev space $W^{1,p}_{\rm loc}(\mathbb{H}^n)$ on the n-dimensional Heisenberg group $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$, where $1 \le p < Q$ and $Q = 2n + 2$ is the homogeneous dimension of $\mathbb{H}^n$. Suppose that the subelliptic gradient is gloablly L^p integrable, i.e., $\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} f|^p du$ is finite. We prove a Poincaré inequality for f on the entire space $\mathbb{H}^n$. Using this inequality we prove that the function f subtracting a certain constant is in the nonisotropic Sobolev space formed by the completion of $C_0^\infty(\mathbb{H}^n)$ under the norm of

$$\left(\int_{\mathbb{H}^n} |f|^{\frac{Qp}{Q-p}}\right)^{\frac{Q-p}{Qp}} + \left(\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} f|^p\right)^{\frac{1}{p}}.$$

We will also prove that the best constants and extremals for such Poincaré inequalities on $\mathbb{H}^n$ are the same as those for Sobolev inequalities on $\mathbb{H}^n$. Using the results of Jerison and Lee on the sharp constant and extremals for L^2 to $L^{\frac{2Q}{Q-2}}$ Sobolev inequality on the Heisenberg group, we thus arrive at the explicit best constant for the aforementioned Poincaré inequality on $\mathbb{H}^n$ when $p = 2$. We also derive the lower bound of the best constants for local Poincaré inequalities over metric balls on the Heisenberg group $\mathbb{H}^n$.

Keywords Heisenberg group, Sobolev inequalities, Poincaré inequalities, best constants
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1 Introduction

Let $\mathbb{H}^n$ be the Heisenberg group $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$, whose group structure is given by

$$(z, t) \cdot (z', t') = (z + z', t + t' + 2\mathrm{Im}(z \cdot \overline{z'})),$$

for any two points (z, t) and (z', t') in $\mathbb{H}^n$.

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The Lie algebra of $\mathbb{H}^n$ is generated by the left invariant vector fields
$$T = \frac{\partial}{\partial t}, \quad X_i = \frac{\partial}{\partial x_i} + 2y_i\frac{\partial}{\partial t}, \quad Y_i = \frac{\partial}{\partial y_i} - 2x_i\frac{\partial}{\partial t},$$
for $i = 1, \dots, n$. These generators satisfy the non-commutative relationship $[X_i, Y_j] = -4\delta_{ij}T$. Moreover, all the commutators of length greater than two vanish, and thus this is a nilpotent, graded, and stratified group of step two.

For each real number $r \in \mathbb{R}$, there is a dilation naturally associated with the Heisenberg group struture which is usually denoted as $\delta_r u = \delta_r(z,t) = (rz, r^2t)$. However, for simplicity we will write ru to denote $\delta_r u$. The Jacobian determinant of δ_r is r^Q, where $Q = 2n+2$ is the homogeneous dimension of $\mathbb{H}^n$.

The anisotropic dilation structure on $\mathbb{H}^n$ introduces a homogeneous norm
$$|u| = |(z,t)| = (|z|^4 + t^2)^{\frac{1}{4}}.$$
With this norm, we can define the Heisenberg ball centered at $u = (z,t)$ with radius R
$$B(u,R) = \{v \in \mathbb{H}^n : |u^{-1}\cdot v| < R\}.$$
The volume of such a ball is $C_Q R^Q$ for some constant depending on Q.

Recall that, on the Heisenberg group $\mathbb{H}^n$, the subelliptic gradient is a vector given by
$$\nabla_{\mathbb{H}^n} f(z,t) = \sum_{k=1}^n (X_k f(z,t))\, X_k + \sum_{k=1}^n (Y_k f(z,t))\, Y_k.$$

It has been known for years that the following Sobolev inequality holds for $f \in C_0^\infty(\mathbb{H}^n)$, i.e., a function with compact support:
$$\left(\int_{\mathbb{H}^n} |f(z,t)|^q dzdt\right)^{\frac{1}{q}} \le D(p,Q)\left(\int_{\mathbb{H}^n} |\bigtriangledown_{\mathbb{H}^n} f(z,t)|^p dzdt\right)^{\frac{1}{p}} \tag{1.1}$$
provided that $1 \le p < Q = 2n+2$ and $\frac{1}{p} - \frac{1}{q} = \frac{1}{Q}$. In the above inequality, we have used $|\bigtriangledown_{\mathbb{H}^n} f|$ to express the (Euclidean) norm of the subelliptic gradient of f:
$$|\bigtriangledown_{\mathbb{H}^n} f| = \sum_{i=1}^n \left((X_i f)^2 + (Y_i f)^2\right)^{\frac{1}{2}}.$$
It is clear that the above inequality is also true for functions in the anisotropic Sobolev space $W_0^{1,p}(\mathbb{H}^n)$ $(p \ge 1)$, where $W_0^{1,p}(\Omega)$ for open set $\Omega \subset \mathbb{H}^n$ is the completion of $C_0^\infty(\Omega)$ under the norm $||f||_{L^p(\Omega)} + ||\bigtriangledown_{\mathbb{H}^n} f||_{L^p(\Omega)}$.

Poincaré inequalities in the subelliptic setting have been studied extensively in recent years. In particular, as a special case of sharp Poincaré inequalities proved for vector fields satisfying Hörmander's condition in [1, 2] for $p > 1$ and in [3, 4] for all $p \ge 1$, we have, on the Heisenberg group,
$$\left(\int_B |f(z,t) - f_B|^q dzdt\right)^{\frac{1}{q}} \le C(p,Q)\left(\int_B |\bigtriangledown_{\mathbb{H}^n} f(z,t)|^p dzdt\right)^{\frac{1}{p}} \tag{1.2}$$
all $1 \le p < Q$ and $q = \frac{pQ}{Q-p}$, where $B \subset \mathbb{H}^n$ is any metric ball and f_B is the integral average of f over B.

One of the main theorems of this paper is to extend the above (1.2) to over the whole space for functions neither necessarily with compact support, nor with global L^p integrability.

Theorem 1.1 *Let $1 \le p < Q$. Then, for any $f \in L^p_{\mathrm{loc}}(\mathbb{H}^n)$ and $|\nabla_{\mathbb{H}^n} f| \in L^p(\mathbb{H}^n)$, there is a unique finite constant f_∞ such that the following inequality holds:*
$$||f - f_\infty||_{Qp/(Q-p)} \le C(p,Q)||\nabla_{\mathbb{H}^n} f||_p, \tag{1.3}$$
where $C(p,Q)$ is a constant independent of f.

The following remarks are in order. In the Euclidean space $\mathbb{R}^n$, the above inequality was proved by Sedov in [5] for $p > 1$ in 1960s and by Hajlasz and Kalamjska with a more elementary proof in [6] for all $1 \le p < n$ in 1995. Recently, Lu and Ou provided another elementary proof in [7] by proving the limit of $f_R = \frac{1}{|B_R|}\int_{B_R} f$ (the integral average of f over the ball B_R) converges to a finite constant as $R \to \infty$, and showing that the constant f_∞ is exactly this limit.

On the Heisenberg group $\mathbb{H}^n$, we will use the following notation. Let $ru = (rz, r^2t)$ denote the dilation on $\mathbb{H}^n$ and given any $u = (z,t)$ set $z^* = \frac{z}{|u|}$, $t^* = \frac{t}{|u|^2}$ and $u^* = (z^*, t^*)$. Thus, for any $u \in \mathbb{H}^n$ and $u \ne 0$, we have $u^* \in \Sigma = \{u \in \mathbb{H}^n : |u| = 1\}$, the Heisenberg sphere.

The main ingredient of proving Theorem 1.1 is the following:

Lemma 1.2 *Suppose $p > \frac{2n}{2n-1}$. Let $f \in W^{1,p}_{\mathrm{loc}}(\mathbb{H}^n)$ and $|\nabla_{\mathbb{H}^n} f| \in L^p(\mathbb{H}^n)$. Then there exists a finite constant f_∞ such that*

$$\lim_{R\to\infty} \frac{1}{|B_R|}\int_{B_R} f(u)du = f_\infty.$$

To prove Lemma 1.2, we need to have the following representation formula, which is of its independent interest:

Lemma 1.3 *Let $0 < R_1 < R_2 < \infty$ and assume $f \in C^1(B_{R_2}\backslash B_{R_1})$. Then*

$$\int_\Sigma f(R_2u^*)d\mu - \int_\Sigma f(R_1u^*)d\mu = \int_{B_{R_2}\backslash B_{R_1}} \frac{1}{|z|^2|u|^Q} < \nabla_{\mathbb{H}^n} f(u), \nabla_{\mathbb{H}^n}\left(\frac{|u|^4}{4}\right) > du.$$

The proof of Lemma 1.3 is rather interesting and very elementary. It adapts ideas from our earlier paper by Cohn and Lu [8] on representation formulas for functions with compact support. Though we do not assume here that the functions f under consideration have compact support, we are able to carry out similar calculations to those given in [9] by carefully using the fundamental theorem of calculus. This proof also applies to groups of Heisenberg type by using the method in [9] and we thus are able to get results similar to that of Lemma 1.3.

To get Lemma 1.2 from Lemma 1.3, we need to assume $p > \frac{2n}{2n-1}$. However, Theorem 1.1 holds for all $1 \le p < Q$ even on more general Carnot groups. This has been done in [10] without using the full strength of Lemma 1.2.

We now recall the result concerning the best constant and extremal functions for L^2 to $L^{\frac{2Q}{Q-2}}$ Sobolev inequality (1.1) on the Heisenberg group $\mathbb{H}^n$ due to Jerison and Lee [11–13].

Though a fairly complete study of sharp constants and extremal functions has been given for classical Sobolev inequalities in Euclidean space, much less is known about sharp constants for Sobolev inequalities for the Heisenberg group than for Euclidean space. In fact, the first major breakthrough came after the works by Jerison and Lee [11–13] on the sharp constants for the Sobolev inequality and extremal functions on the Heisenberg group in conjunction with the solution to the CR Yamabe problem

In [13], the best constant $D(2,Q)$ for the Sobolev inequality (1.1) on $\mathbb{H}^n$ for $p = 2$ was found to be $D(2,Q) = (4\pi)^{-1}n^{-2}\left[\Gamma(n+1)\right]^{\frac{1}{n+1}}$ and it is also shown that all the extremals of (1.1) are obtained by dilations and left translations of the function $K|\left(t + i(|z|^2+1)\right)|^{-n}$. Furthermore, Jerison and Lee proved [11, 12] that the extremals in (1.1) are constant multiples of images under the Cayley transform of extremals for the Yamabe functional on the sphere $\mathbb{S}^{2n+1}$ in $\mathbb{C}^{n+1}$.

We will show that the best constant and extremal functions for the L^2 to $L^{\frac{2Q}{Q-2}}$ global Poincaré inequality (1.3) are the same as those for the Sobolev inequality (1.1), namely, $D(2,Q)$.

More precisely, we have,

Theorem 1.4 *The best constant $C(p,Q)$ in (1.3) is $D(p,Q)$. When $p = 2$, the inequality (1.3) becomes an equality when f are dilations and translations of the function*

$$K|\left(t + i(|z|^2+1)\right)|^{-n}.$$

The proof of Theorem 1.4 relies on the following density theorem of $C_0^\infty(\mathbb{H}^n)$ functions in the space

$$V^{1,p}(\mathbb{H}^n) = \{f : f \in L^p_{\mathrm{loc}}, |\nabla_{\mathbb{H}^n} f| \in L^p(\mathbb{H}^n)\},$$

under the norm

$$||f|| \equiv \left(\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} f(u)|^p du + |f_\infty|^p \right)^{1/p}.$$

This density theorem is of independent interest and we state it as

Theorem 1.5 *Let $\epsilon > 0$ and $f \in V^{1,p}(\mathbb{H}^n)$. Then there exists some function $\phi \in C_0^\infty(\mathbb{H}^n)$ such that*

$$||f - f_\infty - \phi||_{\frac{Qp}{Q-p}} + ||\nabla_{\mathbb{H}^n} f - \nabla_{\mathbb{H}^n} \phi||_p < \epsilon.$$

Moreover, the completion of $C_0^\infty(\mathbb{H}^n)$ under the norm has codimension 1 in $V^{1,p}(\mathbb{H}^n)$.

A natural question is then what are the best constants for the local Poincaré inequality (1.2)? It is easy to get an upper bound for the sharp constant in (1.2) since any bound $C(p,Q)$ will serve the purpose. However, it is more difficult to find a good lower bound. Surprisingly, using the sharp constants of global Poincaré inequality (1.3) (namely those for the Sobolev inequality (1.1) by Theorem 1.4), we can easily get a lower bound for the local Poincaré inequality (1.2).

Theorem 1.6 *Suppose that the local Poincaré inequality (1.2) holds over any ball $B \subset \mathbb{H}^n$, namely,*

$$\left(\int_B |f - f_B|^{\frac{Qp}{Q-p}} \right)^{\frac{Q-p}{Qp}} \le C(p,Q) \left(\int_B |\nabla_{\mathbb{H}^n} f|^p \right)^{\frac{1}{p}}, \tag{1.4}$$

where we have assumed that $f \in W^{1,p}(B)$, $1 \le p < Q$, $u_B = |B|^{-1} \int_B f$. Thus,

$$C(p,Q) \ge D(p,Q).$$

We conclude this introduction with the following remarks. This paper is motivated by results in Euclidean space (see [5, 6] and [7]). We have chosen to present only some of the theorems on the Heisenberg group. The novelty here is to emphasize the method of proofs. The global Poincaré inequalities on more gerenral Carnot groups, global Poincaré inequalities of higher orders, density theorems and inequalities on unbounded exterior domains are all valid. These will be addressed in a forthcoming paper [10].

The organization of the paper is as follows: Section 2 provides the proof of a representation formula on the Heisenberg group $\mathbb{H}^n$ for functions without compact support (Lemma 2.2). Using this lemma we can show that the average of the integral over a ball converges as the radius approaches ∞ (Lemma 2.1). Then we prove the global Poincaré inequality (Theorem 1.1). Section 3 proves the density theorem (Theorem 1.5). In Section 4, we deal with the best constants and extremal functions for global Poincaré inequalities on $\mathbb{H}^n$ and provide a proof of the lower bound of best constants for local Poincaré inequalities over balls in $\mathbb{H}^n$. Finally, in Section 5 (Appendix) we give some remarks concerning global Poincaré inequalities in $\mathbb{R}^n$, and local Poincaré inequalities over balls in $\mathbb{R}^n$.

2 A Representation Formula and Global Poincaré Inequality on $\mathbb{H}^n$

The main purpose of this section is to show the global Poincaré inequality on $\mathbb{H}^n$, namely, Theorem 1.1. To this end, we first prove that the average of the integral of the function over a ball centered at 0 with radius R converges as $R \to \infty$.

Recalling the Heisenberg group $\mathbb{H}^n$, we will use the following notation. Let $ru = (rz, r^2 t)$ denote the dilation on $\mathbb{H}^n$ and, given any $u = (z,t)$, set $z^* = \frac{z}{|u|}$, $t^* = \frac{t}{|u|^2}$ and $u^* = (z^*, t^*)$. For any $u \in \mathbb{H}^n$ and $u \neq 0$, we have $u^* \in \Sigma = \{u \in \mathbb{H}^n : |u| = 1\}$, the Heisenberg sphere.

Lemma 2.1 *Suppose $p > \frac{2n}{2n-1}$. Let $f \in W^{1,p}_{\mathrm{loc}}(\mathbb{H}^n)$ and $|\nabla_{\mathbb{H}^n} f| \in L^p(\mathbb{H}^n)$. Then there exists a finite constant f_∞ such that*

$$\lim_{R \to \infty} \frac{1}{|B_R|} \int_{B_R} f(u) du = f_\infty.$$

To prove Lemma 2.1, we need to have the following:

Lemma 2.2

$$\int_{\Sigma} f(R_2u^*)d\mu - \int_{\Sigma} f(R_1u^*)d\mu = \int_{B_{R_2}\setminus B_{R_1}} \frac{1}{|z|^2|u|^Q} < \nabla_{\mathbb{H}^n} f(u), \nabla_{\mathbb{H}^n}\left(\frac{|u|^4}{4}\right) > du.$$

Proof of Lemma 2.2 Let u^* be a point on the Heisenberg sphere, that is $u^* = (z^*, t^*)$, where $|z^*|^4 + (t^*)^2 = 1$. We first show that the average over the sphere converges. To see this, we consider for $0 < R_1 < R_2$ the following difference using the fundamental theorem of Calculus:

$$\begin{aligned}
&\int_{\Sigma} f(R_2u^*)d\mu - \int_{\Sigma} f(R_1u^*)d\mu \\
&\quad = \int_{R_1}^{R_2}\int_{\Sigma} \frac{d}{dr} f(ru^*)d\mu dr \\
&\quad = \int_{R_1}^{R_2}\int_{\Sigma} \sum_{j=1}^{n}\left(\frac{x_j}{r}\frac{\partial f}{\partial x_j}(ru^*) + \frac{y_j}{r}\frac{\partial f}{\partial y_j}(ru^*)\right) + \frac{2t}{r}\frac{\partial f}{\partial t}(ru^*)d\mu dr \\
&\quad = \int_{\Sigma}\int_{R_1}^{R_2} \sum_{j=1}^{n}\left(\frac{x_j}{r}\frac{\partial f}{\partial x_j}(ru^*) + \frac{y_j}{r}\frac{\partial f}{\partial y_j}(ru^*) + \frac{y_j^2+x_j^2}{|z|^2}\frac{2t}{r}\frac{\partial f}{\partial t}(ru^*)\right)drd\mu,
\end{aligned}$$

where $u = ru^* = (x_1 + iy_1, \ldots, x_n + iy_n, t) = (rz^*, r^2t^*)$.

Rewriting the last expression into a solid integral using the polar coordinates over $\mathbb{H}^n$, we get

$$\begin{aligned}
&\int_{B_{R_2}\setminus B_{R_1}} \frac{1}{|u|^Q}\sum_{j=1}^{n}\left(x_j\frac{\partial f}{\partial x_j}(u) + y_j\frac{\partial f}{\partial y_j}(u) + 2t\left(\frac{y_j^2+x_j^2}{|z|^2}\right)\frac{\partial f}{\partial t}(u)\right)dzdt \\
&= \int_{B_{R_2}\setminus B_{R_1}} \frac{1}{|z|^2|u|^Q}\sum_{j=1}^{n}\left(\left(\frac{\partial f}{\partial x_j} + 2y_j\frac{\partial f}{\partial t}\right)(|z|^2x_j + y_jt) + \left(\frac{\partial f}{\partial y_j} - 2x_j\frac{\partial f}{\partial t}\right)(|z|^2y_j - x_jt)\right)du \\
&\quad - \int_{B_{R_2}\setminus B_{R_1}} \frac{t}{|z|^2|u|^Q}\sum_{j=1}^{n}\left(y_j\frac{\partial f}{\partial x_j} - x_j\frac{\partial f}{\partial y_j}\right)du \\
&= \int_{B_{R_2}\setminus B_{R_1}} \frac{1}{|z|^2|u|^Q}\sum_{j=1}^{n}\big((X_jf)(|z|^2x_j + y_jt) + (Y_jf)(|z|^2y_j - x_jt)\big)du \\
&\quad - \int_{B_{R_2}\setminus B_{R_1}} \frac{t}{|z|^2|u|^Q}\sum_{j=1}^{n}\left(y_j\frac{\partial f}{\partial x_j} - x_j\frac{\partial f}{\partial y_j}\right)du.
\end{aligned}$$

It is easy to calculate $X_j(|u|^4) = 4|z|^2x_j + 4y_jt$ and $Y_j(|u|^4) = 4|z|^2y_j - 4x_jt$. Therefore we have derived

$$\begin{aligned}
\int_{\Sigma} f(R_2u^*)d\mu - \int_{\Sigma} f(R_1u^*)d\mu &= \int_{B_{R_2}\setminus B_{R_1}} \frac{1}{|z|^2|u|^Q} < \nabla_{\mathbb{H}^n} f(u), \nabla_{\mathbb{H}^n}\left(\frac{|u|^4}{4}\right) \\
&\qquad > du - \int_{B_{R_2}\setminus B_{R_1}} \frac{t}{|z|^2|u|^Q}\sum_{j=1}^{n}\left(y_j\frac{\partial f}{\partial x_j} - x_j\frac{\partial f}{\partial y_j}\right)du.
\end{aligned}$$

Thus, the formula of Lemma 2.2 will follow if we prove the assertion that the second integral on the right-hand side in the last equation vanishes. To see this, for each j, let

$$T_jf = y_j\frac{\partial f}{\partial x_j} - x_j\frac{\partial f}{\partial y_j}.$$

Notice that the integrand in the second integral is absolutely integrable. Using the Gauss–Green formula, this integral is equal to

$$-\int_{B_{R_2}\backslash B_{R_1}} \sum_{j=1}^{n} T_j\left(\frac{1}{|z|^2|u|^Q}\right) f du + \int_{\partial(B_{R_2}\backslash B_{R_1})} \frac{1}{|z|^2|u|^Q} f\langle T_j, \nu\rangle d\mu,$$

where ν is the outer unit normal vector to the boundary.

Since T_j annihilates functions of $|z|$, the first term of the above integral vanishes.

We also note that $< T_j, \nu >= 0$ over the boundary of $B_{R_2}\backslash B_{R_1}$, then the assertion follows and so does the lemma.

Note that $| \bigtriangledown_{\mathbb{H}^n} (|u|^4)| = 4|z||u|^2$.

Using the pointwise Schwartz inequality we get

Proposition 2.3

$$\begin{aligned}\left|\int_\Sigma f(R_2u^*)d\mu - \int_\Sigma f(R_1u^*)d\mu\right| &= \left|\int_{B_{R_2}\backslash B_{R_1}} \frac{1}{|z|^2|u|^Q}\left\langle \nabla_{\mathbb{H}^n} f(u), \nabla_{\mathbb{H}^n}\left(\frac{|u|^4}{4}\right)\right\rangle du\right| \\ &\leq \int_{B_{R_2}\backslash B_{R_1}} \frac{1}{|z||u|^{Q-2}}|\nabla_{\mathbb{H}^n} f(u)|du.\end{aligned}$$

Before we come to the proof of Lemma 2.1, we need to use the following result proved by Cohn and Lu [8]:

Proposition 2.4 *Let $\omega_{2n-1} = \frac{2\pi^n}{\Gamma(n)}$ be the surface area of the unit sphere in $\mathbb{C}^n = \mathbb{R}^{2n}$ and, for $\beta > -2n$, let*

$$c_\beta = \int_\Sigma |z^*|^\beta d\mu.$$

Then

$$c_\beta = \frac{\omega_{2n-1}\Gamma(\frac{1}{2})\Gamma(\frac{Q-2+\beta}{4})}{\Gamma(\frac{Q+\beta}{4})}.$$

We are now ready to give the

Proof of Lemma 2.1 We note from Lemma 2.3 that

$$\begin{aligned}\left|\int_\Sigma f(R_2u^*)d\mu - \int_\Sigma f(R_1u^*)d\mu\right| &\leq \int_{B_{R_2}\backslash B_{R_1}} \frac{1}{|z||u|^{Q-2}}|\nabla_{\mathbb{H}^n} f(u)|du \\ &\leq \left(\int_{B_{R_2}\backslash B_{R_1}} |\nabla_{\mathbb{H}^n} f(u)|^p du\right)^{1/p}\left(\int_{B_{R_2}\backslash B_{R_1}} \left(\frac{1}{|z||u|^{Q-2}}\right)^q |du\right)^{1/q},\end{aligned}$$

where $q = \frac{p}{p-1}$.

It is easy to see

$$\left(\int_{B_{R_2}\backslash B_{R_1}} \left(\frac{1}{|z||u|^{Q-2}}\right)^q du\right)^{1/q} = \left(\int_{R_1}^{R_2} r^{(Q-1)(1-q)}dr\right)^{1/q} \cdot \left(\int_\Sigma |z^*|^{-q}d\mu\right)^{1/q} < \infty$$

provided $q < 2n$ by Proposition 2.4, namely $p > \frac{2n}{2n-1}$.

Thus,

$$\left|\int_\Sigma f(R_2u^*)d\mu - \int_\Sigma f(R_1u^*)d\mu\right| \to 0,$$

as $R_1, R_2 \to \infty$.

This shows $\frac{1}{|\Sigma|}\int_\Sigma f(R_2u^*)d\mu$ converges to a finite constant f_∞ as $R \to \infty$ provided $p > \frac{2n}{2n-1}$. This completes the proof of Lemma 2.1.

Finally, we are ready to give the

Proof of Theorem 1.1 Recall from the local Poincaré inequality (1.2) that

$$\left(\int_B |f(z,t) - f_B|^q dzdt\right)^{\frac{1}{q}} \leq C(p,Q)\left(\int_B | \bigtriangledown_{\mathbb{H}^n} f(z,t)|^p dzdt\right)^{\frac{1}{p}},$$

for all $1 \le p < Q$ and $q = \frac{pQ}{Q-p}$, where $B \subset \mathbb{H}^n$ is any metric ball and f_B is the integral average of f over B.

Now, for any $0 < R_1 < R_2$, we have

$$\left(\int_{B_{R_1}} |f(z,t) - f_{B_{R_2}}|^q dzdt\right)^{\frac{1}{q}} \le \left(\int_{B_{R_2}} |f(z,t) - f_{B_{R_2}}|^q dzdt\right)^{\frac{1}{q}}$$
$$\le C(p,Q)\left(\int_{B_{R_2}} |\nabla_{\mathbb{H}^n} f(z,t)|^p dzdt\right)^{\frac{1}{p}}.$$

For any $p > \frac{2n}{2n-1}$, letting $R_2 \to \infty$ in the above inequality, we obtain

$$\left(\int_{B_{R_1}} |f(z,t) - f_\infty|^q dzdt\right)^{\frac{1}{q}} \le C(p,Q)\left(\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} f(z,t)|^p dzdt\right)^{\frac{1}{p}}.$$

We thus have shown Theorem 1.1 by letting $R_1 \to \infty$.

3 A Density Theorem on $\mathbb{H}^n$

As we mentioned earlier, the proof of Theorem 1.4 relies on the density theorem (Theorem 1.5) of $C_0^\infty(\mathbb{H}^n)$ functions in the space

$$V^{1,p}(\mathbb{H}^n) = \{f : f \in L^p_{\text{loc}}(\mathbb{H}^n), |\nabla_{\mathbb{H}^n} f| \in L^p(\mathbb{H}^n)\} \tag{3.1}$$

under the norm

$$||f|| \equiv \left(\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} f(u)|^p du + |f_\infty|^p\right)^{1/p}. \tag{3.2}$$

Lemma 3.1 *The linear space $V^{1,p}(\mathbb{H}^n)$ consisting of functions $f(x)$ satisfying* (3.1) *is a complete Banach space with the norm* (3.2), *where f_∞ is the limit whose existence is guaranteed by Theorem* 1.1.

Proof The proof is rather easy and similar to the one in the Euclidean case given in [7]. We need to prove only the completeness. Let $\{f^i\}$ be a Cauchy sequence. Let $w^i = f^i - (f^i)_\infty$, $i = 1, 2, \ldots$. By Theorem 1.1,

$$||w^i - w^j||_{Qp/(Q-p)} = ||(f^i - f^j) - (f^i - f^j)_\infty||_{Qp/(Q-p)} \le C(p,Q)||\nabla_{\mathbb{H}^n}(f^i - f^j)||_p.$$

Also, $||\nabla_{\mathbb{H}^n} w^i - \nabla_{\mathbb{H}^n} w^j||_p = ||\nabla_{\mathbb{H}^n} f^i - \nabla_{\mathbb{H}^n} f^j||_p$. Thus the sequence of w^i has a limit w such that w is in $L^{Qp/(Q-p)}$. We note each w_i is in $V^{1,p}(\mathbb{H}^n)$ and thus is in $W^{1,p}_{loc}(\mathbb{H}^n)$. Therefore, by a standard argument of integration by parts, we have $\nabla_{\mathbb{H}^n} w$ is in $L^p(\mathbb{H}^n)$, and

$$||w^i - w||_{Qp/(Q-p)} + ||\nabla_{\mathbb{H}^n} w^i - \nabla_{\mathbb{H}^n} w||_p \to 0,$$

as i goes to infinity.

Note that $\lim_{i\to\infty}(f^i)_\infty$ exists and $w_\infty = 0$ and take $f = w + \lim_{i\to\infty}(f^i)_\infty$. Then f is in $V^{1,p}(\mathbb{H}^n)$ and is the limit of the sequence of f^i in $V^{1,p}(\mathbb{H}^n)$ with the norm (3.2). The completeness is then proved.

Lemma 3.2 *Suppose that w is in $W^{1,p}_{loc}(\mathbb{H}^n)$ and satisfies*

$$||w||_{Qp/(Q-p)} + ||\nabla_{\mathbb{H}^n} w||_p < \infty. \tag{3.3}$$

Then, for any $\epsilon > 0$, there is a smooth function $\phi(x) \in C_0^\infty(\mathbb{H}^n)$ such that

$$||w - \phi||_{Qp/(Q-p)} + ||\nabla_{\mathbb{H}^n} w - \nabla_{\mathbb{H}^n} \phi||_p < \epsilon. \tag{3.4}$$

Proof Let $R > 0$ and let $\psi_R(u)$ be a cut-off function satisfying

$$\begin{cases} \psi_R(u) = 1 & \text{if } |u| \le R, \\ \psi_R(u) = 0 & \text{if } |u| \ge 2R, \\ |\psi_R(u)| \le 1 & \text{for all } u, \\ |\nabla_{\mathbb{H}^n} \psi_R(u)| \le C/R & \text{for all } u. \end{cases}$$

By choosing $w_R(u) = w(u)\psi(u/R)$ we can show
$$||w - w_R||_{Qp/(Q-p)} + ||\nabla_{\mathbb{H}^n} w - \nabla_{\mathbb{H}^n} w_R||_p$$
is small provided R is large. This suffices to show Lemma 3.2.

We are now ready to show the density theorem (Theorem 1.5).

Proof of Theorem 1.5 By Theorem 1.1, for every f in $V^{1,p}(\mathbb{H}^n)$ the function $w = f - f_\infty$ satisfies the condition of Lemma 3.2. Thus, for every $\epsilon > 0$, there is a smooth function ϕ with compact support and satisfying $||w - \phi||_{\frac{Qp}{Q-p}} + ||\nabla_{\mathbb{H}^n} w - \nabla_{\mathbb{H}^n} \phi||_p < \epsilon$.

4 Lower Bound for the Local Poincaré Inequality on $\mathbb{H}^n$

In this section, we prove Theorems 1.4 and 1.6. We will use Theorems 1.1 and 1.5 to achieve the goal.

Proof of Theorem 1.4 We first observe that, if $f \in L^{\frac{pQ}{Q-p}}(\mathbb{H}^n)$ and $|\nabla_{\mathbb{H}^n} f| \in L^p(\mathbb{H}^n)$, then $f_\infty = 0$. Therefore, the inequality (1.3) reduces to the Sobolev inequality (1.1). Thus, the best constant $C(p,n)$ in (1.3) is not less than the best constant $D(p,n)$ in the Sobolev inequality (1.1). Using the density Theorem 1.5, functions in $C_0^\infty(R^n)$ are dense in the sense that, for any $\epsilon > 0$ and $f \in V^{1,p}(\mathbb{H}^n)$, there exists some function $\phi \in C_0^\infty(\mathbb{H}^n)$ such that
$$||f - f_\infty - \phi||_{\frac{Qp}{Q-p}} + ||\nabla_{\mathbb{H}^n} f - \nabla_{\mathbb{H}^n} \phi||_p < \epsilon.$$
Since
$$\left(\int_{\mathbb{H}^n} |\phi|^{\frac{Qp}{Q-p}}\right)^{\frac{Q-p}{Qp}} \le D(p,Q)\left(\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} \phi|^p\right)^{\frac{1}{p}},$$
we then conclude that we must have $C(p,Q) = D(p,Q)$.

Clearly, those extremals for the Sobolev inequality (1.1) are also extremals for the global Poincaré inequality (1.3).

Proof of Theorem 1.6 First of all, it is easy to observe that the local Poincaré inequality (1.2) is dilation invariant. Namely, the constant $C(p,Q)$ in (1.2) is independent of the radius R of the ball. For $f \in V^{1,p}(\mathbb{H}^n)$, then enlarging R to ∞ on the right-hand side of (1.2) we get
$$\left(\int_{B_R} |f - f_{B_R}|^{\frac{Qp}{Q-p}}\right)^{\frac{Q-p}{Qp}} \le C(p,Q)\left(\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} f|^p\right)^{\frac{1}{p}}.$$
Taking $R \to \infty$ on the left-hand side in the above inequality, we can get
$$||f - f_\infty||_{Qp/(Q-p)} \le C(p,Q)||\nabla_{\mathbb{H}^n} f||_p$$
with the same $C(p,Q)$ as in (1.2). But the best constant in the above inequality is $D(p,n)$, therefore we conclude that $C(p,n) \ge D(p,n)$ in the local Poincaré inequality (1.2).

Remark While the constant $D(p,Q)$ is clearly not sharp in the local Poincaré inequality (1.2), we have concluded that the sharp constant in (1.2) is at least $D(p,Q)$ by a very elementary argument as shown above without involving any complicated or lengthy calculation. This is the main motivation for providing this theorem here.

5 Appendix: Some Remarks in Euclidean Space

In this section, we make some remarks concerning Sobolev and global Poincaré inequalities in Euclidean space.

It is well known that the following local Poincaré inequality holds over any ball $B \subset R^n$:
$$\left(\int_B |u - u_B|^{\frac{np}{n-p}}\right)^{\frac{n-p}{np}} \le C(p,n)\left(\int_B |\nabla u|^p\right)^{\frac{1}{p}}, \tag{5.1}$$
where we have assumed that $u \in W^{1,p}(B)$, $1 \le p < n$, $u_B = |B|^{-1}\int_B u$ and $C(p,n)$ is a constant depending only on p and n.

We refer to the books [14], [15] and [16] for a thorough study of classical Poincaré and Sobolev inequalities in Euclidean space, while for functions $u \in W_0^{1,p}(R^n)$, the following global Sobolev inequality holds in R^n:

$$\left(\int_{R^n} |u|^{\frac{np}{n-p}}\right)^{\frac{n-p}{np}} \le D(p,n)\left(\int_{R^n} |\nabla u|^p\right)^{\frac{1}{p}}. \tag{5.2}$$

It is a known fact that (5.2) also holds for $u \in W^{1,p}(R^n)$ since $W_0^{1,p}(R^n) = W^{1,p}(R^n)$ and there is an explicit sharp constant $D(p,n)$ in (5.2) due to Talenti [17] and Aubin [18] for $p > 1$, and Federer–Fleming, Fleming–Rishel for $p = 1$ ([19], [20]), where

$$D(p,n) = \pi^{-\frac{1}{2}} m^{-\frac{1}{p}} \left(\frac{p-1}{m-p}\right)^{1-\frac{1}{p}} \left\{\frac{\Gamma(1+\frac{m}{2})\Gamma(m)}{\Gamma(\frac{m}{p})\Gamma(1+m-\frac{m}{p})}\right\}^{\frac{1}{m}},$$

for $1 < p < n$, and in the case $p = 1$, the best constant $C(1,n)$ is the limit of $C(p,n)$ as $p \to 1$, namely, $C(1,n) = \frac{(\Gamma(1+\frac{m}{2}))^{\frac{1}{m}}}{\sqrt{\pi}m}$.

We point out that $u \in W^{1,p}(R^n)$ virtually requires both u and $|\nabla u|$ are L^p-integrable over the entire space R^n.

Having considered (5.1) and (5.2) carefully, some natural questions arise:

Question 1 What is the analogue of the local Poincaré inequality (5.1) over the entire space R^n when we assume $|\nabla u| \in L^p(R^n)$ but only assume $u \in L^p_{\text{loc}}(R^n)$ (instead of $u \in L^p(R^n)$)?

This question has been answered in a number of papers, see [5] for $p > 1$, [6] for all $1 \le p < n$ and also in [7]. Namely the following holds:

Theorem 5.1 *There is a constant $C(p,n)$ such that, for any $f \in L^p_{\text{loc}}(R^n)$ and $|\nabla f| \in L^p(R^n)$, we have a unique constant f_∞ and the following inequality holds*:

$$||f - f_\infty||_{np/(n-p)} \le C(p,n)||\nabla u||_p. \tag{5.3}$$

After we have answered Question 1, we further ask:

Question 2 What are the best constants $C(p,n)$ and the extremals for (5.3)?

The answer to Question 2 is the same as that of (5.2). More explicitly, we have

Theorem 5.2 *The best constants in* (5.3) *are $C(p,n) = D(p,n)$. The inequality* (5.3) *becomes an equality for $p > 1$ when u has the form $f(x) = [a+b|x|^{\frac{p}{p-1}}]^{1-\frac{n}{p}}$, where $|x| = (x_1^2+\cdots+|x_n|^2)^{\frac{1}{2}}$ and a, b are positive constants*; *and the extremals are the characteristic functions of balls* (*we interpret the right-hand side of* (5.3) *as the bounded variation of f when $p = 1$*).

The proof of (5.1) relies on the following density theorem of $C_0^\infty(R^n)$ functions in the space

$$U^{1,p}(R^n) = \{f : f \in L^p_{\text{loc}}, |\nabla f| \in L^p(R^n)\}$$

under the norm

$$||u|| \equiv \left(\int_{R^n} |\nabla u|^p dx + |(u)_\infty|^p\right)^{1/p}.$$

This density theorem was proved in [21], [22], [23], [6] and [7], and we state it here as follows:

Theorem 5.3 *Let $\epsilon > 0$ and $f \in U^{1,p}(R^n)$. Then there exists some function $\phi \in C_0^\infty(R^n)$ such that $||f - f_\infty - \phi||_{\frac{np}{n-p}} + ||\nabla f - \nabla\phi||_p < \epsilon$. Moreover, the completion of $C_0^\infty(R^n)$ under the norm has codimension* 1 *in $U^{1,p}(R^n)$.*

We mention in passing that the sharp constants $C(p,n)$ in the inequality (5.1) are still not known (see some sharp constant result for the Poincaré inequality on rectangles in [24]). It appears that they are more difficult to seek than the best constants in the global Sobolev inequality (5.2). Using the sharp constants derived in Theorem 5.2, we will be able to estimate the lower bound for the local Poincaré inequality (5.1) rather easily.

Theorem 5.4 *Suppose that the local Poincaré inequality* (5.1) *holds over any ball* $B \subset R^n$*, namely,*

$$\left(\int_B |u-u_B|^{\frac{np}{n-p}}\right)^{\frac{n-p}{np}} \le C(p,n)\left(\int_B |\nabla u|^p\right)^{\frac{1}{p}}, \tag{5.4}$$

where we have assumed that $u \in W^{1,p}(B)$*,* $1 \le p < n$*,* $u_B = |B|^{-1}\int_B u$*. Thus,* $C(p,n) \ge D(p,n)$*.*

The ideas of the proofs of Theorems 5.2 and 5.4 are similar to those given in Section 4 on the Heisenberg group and we shall not repeat them here.

References

[1] Lu, G.: Weighted Poincaré and Sobolev inequalities for vector fields satisfying Hörmander's condition and applications. *Revista Mat. Iberoamericana*, **8**, 367–439 (1992)

[2] Lu, G.: The sharp Poincaré inequality for free vector fields: an endpoint result. *Revista Mat. Iberoamericana*, **10**(3), 453–466 (1994)

[3] Franchi, B., Lu, G., Wheeden, R. L.: Representation formulas and weighted Poincaré inequalities for Hörmander vector fields. *Ann. Inst. Fourier* (*Grenoble*), **45**, 577–604 (1995)

[4] Maheux, P., L. Saloff–Coste, L.: Analyse sur les boules d'un opratteur sous-elliptique (French) [Analysis on the balls of a subelliptic operator]. *Math. Ann.*, **303**(4), 713–740 (1995)

[5] Sedov, V. N.: On functions tending to a polynomial at infinity, in Imbedding Theorems and Their Applications. Proc. Sympos. Imbedding Theorems, Baku, (1966), Nauka, Moscow, 204–212 (1970), (in Russian)

[6] Hajlasz, P., Kalamajska, A.: Polynomial asymptotics and approximation of Sobolev functions. *Studia Mathematica*, **113**(1), 55–64 (1995)

[7] Lu, G., Ou, B.: A Poincaré inequality on an entire space and its application to potential fluid flows in space. *Comm. Nonlinear Appl Anal.*, **12**(1), 1–24 (2005)

[8] Cohn, W., Lu, G.: Best constants for Moser-Trudinger inequalities on the Heisenberg group. *Indiana University Mathematics Journal*, **50**(4), 1567–1591 (2001)

[9] Cohn, W., Lu, G.: Best constants for Moser-Trudinger inequalities, Fundamental solutions and one-parameter representation formulas on groups of Heisenberg type. *Acta Mathematica Sinica, English Series*, **18**(2), 375–390 (2002)

[10] Lu, G., Ou, B.: Poincaré inequalities on unbounded domains on stratified nilpotent Lie groups, to appear

[11] Jerison, D., Lee, J. M.: The Yamabe problem on CR manifolds. *J. Diff. Geom.*, **25**, 167–197 (1987)

[12] Jerison, D., Lee, J. M.: Intrinsic CR coordinates and the CR Yamabe problem. *J. Diff. Geom.*, **29**, 303–343 (1989)

[13] Jerison, D., Lee, J. M.: Extremals for the Sobolev inequality on the Heisenberg group and the CR Yamabe problem. *J. Amer. Math. Soc.*, **1**, 1–13 (1988)

[14] Evans, L. C., Gariepy, R. F.: Measure Theory and Fine Properties of Functions, CRC Press, Ann Arbor, 1992

[15] Maz′ya, V. G.: Sobolev spaces, Springer-Verlag, 1985

[16] Ziemer, W.: Weakly differentiable functions, Springer-Verlag, 1989

[17] Talenti, G.: Best constant in Sobolev inequality. *Ann. Mat. Pura Appl.*, **110**, 353–372 (1976)

[18] Aubin, T.: Best constants in the Sobolev imbedding theorem: the Yamabe problem, in Seminar on Differential Geometry, S. T. Yau, ed., Princeton University, Princeton, 173–184, 1982

[19] Federer, H., Fleming, W.: Normal and integral currents. *Annals of Math.*, **72**, 458–520 (1960)

[20] Fleming, W., Rishel, R.: An integral formula for total gradient variation. *Arch. Math.*, **11**, 218–222 (1960)

[21] Sobolev, S. L.: The density of C_0^∞ functions in the $L_p^{(m)}$ space. *Dokl. Akad. Nauk SSSR*, **149**, 40–43 (1963), (in Russian); *English translation*: *Soviet Math. Dokl.*, **4**, 313–316 (1963)

[22] Sobolev, S. L.: The density of C_0^∞ functions in the $L_p^{(m)}$ space. *Sibirsk. Mat. Zh.*, **4**, 673–682 (1963) (in Russian)

[23] Besov, O. V., Il'in V. P., Nikol'skii, S. M.: Integral representations of functions and imbedding theorems, Nauka, Moscow, 1975 (in Russian)

[24] Lu, G., Wheeden, R.: Poincaré inequalities, isoperimetric estimates and representation formulas on product spaces. *Indiana University Math. Jour.*, **47**(1), 123–151 (1998)