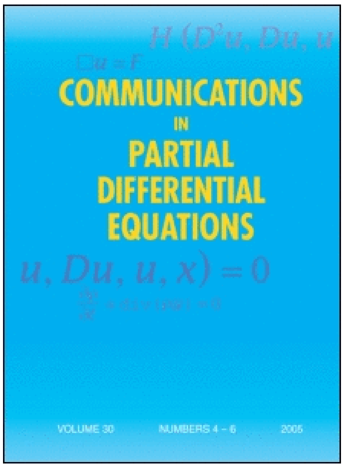

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A PDE Perspective of the Normalized Infinity Laplacian

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A PDE Perspective of the Normalized Infinity Laplacian

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The inhomogeneous normalized infinity Laplace equation was derived from the tug-of-war game in [21] with the positive right-hand-side as a running payoff. The existence, uniqueness and comparison with polar quadratic functions were proved in [21] by the game theory. In this paper, the normalized infinity Laplacian, formally written as $\triangle_\infty^N u = |\nabla u|^{-2} \sum_{i,j=1}^n \partial_{x_i} u \partial_{x_j} u \partial^2 x_i x_j u$, *is defined in a canonical way with the second derivatives in the local maximum and minimum directions, and understood analytically by a dichotomy. A comparison with polar quadratic polynomials property, the counterpart of the comparison with cones property, is proved to characterize the viscosity solutions of the inhomogeneous normalized infinity Laplace equation. We also prove that there is exactly one viscosity solution of the boundary value problem for the infinity Laplace equation*

$$\triangle_\infty^N u = f \quad \textit{with positive } f$$

in a bounded open subset of $\mathbf{R}^n$. *The stability of the inhomogeneous infinity Laplace equation* $\triangle_\infty^N u = f$ *with strictly positive* f *and of the homogeneous equation* $\triangle_\infty^N u = 0$ *by small perturbation of the right-hand-side and the boundary data is established in the last part of the work.*

Our PDE method approach is quite different from those in [21].

Keywords Comparison property; Inhomogeneous equation; Normalized infinity Laplacian; Stability of solutions; Tug-of-war game; Viscosity solutions; Well-posedness.

Mathematics Subject Classification 35J70; 35B35.

Introduction

The homogeneous infinity Laplace equation

$$\triangle_\infty u := \sum_{i,j=1}^n \partial_{x_i} u \partial_{x_j} u \partial^2_{x_i x_j} u = 0,$$

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a quasi-linear degenerate elliptic partial differential equation, has received extensive study since 1960s. To help the reader to trace the development of the theory, the authors would like to just list a few references such as [1–6, 8–11, 13, 15–17, 19]. The viscosity solutions of the infinity Laplace equation $\triangle_\infty u = 0$ in the sense defined by Crandall, Evans and Lions have been adopted due to the absence of classical solutions in general settings. The well-posedness, initially proposed by Hadamard for any partial differential equation, of the homogeneous infinity Laplace equation in bounded domains in $\mathbf{R}^n$ or other settings is well developed. See, for example, the manuscripts [4] and [16]. The existence (see [4]) and uniqueness (see [11]) of a viscosity solution with prescribed growth rate of the homogeneous infinity Laplace equation in an unbounded domain are also known, at least to a large extent.

Study of the singular inhomogeneous normalized infinity Laplace equation $\triangle_\infty^N u = |\nabla u|^{-2}\sum_{i,j=1}^n \partial_{x_i}u\partial_{x_j}u\partial^2_{x_ix_j}u = f$ in this work was inspired by recent works, [21] and [18, 20], on the inhomogeneous infinity Laplace equation by the theory of partial differential equations, and on its normalized counterpart by the game theory.

In [20], the well-posedness problem of the inhomogeneous infinity Laplace equation $\triangle_\infty u = f$ was investigated. A counterexample shows that the well-posedness fails if f changes its sign. On the other hand, under the assumption $\inf_\Omega f > 0$, existence and uniqueness of a viscosity solution of the Dirichlet problem for the inhomogeneous infinity Laplace equation $\triangle_\infty u = f$ in a bounded domain were proved. In addition, a family of special solutions of $\triangle_\infty u = 1$ were found to fully characterize the viscosity solutions of the equation $\triangle_\infty u = f$ via a comparison principle with those cone-like special solutions. Stability of the inhomogeneous infinity Laplace equation $\triangle_\infty u = f$ with $\inf_\Omega f > 0$ and of the homogeneous infinity Laplace equation $\triangle_\infty u = 0$ has been justified with the existence, uniqueness and comparison with the cone-like functions results being employed. Moreover, the restriction $\inf_\Omega f > 0$ can be relaxed to $f > 0$ in Ω in most results in the work except in the stability problem of $\triangle_\infty u = f$.

In the manuscripts by Kohn and Serfaty [18] and by Peres et al. [21], they studied the differential games and their connection to normalized inhomogeneous infinity Laplace equation. In particular, in the tug-of-war game, the underlying equation that the value u of the game verifies is the discrete infinity Laplace equation

$$\Big(\sup_{(x,y)\in E} u(y) + \inf_{(x,y)\in E} u(y)\Big) - 2u(x) = -2f(x),$$

where E is the transition graph and f is the payoff function. Let E^ε be the transition graph that admits all pairs (x, y) with $d(x, y) < \varepsilon$ for any $\varepsilon > 0$. The limiting value $u = \lim_{\varepsilon\to 0} u^\varepsilon$ if existing verifies the normalized infinity Laplace equation $\triangle_\infty^N u = -2f$. This continuum value u is proved to be unique as long as the payoff function f stays strictly positive. A counter-example was provided to show this assumption is necessary. We also refer the reader to the recent works of Barron et al. [7], and of Evans [14], in which nice interpretations of relevance of infinity Laplace equation to two-person game theory with random order of play, rapid switching of states in control problems, etc. are given.

Motivated by the method we used for the non-normalized inhomogeneous infinity Laplacian $\triangle_\infty u$ in [20], we will revisit the normalized infinity Laplacian

$\triangle_\infty^N u$ studied earlier in [21] using game theory. Our approach is based on the PDE method and is thus quite different from those in [21]. We first give a definition of the normalized infinity Laplacian $\triangle_\infty^N u$ as a possibly multi-valued function by applying the Hessian matrix to the maximum and minimum directions. This definition is consistent with the one used in [21] and other places in the literature. The idea using the maximum and minimum directions in the definition appears to be new. The dichotomy that follows inevitably makes the proofs of uniqueness and existence considerably more difficult. However, the comparison with cone-like functions property which characterizes the viscosity solutions of the normalized equation $\triangle_\infty^N$ is neater than that for the non-normalized equation considered in [20] in the sense that we do not require here the assumption $|Du| > 0$ for the normalized equation. We also improve the approach given in [20] so that we can relax the restriction $\inf_\Omega f > 0$ on the right-hand-side of the equation $\triangle_\infty^N u = f$ to $f > 0$ in Ω in most of the theory except the stability for $\triangle_\infty^N u = f$. This improvement also applies to the results in [20]. Our current work also helps to build a further connection between the partial differential equation theory and the differential game theory about the infinity Laplacian. Such a connection has already been recently explored in [7] and [14].

This paper is organized in the following order. In Section 1, we introduce the normalized infinity Laplacian $\triangle_\infty^N u$. We prove a comparison with polar quadratic functions for a viscosity solution of the inhomogeneous infinity Laplace equation in Section 2. In Section 3, we prove a strict comparison principle which is by itself a very useful tool in our approach and which immediately implies the uniqueness theorem. Section 4 is devoted to the proof of the existence theorem. In the last section, we establish the uniform convergence of the viscosity solutions of the uniformly perturbed inhomogeneous infinity Laplace equation to the viscosity solution of the homogeneous infinity Laplace equation.

1. Definition of $\triangle_\infty^N u$ and the Main Results

We adopt the standard notations in analysis and the set theory. For example, $\partial\Omega$ and $\overline{\Omega}$ mean the boundary and closure of a set Ω respectively, while $\partial_{x_i} u$ denotes the partial derivative of u with respect to x_i. $V \subset\subset \Omega$ means V is compactly contained in Ω, i.e., V is a subset of Ω whose closure is also contained in Ω. Also, for two positive numbers λ and μ, $\lambda \ll \mu$ means λ is bounded above by a sufficiently small multiple of μ. $\circ(\varepsilon)$ denotes quantities whose quotients by ε approach 0 as ε does, while $O(\varepsilon)$ denotes quantities that are comparable to ε.

For two vectors $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n) \in \mathbf{R}^n$, $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$ is the inner product of x and y, while $x \otimes y$ is the tensor product yx^t, or $[y_i x_j]_{n\times n}$ in the matrix form, of the vectors x and y. For $x \in \mathbf{R}^n$, $|x|$ denotes the Euclidean norm $\langle x, x\rangle^{\frac{1}{2}}$ of x and $\hat{x} = \frac{x}{|x|}$ denotes the normalized vector for $x \neq 0$. $B_r(x)$ denotes the ball with radius r and centered at x. S^1 denotes the unit sphere of the Euclidean space $\mathbf{R}^n$.

Suppose S is a subset of $\mathbf{R}^n$. A function $f : S \to \mathbf{R}$ is said to be Lipschitz continuous on S if there is a constant L such that

$$|f(x) - f(y)| \le L|x - y|,$$

for any x and y in S. The least of such constants is denoted by $L_f(S)$. If S is an open subset Ω of $\mathbf{R}^n$, we use the symbol $Lip(\Omega)$ to denote the set of all Lipschitz continuous functions on Ω. If instead $S = \partial\Omega$ is the boundary of an open subset Ω of $\mathbf{R}^n$, we use the symbol $Lip_\partial(\Omega)$ to denote the set of all Lipschitz continuous functions on $\partial\Omega$. Ω always denotes an open subset of $\mathbf{R}^n$ and is usually bounded. $C(\Omega)$ denotes the set of continuous functions defined on Ω and $C(\overline{\Omega})$ denotes the set of continuous functions on $\overline{\Omega}$. $C^2(\Omega)$ denotes the set of functions which are continuously twice differentiable on Ω. A smooth function usually means a C^2 function in this paper. If $f \in C(\Omega)$, then $\|f\|_{L^\infty(\Omega)} := \sup_{x\in\Omega} |f(x)|$ denotes the L^∞-norm of f on Ω.

$\mathcal{S}_{n\times n}$ denotes the set of all $n \times n$ symmetric matrices with real entries. We use I to denote the identity matrix in $\mathcal{S}_{n\times n}$. For an element $S \in \mathcal{S}_{n\times n}$, $\|S\|$ denotes its operator norm, namely $\|S\| = \sup_{x\in\mathbf{R}^n\setminus\{0\}} \frac{\langle Sx,x\rangle}{|x|^2}$. And we use $\lambda_1(S), \lambda_2(S), \ldots, \lambda_n(S)$ to denote the eigenvalues of an $n \times n$ symmetric matrix S.

$u \prec_{x_0} \varphi$ means $u - \varphi$ has a local maximum at x_0. In this case, we say φ touches u by above at x_0. Almost always in this paper, $u \prec_{x_0} \varphi$ is understood as $u(x) \le \varphi(x)$ for all $x \in \Omega$ in interest and $u(x_0) = \varphi(x_0)$, as subtracting a constant from φ does not cause any problem in the standard viscosity solution argument applied in the paper. On the other hand, if $\varphi \prec_{x_0} u$, we say φ touches u by below at x_0.

For $u \in C(\Omega)$, $x_0 \in \Omega$, and $r > 0$ with $\overline{B}_r(x_0) \subset \Omega$, we define $g(r) = \max_{|x-x_0|=r} u(x)$ and $h(r) = \min_{|x-x_0|=r} u(x)$. In addition, x_r^+ denotes any point with $|x_r^+ - x_0| = r$ such that $u(x_r^+) = g(r)$, while x_r^- denotes any point with $|x_r^- - x_0| = r$ such that $u(x_r^-) = h(r)$.

If $x_0 \in \Omega$ and $u \in C(\Omega)$ such that u is twice differentiable at x_0, we define the *set of maximum directions* of u at x_0 to be the set

$$E^+(x_0) = \left\{ e \in S^1 : e = \lim_k \frac{x_{r_k}^+ - x_0}{r_k} \text{ for some sequence } r_k \downarrow 0 \right\}$$

and the *set of minimum directions* of u at x_0 to be the set

$$E^-(x_0) = \left\{ e \in S^1 : e = \lim_k \frac{x_{r_k}^- - x_0}{r_k} \text{ for some sequence } r_k \downarrow 0 \right\}.$$

Definition 1.1. If $u \in C(\Omega)$ is twice differentiable at x_0, we define the *upper infinity Laplacian* of u at x_0 to be the set $\triangle_\infty^+ u(x_0) = \{\langle D^2u(x_0)e, e\rangle : e \in E^+(x_0)\}$.

Similarly, the *lower infinity Laplacian* of u at x_0 is defined to be the set $\triangle_\infty^- u(x_0) = \{\langle D^2u(x_0)e, e\rangle : e \in E^-(x_0)\}$.

Proposition 1.2. *Suppose $u \in C(\Omega)$ is twice differentiable at x_0.*

(a) *If $\nabla u(x_0) \neq 0$, then*

$$\triangle_\infty^+ u(x_0) = \triangle_\infty^- u(x_0) = \{|\nabla u(x_0)|^{-2}\langle D^2u(x_0)\nabla u(x_0), \nabla u(x_0)\rangle\}.$$

(b) *If $\nabla u(x_0) = 0$, then both $\triangle_\infty^+ u(x_0)$ and $\triangle_\infty^- u(x_0)$ contain a single element.*

Proof. (a) There exists a positive-valued function ρ with $\rho(r) \to 0$ as $r \downarrow 0$, defined for all small positive numbers r, such that

$$|u(x) - u(x_0) - \nabla u(x_0) \cdot (x - x_0)| \leq \rho(r)r \tag{1.1}$$

for all x with $|x - x_0| = r$.

Take $\tilde{x}_r^+ = x_0 + r\widehat{\nabla u}(x_0)$. Then

$$u(x_0) + \nabla u(x_0) \cdot (x_r^+ - x_0) - \rho(r)r \leq u(x_r^+) \leq u(x_0) + \nabla u(x_0) \cdot (\tilde{x}_r^+ - x_0) + \rho(r)r.$$

The second inequality is due to the choice of $\tilde{x}_r^+$.

So $\nabla u(x_0) \cdot (x_r^+ - \tilde{x}_r^+) \leq 2\rho(r)r$.

On the other hand, the chain of inequalities

$$\begin{aligned} u(x_0) + \nabla u(x_0) \cdot (\tilde{x}_r^+ - x_0) - \rho(r)r &\leq u(\tilde{x}_r^+) \\ \leq u(x_r^+) &\leq u(x_0) + \nabla u(x_0) \cdot (x_r^+ - x_0) + \rho(r)r \end{aligned}$$

implies $\nabla u(x_0) \cdot (x_r^+ - \tilde{x}_r^+) \geq -2\rho(r)r$. So

$$| \nabla u(x_0) \cdot (x_r^+ - \tilde{x}_r^+)| \leq 2\rho(r)r. \tag{1.2}$$

If we denote the angle between $x_r^+ - x_0$ and $\tilde{x}_r^+ - x_0$ by $\alpha(r)$, then the above estimate (1.2) gives

$$| \nabla u(x_0)|(1 - \cos\alpha(r)) \leq 2\rho(r), \tag{1.3}$$

or equivalently

$$\sin\left(\frac{1}{2}\alpha(r)\right) \leq \left(\frac{\rho(r)}{| \nabla u(x_0)|}\right)^{\frac{1}{2}}. \tag{1.4}$$

So $\alpha(r) \to 0$ as $r \downarrow 0$. So for any $e \in E^+(x_0)$, $e = \widehat{\nabla u}(x_0)$ holds. Similarly, $E^-(x_0) = \{-\widehat{\nabla u}(x_0)\}$. Therefore

$$\triangle_\infty^+ u(x_0) = \triangle_\infty^- u(x_0) = \{| \nabla u(x_0)|^{-2}\langle D^2u(x_0) \nabla u(x_0), \nabla u(x_0)\rangle\}.$$

(b) If $\nabla u(x_0) = 0$, we denote $D^2u(x_0)$ by S. Clearly,

$$u(x) = u(x_0) + \frac{1}{2}\langle S(x - x_0), x - x_0\rangle + \circ(|x - x_0|^2). \tag{1.5}$$

Let $\lambda_{\max}(S)$ and $\lambda_{\min}(S)$ be the maximum and minimum eigenvalues of S. It is not difficult to provide a proof similar to that of (a) to show that $E^+(x_0)$ and $E^-(x_0)$ are precisely the sets of normalized eigenvectors corresponding to $\lambda_{\max}(S)$ and $\lambda_{\min}(S)$ respectively.

So $\triangle_\infty^+ u(x_0) = \{\lambda_{\max}(S)\}$ and $\triangle_\infty^- u(x_0) = \{\lambda_{\min}(S)\}$. □

From now on, we do not distinguish $\triangle_\infty^+ u(x_0)$ or $\triangle_\infty^- u(x_0)$ from its single element. It is implied by the above proof that $\triangle_\infty^+ u(x_0) \geq \triangle_\infty^- u(x_0)$.

Definition 1.3. Suppose $u \in C(\Omega)$ is twice differentiable at x_0. We define the *normalized infinity Laplacian* of u at x_0 to be the closed interval

$$\triangle_\infty^N u(x_0) = \left[\, \triangle_\infty^- u(x_0), \triangle_\infty^+ u(x_0)\right], \tag{1.6}$$

and if $\triangle_\infty^N u(x_0)$ contains only one real number, we do not distinguish $\triangle_\infty^N u(x_0)$ from its single element.

Remark 1.4. If u is twice differentiable at x_0 with $\nabla u(x_0) \neq 0$, clearly

$$\triangle_\infty^N u(x_0) = |\nabla u(x_0)|^{-2} \langle D^2 u(x_0) \nabla u(x_0), \nabla u(x_0)\rangle \tag{1.7}$$

which is the usual normalized infinity Laplacian.

If $u \in C^2(\Omega)$, then $\triangle_\infty^N u(x)$ is defined at every point in Ω.

We define viscosity solutions of the normalized infinity Laplacian as follows.

Definition 1.5. A continuous function u defined in an open subset Ω of $\mathbf{R}^n$ is called a *viscosity sub-solution*, or simply a *sub-solution*, of the partial differential equation $\triangle_\infty^N u(x) = f(x)$ in Ω, if

$$\triangle_\infty^+ \varphi(x_0) \geq f(x_0), \tag{1.8}$$

whenever $u \prec_{x_0} \varphi$ for any $x_0 \in \Omega$ and any C^2 test function φ.

Similarly, u is called a *viscosity super-solution*, or simply a *super-solution*, of the partial differential equation $\triangle_\infty^N u(x) = f(x)$ in Ω, if

$$\triangle_\infty^- \varphi(x_0) \leq f(x_0), \tag{1.9}$$

whenever $\varphi \prec_{x_0} u$ for any $x_0 \in \Omega$ and any C^2 test function φ.

A *viscosity solution*, or simply a *solution*, of the partial differential equation $\triangle_\infty^N u(x) = f(x)$ in Ω is both a viscosity sub-solution and super-solution of the equation.

According to this definition, for a C^2 function u, $\triangle_\infty^N u = f$ simply means $f(x) \in \triangle_\infty^N u(x)$ for every x in consideration.

We will need the concepts of superjets and subjets in our approach.

Definition 1.6. Suppose $u \in C(\Omega)$.

The second-order *superjet* of u at x_0 is defined to be the set

$$J_\Omega^{2,+} u(x_0) = \{(D\varphi(x_0), D^2\varphi(x_0)) : \varphi \text{ is } C^2 \text{ and } u \prec_{x_0} \varphi\},$$

whose closure is defined to be

$$\begin{aligned}\overline{J}_\Omega^{2,+} u(x_0) = \{(p, X) \in \mathbf{R}^n \times \mathcal{S}_{n\times n} : \exists (x_n, p_n, X_n) \in \Omega \times \mathbf{R}^n \times \mathcal{S}_{n\times n} \text{ such that}\\ (p_n, X_n) \in J_\Omega^{2,+} u(x_n) \text{ and } (x_n, u(x_n), p_n, X_n) \to (x_0, u(x_0), p, X)\}.\end{aligned}$$

The second-order *subjet* of u at x_0 is defined to be the set

$$J_{\Omega}^{2,-}u(x_0) = \{(D\varphi(x_0), D^2\varphi(x_0)) : \varphi \text{ is } C^2 \text{ and } \varphi \prec_{x_0} u\},$$

whose closure is defined to be

$$\begin{aligned}\overline{J}_{\Omega}^{2,-}u(x_0) = \{(p, X) \in \mathbf{R}^n \times \mathcal{S}_{n\times n} : \exists (x_n, p_n, X_n) \in \Omega \times \mathbf{R}^n \times \mathcal{S}_{n\times n} \text{ such that} \\ (p_n, X_n) \in J_{\Omega}^{2,-}u(x_n) \text{ and } (x_n, u(x_n), p_n, X_n) \to (x_0, u(x_0), p, X)\}.\end{aligned}$$

The following listed are the main theorems of this paper. The first is a characteristic property of the viscosity solutions of the inhomogeneous infinity Laplace equation.

Theorem 1.7. *Assume $u \in C(\Omega)$ and $f \in C(\Omega)$, where Ω is an open subset of $\mathbf{R}^n$. Then*

$$\triangle_{\infty}^{N} u = f(x)$$

in the viscosity sense in Ω if and only if u enjoys comparison with polar quadratic polynomials in Ω with respect to f.

The concept of the comparison with polar quadratic polynomials property will be made clear in Section 2.

The second theorem is about the Dirichlet problem, or equivalently the boundary value problem, for the inhomogeneous infinity Laplace equation.

Theorem 1.8. *Suppose Ω is a bounded open subset of $\mathbf{R}^n$, $f \in C(\Omega)$ with $f > 0$ and $g \in C(\partial\Omega)$.*

Then there exists a unique $u \in C(\overline{\Omega})$ such that $u = g$ on $\partial\Omega$ and

$$\triangle_{\infty}^{N} u(x) = f(x)$$

in Ω in the viscosity sense.

The last is a connection between the homogeneous and inhomogeneous infinity Laplace equations.

Theorem 1.9. *Let Ω be a bounded open subset of $\mathbf{R}^n$. Suppose $\{g_k\}$ is a sequence of functions in $Lip_{\partial}(\Omega)$ which converges to $g \in Lip_{\partial}(\Omega)$ uniformly on $\partial\Omega$, and $\{f_k\}$ is a sequence of continuous functions on Ω which converges uniformly to 0 in Ω. If, for each k, $u_k \in C(\overline{\Omega})$ is a viscosity solution of the Dirichlet problem*

$$\begin{cases} \triangle_{\infty}^{N} u_k = f_k & \text{in } \Omega \\ u_k = g_k & \text{on } \partial\Omega \end{cases} \tag{1.10}$$

and $u \in C(\overline{\Omega})$ is the unique viscosity solution of the Dirichlet problem

$$\begin{cases} \triangle_{\infty}^{N} u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{1.11}$$

Then u_k converges to u uniformly on $\overline{\Omega}$, i.e.,

$$\sup_{\Omega} |u_k - u| \to 0 \tag{1.12}$$

as $k \to \infty$.

Because of the singularity of the normalized infinity Laplacian $\triangle_\infty^N u$ when $\nabla u = 0$ in the viscosity sense, we need to prove the following lemma which does not automatically follow from the standard viscosity solution theory.

Lemma 1.10. *Assume Ω is an open subset of $\mathbf{R}^n$ and $f \in C(\Omega)$. Λ is an index set.*

(a) *Suppose $u(x) = \sup_{\lambda\in\Lambda} u_\lambda(x) < \infty$, $x \in \Omega$, where $\triangle_\infty^N u_\lambda \geq f$ in Ω in the viscosity sense for every $\lambda \in \Lambda$. Then $\triangle_\infty^N u \geq f$ in Ω in the viscosity sense.*
(b) *Suppose $u(x) = \inf_{\lambda\in\Lambda} u_\lambda(x) > -\infty$, $x \in \Omega$, where $\triangle_\infty^N u_\lambda \leq f$ in Ω in the viscosity sense for every $\lambda \in \Lambda$. Then $\triangle_\infty^N u \leq f$ in Ω in the viscosity sense.*

Proof. Because the proof of (b) is similar to that of (a), we only present the proof of (a).

Suppose $\triangle_\infty^N u \geq f$ in the viscosity sense is not true in Ω. Then there exists a function $\varphi \in C^2(\Omega)$ and a point $x_0 \in \Omega$ such that $u \prec_{x_0} \varphi$ and $\triangle_\infty^+ \varphi(x_0) < f(x_0)$. If we replace φ by φ_δ defined by

$$\varphi_\delta(x) = \varphi(x) + \delta|x - x_0|^2, \tag{1.13}$$

then $u \prec_{x_0} \varphi_\delta$ and

$$\triangle_\infty^+ \varphi_\delta(x_0) = \triangle_\infty^+ \varphi(x_0) + O(\delta) < f(x_0) \tag{1.14}$$

if δ is taken small enough. So we may simply assume in addition that the original test function φ satisfies

$$\varphi(x) \geq u(x) + \delta|x - x_0|^2, \tag{1.15}$$

for some $\delta > 0$.

We claim that $\triangle_\infty^+ \varphi(x) < f(x)$ in an open neighborhood $B_r(x_0)$ of x_0.

In fact, we prove the claim via a dichotomy.

If $\nabla\varphi(x_0) \neq 0$, then $\nabla\varphi(x) \neq 0$ in a neighborhood $B_R(x_0)$ of x_0. The continuity of $D^2\varphi$ and of f implies that in a neighborhood $B_r(x_0) \subseteq B_R(x_0)$ of x_0,

$$\triangle_\infty^+ \varphi(x) = \langle D^2\varphi(x)\widehat{D\varphi}(x), \widehat{D\varphi}(x)\rangle < f(x). \tag{1.16}$$

If $\nabla\varphi(x_0) = 0$, then $\lambda_{\max}(D^2\varphi(x_0)) = \triangle_\infty^+ \varphi(x_0) < f(x_0)$. So in a neighborhood $B_r(x_0)$ of x_0, $\lambda_{\max}(D^2\varphi(x)) < f(x)$. As a result,

$$\triangle_\infty^+ \varphi(x) \leq \lambda_{\max}(D^2\varphi(x)) < f(x) \tag{1.17}$$

in this neighborhood $B_r(x_0)$ of x_0.

The claim is proved.

For any ε with $0 < \varepsilon < \delta r^2$, $\exists \lambda \in \Lambda$ such that $u_\lambda(x_0) > u(x_0) - \varepsilon$. Let $\hat{\varphi}(x) = \varphi(x) - \varepsilon$. Then $\hat{\varphi}(x_0) = u(x_0) - \varepsilon < u_\lambda(x_0)$ and, on $\partial B_r(x_0)$,

$$\hat{\varphi}(x) \geq u(x) + \delta r^2 - \varepsilon > u(x) \geq u_\lambda(x). \tag{1.18}$$

So there exists $x_* \in B_r(x_0)$ such that $u_\lambda \prec_{x_*} \hat{\varphi}$. As $\triangle_\infty^N u_\lambda \geq f$ in Ω in the viscosity sense,

$$\triangle_\infty^+ \hat{\varphi}(x_*) \geq f(x_*) \tag{1.19}$$

holds, which is in contradiction with the claim we just have derived,

$$\triangle_\infty^+ \hat{\varphi} = \triangle_\infty^+ \varphi < f \tag{1.20}$$

in $B_r(x_0)$. □

2. Comparison with Polar Quadratic Polynomials

We define the domain $\mathscr{D}(x_0, b)$ of differentiability of a polar quadratic polynomial $\psi(x) = a|x - x_0|^2 + b|x - x_0| + d$, where $a, b, d \in \mathbf{R}$, and $x_0 \in \mathbf{R}^n$, as

$$\mathscr{D}(x_0, b) = \begin{cases} \mathbf{R}^n \backslash \{x_0\}, & \text{if } b \neq 0 \\ \mathbf{R}^n, & \text{if } b = 0. \end{cases} \tag{2.1}$$

If $b = 0$, it is clear that $\triangle_\infty^N \psi(x) = 2a$, for $x \in \mathbf{R}^n$.
If $b \neq 0$, for any $x \in \mathscr{D}(x_0, b)$,

$$D\psi(x) = (2a|x - x_0| + b)(\widehat{x - x_0}) \tag{2.2}$$

and

$$D^2\psi(x) = 2aI + \frac{b}{|x - x_0|}\{I - (\widehat{x - x_0}) \otimes (\widehat{x - x_0})\}. \tag{2.3}$$

If $2a|x - x_0| + b \neq 0$, then $D\psi(x) \neq 0$ and $\triangle_\infty^N \psi(x) = 2a$.
If $2a|x - x_0| + b = 0$, then $D^2\psi(x) = 2a(\widehat{x - x_0}) \otimes (\widehat{x - x_0})$. The eigenvalues of $D^2\psi$ are $\lambda_1 = 2a$ and $\lambda_2 = \cdots = \lambda_n = 0$. In this case, $a \neq 0$ as $b \neq 0$, and

$$\triangle_\infty^N \psi(x) = \begin{cases} [2a, 0], & \text{if } a < 0 \text{ (i.e., } b > 0) \\ [0, 2a], & \text{if } a > 0 \text{ (i.e., } b < 0). \end{cases} \tag{2.4}$$

In particular, we have proven the following lemma:

Lemma 2.1. *$\psi(x) = a|x - x_0|^2 + b|x - x_0| + d$ is a viscosity solution of $\triangle_\infty^N \psi = 2a$ in $\mathscr{D}(x_0, b)$.*

Proof. The fact that a classical solution is a viscosity solution follows easily from the definition of a viscosity solution. □

One also finds that the constant function $x \mapsto 2a$ is the only continuous value of $\triangle_\infty^N \psi$ in $\mathcal{D}(x_0, b)$.

For a continuous function u defined in Ω and any open set $V \subset\subset \Omega$, we use the notation $u \in \operatorname{Max} P(V)$ to denote the fact that u verifies the weak maximum principle

$$\sup_V u = \max_{\partial V} u \tag{2.5}$$

on $\overline{V}$. Similarly, $u \in \operatorname{Min} P(V)$ means u verifies the weak minimum principle

$$\inf_V u = \min_{\partial V} u \tag{2.6}$$

on $\overline{V}$.

We now prove a one-sided comparison principle for viscosity sub-solutions of the inhomogeneous normalized infinity Laplace equation $\triangle_\infty^N u = f$ with continuous right-hand-side.

Theorem 2.2. *Assume $u \in C(\Omega)$ and $f \in C(\Omega)$, where Ω is an open subset of $\mathbf{R}^n$.*

(a) *If $\triangle_\infty^N u \geq f$ in the viscosity sense in Ω, then $u - \psi \in \operatorname{Max} P(V)$ for any polar quadratic polynomial $\psi(x) = a|x - x_0|^2 + b|x - x_0| + d$ and any open set $V \subset\subset \Omega$ with $V \subseteq \mathcal{D}(x_0, b)$, where $a, b, d \in \mathbf{R}$ with $a \leq \frac{1}{2}\inf_V f$ and $x_0 \in \mathbf{R}^n$.*
(b) *If for any polar quadratic polynomial $\psi(x) = a|x - x_0|^2 + b|x - x_0| + d$, where $x_0 \in \mathbf{R}^n$, $a \leq \frac{1}{2}\inf_V f$ and $b, d \in \mathbf{R}$, and for any open set $V \subset\subset \Omega$ with $V \subseteq \mathcal{D}(x_0, b)$, the maximum principle $u - \psi \in \operatorname{Max} P(V)$ holds,*

then $\triangle_\infty^N u(x) \geq f(x)$ in Ω in the viscosity sense.

Remark 2.3. We say u enjoys comparison with polar quadratic polynomials from above in Ω with respect to f if the hypothesis in part (b) of the theorem holds.

Proof. Part (a) in the case $a < \frac{1}{2}\inf_V f$ follows directly from the strict comparison principle, Theorem 3.1, the proof of which is independent of the results in this section, in the next section and the preceding lemma. If $a = \frac{1}{2}\inf_V f$, then

$$\sup_V (u - \psi) \leq \max_{\partial V}(u - \psi) \tag{2.7}$$

is the limiting result of

$$\sup_V (u - \psi_k) \leq \max_{\partial V}(u - \psi_k), \tag{2.8}$$

where $\psi_k(x) = a_k|x - x_0|^2 + b|x - x_0| + d$ and $\{a_k\}$ is a sequence that increases to $a = \frac{1}{2}\inf_V f$.

We turn to the proof of (b). Assume u enjoys comparison with polar quadratic polynomials from above in Ω. Suppose that $\triangle_\infty^N u(x) \geq f(x)$ does not hold in Ω in the viscosity sense. So there exist a C^2 function φ and a point $x_* \in \Omega$ such that $u \prec_{x_*} \varphi$ and $\triangle_\infty^+ \varphi(x_*) < f(x_*)$.

We will construct a polar quadratic polynomial $\psi(x) = a|x - x_0|^2 + b|x - x_0| + d$ and an open neighborhood V of x_* satisfying the assumption specified in (b) such that x_* is a strict maximum point of $u - \psi$ in V. Here x_0 is taken to be different from x_*. Then $u - \psi$ violates the weak maximum principle in the small open neighborhood V of x_*. In order to make x_* a strict maximum point of $u - \psi$ in V, we simply construct ψ so that x_* is a strict maximum point of $\varphi - \psi$ in V. It suffices to make x_* a strict local maximum point of $\varphi - \psi$ if ψ verifies $\nabla\psi(x_*) = \nabla\varphi(x_*)$ and $D^2\psi(x_*) > D^2\varphi(x_*)$.

Without loss of generality, we assume $x_* = 0$. Denote $p = D\varphi(0)$ and $S = D^2\varphi(0)$.

Again, we handle the problem with a dichotomy.

If $\nabla\varphi(0) = 0$, then the largest eigenvalue of $D^2\varphi(0)$, $\lambda_{\max}(D^2\varphi(0))$, verifies

$$\lambda_{\max}(D^2\varphi(0)) < f(0). \tag{2.9}$$

Then there exists an open neighborhood $V = B_r(0)$ of 0 such that $\inf_{x\in V} f(x) > \lambda_{\max}(D^2\varphi(0))$ as f is continuous at 0. If we take $\psi(x) = \varphi(0) + a|x|^2$ where $a = \frac{1}{2}\inf_{x\in V} f(x)$, then $\nabla\psi(0) = 0$ and

$$D^2\psi(0) = 2aI > \lambda_{\max}(D^2\varphi(0))I \geq D^2\varphi(0). \tag{2.10}$$

Then 0 is a strict local maximum point of $\varphi - \psi$ in V. Taking a smaller neighborhood of 0 as V if necessary, we have shown that $u - \psi \notin \operatorname{Max} P(V)$ while still keeping $a \leq \frac{1}{2}\inf_V f$ for the new neighborhood V of 0.

If $p := \nabla\varphi(0) \neq 0$ and $S := D^2\varphi(0)$, then

$$\triangle_\infty^+\varphi(0) = \langle S\hat{p}, \hat{p}\rangle < f(0). \tag{2.11}$$

We take $a \in \mathbf{R}$ so that $\langle S\hat{p}, \hat{p}\rangle < 2a < f(0)$ and $a \neq 0$.

We will take $b \in \mathbf{R}$ and $x_0 \in \mathbf{R}^n\backslash\{0\}$ so that a polar quadratic polynomial $\psi(x) = a|x - x_0|^2 + b|x - x_0| + \varphi(0)$ satisfies $\nabla\psi(0) = p$ and $D^2\psi(0) > S$.

Let $r = |x_0|$. Then $\nabla\psi(0) = -(2ar + b)\hat{x}_0$ and $D^2\psi(0) = \big(2a + \frac{b}{r}\big)I - \frac{b}{r}\hat{x}_0 \otimes \hat{x}_0$. So $\nabla\psi(0) = p$ and $D^2\psi(0) > S$ are equivalent to

$$\begin{cases} -(2ar + b)\hat{x}_0 = p \\ \left(2a + \dfrac{b}{r}\right)I - \dfrac{b}{r}\hat{x}_0 \otimes \hat{x}_0 > S. \end{cases} \tag{2.12}$$

In order to verify the first equality, we take $\hat{x}_0 = -\hat{p}$, and b and $r > 0$ must be taken so that $2ar + b = |p|$.

In order to prove the second condition, we write $x = \alpha\hat{p} + y^1$ with $\langle\hat{p}, y^1\rangle = 0$ and show that

$$\langle D^2\psi(0)x, x\rangle > \langle Sx, x\rangle \tag{2.13}$$

for any $x \in \mathbf{R}^n\backslash\{0\}$.

Clearly,

$$\langle D^2\psi(0)x, x\rangle = 2a\alpha^2 + \left(2a + \frac{b}{r}\right)|y^1|^2. \tag{2.14}$$

On the other hand, if $x = \alpha\hat{p} + y^1 \neq 0$,

$$\begin{aligned}
\langle Sx, x\rangle &= \alpha^2\langle S\hat{p}, \hat{p}\rangle + 2\alpha\langle S\hat{p}, y^1\rangle + \langle Sy^1, y^1\rangle \\
&\leq \alpha^2\langle S\hat{p}, \hat{p}\rangle + \alpha^2\varepsilon|S\hat{p}|^2 + \frac{1}{\varepsilon}|y^1|^2 + \langle Sy^1, y^1\rangle \\
&\leq \alpha^2(\langle S\hat{p}, \hat{p}\rangle + \varepsilon|S\hat{p}|^2) + \left(\frac{1}{\varepsilon} + \|S\|\right)|y^1|^2 \\
&\leq 2a\alpha^2 + \left(\frac{1}{\varepsilon} + \|S\|\right)|y^1|^2 \quad \text{as } \langle S\hat{p}, \hat{p}\rangle < 2a \text{ and } \varepsilon \text{ is small.} \\
&< 2a\alpha^2 + \left(2a + \frac{b}{r}\right)|y^1|^2,
\end{aligned}$$

as $2a + \frac{b}{r} = \frac{|p|}{r} \to +\infty$ as $r \downarrow 0$. So if we take $r > 0$ small enough and $b = |p| - 2ar$, then $\langle Sx, x\rangle < \langle D^2\psi(0)x, x\rangle$ for all $x \in \mathbf{R}^n\backslash\{0\}$.

So 0 is a strict local maximum point of $\varphi - \psi$ and $a < \frac{1}{2}f(0)$. Therefore, there is a small open neighborhood V of 0 such that $a \leq \frac{1}{2}\inf_V f$ and 0 is a strict maximum point of $\varphi - \psi$ due to the continuity of f.

The proof is complete. □

The duality $u = -v$ between the viscosity sub-solutions of $\triangle_\infty^N v = -f$ and the viscosity super-solutions of $\triangle_\infty^N u = f$ leads to a comparison principle for viscosity super-solutions of $\triangle_\infty^N u = f$.

Theorem 2.4. *Assume $u \in C(\Omega)$. Then*

$$\triangle_\infty^N u \leq f(x)$$

in the viscosity sense in Ω if and only if the minimum principle $u - \psi \in \operatorname{Min} P(V)$ holds for any polar quadratic polynomial $\psi(x) = a|x - x_0|^2 + b|x - x_0| + d$, where $x_0 \in \mathbf{R}^n$, $a \geq \frac{1}{2}\sup_V f$, b and $d \in \mathbf{R}$, and for any open set $V \subset\subset \Omega$ with $V \subseteq \mathcal{D}(x_0, b)$.

We say u enjoys comparison with polar quadratic polynomials from below in Ω with respect to f if the condition $u - \psi \in \operatorname{Min} P(V)$ holds for any ψ and V as specified in the theorem. If u enjoys comparison with polar quadratic polynomials from above and from below in Ω, we simply say u enjoys comparison with polar quadratic polynomials in Ω with respect to f.

The two-sided comparison principle with polar quadratic polynomials, Theorem 1.7, follows from the above two one-sided theorems.

Theorem 1.7. *Assume $u \in C(\Omega)$ and $f \in C(\Omega)$, where Ω is an open subset of $\mathbf{R}^n$. Then*

$$\triangle_\infty^N u = f(x)$$

in the viscosity sense in Ω if and only if u enjoys comparison with polar quadratic polynomials in Ω with respect to f.

3. A Strict Comparison Principle

In this section, Ω always denotes a bounded open subset of $\mathbf{R}^n$.

The main theorem in this section is the following strict comparison principle.

Theorem 3.1. *For $j = 1, 2$, suppose $u_j \in C(\overline{\Omega})$ and*

$$\triangle_\infty^N u_1 \leq f_1 \quad \text{and} \quad \triangle_\infty^N u_2 \geq f_2$$

in Ω, where $f_1 < f_2$, and $f_j \in C(\Omega)$.

Then $\sup_\Omega (u_2 - u_1) \leq \max_{\partial\Omega}(u_2 - u_1)$.

Proof. Without the loss of generality, we may assume $u_2 \leq u_1$ on $\partial\Omega$ and intend to prove $u_2 \leq u_1$ in Ω. Furthermore, for any small $\delta > 0$, let $u_\delta = u_2 - \delta$. Then $u_\delta < u_1$ on $\partial\Omega$ and $\triangle_\infty^N u_\delta \geq f_2$ in Ω. If we can show that $u_\delta \leq u_1$ in Ω for every small $\delta > 0$, then it follows that $u_2 \leq u_1$ in Ω. So we may additionally assume $u_2 < u_1$ on $\partial\Omega$ in the following proof.

We apply the sup- and inf-convolution technique here. Take any $A \geq \max\{\|u_1\|_{L^\infty(\Omega)}, \|u_2\|_{L^\infty(\Omega)}\}$. For any sufficiently small real number $\varepsilon > 0$, we take $\delta = 3\sqrt{A\varepsilon}$ and $\Omega_\delta = \{x \in \Omega : dist(x, \partial\Omega) > \delta\}$. We define, on $\mathbf{R}^n$,

$$u_{1,\varepsilon}(x) = \inf_{y\in\Omega}\left(u_1(y) + \frac{1}{2\varepsilon}|x-y|^2\right) \tag{3.1}$$

and

$$u_2^\varepsilon(x) = \sup_{y\in\Omega}\left(u_2(y) - \frac{1}{2\varepsilon}|x-y|^2\right). \tag{3.2}$$

For any $y \in \Omega$ such that $|y - x| \geq 2\sqrt{A\varepsilon}$, $u_1(y) + \frac{1}{2\varepsilon}|x-y|^2 \geq u_1(x)$ holds. So, in Ω_δ,

$$u_{1,\varepsilon}(x) = \inf_{y\in\Omega, |x-y|\leq 2\sqrt{A\varepsilon}}\left(u_1(y) + \frac{1}{2\varepsilon}|x-y|^2\right) = \inf_{|z|\leq 2\sqrt{A\varepsilon}}\left(u_1(x+z) + \frac{1}{2\varepsilon}|z|^2\right), \tag{3.3}$$

as $x + z \in \Omega$ for any $x \in \Omega_\delta$ and $|z| \leq 2\sqrt{A\varepsilon}$. Similarly, for $x \in \Omega_\delta$,

$$u_2^\varepsilon(x) = \sup_{y\in\Omega, |x-y|\leq 2\sqrt{A\varepsilon}}\left(u_2(y) - \frac{1}{2\varepsilon}|x-y|^2\right) = \sup_{|z|\leq 2\sqrt{A\varepsilon}}\left(u_2(x+z) - \frac{1}{2\varepsilon}|z|^2\right). \tag{3.4}$$

Let

$$f_1^\varepsilon(x) = \sup_{x+z\in\Omega, |z|\leq 2\sqrt{A\varepsilon}} f_1(x+z) = \sup_{|z|\leq 2\sqrt{A\varepsilon}} f_1(x+z) \tag{3.5}$$

and

$$f_{2,\varepsilon}(x) = \inf_{x+z\in\Omega, |z|\leq 2\sqrt{A\varepsilon}} f_2(x+z) = \inf_{|z|\leq 2\sqrt{A\varepsilon}} f_2(x+z), \tag{3.6}$$

for $x \in \Omega_\delta$. Clearly, f_1^ε is upper-semicontinuous. It is continuous due to the equicontinuity of the one parameter family of the functions $x \mapsto f_1(x+z)$ in any compact subset of Ω. $f_{2,\varepsilon}$ is continuous for a similar reason.

We notice that, for every z with $|z| \leq 2\sqrt{A\varepsilon}$ and $x \in \Omega_\delta$,

$$\triangle_\infty^N\left(u_1(x+z) + \frac{1}{2\varepsilon}|z|^2\right) \leq f_1(x+z) \leq f_1^\varepsilon(x) \tag{3.7}$$

and

$$\triangle_\infty^N\left(u_2(x+z) - \frac{1}{2\varepsilon}|z|^2\right) \geq f_2(x+z) \geq f_{2,\varepsilon}(x). \tag{3.8}$$

Lemma 1.10 implies that $\triangle_\infty^N u_{1,\varepsilon} \leq f_1^\varepsilon$ and $\triangle_\infty^N u_2^\varepsilon \geq f_{2,\varepsilon}$ in Ω_δ in the viscosity sense. The rest properties of $u_{1,\varepsilon}$ and u_2^ε are summarized in the following proposition, the proof of which is well known (see, for example, [4] Proposition 6.4.).

Proposition 3.2. *$-u_{1,\varepsilon}$ and u_2^ε are semi-convex in $\mathbf{R}^n$. $u_{1,\varepsilon} \leq u_1$ and $u_2^\varepsilon \geq u_2$ in Ω. $u_{1,\varepsilon}$ and u_2^ε converge locally uniformly to u_1 and u_2 in Ω, as $\varepsilon \to 0$. $u_{1,\varepsilon}$ and u_2^ε are both differentiable at the maximum points of $u_2^\varepsilon - u_{1,\varepsilon}$.*

As a result, if we take the value of ε smaller if necessary, then $u_{1,\varepsilon} > u_2^\varepsilon$ on $\partial\Omega_\delta$, $\triangle_\infty^N u_{1,\varepsilon} \leq f_1^\varepsilon$ and $\triangle_\infty^N u_2^\varepsilon \geq f_{2,\varepsilon}$ in Ω_δ, and $f_1^\varepsilon < f_{2,\varepsilon}$ in Ω_δ.

If we can prove $u_2^\varepsilon \leq u_{1,\varepsilon}$ in Ω_δ for any small $\varepsilon > 0$ and $\delta = 3\sqrt{A\varepsilon}$, then $u_2 \leq u_1$ in Ω holds. So we may without loss of generality assume that $-u_1$ and u_2 are semi-convex in $\mathbf{R}^n$.

Suppose $u_1(x_0) < u_2(x_0)$ for some $x_0 \in \Omega$. Without the loss of generality, we may assume that $u_2(x_0) - u_1(x_0) = \max_{\overline{\Omega}}(u_2 - u_1)$. Then $\exists \delta > 0$ such that for any $h \in \mathbf{R}^n$ with $|h| < \delta$, we have $u_1(x_0) < u_2(x_0 + h)$, while $u_2(\cdot + h) < u_1(\cdot)$ in $\Omega \backslash \Omega_\delta$, and $f_2(x+h) > f_1(x)$, $\forall x \in \Omega_\delta$. For any small positive number ε and $h \in \mathbf{R}^n$ with $|h| < \delta$, we define

$$w_{\varepsilon,h}(x,y) = u_2(x+h) - u_1(y) - \frac{1}{2\varepsilon}|x-y|^2, \tag{3.9}$$

$\forall (x,y) \in \overline{\Omega}_\delta \times \overline{\Omega}_\delta$.

Let

$$M_0 = \max_{\overline{\Omega}}(u_2 - u_1), \tag{3.10}$$

$$M_h = \max_{\overline{\Omega}_\delta}(u_2(\cdot + h) - u_1(\cdot)) \tag{3.11}$$

and

$$M_{\varepsilon,h} = \max_{\overline{\Omega}_\delta \times \overline{\Omega}_\delta} w_{\varepsilon,h} = u_2(x_{\varepsilon,h} + h) - u_1(y_{\varepsilon,h}) - \frac{1}{2\varepsilon}|x_{\varepsilon,h} - y_{\varepsilon,h}|^2 \tag{3.12}$$

for some $(x_{\varepsilon,h}, y_{\varepsilon,h}) \in \overline{\Omega}_\delta \times \overline{\Omega}_\delta$. Our assumption implies $M_h > 0$ for all h with $0 \leq |h| < \delta$, and clearly $\lim_{h\to 0} M_h = M_0$.

As the semi-convex functions $u_2(\cdot+h)$ and $-u_1$ are locally Lipschitz continuous, the function M_h is Lipschitz continuous in $h \in \mathbf{R}^n$ with $|h| < \delta$, if δ is taken smaller.

By Lemma 3.1 of [12], we know

$$\lim_{\varepsilon \downarrow 0} M_{\varepsilon,h} = M_h, \tag{3.13}$$

$$\lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon}|x_{\varepsilon,h} - y_{\varepsilon,h}|^2 = 0 \tag{3.14}$$

and

$$\lim_{\varepsilon \downarrow 0}(u_2(x_{\varepsilon,h}+h) - u_1(y_{\varepsilon,h})) = M_h. \tag{3.15}$$

As a result of the second equality, $\lim_{\varepsilon \downarrow 0}|x_{\varepsilon,h} - y_{\varepsilon,h}| = 0$.

As $M_h > 0 \geq \max_{\partial\Omega_\delta}(u_2(\cdot+h) - u_1(\cdot))$, we know $x_{\varepsilon,h}$, $y_{\varepsilon,h} \in \Omega_1$ for some $\Omega_1 \subset\subset \Omega_\delta$ and all small $\varepsilon > 0$.

Theorem 3.2 of [12] implies that there exist $X = X_{\varepsilon,h}$, $Y = Y_{\varepsilon,h} \in \mathcal{S}_{n\times n}$ such that $(\frac{x_{\varepsilon,h}-y_{\varepsilon,h}}{\varepsilon}, X) \in \overline{J}_{\Omega}^{2,+}u_2(x_\varepsilon+h)$, $(\frac{x_{\varepsilon,h}-y_{\varepsilon,h}}{\varepsilon}, Y) \in \overline{J}_{\Omega}^{2,-}u_1(y_\varepsilon)$ and

$$-\frac{3}{\varepsilon}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \frac{3}{\varepsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}. \tag{3.16}$$

In particular, $X \leq Y$.

Again, we solve the problem via a dichotomy.

Case 1. Suppose that $\exists h$ with $|h| < \delta$, and $\varepsilon_k \to 0$ such that $x_{\varepsilon_k,h} \neq y_{\varepsilon_k,h}$. Then it is easy to see that

$$f_2(x_{\varepsilon_k,h}) \leq \left\langle X\left(\widehat{\frac{x_{\varepsilon_k,h} - y_{\varepsilon_k,h}}{\varepsilon_k}}\right), \left(\widehat{\frac{x_{\varepsilon_k,h} - y_{\varepsilon_k,h}}{\varepsilon_k}}\right)\right\rangle \tag{3.17}$$

$$\leq \left\langle Y\left(\widehat{\frac{x_{\varepsilon_k,h} - y_{\varepsilon_k,h}}{\varepsilon_k}}\right), \left(\widehat{\frac{x_{\varepsilon_k,h} - y_{\varepsilon_k,h}}{\varepsilon_k}}\right)\right\rangle \tag{3.18}$$

$$\leq f_1(y_{\varepsilon_k,h}). \tag{3.19}$$

For a subsequence of $\{\varepsilon_k\}$, $x_{\varepsilon_k,h} \to x_h$ and $y_{\varepsilon_k,h} \to y_h$. As $\lim_{\varepsilon\downarrow 0}|x_{\varepsilon_k,h} - y_{\varepsilon_k,h}| = 0$, we know that $x_h = y_h$, which leads to a contradiction with the assumption $f_1(x_h) < f_2(x_h)$.

Case 2. For every $h \in \mathbf{R}^n$ with $|h| < \delta$, $x_{\varepsilon,h} = y_{\varepsilon,h}$ holds for every small $\varepsilon > 0$.

Then $M_{\varepsilon,h} = u_2(x_{\varepsilon,h}+h) - u_1(y_{\varepsilon,h}) = M_h$. We simply write $x_{\varepsilon,h} = y_{\varepsilon,h} = x_h$.

The semi-convexity of $u_2(\cdot+h)$ and $-u_1(\cdot)$ implies that the two functions are differentiable at the maximum point x_h of their sum. The definition of x_h shows that

$$u_2(x_h+h) - u_1(x_h) \geq u_2(y+h) - u_1(x_h) - \frac{1}{2\varepsilon}|x_h - y|^2, \tag{3.20}$$

which in turn implies

$$u_2(x_h + h) \geq u_2(y + h) - \frac{1}{2\varepsilon}|x_h - y|^2, \tag{3.21}$$

for small $\varepsilon > 0$. So $\triangledown u_2(x_h + h) = \triangledown u_1(x_h) = 0$.

For small $h, k \in \mathbf{R}^n$,

$$\begin{aligned} M_h &= u_2(x_h + h) - u_1(x_h) \\ &\geq u_2(x_k + h) - u_1(x_k) \\ &= M_k + u_2(x_k + h) - u_2(x_k + k) \\ &\geq M_k - \circ(|h - k|), \quad \text{as } \triangledown u_2(x_k + k) = 0. \end{aligned}$$

So $DM_h = 0$ a.e. as M_h is Lipschitz continuous, which implies $M_h = M_0$ for all small $h \in \mathbf{R}^n$.

At x_0, either $f_1(x_0) < 0$ or $f_2(x_0) > 0$ holds due to the fact $f_1 < f_2$. Without loss of generality, we assume that $f_2(x_0) > 0$. The proof for the case $f_1(x_0) < 0$ is parallel. So u_2 is ∞-subharmonic in a neighborhood of x_0.

For any h with $|h| < \delta$,

$$u_2(x_0 + h) - u_1(x_0) \leq u_2(x_h + h) - u_1(x_h) = u_2(x_0) - u_1(x_0). \tag{3.22}$$

So $u_2(x_0)$ is a local maximum of u_2. As $\triangle_\infty u_2 \geq 0$, the maximum principle for infinity harmonic functions implies that u_2 is constant near x_0.

So, if we denote the largest eigenvalue of a symmetric matrix M by $\lambda_{\max}(M)$, we have

$$\triangle_\infty^N u_2(x_0) = \lambda_{\max}(D^2 u_2(x_0)) = 0 < f_2(x_0), \tag{3.23}$$

which is a contradiction. $\square$

To prove the uniqueness of viscosity solutions to the Dirichlet problem, we need to prove the following comparison principle which follows fairly easily from the strict comparison principle.

Theorem 3.3. *Suppose $u, v \in C(\overline{\Omega})$ satisfy*

$$\triangle_\infty^N u \geq f(x) \tag{3.24}$$

and

$$\triangle_\infty^N v \leq f(x) \tag{3.25}$$

in the viscosity sense in the domain Ω, where f is a continuous positive function defined on Ω.

Then

$$\sup_{\Omega}(u - v) \leq \max_{\partial\Omega}(u - v). \tag{3.26}$$

Proof. Without loss of generality, we may assume that $u \leq v$ on $\partial\Omega$ and intend to prove $u \leq v$ in Ω.

For every small $\delta > 0$, we take

$$u_\delta(x) = (1+\delta)u(x) - \delta\|u\|_{L^\infty(\partial\Omega)}. \tag{3.27}$$

Then $u_\delta \leq u \leq v$ on $\partial\Omega$, and it is easily checked by the standard viscosity solution theory that

$$\triangle_\infty^N u_\delta(x) = (1+\delta)\,\triangle_\infty^N u(x) \geq (1+\delta)f(x) > f(x) \geq \triangle_\infty^N v(x) \tag{3.28}$$

in Ω in the viscosity sense.

Applying the preceding strict comparison theorem to v and u_δ, we have $u_\delta \leq v$ in Ω for any small $\delta > 0$. Sending δ to 0, we have $u \leq v$ in Ω as desired. □

It is obvious that the theorem is true if the condition $f > 0$ in Ω is replaced by the condition $f < 0$ in Ω.

The uniqueness theorem follows as a direct corollary of the preceding comparison principle.

Theorem 3.4. *Suppose Ω is a bounded open subset of $\mathbf{R}^n$, and u and $v \in C(\overline{\Omega})$ are both viscosity solutions of the inhomogeneous normalized infinity Laplace equation $\triangle_\infty^N w = f(x)$ in Ω, where f is a continuous function defined on Ω such that either $f > 0$ in Ω or $f < 0$ in Ω holds. If, in addition, $u = v$ on $\partial\Omega$, then $u = v$ in Ω.*

The condition that f does not change sign in Ω is indispensable, as a counterexample provided in [21] shows uniqueness fails without such a condition.

4. Proof of Existence Theorem

We prove the existence of a viscosity solution of the normalized infinity Laplace equation by constructing a solution as the infimum of a family of admissible super-solutions.

Theorem 4.1. *Suppose Ω is a bounded open subset of $\mathbf{R}^n$, $f \in C(\Omega)$ with $f > 0$ and $g \in C(\partial\Omega)$.*

Then there exists $u \in C(\overline{\Omega})$ such that $u = g$ on $\partial\Omega$ and

$$\triangle_\infty^N u(x) = f(x)$$

in Ω in the viscosity sense.

Proof. We define the admissible set to be

$$\mathcal{A}_{f,g} = \left\{ v \in C(\overline{\Omega}) : \triangle_\infty^N v \leq f(x) \text{ in } \Omega, \text{ and } v \geq g \text{ on } \partial\Omega \right\}. \tag{4.1}$$

Here the differential inequality $\triangle_\infty^N v(x) \leq f(x)$ is verified in the viscosity sense.

Take

$$u(x) = \inf_{v \in \mathcal{A}_{f,g}} v(x), \quad x \in \overline{\Omega}. \tag{4.2}$$

We may take a constant function which is bigger than the supremum of g on $\partial\Omega$. This constant function is clearly an element of $\mathscr{A}_{f,g}$. So the admissible set $\mathscr{A}_{f,g}$ is nonempty. In addition, we take any $x_0 \notin \Omega$ and $b < 0$ and $d \in \mathbf{R}$, and define a polar quadratic polynomial $\psi_{x_0,bd}(x) = \frac{1}{2}|x - x_0|^2 + b|x - x_0| + d$. For any $v \in \mathscr{A}_{f,g}$, $v - (\inf_\Omega f)\psi_{x_0,bd} \in \operatorname{Min} P(\Omega)$. In particular, $v \in \mathscr{A}_{f,g}$ is locally uniformly bounded below in Ω. So $\inf_{v\in\mathscr{A}_{f,g}} v(x) > -\infty$ for any $x \in \Omega$.

Clearly, $u \geq g$ on $\partial\Omega$. As the infimum of a family of continuous functions, u is upper-semicontinuous on $\overline{\Omega}$.

Lemma 1.10 implies u is a viscosity super-solution of $\triangle_\infty^N u(x) = f(x)$ in Ω.

We next prove $\triangle_\infty^N u(x) \geq f(x)$ in Ω in the viscosity sense. Suppose not, there exists a C^2 function φ and a point $x_0 \in \Omega$ such that

$$u \prec_{x_0} \varphi,$$

and $\triangle_\infty^+ \varphi(x_0) < f(x_0)$.

For any small $\varepsilon > 0$, we define

$$\varphi_\varepsilon(x) = \varphi(x_0) + \nabla\varphi(x_0)\cdot(x - x_0) + \frac{1}{2}\langle D^2\varphi(x_0)(x - x_0), x - x_0\rangle + \varepsilon|x - x_0|^2. \tag{4.3}$$

Clearly, x_0 is a strict local maximum point of $u - \varphi_\varepsilon$ in the sense that $\varphi_\varepsilon(x_0) = u(x_0)$ and $\varphi_\varepsilon(x) > u(x)$ for x near x_0 but $x \neq x_0$.

We claim that $\triangle_\infty^+\varphi_\varepsilon(x) < f(x)$ for all x close to x_0 including x_0 and for ε sufficiently small.

As before, we prove the claim by way of a dichotomy.

If $\nabla\varphi(x_0) \neq 0$, then $\nabla\varphi(x) \neq 0$ in a neighborhood of x_0. So for all x close to x_0,

$$\triangle_\infty^+\varphi_\varepsilon(x) = \langle D^2\varphi_\varepsilon(x)\widehat{\nabla\varphi_\varepsilon}(x), \widehat{\nabla\varphi_\varepsilon}(x)\rangle \tag{4.4}$$

$$= \triangle_\infty^+\varphi(x_0) + O(\varepsilon). \tag{4.5}$$

The continuity of $\triangle_\infty^+\varphi$ and f implies that for $\varepsilon > 0$ sufficiently small and x sufficiently close to x_0, $\triangle_\infty^+\varphi_\varepsilon(x) < f(x)$ as a result of $\triangle_\infty^+\varphi(x_0) < f(x_0)$.

On the other hand, if $\nabla\varphi(x_0) = 0$, then $\lambda_{\max}(D^2\varphi(x_0)) = \triangle_\infty^+\varphi(x_0) < f(x_0)$. As $\lambda_{\max}(D^2\varphi_\varepsilon(x)) \leq \lambda_{\max}(D^2\varphi(x)) + C\varepsilon$, $\lambda_{\max}(D^2\varphi_\varepsilon(x)) < f(x)$ for sufficiently small ε and all x near x_0 due to the continuity of f and the C^2 regularity of φ.

Therefore

$$\triangle_\infty^+\varphi_\varepsilon(x) \leq \lambda_{\max}(D^2\varphi_\varepsilon(x)) < f(x) \tag{4.6}$$

as

$$\triangle_\infty^+\varphi_\varepsilon(x) = \langle D^2\varphi_\varepsilon(x)\widehat{D\varphi_\varepsilon}(x), \widehat{D\varphi_\varepsilon}(x)\rangle \leq \lambda_{\max}(D^2\varphi_\varepsilon(x)) \tag{4.7}$$

if $\nabla\varphi_\varepsilon(x) \neq 0$. The claim is proved.

Fix the value of ε so that the claim holds. We take $\delta > 0$ so small that a newly defined function $\hat{\varphi}(x) = \varphi_\varepsilon(x) - \delta$ satisfies $\hat{\varphi} < u$ in a neighborhood of x_0 which is an open subset of the set $\{x \in \Omega : \triangle_\infty^+\varphi_\varepsilon(x) < f(x)\}$, and $\hat{\varphi} \geq u$ outside this neighborhood of x_0.

Take $\hat{v} = \min\{\hat{\varphi}, u\}$. Then $\hat{v} = \hat{\varphi}$ in a neighborhood of x_0 and $\hat{v} = u$ otherwise. So $\hat{v} \geq g$ on $\partial\Omega$ and $\hat{v} \in C(\overline{\Omega})$.

Suppose a C^2 function ψ satisfies $\psi \prec_z \hat{v}$ for some $z \in \Omega$. Then either $\psi \prec_z \hat{\varphi}$ or $\psi \prec_z u$ depending on the location of z. In either case, $\triangle_\infty^+ \psi(z) \leq f(z)$. So $\hat{v} \in \mathscr{A}_{f,g}$. But $\hat{v} = \hat{\varphi} < u$ in a neighborhood of x_0. This is a contradiction to the definition of u.

So $\triangle_\infty^N u(x) \geq f(x)$ in Ω in the viscosity sense.

We now show $u = g$ on $\partial\Omega$. For any point $z \in \partial\Omega$, and any $\varepsilon > 0$, there is a neighborhood $B_r(z)$ of z such that $|g(x) - g(z)| < \varepsilon$ for all $x \in B_r(z)$. Take a large number $C > 0$ such that $Cr > 2\|g\|_{L^\infty(\partial\Omega)}$. We define

$$v(x) = g(z) + \varepsilon + C|x - z|, \tag{4.8}$$

for $x \in \overline{\Omega}$. For $|x - z| < r$ and $x \in \partial\Omega$, $v(x) \geq g(z) + \varepsilon \geq g(x)$; while for $|x - z| \geq r$ and $x \in \partial\Omega$, $v(x) \geq g(z) + \varepsilon + Cr \geq \|g\|_{L^\infty(\partial\Omega)} \geq g(x)$. In addition, for any $x \in \Omega$, $\nabla v(x) = C\widehat{(x - z)} \neq 0$ and $\triangle_\infty v = 0$. So, $\forall x \in \Omega$, $\triangle_\infty^N v(x) = 0 \leq f(x)$. So $v \in \mathscr{A}_{f,g}$ and $v(z) = g(z) + \varepsilon$. Therefore

$$g(z) \leq u(z) \leq g(z) + \varepsilon, \tag{4.9}$$

$\forall \varepsilon > 0$. So $u(z) = g(z)$, $\forall z \in \partial\Omega$.

Our proof is complete if we can show $u \in C(\overline{\Omega})$.

We know that $\triangle_\infty u \geq 0$ as $\triangle_\infty^N u \geq f > 0$. So u is locally Lipschitz continuous in Ω. So it suffices to show that, $\forall z \in \partial\Omega$,

$$\lim_{x \in \Omega \to z} u(x) = g(z). \tag{4.10}$$

Let us construct another set of admissible functions by defining

$$\mathscr{S}_{f,g} = \{w \in C(\overline{\Omega}) : \triangle_\infty^N w \geq f(x) \text{ in } \Omega, \text{ and } w \leq g \text{ on } \partial\Omega\}. \tag{4.11}$$

Again $\triangle_\infty^N w \geq f(x)$ is satisfied in the viscosity sense. $\mathscr{S}_{f,g}$ is nonempty with a particular element $\psi(x) := C|x - z|^2 - D$, where $z \notin \overline{\Omega}$ and C and D are so large that $2C > \|f\|_{L^\infty(\Omega)}$ and $\psi \leq g$ on $\partial\Omega$, since $\triangle_\infty^N \psi = 2C$.

We take

$$\bar{u}(x) = \sup_{w \in \mathscr{S}_{f,g}} w(x) \tag{4.12}$$

for every $x \in \overline{\Omega}$. Clearly, $\bar{u}$ is lower-semicontinuous in $\overline{\Omega}$ and $\bar{u}(z) \leq g(z)$ for any $z \in \partial\Omega$.

We claim that

$$\liminf_{x \in \Omega \to z} \bar{u}(x) \geq g(z), \quad \forall z \in \partial\Omega. \tag{4.13}$$

$\forall z \in \partial\Omega$ and $\forall \varepsilon > 0$, $\exists r > 0$ such that $|g(x) - g(z)| < \varepsilon$ for all $x \in \Omega \cap B_r(z)$. We take a large number A such that

$$A > \sup_{x \in \Omega} |x - z| \tag{4.14}$$

and a large number $C \geq \|f\|_{L^\infty(\Omega)}$ such that

$$C\{A^2 - (A-r)^2\} \geq 2\|g\|_{L^\infty(\partial\Omega)}. \tag{4.15}$$

One may define

$$w(x) = g(z) - \varepsilon - C\{A^2 - (A - |x-z|)^2\}, \quad x \in \overline{\Omega}. \tag{4.16}$$

For $x \in \Omega$,

$$\nabla w(x) = 2C(|x-z| - A)(\widehat{x-z}) \neq 0 \tag{4.17}$$

and

$$\triangle_\infty^N w(x) = \langle D^2 w(x)\widehat{\nabla w}(x), \widehat{\nabla w}(x)\rangle = 2C \geq \|f\|_{L^\infty(\Omega)} \geq f(x). \tag{4.18}$$

On $\partial\Omega \cap B_r(z)$, $w(x) \leq g(z) - \varepsilon < g(x)$, while on $\partial\Omega \backslash B_r(z)$, $w(x) \leq g(z) - \varepsilon - C\{A^2 - (A-r)^2\} \leq g(z) - 2\|g\|_{L^\infty(\partial\Omega)} \leq g(x)$. So $w \in \mathscr{S}_{f,g}$ and

$$\bar{u}(z) \geq w(z) = g(z) - \varepsilon, \tag{4.19}$$

$\forall \varepsilon > 0$, which implies $\bar{u}(z) \geq g(z)$.

As the supremum of a family of continuous functions on $\overline{\Omega}$, $\bar{u}$ is lower semi-continuous on $\overline{\Omega}$. Therefore

$$\liminf_{x\in\Omega\to z} \bar{u}(x) \geq \bar{u}(z) \geq g(z), \quad \forall z \in \partial\Omega. \tag{4.20}$$

The claim is proved.

The comparison principle, Theorem 3.3, implies $w \leq v$ on $\overline{\Omega}$ for any $w \in \mathscr{S}_{f,g}$ and $v \in \mathscr{A}_{f,g}$. In particular, $\bar{u}(x) \leq u(x)$, $\forall x \in \Omega$. So

$$\liminf_{x\in\Omega\to z} u(x) \geq \liminf_{x\in\Omega\to z} \bar{u}(x) \geq g(z), \quad \forall z \in \partial\Omega. \tag{4.21}$$

On the other hand, the upper semi-continuity of u on $\overline{\Omega}$ implies that

$$\limsup_{x\in\Omega\to z} u(x) \leq u(z) = g(z), \quad \forall z \in \partial\Omega. \tag{4.22}$$

So $\lim_{x\in\Omega\to z} u(x) = g(z)$, $\forall z \in \partial\Omega$. This completes the proof. □

The following theorem is obtained from the above one by considering $v = -u$ and the proof is clear.

Theorem 4.2. *Suppose Ω is a bounded open subset of $\mathbf{R}^n$, $f \in C(\Omega)$ with $f < 0$ and $g \in C(\partial\Omega)$.*

Then there exists $u \in C(\overline{\Omega})$ such that $u = g$ on $\partial\Omega$ and

$$\triangle_\infty^N u(x) = f(x)$$

in Ω in the viscosity sense.

5. A Connection Between the Homogeneous and Inhomogeneous Equations

This section deals with the stability of the inhomogeneous and homogeneous normalized infinity Laplace equations subject to the boundary value condition. The general approach in this section is similar in spirit to a scheme that we developed in our earlier work [20] for the non-normalized infinity Laplace equations. Nevertheless, we have to take extra care of the difficulties arising from the normalized infinity Laplacian.

In this section, Ω again denotes a bounded open subset of $\mathbf{R}^n$.

Lemma 5.1. *Assume $f \in C(\Omega)$ such that either $f > 0$ in Ω or $f < 0$ in Ω. For $j = 1, 2$, suppose $c_j > 0$, $g_j \in C(\partial\Omega)$ and $u_j \in C(\overline{\Omega})$ is the viscosity solution of the Dirichlet problem*

$$\begin{cases} \triangle_\infty^N u_j = c_j f & \text{in } \Omega \\ u_j = g_j & \text{on } \partial\Omega. \end{cases} \tag{5.1}$$

Then

$$\left\| \frac{u_1}{c_1} - \frac{u_2}{c_2} \right\|_{L^\infty(\Omega)} \le \left\| \frac{g_1}{c_1} - \frac{g_2}{c_2} \right\|_{L^\infty(\partial\Omega)}. \tag{5.2}$$

If, in particular $g_1 = g_2 = g \in C(\partial\Omega)$, then

$$\left\| \frac{u_1}{c_1} - \frac{u_2}{c_2} \right\|_{L^\infty(\Omega)} \le \left| \frac{1}{c_1} - \frac{1}{c_2} \right| \|g\|_{L^\infty(\partial\Omega)}. \tag{5.3}$$

Proof. Let

$$v_j = \frac{1}{c_j} u_j.$$

Then v_j is the viscosity solution of the Dirichlet problem

$$\begin{cases} \triangle_\infty^N v_j = f & \text{in } \Omega \\ v_j = \dfrac{1}{c_j} g_j & \text{on } \partial\Omega, \end{cases} \tag{5.4}$$

$j = 1, 2$. Applying the maximum principle, Theorem 3.3, and the remark following it, one obtains

$$\|v_1 - v_2\|_{L^\infty(\Omega)} \le \left\| \frac{g_1}{c_1} - \frac{g_2}{c_2} \right\|_{L^\infty(\partial\Omega)}, \tag{5.5}$$

which implies the desired inequality. □

Lemma 5.2. *Assume $f \in C(\Omega)$ such that either $f > 0$ in Ω or $f < 0$ in Ω. Suppose $c_k \to 0$, g_k, $g \in C(\partial\Omega)$ such that $\|g_k - g\|_{L^\infty(\partial\Omega)} \to 0$, and u_k and u in $C(\overline{\Omega})$ are the* *viscosity solutions of the following Dirichlet problems respectively*

$$\begin{cases} \triangle_\infty^N u_k = (1+c_k)f & \text{in } \Omega \\ u_k = g_k & \text{on } \partial\Omega \end{cases} \tag{5.6}$$

and

$$\begin{cases} \triangle_\infty^N u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{5.7}$$

Then

$$\sup_\Omega (u_k - u) \to 0 \quad \text{as } k \to \infty. \tag{5.8}$$

Proof. The preceding lemma implies

$$\left\| \frac{1}{1+c_k} u_k - u \right\|_{L^\infty(\Omega)} \le \left\| \frac{g_k}{1+c_k} - g \right\|_{L^\infty(\partial\Omega)},$$

which in turn implies

$$\frac{1}{1+c_k} \|u_k - u\|_{L^\infty(\Omega)} - \frac{|c_k|}{1+c_k} \|u\|_{L^\infty(\Omega)} \le \frac{1}{1+c_k} \|g_k - g\|_{L^\infty} + \frac{|c_k|}{1+c_k} \|g\|_{L^\infty(\partial\Omega)}.$$

Therefore

$$\|u_k - u\|_{L^\infty(\Omega)} \le |c_k| (\|u\|_{L^\infty(\Omega)} + \|g\|_{L^\infty(\Omega)}) + \|g_k - g\|_{L^\infty(\partial\Omega)}. \tag{5.9}$$

So $\lim_{k\to\infty} \|u_k - u\|_{L^\infty(\Omega)} = 0$. □

The following perturbation theorem secures the stability of the inhomogeneous normalized infinity Laplace equation with positive or negative right-hand-side.

Theorem 5.3. *Suppose $\{f_k\}$ is a sequence of continuous functions in $C(\Omega)$ which converges uniformly in Ω to $f \in C(\Omega)$ and either $\inf_\Omega f > 0$ or $\sup_\Omega f < 0$. Furthermore, $\{g_k\}$ is a sequence of functions in $C(\partial\Omega)$ which converges uniformly on $\partial\Omega$ to $g \in C(\partial\Omega)$. Assume $u_k \in C(\overline{\Omega})$ is a viscosity solution of the Dirichlet problem*

$$\begin{cases} \triangle_\infty^N u_k = f_k & \text{in } \Omega \\ u_k = g_k & \text{on } \partial\Omega \end{cases} \tag{5.10}$$

while $u \in C(\overline{\Omega})$ is the unique viscosity solution of the Dirichlet problem

$$\begin{cases} \triangle_\infty^N u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{5.11}$$

Then $\sup_\Omega |u_k - u| \to 0$ as $k \to \infty$.

Proof. Without loss of generality, we assume $\inf_\Omega f > 0$.

Let $\varepsilon_k = \sup_\Omega |f_k - f|$. Then $\varepsilon_k \to 0$ as $k \to \infty$ and

$$f(x) - \varepsilon_k \le f_k(x) \le f(x) + \varepsilon_k \quad \text{for all } x \in \Omega. \tag{5.12}$$

To forbid $\varepsilon_k = 0$, we replace ε_k by $\varepsilon_k + \frac{1}{k}$ and still denote the new quantity by ε_k, as the new $\varepsilon_k \to 0$. And now

$$f(x) - \varepsilon_k < f_k(x) < f(x) + \varepsilon_k \quad \text{for all } x \in \Omega. \tag{5.13}$$

Since $\inf_\Omega f > 0$, the sequence $\{c_k\}$ defined by $c_k = \frac{\varepsilon_k}{\inf_\Omega f}$ converges to 0 but never equals 0. So, for all $x \in \Omega$,

$$(1 - c_k)f(x) < f_k(x) < (1 + c_k)f(x), \tag{5.14}$$

as a result of $\varepsilon_k \le c_k f(x)$.

We define u_k^1 and u_k^2 to be the viscosity solutions of the following Dirichlet problems respectively

$$\begin{cases} \triangle_\infty^N u_k^1 = (1 - c_k)f & \text{in } \Omega \\ u_k^1 = g_k & \text{on } \partial\Omega \end{cases} \tag{5.15}$$

and

$$\begin{cases} \triangle_\infty^N u_k^2 = (1 + c_k)f & \text{in } \Omega \\ u_k^2 = g_k & \text{on } \partial\Omega. \end{cases} \tag{5.16}$$

By Theorem 3.1, we know that $u_k^2 \le u_k \le u_k^1$ on $\overline{\Omega}$, since $(1 - c_k)f(x) < f_k(x) < (1 + c_k)f(x)$ for all $x \in \Omega$. In addition, the preceding Lemma 5.2 implies $\sup_\Omega |u_k^j - u| \to 0$ for $j = 1, 2$. Consequently,

$$\sup_\Omega |u_k - u| \to 0$$

as $k \to \infty$. □

Now we are at a position to prove the last main theorem of this paper, Theorem 1.9. For simplicity, we fix the boundary data for the time being.

Theorem 5.4. *Suppose Ω is a bounded open subset of $\mathbf{R}^n$. Suppose $g \in Lip_\partial(\Omega)$ and $\{f_k\}$ is a sequence of continuous functions on Ω which converges uniformly to 0 in Ω. If $u_k \in C(\overline{\Omega})$ is a viscosity solution of the Dirichlet problem*

$$\begin{cases} \triangle_\infty^N u_k = f_k & \text{in } \Omega \\ u_k = g & \text{on } \partial\Omega \end{cases} \tag{5.17}$$

and $u \in C(\overline{\Omega})$ is the unique viscosity solution of the Dirichlet problem

$$\begin{cases} \triangle_\infty^N u = 0 & \textit{in } \Omega \\ u = g & \textit{on } \partial\Omega, \end{cases} \tag{5.18}$$

Then u_k converges to u uniformly on $\overline{\Omega}$, i.e.,

$$\sup_\Omega |u_k - u| \to 0$$

as $k \to \infty$.

Proof. Let $c_k = \|f_k\|_{L^\infty(\Omega)}$ and $\{\varepsilon_k\}$ denote a sequence of positive numbers that converges to 0.

Let u_k^1 and $u_k^2 \in C(\overline{\Omega})$ be the respective viscosity solutions of the following Dirichlet problems

$$\begin{cases} \triangle_\infty^N u_k^1 = -c_k - \varepsilon_k & \text{in } \Omega \\ u_k^1 = g & \text{on } \partial\Omega \end{cases} \tag{5.19}$$

and

$$\begin{cases} \triangle_\infty^N u_k^2 = c_k + \varepsilon_k & \text{in } \Omega \\ u_k^2 = g & \text{on } \partial\Omega. \end{cases} \tag{5.20}$$

By Theorem 3.1, we know that

$$u_k^2 \le u_k \le u_k^1 \quad \text{on } \overline{\Omega}. \tag{5.21}$$

So it suffices to show that $\sup_\Omega |u_k^j - u| \to 0$ as $k \to \infty$, for both $j = 1$ and 2. As the proof of either case of the above convergence implies that of the other, we will only prove $\sup_\Omega (u - u_k^2) \to 0$ as $k \to \infty$. The proof of $\sup_\Omega (u_k^1 - u) \to 0$ follows when one considers $-u_k^1$ and $-u$. In other words, we reduce the problem to the case in which u_k is a viscosity solution of the Dirichlet problem

$$\begin{cases} \triangle_\infty^N u_k = \delta_k & \text{in } \Omega \\ u_k = g & \text{on } \partial\Omega \end{cases} \tag{5.22}$$

where $\delta_k > 0$ and $\delta_k \to 0$ as $k \to \infty$, and our goal is to prove

$$\sup_\Omega (u - u_k) \to 0 \quad \text{as } k \to \infty, \tag{5.23}$$

since $u_k \le u$ is clear. For simplicity, we omitted the superscript 2 in the above and will do the same in the following.

We use argument by contradiction. Suppose there is an $\varepsilon_0 > 0$ and a subsequence $\{u_{k_j}\}$ such that $\sup_\Omega (u - u_{k_j}) \ge \varepsilon_0$, for all $j = 1, 2, 3, \ldots$. In addition, we may assume $\{\delta_{k_j}\}$ is a strictly decreasing sequence that converges to 0.

Without further confusion, we will abuse our notation by using $\{u_k\}$ for the subsequence $\{u_{k_j}\}$ and δ_k for δ_{k_j}.

So we will derive a contradiction from the fact

$$\sup_{\Omega}(u - u_k) \geq \varepsilon_0 > 0, \quad \forall k, \tag{5.24}$$

where $u_k \in C(\overline{\Omega})$ is a viscosity solution of the Dirichlet problem

$$\begin{cases} \triangle_\infty^N u_k = \delta_k & \text{in } \Omega \\ u_k = g & \text{on } \partial\Omega \end{cases} \tag{5.25}$$

and $\{\delta_k\}$ decreases to 0.

By Theorem 3.1, one obtains

$$u_k \leq u_{k+1} \leq u \quad \text{in } \Omega, \ \forall k. \tag{5.26}$$

So $\{u_k\}$ converges pointwise on $\overline{\Omega}$, as $u_k = g$ on $\partial\Omega$.

Moreover, $\{u_k\}$ is equicontinuous on any compact subset of Ω. In fact, let K be any compact subset of Ω. Then the distance from K to $\partial\Omega$ defined by

$$dist(K, \partial\Omega) = \inf\{dist(x, \partial\Omega) : x \in K\},$$

must equal to some positive number ε. Take $R > 0$ such that $4R < \varepsilon$. Then $B_{4R}(z) \subset \Omega$ for any $z \in K$. Since u_k is infinity sub-harmonic in Ω, i.e., $\triangle_\infty u_k \geq 0$ in the viscosity sense, it is well-known, e.g., by [4] Lemma 2.9, that

$$|u_k(x) - u_k(y)| \leq \left(\sup_{B_{4R}(z)} u_k - \sup_{B_R(z)} u_k \right) \frac{|x-y|}{R}, \tag{5.27}$$

for any $x, y \in B_{4R}(z)$. As $u_1 \leq u_k \leq u$ in Ω, we have

$$|u_k(x) - u_k(y)| \leq \left(\sup_{B_{4R}(z)} u - \sup_{B_R(z)} u_1 \right) \frac{|x-y|}{R} \leq L_R \frac{|x-y|}{R}, \tag{5.28}$$

where $L_R = \sup_\Omega u - \inf_\Omega u_1 \geq 0$, which is independent of k. As K can be covered by finitely many balls $B_{4R}(z)$, where $z \in K$, $\{u_k\}$ must be equicontinuous on K.

Therefore a subsequence of $\{u_k\}$ converges locally uniformly in Ω to some function $\bar{u} \in C(\Omega)$. We once again abuse our notation by denoting the convergent subsequence by $\{u_k\}$.

We claim that $\bar{u}$ verifies

(i) $\triangle_\infty^N \bar{u} = 0$ in the viscosity sense in Ω,
(ii) $\forall x_0 \in \partial\Omega$, $\lim_{x\in\Omega\to x_0} \bar{u}(x) = g(x_0)$, and
(iii) $\bar{u} \in C(\overline{\Omega})$ if we extend the definition of $\bar{u}$ to $\partial\Omega$ by defining $\bar{u}|_{\partial\Omega} = g$.

One side of (i), $\triangle_\infty^N u \geq 0$, is implied by Lemma 1.10 as $\bar{u}(x) = \sup_k u_k(x)$ and $\triangle_\infty^N u_k \geq 0$.

In order to prove $\triangle_\infty^N u \leq 0$, we suppose $\varphi \in C^2(\Omega)$ touches u from below at $x_0 \in \Omega$. Then, for any small $\varepsilon > 0$, the function $x \mapsto u(x) - (\varphi(x) - \frac{\varepsilon}{2}|x-x_0|^2)$ has a strict minimum at x_0. In particular,

$$u(x_0) - \varphi(x_0) < \min_{y \in \partial B_r(x_0)} \left(u(y) - \left(\varphi(y) - \frac{\varepsilon}{2}|y-x_0|^2 \right) \right) \tag{5.29}$$

for all small $r > 0$ and $B_r(x_0) \subset\subset \Omega$.

As $\{u_k\}$ converges to u uniformly on $\overline{B}_r(x_0)$, for all large k,

$$\begin{aligned} \inf_{x \in B_r(x_0)} \left(u_k(x) - \left(\varphi(x) - \frac{\varepsilon}{2}|x-x_0|^2 \right) \right) &\leq u_k(x_0) - \varphi(x_0) \\ &< \min_{y \in \partial B_r(x_0)} \left(u_k(y) - (\varphi(y) - \frac{\varphi}{2}|y-x_0|^2) \right). \end{aligned}$$

So the function $x \mapsto u_k(x) - (\varphi(x) - \frac{\varepsilon}{2}|x-x_0|^2)$ assumes its minimum over $\overline{B}_r(x_0)$ at some point $x_k \in B_r(x_0)$.

By the definition of viscosity solutions,

$$\triangle_\infty^N \left(\varphi(x) - \frac{\varepsilon}{2}|x-x_0|^2 \right) \leq \delta_k \tag{5.30}$$

at $x = x_k$, i.e.,

$$\triangle_\infty^N \varphi(x_k) + O(\varepsilon) \leq \delta_k, \tag{5.31}$$

$\forall \varepsilon > 0$ and $\forall k \geq k(r)$, where $k(r) \uparrow \infty$ as $r \downarrow 0$.

If we send r to 0, we obtain $\triangle_\infty^N \varphi(x_0) \leq O(\varepsilon)$ for any $\varepsilon > 0$, which implies $\triangle_\infty^N \varphi(x_0) \leq 0$, i.e., $\triangle_\infty^N u(x_0) \leq 0$ in the viscosity sense.

As the local uniform limit of $\{u_k\}$ in Ω, $\bar{u}$ is clearly in $C(\Omega)$. In order to prove (ii) and (iii), we will apply the comparison with polar quadratic polynomials properties of the viscosity sub- and super-solutions of the equation $\triangle_\infty^N v = 1$.

In fact, $\triangle_\infty^N \frac{u_k}{\delta_k} = 1$ in the viscosity sense in Ω. Fix $x_0 \in \partial\Omega$. The comparison with polar quadratic polynomials from above property states that

$$\frac{u_k(x)}{\delta_k} - (a|x-x_0|^2 + b|x-x_0|) \leq \max_{y \in \partial\Omega} \left(\frac{u_k(y)}{\delta_k} - (a|y-x_0|^2 + b|y-x_0|) \right) \tag{5.32}$$

for all $x \in \Omega$, $a \leq \frac{1}{2}$ and $b \in \mathbf{R}$, or equivalently

$$u_k(x) - a\delta_k|x-x_0|^2 - b\delta_k|x-x_0| \leq \max_{y \in \partial\Omega}(g(y) - a\delta_k|y-x_0|^2 - b\delta_k|y-x_0|). \tag{5.33}$$

Fix the value of a, say $a = \frac{1}{2}$. Take $b = b_k > 0$ large enough so that $b \geq 2|a| \max_{y \in \overline{\Omega}} |y-x_0|$ and $b\delta_k = CL_g(\partial\Omega)$ for some universal constant $C \gg 1$. So, for $y \in \partial\Omega$,

$$g(y) - a\delta_k|y-x_0|^2 - b\delta_k|y-x_0| \leq g(y) - \frac{1}{2}b\delta_k|y-x_0| \leq g(x_0), \tag{5.34}$$

$$\text{i.e., } \max_{y \in \partial\Omega}(g(y) - a\delta_k|y-x_0|^2 - b\delta_k|y-x_0|) = g(x_0). \tag{5.35}$$

As a result, for $x \in \Omega$ near x_0,

$$u_k(x) \le g(x_0) + a\delta_k|x - x_0|^2 + b\delta_k|x - x_0| \tag{5.36}$$

$$\le g(x_0) + CL_g(\partial\Omega)|x - x_0|. \tag{5.37}$$

On the other hand, the comparison with polar quadratic polynomials from below property implies that, for any $a \ge \frac{1}{2}$, $b \in \mathbf{R}$ and all x in Ω,

$$\frac{u_k(x)}{\delta_k} - a|x - x_0|^2 - b|x - x_0| \ge \min_{y \in \partial\Omega}\left(\frac{u_k(y)}{\delta_k} - a|x - x_0|^2 - b|x - x_0|\right), \tag{5.38}$$

or equivalently

$$u_k(x) - a\delta_k|x - x_0|^2 - b\delta_k|x - x_0| \ge \min_{y \in \partial\Omega}(g(y) - a\delta_k|y - x_0|^2 - b\delta_k|y - x_0|). \tag{5.39}$$

Fix the value of a. Take $b = b_k < 0$ so that $-b > 2a \max_{x \in \overline{\Omega}} |x - x_0|$ and $-b\delta_k = CL_g(\partial\Omega)$ for some universal constant $C \gg 1$.

As a result, for $y \in \partial\Omega$,

$$g(y) - a\delta_k|y - x_0|^2 - b\delta_k|y - x_0| \ge g(y) - \frac{1}{2}b\delta_k|y - x_0| \ge g(x_0), \tag{5.40}$$

which means

$$\min_{\partial\Omega}(g(x) - a\delta_k|x - x_0|^2 - b|x - x_0|) = g(x_0). \tag{5.41}$$

So, for $x \in \Omega$ near x_0,

$$u_k(x) \ge g(x_0) + a\delta_k|x - x_0|^2 + b\delta_k|x - x_0| \tag{5.42}$$

$$\ge g(x_0) - CL_g(\partial\Omega)|x - x_0|. \tag{5.43}$$

Therefore, for some $C \gg 1$ independent of k,

$$g(x_0) - CL_g(\partial\Omega)|x - x_0| \le u_k(x) \le g(x_0) + CL_g(\partial\Omega)|x - x_0|, \tag{5.44}$$

for all k and all $x \in \Omega$ near x_0.

Sending k to ∞, we have

$$g(x_0) - CL_g(\partial\Omega)|x - x_0| \le \bar{u}(x) \le g(x_0) + CL_g(\partial\Omega)|x - x_0|, \tag{5.45}$$

for all k and all $x \in \Omega$ near x_0.

Now it is clear that (ii) and (iii) hold.

The uniqueness of a viscosity solution in $C(\overline{\Omega})$ of the Dirichlet problem for homogeneous equation $\triangle_\infty^N u = 0$ in Ω under $u|_{\partial\Omega} = g$ implies that $\bar{u} = u$ on $\overline{\Omega}$. As a result, $\{u_k\}$ converges to u locally uniformly in Ω.

Recall that $\sup_\Omega(u - u_k) > \varepsilon_0$. There exists, for each k, an $x_k \in \Omega$ such that $u(x_k) > u_k(x_k) + \varepsilon_0$ and x_k approaches the boundary $\partial\Omega$, since $\{u_k\}$ converges to u locally uniformly in Ω. Without loss of generality, we assume $x_k \to x_0 \in \partial\Omega$.

Then we will have the following contradiction by previous estimate on $u_k(x)$ for $x \in \Omega$ near x_0.

$$\begin{aligned} g(x_0) &= \lim_k u(x_k) \geq \limsup_k u_k(x_k) + \varepsilon_0 \\ &\geq \limsup_k (g(x_0) - CL_g(\partial\Omega)|x_k - x_0| + \varepsilon_0) \\ &= g(x_0) + \varepsilon_0. \end{aligned}$$

This completes the proof. □

We may also perturb the boundary data and still have the uniform convergence desired. This is the content of Theorem 1.9. Now we prove Theorem 1.9.

Proof of Theorem 1.9. Let $c_k = \|f_k\|_{L^\infty(\Omega)}$ and $\{\varepsilon_k\}$ denotes a sequence of positive numbers that converges to 0.

Proceeding as in the proof of the preceding theorem, we let u_k^1 and $u_k^2 \in C(\overline{\Omega})$ be the respective viscosity solutions of the following Dirichlet problems

$$\begin{cases} \triangle_\infty^N u_k^1 = -c_k - \varepsilon_k & \text{in } \Omega \\ u_k^1 = g_k & \text{on } \partial\Omega \end{cases} \tag{5.46}$$

and

$$\begin{cases} \triangle_\infty^N u_k^2 = c_k + \varepsilon_k & \text{in } \Omega \\ u_k^2 = g_k & \text{on } \partial\Omega. \end{cases} \tag{5.47}$$

By Lemma 1.10, we know that

$$u_k^2 \leq u_k \leq u_k^1 \quad \text{on } \overline{\Omega}. \tag{5.48}$$

So it suffices to show that $\sup_{\overline{\Omega}}(u_k^j - u) \to 0$ as $k \to \infty$, for both $j = 1$ and 2.

We introduce v_k^1 and $v_k^2 \in C(\overline{\Omega})$ as the viscosity solutions of the following Dirichlet problems respectively

$$\begin{cases} \triangle_\infty^N v_k^1 = -c_k - \varepsilon_k & \text{in } \Omega \\ v_k^1 = g & \text{on } \partial\Omega \end{cases} \tag{5.49}$$

and

$$\begin{cases} \triangle_\infty^N v_k^2 = c_k + \varepsilon_k & \text{in } \Omega \\ v_k^2 = g & \text{on } \partial\Omega. \end{cases} \tag{5.50}$$

The comparison principle, Theorem 3.3, implies that

$$\sup_{\overline{\Omega}} |u_k^j - v_k^j| \leq \max_{\partial\Omega} |g_k - g| \to 0, \quad \text{as } k \to \infty, \tag{5.51}$$

for $j = 1, 2$.

The preceding Theorem 5.4 implies that

$$\sup_{\overline{\Omega}} |v_k^j - u| \to 0, \quad \text{as } k \to \infty,$$

for $j = 1, 2$.

Therefore we have

$$\sup_{\overline{\Omega}} |u_k^j - u| \to 0, \quad \text{as } k \to \infty,$$

for $j = 1, 2$, as expected. □

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References

[1] Aronsson, G. (1965). Minimization problems for the functional $\sup_x F(x; f(x); f'(x))$. *Ark. Mat.* 6:33–53.
[2] Aronsson, G. (1966). Minimization problems for the functional $\sup_x F(x; f(x); f'(x))$. II. *Ark. Mat.* 6:409–431.
[3] Aronsson, G. (1967). Extension of functions satisfying Lipschitz conditions. *Ark. Mat.* 6:551–561.
[4] Aronsson, G., Crandall, M., Juutinen, P. (2004). A tour of the theory of absolute minimizing functions. *Bull. A.M.S.* 41:439–505.
[5] Barles, G., Busca, J. (2001). Existence and comparison results for fully nonlinear degenerate elliptic equations without zeroth-order term. *Comm. Partial Diff. Equations* 26:2323–2337.
[6] Barron, E. N., Jensen, R. R., Wang, C. Y. (2001). The Euler equation and absolute minimizers of L^∞ functionals. *Arch. Ration. Mech. Anal.* 157:255–283.
[7] Barron, E. N., Evans, L. C., Jensen, R. R. (2008). The infinity Laplacian, Aronsson's equation and their generalizations. *Trans. Amer. Math. Soc.* 360:77–101.
[8] Crandall, M. G. (2003). An efficient derivation of the Aronsson equation. *Arch. Rational Mech. Anal.* 167:271–279.
[9] Crandall, M. G., Evans, L. C. (2001). A remark on infinity harmonic functions. Proceedings of the USA-Chile Workshop on Nonlinear Analysis (Viã del Mar-Valparaiso, 2000). *Electon. J. Differ. Equ. Conf.* 6:123–129.
[10] Crandall, M. G., Evans, L. C., Gariepy, R. F. (2001). Optimal Lipschitz extensions and the infinity Laplacian. *Calc. Var. Partial Differential Equations* 13:123–139.
[11] Crandall, M. G., Gunnarsson, G., Wang, P. (2007). Uniqueness of ∞-harmonic functions and the eikonal equation. *Comm. Partial Diff. Equations* 32:1–29.
[12] Crandall, M. G., Ishii, H., Lions, P. L. (1992). User's guide to viscosity solutions of second-order partial differential equations. *Bull. A.M.S.* 27:1–67.
[13] Crandall, M. G., Wang, P. (2004). Another way to say caloric. *J. Evol. Equ.* 3:653–672.
[14] Evans, L. C. (2007). The 1-Laplacian, the infinity Laplacian and differential games. *Contemp. Math.* 446:245–254.
[15] Evans, L. C., Yu, Y. (2005). Various properties of solutions of the infinity-Laplacian equation. *Comm. Partial Differential Equations* 30:1401–1428.
[16] Jensen, R. (1993). Uniqueness of Lipschitz extensions: minimizing the sup norm of the gradient. *Arch. Rational Mech. Anal.* 123:51–74.

[17] Juutinen, P. (1998). Minimization problems for Lipschitz functions via viscosity solutions. *Ann. Acad. Sci. Fenn. Math. Diss.* 115:53.

[18] Kohn, R. V., Serfaty, S. (2006). A deterministic-control-based approach to motion by curvature. *Comm. Pure Appl. Math.* 59:344–407.

[19] Lindqvist, P., Manfredi, J. (1995). The Harnack inequality for 1-harmonic functions. *Electron. J. Differential Equations* 1995(4):5 pp.

[20] Lu, G., Wang, P. (2008). Inhomogeneous infinity Laplace equation. *Advances in Mathematics* 217(4):1838–1868.

[21] Peres, Y., Schramm, O., Sheffield, S., Wilson, D. (2006). Tug of war and the infinity Laplacian. arXiv:math.AP/0605002 v1 28 Apr 2006.