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Sub-elliptic global high order Poincaré inequalities in stratified Lie groups and applications

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Abstract

Sharp Poincaré inequalities on balls or chain type bounded domains have been extensively studied both in classical Euclidean space and Carnot–Carathéodory spaces associated with sub-elliptic vector fields (e.g., vector fields satisfying Hörmander's condition). In this paper, we investigate the validity of sharp global Poincaré inequalities of both first order and higher order on the entire nilpotent stratified Lie groups or on unbounded extension domains in such groups. We will show that simultaneous sharp global Poincaré inequalities also hold and weighted versions of such results remain to be true. More precisely, let $\mathbb{G}$ be a nilpotent stratified Lie group and f be in the localized non-isotropic Sobolev space $W^{m,p}_{\mathrm{loc}}(\mathbb{G})$, where $1 \leqslant p < Q/m$ and Q is the homogeneous dimension of the Lie group $\mathbb{G}$. Suppose that the mth sub-elliptic derivatives of f is globally L^p integrable; i.e., $\int_{\mathbb{G}} |X^m f(x)|^p\,dx$ is finite (but assume that lower order sub-elliptic derivatives are only locally L^p integrable). We denote the space of such functions as $B^{m,p}(\mathbb{G})$. We prove a high order Poincaré inequality for f minus a polynomial of order $m-1$ over the entire space $\mathbb{G}$ or unbounded extension domains. As applications, we will prove a density theorem stating that smooth functions with compact support are dense in $B^{m,p}(\mathbb{G})$ modulus a finite-dimensional subspace.

Keywords: High order Poincaré inequality; Entire space; Stratified groups; Density theorem; Unbounded extension domains

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1. Introduction

We first recall some preliminaries concerning stratified Lie groups (or so-called Carnot groups). We refer the reader to the books [1,6,30] for analysis on stratified groups. Let $\mathcal{G}$ be a finite-dimensional, stratified, nilpotent Lie algebra. Assume that

$$\mathcal{G} = \bigoplus_{i=1}^{s} V_i,$$

with $[V_i, V_j] \subset V_{i+j}$ for $i + j \leqslant s$ and $[V_i, V_j] = 0$ for $i + j > s$. Let $X_1, \ldots, X_l$ be a basis for V_1 and suppose that $X_1, \ldots, X_l$ generate $\mathcal{G}$ as a Lie algebra. Then for $2 \leqslant j \leqslant s$, we can choose a basis $\{X_{ij}\}$, $1 \leqslant i \leqslant k_j$, for V_j consisting of commutators of length j. We set $X_{i1} = X_i$, $i = 1, \ldots, l$ and $k_1 = l$, and we call X_{i1} a commutator of length 1.

If $\mathbb{G}$ is the simply connected Lie group associated with $\mathcal{G}$, then the exponential mapping is a global diffeomorphism from $\mathcal{G}$ to $\mathbb{G}$. Thus, for each $g \in \mathbb{G}$, there is $x = (x_{ij}) \in R^N$ for $1 \leqslant i \leqslant k_j$, $1 \leqslant j \leqslant s$ and $N = \sum_{j=1}^{s} k_j$ such that

$$g = \exp\Bigl(\sum x_{ij} X_{ij}\Bigr).$$

A homogeneous norm function $|\cdot|$ on $\mathbb{G}$ is defined by

$$|g| = \Bigl(\sum |x_{ij}|^{2s!/j}\Bigr)^{\frac{1}{2s!}},$$

and $Q = \sum_{j=1}^{s} j k_j$ is said to be the *homogeneous dimension* of $\mathbb{G}$. The dilation δ_r on $\mathbb{G}$ is defined by

$$\delta_r(g) = \exp\Bigl(\sum r^j x_{ij} X_{ij}\Bigr) \quad \text{if } g = \exp\Bigl(\sum x_{ij} X_{ij}\Bigr).$$

The convolution operation on $\mathbb{G}$ is defined by

$$f * h(x) = \int_{\mathbb{G}} f\bigl(xy^{-1}\bigr) h(y)\, dy = \int_{\mathbb{G}} f(y) h\bigl(y^{-1}x\bigr)\, dy,$$

where y^{-1} is the inverse of y and xy^{-1} denotes group multiplication of x by y^{-1}. It is known that for any left-invariant vector field X on $\mathbb{G}$,

$$X(f * h) = f * (Xh).$$

We call a curve $\gamma : [a, b] \to \mathbb{G}$ "a horizontal curve" connecting two points $x, y \in \mathbb{G}$ if $\gamma(a) = x$, $\gamma(b) = y$ and $\gamma'(t) \in V_1$ for all t. Then the Carnot–Caratheodory distance between x, y is defined as

$$d_{cc}(x, y) = \inf_{\gamma} \int_a^b \bigl\langle \gamma'(t), \gamma'(t)\bigr\rangle^{\frac{1}{2}}\, dt,$$

where the infimum is taken over all horizontal curves γ connecting x and y. It is known that any two points x, y on $\mathbb{G}$ can be joined by a horizontal curve of finite length and then d_{cc} is a left-invariant metric on $\mathbb{G}$. We can define the metric ball centered at x and with radius r associated with this metric by

$$B_{cc}(x,r)=\{y\colon\ d_{cc}(x,y)<r\}.$$

We must notice that this metric d_{cc} is equivalent to the pseudo-metric $\rho(x,y)=|x^{-1}y|$ defined by the homogeneous norm $|\cdot|$ in the following sense (see [6]):

$$C\rho(x,y)\leqslant d_{cc}(x,y)\leqslant C\rho(x,y).$$

We denote the metric ball associated with ρ as $D(x,r)=\{y\in\mathbb{G}\colon\ \rho(x,y)<r\}$. An important feature of both of these distance functions is that these distances and thus the associated metric balls are left-invariant, namely,

$$d_{cc}(zx,zy)=d(x,y),\qquad B_{cc}(x,r)=xB_{cc}(0,r)$$

and

$$\rho(zx,zy)=\rho(x,y),\qquad D(x,r)=xD(0,r).$$

From now on, we will always use the metric d_{cc} and drop the subscript from d_{cc}. Similarly, we will use $B(x,r)$ to denote $B_{cc}(x,r)$.

We now recall the definition of the class of polynomials on $\mathbb{G}$ given by Folland and Stein in [6]. Let $X_1,\dots,X_l$ in V_1 be the generators of the Lie algebra $\mathcal{G}$, and let $X_1,\dots,X_l,\dots,X_N$ be a basis of $\mathcal{G}$. We denote $d(X_j)=d_j$ to be the length of X_j as a commutator, and we arrange the order so that $1\leqslant d_1\leqslant\cdots\leqslant d_N$. Then it is easy to see that $d_j=1$ for $j=1,\dots,l$. Let $\xi_1,\dots,\xi_N$ be the dual basis for $\mathcal{G}^*$, and let $\eta_i=\xi_i\circ\exp^{-1}$. Each η_i is a real-valued function on $\mathbb{G}$, and $\eta_1,\dots,\eta_N$ gives a system of global coordinates on $\mathbb{G}$. A function P on $\mathbb{G}$ is said to be a polynomial on $\mathbb{G}$ if $P\circ\exp$ is a polynomial on $\mathcal{G}$. Every polynomial on $\mathbb{G}$ can be written uniquely as

$$P(x)=\sum_I a_I\eta^I(x),\qquad \eta^I=\eta_1^{i_1}\cdots\eta_N^{i_N},$$

where all but finitely many of the coefficients a_I vanish. Clearly η^I is homogeneous of degree $d(I)=\sum_{j=1}^N i_jd_j$, i.e., $\eta^I(\delta_r x)=r^{d(I)}\eta_i(x)$. If $P=\sum_I a_I\eta^I$, then we define the homogeneous degree (or order) of P to be $\max\{d(I)\colon\ a_I\neq 0\}$.

Throughout this paper, we use $\mathcal{P}_k$ to denote polynomials of homogeneous degree less than k for each positive integer k.

We also adopt the following multi-index notation for higher order derivatives. If we set

$$X^I=X_1^{i_1}\cdot X_2^{i_2}\cdots X_N^{i_N}.$$

By the Poincaré–Birkhoff–Witt theorem (cf. Bourbaki [2, I.3.7]), the differential operators X^I form a basis for the algebra of left-invariant differential operators in $\mathbb{G}$. Furthermore, we set

$$|I| = i_1 + i_2 + \cdots + i_N, \qquad d(I) = d_1 i_1 + d_2 i_2 + \cdots + d_N i_N.$$

Thus, $|I|$ is the order of the differential operator X^I, and $d(I)$ is its degree of homogeneity; $d(I)$ is called the homogeneous degree of X^I. We will also use the notation

$$\left|X^m f\right| = \left(\sum_{I:\, d(I)=m} \left|X^I f\right|^2\right)^{\frac{1}{2}}$$

for any positive integer $m > 1$ and

$$|Xf| = \left|X^1 f\right| = \left(\sum_{j=1}^{m} |X_j f|^2\right)^{\frac{1}{2}}.$$

Let m be a positive integer, $1 \leqslant p < \infty$, and Ω be an open set in $\mathbb{G}$. The Folland–Stein Sobolev space $W^{m,p}(\Omega)$ associated with the vector fields $X_1, \ldots, X_l$ is defined to consist of all functions $f \in L^p(\Omega)$ with distributional derivatives $X^I f \in L^p(\Omega)$ for every X^I defined above with $d(I) \leqslant m$. Here, we say that the distributional derivative $X^I f$ exists and equals a locally integrable function g_I if for every $\phi \in C_0^\infty(\Omega)$,

$$\int_\Omega f X^I \phi \, dx = (-1)^{d(I)} \int_\Omega g_I \phi \, dx.$$

$W^{m,p}(\Omega)$ is equipped with the norm

$$\|f\|_{W^{m,p}(\Omega)} = \|f\|_{L^p(\Omega)} + \sum_{1 \leqslant d(I) \leqslant m} \left\|X^I f\right\|_{L^p(\Omega)}.$$

When $\Omega = \mathbb{G}$, we sometimes use $\|f\|_{m,p}$ to denote $\|f\|_{W^{m,p}(\mathbb{G})}$. We also sometimes use $\|f\|_{m,p;\Omega}$ to denote $\|f\|_{W^{m,p}(\Omega)}$.

We also use $W^{m,p}_{\mathrm{loc}}(\mathbb{G})$ to denote the space of functions which are in $W^{m,p}(\Omega)$ for any bounded open set $\Omega \subset \mathbb{G}$.

We are now ready to state the main theorems of this paper.

Theorem 1.1. *Let m be a positive integer, $p \geqslant 1$, $f \in W^{m,p}_{\mathrm{loc}}(\mathbb{G})$ and $|X^m f| \in L^p(\mathbb{G})$. Then there exists a unique polynomial $P \in \mathcal{P}_m$ such that for any integer j with $0 \leqslant j < m$,*

$$\left(\int_{\mathbb{G}} \left|X^j (f-P)(x)\right|^{q_{mj}} dx\right)^{\frac{1}{q_{mj}}} \leqslant C\left(\int_{\mathbb{G}} \left|X^m f(x)\right|^p dx\right)^{\frac{1}{p}}$$

for all $1 \leqslant p < \frac{Q}{m-j}$ and $q_{mj} = \frac{pQ}{Q-(m-j)p}$, where C is independent of f.

Theorem 1.2. *Let m and i be positive integers and $0 < i \leqslant m$, $p \geqslant 1$ and $f \in W^{m,p}_{\text{loc}}(\mathbb{G})$ and $|X^i f| \in L^p(\mathbb{G})$. Then there exists a unique polynomial $P \in \mathcal{P}_m$ such that for any integer j with $0 \leqslant j < i \leqslant m$,*

$$\left(\int_{\mathbb{G}} \left|X^j(f-P)(x)\right|^{q_{ij}}\,dx\right)^{\frac{1}{q_{ij}}} \leqslant C\left(\int_{\mathbb{G}} \left|X^i f(x)\right|^p\,dx\right)^{\frac{1}{p}}$$

for all $1 \leqslant p < \frac{Q}{i-j}$ and $q_{ij} = \frac{pQ}{Q-(i-j)p}$, where C is independent of f.

To see the special case of Theorem 1.1 when $m = 1$ and $j = 0$, we now let f satisfy

$$f \in W^{1,p}_{\text{loc}}(\mathbb{G}) \quad \text{and} \quad \int_{\mathbb{G}} |Xf|^p\,dx < \infty. \tag{1.1}$$

We note that functions satisfying (1.1) are not necessarily in the Sobolev space $W^{1,p}(\mathbb{G})$ as defined above because we only assume that $f \in L^p_{\text{loc}}(\mathbb{G})$ instead of $f \in L^p(\mathbb{G})$.

Assume $1 \leqslant p < Q$. Then the following Poincaré inequality holds for functions satisfying (1.1):

$$\left(\int_{\mathbb{G}} \left|f(x)-(f)_\infty\right|^{\frac{Qp}{Q-p}}\,dx\right)^{\frac{Q-p}{Qp}} \leqslant C(p,Q)\left(\int_{\mathbb{G}} |Xf|^p\,dx\right)^{\frac{1}{p}}.$$

Here $C(p,Q)$ is a positive constant depending on p, Q only, and $(f)_\infty$ is the limit of $(f)_R$, the average value of f on the ball B_R centered at the origin and with radius R, as R approaches infinity. This is proved in Theorem 2.1 in Section 2.

As an application of this case, we show that the linear space consisting of functions satisfying (1.1) is a complete Banach space under the norm

$$\|f\| = \left(\int_{\mathbb{G}} |Xf|^p\,dx + \left|(f)_\infty\right|^p\right)^{\frac{1}{p}}.$$

We refer the reader to Section 3 for more details. Moreover, in Section 3 we will show that for any $\epsilon > 0$ there exists a $C_0^\infty(\mathbb{G})$ function ϕ such that

$$\left(\int_{\mathbb{G}} \left|f(x)-(f)_\infty-\phi\right|^{\frac{Qp}{Q-p}}\,dx\right)^{\frac{Q-p}{Qp}} + \left(\int_{\mathbb{G}} \left|X(f-\phi)\right|^p\,dx\right)^{\frac{1}{p}} < \epsilon.$$

In fact, we will show the following more general theorems for higher order Sobolev spaces consisting of functions satisfying

$$f \in W^{m,p}_{\text{loc}}(\mathbb{G}) \quad \text{and} \quad \left\|X^m f\right\|_{L^p(\mathbb{G})} < \infty. \tag{1.2}$$

Theorem 1.3. *Let $B^{m,p}(\mathbb{G})$ be the function space consisting of all functions satisfying* (1.2). *For each element $f \in B^{m,p}(\mathbb{G})$, let $P \in \mathcal{P}_m$ be the unique polynomial associated with f in Theorem* 1.1 *and let $P(x) = \sum_{I:\, d(I)\leqslant m-1} a_I \eta^I(x)$. Then $B^{m,p}(\mathbb{G})$ is a complete Banach space with the norm*

$$\|f\| = \left(\int_{\mathbb{G}} \left|X^m f\right|^p dx + \sum_{d(I)\leqslant m-1} |a_I|^p \right)^{\frac{1}{p}}.$$

Theorem 1.4. *Functions in $C_0^\infty(\mathbb{G})$ are dense in $B^{m,p}(\mathbb{G})$ modulus a finite-dimensional subspace. Namely, given any function $f \in B^{m,p}(\mathbb{G})$ and any $\epsilon > 0$, there exist a polynomial P of degree no more than $m-1$, and a function $\phi \in C_0^\infty(\mathbb{G})$ such that*

$$\sum_{j=0}^{m-1} \left\| X^j (f - P - \phi) \right\|_{q_{mj}} + \left\| X^m (f - P - \phi) \right\|_p < \epsilon,$$

where $q_{mj} = \frac{pQ}{Q-(m-j)p}$, $0 \leqslant j < m$. Consequently, the codimension of this subspace equals the dimension of the linear space consisting of all polynomials of order less than m.

Using the global Poincaré inequalities proved on the entire group $\mathbb{G}$ together with Sobolev extension theorems, we may further derive Poincaré inequalities on unbounded (ϵ, ∞) domains (see Section 6 for definitions and more details). We will prove in Section 6 the following theorem.

Theorem 1.5. *Assume that $\Omega \subset \mathbb{G}$ is an unbounded (ϵ, ∞) extension domain. Let m be a positive integer, $p \geqslant 1$, and $f \in W^{m,p}_{\mathrm{loc}}(\Omega)$ and $|X^m f| \in L^p(\Omega)$. Then there exists a polynomial $P \in \mathcal{P}_m$ such that for any integer j with $0 \leqslant j < m$,*

$$\left(\int_{\Omega} \left|X^j (f - P)(x)\right|^{q_{mj}} dx \right)^{\frac{1}{q_{mj}}} \leqslant C \left(\int_{\Omega} \left|X^m f(x)\right|^p dx \right)^{\frac{1}{p}}$$

for all $1 \leqslant p < \frac{Q}{m-j}$ and $q_{mj} = \frac{pQ}{Q-(m-j)p}$, where C is independent of f.

Similarly, we have

Theorem 1.6. *Assume that $\Omega \subset \mathbb{G}$ is an unbounded (ϵ, ∞) extension domain. Let m and i be positive integers and $0 < i \leqslant m$, $p \geqslant 1$ and $f \in W^{m,p}_{\mathrm{loc}}(\Omega)$ and $|X^i f| \in L^p(\Omega)$. Then there exists a unique polynomial $P \in \mathcal{P}_m$ such that for any integer j with $0 \leqslant j < i \leqslant m$,*

$$\left(\int_{\Omega} \left|X^j (f - P)(x)\right|^{q_{ij}} dx \right)^{\frac{1}{q_{ij}}} \leqslant C \left(\int_{\Omega} \left|X^i f(x)\right|^p dx \right)^{\frac{1}{p}}$$

for all $1 \leqslant p < \frac{Q}{i-j}$ and $q_{ij} = \frac{pQ}{Q-(i-j)p}$, where C is independent of f.

We can also obtain the following results.

Theorem 1.7. *Assume that $\Omega \subset \mathbb{G}$ is an unbounded (ϵ, ∞) extension domain. Let $B^{m,p}(\Omega)$ be the function space consisting of all functions satisfying*

$$f \in W^{m,p}_{\mathrm{loc}}(\Omega) \quad \text{and} \quad \left\|X^m f\right\|_{L^p(\Omega)} < \infty.$$

For each element $f \in B^{m,p}(\Omega)$ let $P \in \mathcal{P}_m$ be the polynomial associated with f in Theorem 1.5 *and let $P(x) = \sum_{I:\, d(I) \leqslant m-1} a_I \eta^I(x)$. Then $B^{m,p}(\Omega)$ is a complete Banach space with the norm*

$$\|f\| = \left(\int_{\Omega} \left|X^m f\right|^p dx + \sum_{d(I) \leqslant m-1} |a_I|^p\right)^{\frac{1}{p}}.$$

Moreover, we have

Theorem 1.8. *If f satisfies*

$$f \in W^{m,p}_{\mathrm{loc}}(\Omega) \quad \text{and} \quad \left\|X^m f\right\|_{L^p(\Omega)} < \infty,$$

then for any $\epsilon > 0$ there exist a polynomial $P \in \mathcal{P}_m$ and a function $\phi \in C_0^\infty(\mathbb{G})$ such that

$$\sum_{j=0}^{m-1} \left\|X^j(f - P - \phi)\right\|_{L^{q_{mj}}(\Omega)} + \left\|X^m(f - P - \phi)\right\|_{L^p(\Omega)} < \epsilon,$$

where $q_{mj} = \frac{pQ}{Q-(m-j)p}$. If we further define $B_0^{m,p}(\Omega)$ as the closed subspace of $B^{m,p}(\mathbb{G})$ which contains the $C_0^\infty(\mathbb{G})$ functions as a dense subset. Then the co-dimension of this subspace equals the dimension of the linear space consisting of all polynomials of order less than m.

Weighted Poincaré inequalities on the entire group $\mathbb{G}$ or unbounded extension domains $\Omega \subset \mathbb{G}$ can also be derived. However, we will not state them here, but refer the reader to Sections 5 and 6 for statements and proofs.

The following remarks are in order. First, in the classical Euclidean space, global high order Poincaré inequalities as in Theorems 1.1 and 1.3 were shown to hold in the recent paper of Lu and Ou [18] in the special case of $j = 0$ (see also the results in this line of the first order [9,26–29] and applications in incompressible flow given in [4,18,22]). Such results were also extended to the case of Heisenberg group in [5] by showing the constant $(f)_\infty$ is actually the limit of the average integral of f over balls when the radius goes to infinity. In [5], the derived inequality, together with a density theorem proved in [5], was also used to prove that the sharp constant for the global Poincaré inequality is the same as the sharp constant for Sobolev inequality established by Jerison and Lee [10–12]. Simultaneous global Poincaré inequalities of high order are also proved in this paper (see Theorem 1.2) and they are new even in the Euclidean case. Moreover, our proof for Theorem 1.1 of high order is different from those given in [18]. Thus, our proofs also provide another approach even in the Euclidean case. Second, Theorem 1.1 was obtained in Saloff-Coste [23] for the special case of the first order $m = 1$ and $j = 0$ on groups of polynomial growth and on Riemannian manifolds of non-negative curvatures by using the Sobolev inequality for functions with compact support.

The organization of the paper is as follows. The main theorems of this paper are the high order global Poincaré inequalities. However, we will begin with the simpler results of first order. Thus, in Sections 2 and 3, we present the proof of the special case of Theorems 1.1, 1.3, 1.4 when $m = 1$ and $j = 0$. We do this for the clarity of presentation and also for the convenience of the reader who is only interested in the first order Poincaré inequality on the entire group $\mathbb{G}$. Moreover, the proof of this first order case is relatively simpler since it does not involve any polynomials and those complicated arguments for high order cases in later sections are not needed. However, proofs in these two sections are of their independent interest and seem to be new even in the Euclidean case. Section 4 contains the proofs of Theorems 1.1–1.4 in its full strength. In Section 5, we derive the weighted versions of higher order Poincaré inequalities on the entire group $\mathbb{G}$ under a balance condition of weights which was first introduced in [3]. Section 6 deals with higher order Poincaré inequalities on unbounded extension domains. Much of results given in Sections 2–5 can be carried over to unbounded extension domains in both weighted and non-weighted cases using the simultaneous extension theorem proved in [16,17]. All these are carried out in Section 6.

2. The first order Poincaré inequalities on $\mathbb{G}$

Let $f(x)$ be a function satisfying (1.1). Let $(f)_R$ denote the average value of f on the ball B_R; that is,

$$(f)_R \equiv \frac{1}{|B_R|} \int_{B_R} f\,dx.$$

We first establish the following theorem for $f(x)$.

Theorem 2.1. *As R approaches to infinity, $(f)_R$ converges to a finite limit $(f)_\infty$. Moreover,*

$$\left\| f - (f)_\infty \right\|_{\frac{Qp}{Q-p}} \leqslant C(p, Q) \|Xf\|_p \tag{2.1}$$

with $C(p, Q)$ independent of f.

Proof. We first consider any two balls B_1 and B_2 such that $B_1 \subset B_2 \subset \mathbb{G}$. Then

$$\begin{aligned}
\left| \frac{1}{|B_1|} \int_{B_1} f - \frac{1}{|B_2|} \int_{B_2} f \right| &\leqslant \frac{1}{|B_1|} \int_{B_2} |f - f_{B_2}| \\
&\leqslant \frac{|B_2|}{|B_1|} \frac{1}{|B_2|} \int_{B_2} |f - f_{B_2}| \\
&\leqslant \frac{|B_2|}{|B_1|} \left(\frac{1}{|B_2|} \int_{B_2} |f - f_{B_2}|^p \right)^{\frac{1}{p}} \\
&\leqslant C \frac{|B_2|}{|B_1|} r(B_2) \left(\frac{1}{|B_2|} \int_{B_2} |Xf|^p \right)^{\frac{1}{p}}
\end{aligned}$$

$$\leqslant Cr(B_2)\frac{|B_2|^{1-1/p}}{|B_1|}\left(\int_{\mathbb{G}}|Xf|^p\right)^{\frac{1}{p}}.$$

We now take a sequence of concentric balls $\{B_k = B_{2^k}(0)\}$ centered at 0 for $k \geqslant 1$. Then for $j < l$ we have

$$\begin{aligned}\left|\frac{1}{|B_j|}\int_{B_j} f - \frac{1}{|B_l|}\int_{B_l} f\right| &\leqslant \sum_{k=j}^{l-1}\left|\frac{1}{|B_k|}\int_{B_k} f - \frac{1}{|B_{k+1}|}\int_{B_{k+1}} f\right| \\ &\leqslant C\sum_{k=j}^{l-1} r(B_{k+1})\frac{|B_{k+1}|^{1-1/p}}{|B_k|}\left(\int_{\mathbb{G}}|Xf|^p\right)^{\frac{1}{p}} \\ &\leqslant C\sum_{k=j}^{l-1} 2^{k(1-\frac{Q}{p})}\left(\int_{\mathbb{G}}|Xf|^p\right)^{\frac{1}{p}} \\ &\leqslant C2^{j(1-\frac{Q}{p})}\left(\int_{\mathbb{G}}|Xf|^p\right)^{\frac{1}{p}} \to 0\end{aligned}$$

as $j \to \infty$. Therefore, $\frac{1}{|B_k|}\int_{B_k} f$ is a Cauchy sequence and thus it converges. We denote the limit as $(f)_\infty$.

Now we recall the standard Poincaré inequality for any metric ball $B_R = B(0, R)$ (see [7,13, 14,20]):

$$\left(\int_{B_R}\left|f(x) - (f)_R\right|^{\frac{Qp}{Q-p}}dx\right)^{\frac{Q-p}{Qp}} \leqslant C(p,Q)\left(\int_{B_R}|Xf|^p\,dx\right)^{\frac{1}{p}}$$

for some constant $C(p, Q)$ independent of B_R and f.

For any $0 < R_1 < R_2$, we have

$$\begin{aligned}\left(\int_{B_{R_1}}\left|f(x) - (f)_{R_2}\right|^{\frac{Qp}{Q-p}}dx\right)^{\frac{Q-p}{Qp}} &\leqslant \left(\int_{B_{R_2}}\left|f(x) - (f)_{R_2}\right|^{\frac{Qp}{Q-p}}dx\right)^{\frac{Q-p}{Qp}} \\ &\leqslant C(p,Q)\left(\int_{B_{R_2}}|Xf|^p\,dx\right)^{\frac{1}{p}}.\end{aligned}$$

Letting $R_2 \to \infty$ in the inequality above, we obtain

$$\left(\int_{B_{R_1}}\left|f(x) - (f)_\infty\right|^{\frac{Qp}{Q-p}}dx\right)^{\frac{Q-p}{Qp}} \leqslant C(p,Q)\left(\int_{\mathbb{G}}|Xf|^p\,dx\right)^{\frac{1}{p}}.$$

We thus arrive at the global Poincaré inequality (2.1) after letting $R_1 \to \infty$. □

3. The complete Banach space $B^{1,p}(\mathbb{G})$

Let $B^{1,p}(\mathbb{G})$ be the linear space consisting of functions satisfying

$$f(x) \in W^{1,p}_{\mathrm{loc}}(\mathbb{G}) \quad \text{and} \quad \int_{\mathbb{G}} |Xf|^p \, dx < \infty. \tag{3.1}$$

We first note that we do not require here $f \in L^p(\mathbb{G})$, but only require $f \in L^p_{\mathrm{loc}}(\mathbb{G})$. Thus, this space is essentially different from the standard non-isotropic Sobolev space $W^{1,p}(\mathbb{G})$ which consists of functions f satisfying

$$f \in L^p(\mathbb{G}), \qquad |Xf| \in L^p(\mathbb{G}).$$

As the first application of the Poincaré inequality on the entire group $\mathbb{G}$ we have

Theorem 3.1. *The linear space $B^{1,p}(\mathbb{G})$ consisting of functions satisfying* (3.1) *is a complete Banach space with the norm*

$$\|f\| \equiv \left(\int_{\mathbb{G}} |Xf|^p \, dx + \left|(f)_\infty\right|^p \right)^{\frac{1}{p}}, \tag{3.2}$$

where $(f)_\infty$ is the limit of integral average of f over balls B whose existence is guaranteed by Theorem 2.1.

Proof. Suppose that $\|f_k - f_l\| \to 0$ as $k, l \to \infty$. Let $\{f^i\}$ be a Cauchy sequence under the norm $\|\cdot\|$. Set $w^i = f^i - (f^i)_\infty$, $i = 1, 2, \ldots$. Using Theorem 2.1,

$$\|w^i - w^j\|_{\frac{Qp}{Q-p}} = \left\|\left(f^i - f^j\right) - \left(f^i - f^j\right)_\infty\right\|_{\frac{Qp}{Q-p}} \leqslant C(p, Q)\left\|X\left(f^i - f^j\right)\right\|_p.$$

Clearly

$$\left\|X\left(w^i - w^j\right)\right\|_p = \left\|X\left(f^i - f^j\right)\right\|_p.$$

Thus it is a standard argument that the sequence of w^i has a limit w such that w is in $L^{\frac{Qp}{Q-p}}(\mathbb{G})$ and $|Xw|$ is in $L^p(\mathbb{G})$, and moreover

$$\|w^i - w\|_{L^{\frac{Qp}{Q-p}}(\mathbb{G})} + \left\|X\left(w^i - w\right)\right\|_{L^p(\mathbb{G})} \to 0$$

as $i \to \infty$. It is trivial that $(w^i)_\infty = 0$ and

$$\begin{aligned}\left|\left(w - w^i\right)_R\right| &\leqslant \left(\frac{1}{|B_R|} \int_{B_R} \left|w - w^i\right|^{\frac{Qp}{Q-p}} dx \right)^{\frac{Q-p}{Qp}} \\ &\leqslant C(p, Q) R^{-(Q-p)/p} \|w - w^i\|_{\frac{Qp}{Q-p}} \to 0\end{aligned}$$

as $R \to \infty$. But $(w^i)_R \to (w^i)_\infty = 0$ as $R \to \infty$. Thus $(w)_\infty = 0$. Set $u = w + \lim_{i\to\infty}(f^i)_\infty$. Then f is in $B^{1,p}(\mathbb{G})$ and is the limit of the sequence of f^i in $B^{1,p}(\mathbb{G})$ under the norm (3.2). The theorem is thus proved. □

Proposition 3.2. *Suppose w is in $W^{1,p}_{\mathrm{loc}}(\mathbb{G})$ and satisfies*

$$\|w\|_{\frac{Qp}{Q-p}} + \|Xw\|_p < \infty. \tag{3.3}$$

Then for any $\epsilon > 0$ there is a $C_0^\infty(\mathbb{G})$ function $\phi(x)$ such that

$$\|w-\phi\|_{\frac{Qp}{Q-p}} + \big\|X(w-\phi)\big\|_p < \epsilon. \tag{3.4}$$

Proof. Let $R > 0$ and let $\psi_R(x)$ be a $C_0^\infty(\mathbb{G})$ function (see [6]) satisfying

$$\begin{cases} \psi_R(x) = 1 & \text{if } |x| \leqslant R,\\ \psi_R(x) = 0 & \text{if } |x| \geqslant 2R,\\ \big|\psi_R(x)\big| \leqslant 1 & \text{for all } x,\\ \big|X\psi_R(x)\big| \leqslant C/R & \text{for all } x. \end{cases}$$

Notice that

$$\begin{aligned}
\int_{\mathbb{G}} |wX\psi_R|^p\,dx &\leqslant \frac{2^p}{R^p}\int_{B_{2R}\setminus B_R} |w|^p\,dx\\
&= C(p,Q)R^{Q-p}\left(\frac{1}{|B_{2R}\setminus B_R|}\int_{B_{2R}\setminus B_R} |w|^p\,dx\right)\\
&\leqslant C(p,Q)R^{Q-p}\left(\frac{1}{|B_{2R}\setminus B_R|}\int_{B_{2R}\setminus B_R} |w|^{\frac{Qp}{Q-p}}\,dx\right)^{\frac{Q-p}{n}}\\
&\leqslant C(p,Q)\left(\int_{B_{2R}\setminus B_R} |w|^{\frac{Qp}{Q-p}}\,dx\right)^{\frac{Q-p}{n}}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\|w-w\psi_R\|_{\frac{Qp}{Q-p}} + \big\|X(w-w\psi_R)\big\|_p\\
&\quad\leqslant \big\|w(1-\psi_R)\big\|_{\frac{Qp}{Q-p}} + \big\|(Xw)(1-\psi_R)\big\|_p + \|wX\psi_R\|_p\\
&\quad\leqslant \left(\int_{B_R^C} |w|^{\frac{Qp}{Q-p}}\,dx\right)^{\frac{Q-p}{Qp}} + \left(\int_{B_R^C} |Xw|^p\,dx\right)^{\frac{1}{p}}\\
&\qquad + C(p,Q)\left(\int_{B_{2R}\setminus B_R} |w|^{\frac{Qp}{Q-p}}\,dx\right)^{\frac{Q-p}{Qp}} \to 0 \quad \text{as } R \to \infty.
\end{aligned}$$

Apparently, we can choose R so large that

$$\|w - w\psi_R\|_{\frac{Qp}{Q-p}} + \big\|X(w - w\psi_R)\big\|_p < \epsilon/2.$$

Next, since $w\psi_R$ has a bounded support, we can find a $C_0^\infty(\mathbb{G})$ function $\phi(x)$ such that

$$\|w\psi_R - \phi\|_{\frac{Qp}{Q-p}} + \big\|X(w\psi_R - \phi)\big\|_p < \epsilon/2.$$

Then this ϕ satisfies (3.4). The theorem is proved. □

We now define $B_0^{1,p}(\mathbb{G})$ as a closed subspace of $B^{1,p}(\mathbb{G})$ that is the completion of $C_0^\infty(\mathbb{G})$ functions under the norm

$$\|w\|_{\frac{Qp}{Q-p}} + \|Xw\|_p.$$

We will prove that the codimension of this subspace in $B^{1,p}(\mathbb{G})$ is one. Indeed, it is sufficient to show that for every f in $B^{1,p}(\mathbb{G})$ the difference $w = f - (f)_\infty$ is in $B_0^{1,p}(\mathbb{G})$.

Theorem 3.3. *The Sobolev space $B_0^{1,p}(\mathbb{G})$ is a subspace of $B^{1,p}(\mathbb{G})$ with co-dimension one. Equivalently, for any function $f \in B^{1,p}(\mathbb{G})$ satisfying* (1.1) *and for any $\epsilon > 0$, there exists a $C_0^\infty(\mathbb{G})$ function ϕ such that*

$$\left(\int_{\mathbb{G}} \left|f(x) - (f)_\infty - \phi\right|^{\frac{Qp}{Q-p}}\,dx\right)^{\frac{Q-p}{Qp}} + \left(\int_{\mathbb{G}} \left|X(f - \phi)\right|^p\,dx\right)^{\frac{1}{p}} < \epsilon.$$

Proof. First of all, by Theorem 2.1, for every f in $B^{1,p}(\mathbb{G})$ the function $w = f - (f)_\infty$ satisfies the condition of Proposition 3.2. Thus for every $\epsilon > 0$ there is a $C_0^\infty(\mathbb{G})$ function $\phi(x)$ satisfying $\|X(w - \phi)\|_p < \epsilon$. Moreover, it is clear that $(\phi)_\infty = 0$, the norm of $w - \phi$ as defined in (3.2) is less than ϵ. Thus, $w \in B_0^{1,p}(\mathbb{G})$. Therefore, $B_0^{1,p}(\mathbb{G})$ is a subspace of $B^{1,p}(\mathbb{G})$ with co-dimension one. □

4. Global Poincaré inequalities of higher orders

Polynomials on homogeneous groups bear some resemblance to those in the Euclidean spaces. We refer the reader to Folland, Stein [6] for more details.

In what follows, C denotes various positive constants. They may differ even in a same string of estimates. Moreover, sometimes, we will use $C(\alpha, \beta, \ldots)$ instead of C to emphasize that the constant is depending on $\alpha, \beta, \ldots,$ and $r(B)$ denote the radius of a metric ball B.

We start with the following lemma.

Lemma 4.1. *(See [6,17].) For each nonnegative integer k there exists a positive constant $C > 0$ such that for any $x_0 \in \mathbb{G}$, $r > 0$, and $s \geqslant 1$ and P is a polynomial of degree k,*

$$\left|X^I P(x_0)\right| \leqslant C r^{-d(I)} \sup_{x\in B(x_0,r)} \left|P(x)\right|, \quad \textit{for } d(I) \leqslant k,$$

$$\sup_{x\in B(x_0,sr)} \left|P(x)\right| \leqslant C s^k \sup_{x\in B(x_0,r)} \left|P(x)\right|,$$

$$\sup_{x\in B(x_0,r)} \left|P(x)\right| \leqslant C \frac{1}{|B(x_0,r)|} \int\limits_{B(x_0,r)} \left|P(x)\right| dx.$$

The following theorem was proved in [15,17].

Theorem 4.2. *Let $\Omega \subset \mathbb{G}$ be an open set of finite Lebesgue measure. Then given any positive integer m and $f \in W^{m,1}(\Omega)$, there exists a unique polynomial $P = P_m(\Omega, f)$ on $\mathbb{G}$ of degree less than m such that*

$$\int\limits_{\Omega} X^I (f - P) = 0 \quad \textit{for all } I \textit{ with } 0 \leqslant d(I) < m. \tag{4.1}$$

The existence of polynomials satisfying (4.1) was also proved in [21].

Using these theorems, we obtained the following results which were proved in [15,17] concerning higher order Poincaré inequalities.

Theorem 4.3. *Let m be a positive integer, $p \geqslant 1$, B be a ball, and $f \in W^{m,p}(B)$. Then there exists a polynomial $P = P_m(B, f) \in \mathcal{P}_m$ such that for any integer j with $0 \leqslant j < m$,*

$$\left(\frac{1}{|B|}\int\limits_B \left|X^j (f-P)(x)\right|^{q_{mj}} dx\right)^{\frac{1}{q_{mj}}} \leqslant C r(B)^{m-j} \left(\frac{1}{|B|}\int\limits_B \left|X^m f(x)\right|^p dx\right)^{\frac{1}{p}}$$

for all $1 \leqslant p < \frac{Q}{m-j}$ and $q_{mj} = \frac{pQ}{Q-(m-j)p}$, where C is independent of B and f.

In [15,17], a second class of polynomials associated with Sobolev functions is considered. Polynomials in this class are called “projection polynomials” and are described in the next definition.

Definition 4.4. For each $m \in N$ and ball $B \subset \mathbb{G}$, a projection of order m associated with B is defined to be a linear map

$$\pi_m(B, \cdot) : W^{m,1}(B) \to \mathcal{P}_m$$

such that the following two properties hold:

$$\sup_{x\in B} \left|\pi_m(B, f)(x)\right| \leqslant C r(B)^{-Q} \|f\|_{L^1(B)}, \tag{4.2}$$

with C independent of f and B and

$$\pi_m(B, P) = P \quad \text{for all } P \in \mathcal{P}_m. \tag{4.3}$$

We will refer to $\pi_m(B, f)$ as a projection polynomial of order $m - 1$ associated with B and f.

The polynomials constructed in Theorem 4.2 may not satisfy the formula (4.1). The existence of projection polynomials is proved in [17, Theorem 3.6]. A remarkable property of projection polynomials is the following theorem which is essentially in proving the particular extension theorem needed to show our global Poincaré inequalities on extension domains.

Theorem 4.5. *Let $m \in N$ and B be a ball in Ω. Then for any projection $\pi_m(B,\cdot): W^{m,1}(B) \to \mathcal{P}_m$, any q with $1 \leqslant q \leqslant \infty$, and any multiple index I with $d(I) = i \geqslant 0$,*

$$\left\| X^I \pi_m(B, f) \right\|_{L^q(B)} \leqslant C \left\| X^i f \right\|_{L^q(B)}, \tag{4.4}$$

with C independent of f and B.

This shows that a sub-elliptic derivative of $\pi_m(B, f)$ is controlled by the same order sub-elliptic derivative of f. Moreover, if we choose the projection polynomial $\pi_m(B, f)$, then Theorem 4.3 can be improved as follows (see [17, Theorem 6.3]).

Theorem 4.6. *Let m be a positive integer, $p \geqslant 1$, B be a ball, and $f \in W^{m,p}(B)$. Then for any integers i, j with $0 \leqslant j < i \leqslant m$,*

$$\left(\frac{1}{|B|} \int_B \left| X^j \big(f - \pi_m(B, f)\big)(x) \right|^{q_{ij}} dx \right)^{\frac{1}{q_{ij}}} \leqslant C r(B)^{i-j} \left(\frac{1}{|B|} \int_B \left| X^i f(x) \right|^p dx \right)^{\frac{1}{p}}$$

for all $1 \leqslant p < \frac{Q}{i-j}$ and $q_{ij} = \frac{pQ}{Q-(i-j)p}$, where C is independent of B and f.

Let m be an integer and let p satisfy $1 \leqslant p < Q/m$. Let $f(x)$ be a function on $\mathbb{G}$ satisfying

$$f \in W^{m,p}_{\mathrm{loc}}(\mathbb{G}) \quad \text{and} \quad \left\| X^m f \right\|_p < \infty. \tag{4.5}$$

In the inequality above, $\|X^m f\|_p$ denotes the sum of the L^p norm of all the mth sub-elliptic derivatives of f. Namely, $\|X^m f\|_p < \infty$ means

$$\left| X^m f(x) \right| = \left(\sum_{I:\, d(I)=m} \left| X^I f(x) \right|^2 \right)^{\frac{1}{2}} \in L^p(\mathbb{G}).$$

To see the simplest case of the global higher order Poincaré inequality (case $j = 0$ in Theorem 1.1), we first extend Theorem 2.1 to the following theorem of higher order Poincaré inequality.

Theorem 4.7. *Suppose f satisfies* (4.5)*. Then there exists a unique polynomial $P \in \mathcal{P}_m$ such that*

$$\| f - P \|_{Qp/(Q-mp)} \leqslant C(p, Q) \left\| X^m f \right\|_p. \tag{4.6}$$

Before we prove the Poincaré inequality, we need to prove the following proposition.

Proposition 4.8. *If we take a sequence of concentric balls $B_k = B_{2^k}(0)$ centered at 0 for $k \geqslant 1$, then the associated polynomial $P_m(B_k, f) \in \mathcal{P}_m$ given in Theorem 4.2 converges to a polynomial $P \in \mathcal{P}_m$.*

Proof. Given any two balls $B_1 \subset B_2 \subset \mathbb{G}$ which contain the given ball $D = B(0, R_0)$, the ball centered at 0 and with radius R_0. Then for any $x \in D$

$$
\begin{aligned}
&\bigl|P_m(B_1, f)(x) - P_m(B_2, f)(x)\bigr| \\
&\quad \leqslant \frac{1}{|B_1|}\int\limits_{B_1} \bigl|P_m(B_1, f)(y) - P_m(B_2, f)(y)\bigr|\,dy \\
&\quad \leqslant \frac{1}{|B_1|}\int\limits_{B_1} \bigl|P_m(B_1, f)(y) - f(y)\bigr|\,dy + \frac{1}{|B_1|}\int\limits_{B_1} \bigl|P_m(B_2, f)(y) - f(y)\bigr|\,dy \\
&\quad \leqslant \left(\frac{1}{|B_1|}\int\limits_{B_1} \bigl|P_m(B_1, f)(y) - f(y)\bigr|^p\,dy\right)^{\frac{1}{p}} + \left(\frac{1}{|B_1|}\int\limits_{B_1} \bigl|P_m(B_2, f)(y) - f(y)\bigr|^p\,dy\right)^{\frac{1}{p}} \\
&\quad \leqslant Cr(B_1)^m\left(\frac{1}{|B_1|}\int\limits_{B_1} \bigl|X^m f(y)\bigr|^p dy\right)^{\frac{1}{p}} + Cr(B_2)^m \frac{|B_2|^{1-1/p}}{|B_1|}\left(\int\limits_{B_2} \bigl|X^m f(y)\bigr|^p\,dy\right)^{\frac{1}{p}}.
\end{aligned}
$$

We now take a sequence of balls B_k such that $D \subset B_k \subset B_{k+1}$ and $B_k = B_{2^k}(0)$ for $k \geqslant 1$, then

$$
\begin{aligned}
&\bigl|P_m(B_k, f)(x) - P_m(B_{k+1}, f)(x)\bigr| \\
&\quad \leqslant C2^{km}\left(\frac{1}{2^{kQ}}\int\limits_{B_k} \bigl|X^m f(y)\bigr|^p\,dy\right)^{\frac{1}{p}} + C\bigl(2^{k+1}\bigr)^m \frac{2^{(kQ(1-1/p)}}{2^{kQ}}\left(\int\limits_{B_{k+1}} \bigl|X^m f(y)\bigr|^p\,dy\right)^{\frac{1}{p}} \\
&\quad \leqslant C2^{km-\frac{kQ}{p}}\left(\int\limits_{\mathbb{G}} \bigl|X^m f(y)\bigr|^p\,dy\right)^{\frac{1}{p}} + C\bigl(2^{km+kQ(1-1/p)-kQ}\bigr)\left(\int\limits_{\mathbb{G}} \bigl|X^m f(y)\bigr|^p\,dy\right)^{\frac{1}{p}} \\
&\quad = C2^{km-\frac{kQ}{p}}\left(\int\limits_{\mathbb{G}} \bigl|X^m f(y)\bigr|^p\,dy\right)^{\frac{1}{p}}.
\end{aligned}
$$

Next we take a sequence of balls $\{B_k = B_{2^k}(0)\}$ for $k = 0, 1, \ldots.$ Then for $j < l$ we have

$$
\begin{aligned}
\bigl|P_m(B_k, f)(x) - P_m(B_l, f)(x)\bigr| &\leqslant \sum_{j=k}^{l-1} \bigl|P_m(B_j, f)(x) - P_m(B_{j+1}, f)(x)\bigr| \\
&\leqslant C\sum_{j=k}^{l-1} 2^{jm-\frac{jQ}{p}}\left(\int\limits_{\mathbb{G}} \bigl|X^m f(y)\bigr|^p\,dy\right)^{\frac{1}{p}}.
\end{aligned}
$$

Since $m < Q/p$, we have

$$\sum_{j=k}^{l-1} 2^{jm-\frac{jQ}{p}} \leqslant C2^{(k-1)(m-Q/p)} \to 0$$

as $k \to \infty$. Therefore, $P_m(B_k, f)$ converges uniformly on D and we denote the limit as P. Since R_0 can be any large number, thus $\lim_{k\to\infty} P_m(B_k, f)(x) = P(x)$ for all $x \in \mathbb{G}$.

However, it is not an immediate result that $P(x)$ is actually a polynomial. We will show this fact below.

Using the Bernstein's inequality (see Lemma 4.1), we have for any $x \in D = B(0, R_0)$ contained in B_k for large k:

$$\begin{aligned}
&\left|X^I\big(P_m(B_k, f)(x) - P_m(B_l, f)(x)\big)\right| \\
&\quad \leqslant R_0^{-d(I)} \left\| P_m(B_k, f) - P_m(B_l, f)\right\|_{L^\infty(B(x,R_0))} \\
&\quad \leqslant R_0^{-d(I)} \left\| P_m(B_k, f) - P_m(B_l, f)\right\|_{L^\infty(B(0,2R_0))} \\
&\quad \leqslant C R_0^{-d(I)} 2^{(k-1)(m-Q/p)} \left(\int_{\mathbb{G}} \left|X^m f(y)\right|^p dy \right)^{\frac{1}{p}}.
\end{aligned}$$

Again, as $k \to \infty$, $X^I(P_m(B_k, f))$ converges uniformly on D. We assume it converges to $g(x)$. It is a standard argument that $X^I P(x)$ exists for every $x \in D$. Since D can be arbitrarily large, thus for all $x \in \mathbb{G}$ and $g(x) = X^I P(x)$.

Since $X^I P_m(B_k, f) = 0$ for all $d(I) \geqslant m$, we conclude that $X^I P(x) = 0$ for all $d(I) \geqslant m$ and all $x \in \mathbb{G}$. This shows that P is a polynomial of degree no greater than $m - 1$, i.e., $P \in \mathcal{P}_m$. □

We are now ready to prove Theorem 4.7. As a matter of fact, we will prove a more general theorem than that. What we will show below is a simultaneous Poincaré inequality which not only control $f - P$ but also on its sub-elliptic derivatives simultaneously.

Theorem 4.9. *Let m be a positive integer, $p \geqslant 1$, and $f \in W^{m,p}_{\mathrm{loc}}(\mathbb{G})$ and $|X^m f| \in L^p(\mathbb{G})$. Then there exists a polynomial P of degree not greater than $m - 1$ such that for any integer j with $0 \leqslant j < m$,*

$$\left(\int_{\mathbb{G}} \left|X^j(f - P)(x)\right|^{q_{mj}} dx \right)^{\frac{1}{q_{mj}}} \leqslant C\left(\int_{\mathbb{G}} \left|X^m f(x)\right|^p dx \right)^{\frac{1}{p}}$$

for all $1 \leqslant p < \frac{Q}{m-j}$ and $q_{mj} = \frac{pQ}{Q-(m-j)p}$, where C is independent of f.

Proof. Now we recall the standard higher order Poincaré inequality

$$\left(\int_{B} \left|X^j\big(f - P_m(B, f)\big)(x)\right|^{q_{mj}} dx \right)^{\frac{1}{q_{mj}}} \leqslant C\left(\int_{B} \left|X^m f(x)\right|^p dx \right)^{\frac{1}{p}}$$

for all $1 \leqslant p < \frac{Q}{m-j}$ and $q_{mj} = \frac{pQ}{Q-(m-j)p}$.

For any $0 < R_1 < R_2$, we have

$$\begin{aligned}
&\left(\int_{B_{R_1}} \left|X^j\left(f(x) - P_m(B_{R_2}, f)\right)\right|^{q_{mj}}\,dx\right)^{\frac{1}{q_{mj}}} \\
&\quad\leqslant \left(\int_{B_{R_2}} \left|X^j\left(f(x) - P_m(B_{R_2}, f)\right)\right|^{q_{mj}}\,dx\right)^{\frac{1}{q_{mj}}} \\
&\quad\leqslant C(p, Q)\left(\int_{B_{R_2}} \left|X^m f\right|^p\,dx\right)^{\frac{1}{p}}.
\end{aligned}$$

Letting $R_2 \to \infty$ in the inequality above, we obtain by using Proposition 4.8 that there exists a polynomial P of degree less than m such that

$$\left(\int_{B_{R_1}} \left|X^j\left(f(x) - P(x)\right)\right|^{q_{mj}}\,dx\right)^{\frac{1}{q_{mj}}} \leqslant C(p, Q)\left(\int_{\mathbb{G}} \left|X^m f\right|^p\,dx\right)^{\frac{1}{p}}.$$

We then conclude the higher order Poincaré inequalities on the entire group $\mathbb{G}$ after letting $R_1 \to \infty$. $\square$

If we choose the projection polynomial $\pi_m(B, f)$ in the local Poincaré inequality over any ball B, then Theorem 4.9 can be improved as follows.

Theorem 4.10. *Let m and i be positive integers and $0 < i \leqslant m$, $p \geqslant 1$ and $f \in W^{m,p}_{\mathrm{loc}}(\mathbb{G})$ and $|X^i f| \in L^p(\mathbb{G})$. Then there exists a unique polynomial $P \in \mathcal{P}_m$ such that for any integers i, j with $0 \leqslant j < i \leqslant m$,*

$$\left(\int_{\mathbb{G}} \left|X^j(f - P)(x)\right|^{q_{ij}}\,dx\right)^{\frac{1}{q_{ij}}} \leqslant C\left(\int_{\mathbb{G}} \left|X^i f(x)\right|^p\,dx\right)^{\frac{1}{p}}$$

for all $1 \leqslant p < \frac{Q}{i-j}$ and $q_{ij} = \frac{pQ}{Q-(i-j)p}$, where C is independent of f.

Remark. We point out here that Theorems 4.9 and 4.10 are substantially different in the sense that we only assume $|X^i f| \in L^p(\mathbb{G})$ instead of $|X^m f| \in L^p(\mathbb{G})$ in Theorem 4.10. However, $f \in W^{m,p}_{\mathrm{loc}}(\mathbb{G})$ guarantees the existence of polynomial $\pi_m(B, f)$ such that the localized Poincaré inequality holds, and therefore we still can prove the existence of a polynomial of order $m - 1$ such that the global Poincaré inequality holds.

Proof of Theorem 4.10. If we choose the projection polynomial $\pi_m(B, f)$, then we have the Poincaré inequality stated in Theorem 4.6.

Thus for any $0 < R_1 < R_2$, we have

$$\begin{aligned}
&\Bigl(\int\limits_{B_{R_1}} \bigl|X^j\bigl(f(x) - \pi_m(B_{R_2}, f)\bigr)\bigr|^{q_{ij}}\,dx\Bigr)^{\frac{1}{q_{ij}}} \\
&\quad \leqslant \Bigl(\int\limits_{B_{R_2}} \bigl|X^i\bigl(f(x) - \pi_m(B_{R_2}, f)\bigr)\bigr|^{q_{ij}}\,dx\Bigr)^{\frac{1}{q_{ij}}} \\
&\quad \leqslant C(p, Q)\Bigl(\int\limits_{B_{R_2}} \bigl|X^m f\bigr|^p\,dx\Bigr)^{\frac{1}{p}}.
\end{aligned}$$

In the same way we proved Proposition 4.8, we can show for a sequence of balls B_k such that for any given ball $D = B(0, R_0)$ with $D \subset B_k \subset B_{k+1}$ and $B_k = B_{2^k}(0)$ for $k \geqslant 1$ large, we have for $x \in D$

$$\begin{aligned}
&\bigl|\pi_m(B_k, f)(x) - \pi_m(B_{k+1}, f)(x)\bigr| \\
&\quad \leqslant C2^{k(i-j)}\Bigl(\frac{1}{2^{kQ}}\int\limits_{B_k} \bigl|X^i f(y)\bigr|^p\,dy\Bigr)^{\frac{1}{p}} + C\bigl(2^{(k+1)(i-j)}\bigr)\frac{2^{(kQ(1-1/p)}}{2^{kQ}}\Bigl(\int\limits_{B_{k+1}} \bigl|X^i f(y)\bigr|^p\,dy\Bigr)^{\frac{1}{p}} \\
&\quad \leqslant C2^{k(i-j)-\frac{kQ}{p}}\Bigl(\int\limits_{\mathbb{G}} \bigl|X^i f(y)\bigr|^p\,dy\Bigr)^{\frac{1}{p}} + C\bigl(2^{k(i-j)+kQ(1-1/p)-kQ}\bigr)\Bigl(\int\limits_{\mathbb{G}} \bigl|X^i f(y)\bigr|^p\,dy\Bigr)^{\frac{1}{p}} \\
&\quad = C2^{k(i-j)-\frac{kQ}{p}}\Bigl(\int\limits_{\mathbb{G}} \bigl|X^i f(y)\bigr|^p\,dy\Bigr)^{\frac{1}{p}}.
\end{aligned}$$

Next we take a sequence of balls $\{B_k = B_{2^k}(0)\}$ for $k \geqslant 1$. Then for $k < l$ we have

$$\begin{aligned}
&\bigl|\pi_m(B_k, f)(x) - \pi_m(B_l, f)(x)\bigr| \\
&\quad \leqslant \sum_{n=k}^{l-1} \bigl|\pi_m(B_n, f)(x) - \pi_m(B_{n+1}, f)(x)\bigr| \\
&\quad \leqslant C\sum_{n=k}^{l-1} 2^{n(i-j)-\frac{nQ}{p}}\Bigl(\int\limits_{\mathbb{G}} \bigl|X^i f(y)\bigr|^p\,dy\Bigr)^{\frac{1}{p}}.
\end{aligned}$$

Since $1 \leqslant p < \frac{Q}{i-j}$, we have

$$\sum_{n=k}^{l-1} 2^{n(i-j)-\frac{nQ}{p}} \leqslant C2^{(k-1)(i-j-Q/p)} \to 0$$

as $k \to \infty$. Therefore, $\pi_m(B_k, f)$ converges uniformly on D and we denote the limit as $P(x)$. Since R_0 can be any large number, thus $\lim_{k\to\infty} P_m(B_k, f)(x) = P(x)$ for all $x \in \mathbb{G}$.

We argue in the same way as we did in the proof of Proposition 4.8, we conclude that P is also a polynomial.

The remaining proof is similar to that given in the proof of Theorem 4.9 and we shall not give the details. $\square$

Next we extend Theorem 3.1 to the case of higher order.

Theorem 4.11. *Let $B^{m,p}(\mathbb{G})$ be the function space consisting of all functions satisfying* (4.5). *For each element f let $P \in \mathcal{P}_m$ be the polynomial associated to f in Theorem* 4.3 *and let $P(x) = \sum_{I:\, d(I)\leqslant m-1} a_I \eta^I(x)$. Then $B^{m,p}(\mathbb{G})$ is a complete Banach space with the norm*

$$\|f\| = \left(\int_{\mathbb{G}} \left|X^m f\right|^p dx + \sum_{d(I)\leqslant m-1} |a_I|^p\right)^{\frac{1}{p}}.$$

Proof. Let f_k be a Cauchy sequence in $B^{m,p}(\mathbb{G})$ and let $P_m(f_k)(x) = \sum_{d(I)\leqslant m-1} a_I^k \eta^I(x)$ be the polynomial associated to f_k. Then

$$\left(\sum_{d(I)\leqslant m-1} \left|a_I^k - a_I^l\right|^p\right)^{\frac{1}{p}} \to 0$$

as $k, l \to \infty$. Therefore, there exist $\{a_I\}_{I:\, d(I)\leqslant m-1}$ and a subsequence of $\{a_I^k\}$ (still denoted by the same notation) such that for $d(I) \leqslant m-1$ we have $a_I^k \to a_I$ as $k \to \infty$.

Since $f_k \in B^{m,p}(\mathbb{G})$, we have for any integer j with $0 \leqslant j < m$,

$$\left(\int_{\mathbb{G}} \left|X^j\big(f_k - P_m(f_k)\big)(x)\right|^{q_{mj}} dx\right)^{\frac{1}{q_{mj}}} \leqslant C\left(\int_{\mathbb{G}} \left|X^m f_k(x)\right|^p dx\right)^{\frac{1}{p}}$$

for all $1 \leqslant p < \frac{Q}{m-j}$ and $q_{mj} = \frac{pQ}{Q-(m-j)p}$, where C is independent of f.

Given any ball $B = B(0, R)$, we thus have

$$\begin{aligned}&\left(\int_B \left|X^j(f_k - f_l)(x)\right|^{q_{mj}} dx\right)^{\frac{1}{q_{mj}}}\\ &\quad\leqslant C\left(\int_B \left|X^m\big(f_k(x) - f_l(x)\big)\right|^p dx\right)^{\frac{1}{p}} + |B|\left(\sum_{d(I)\leqslant m-1} \left|a_I^k - a_I^l\right|^p\right)^{\frac{1}{p}}.\end{aligned}$$

Thus,

$$\left(\int_B \left|X^j(f_k - f_l)(x)\right|^{q_{mj}} dx\right)^{\frac{1}{q_{mj}}} \to 0$$

as $k, l \to \infty$. It is thus a standard argument that there exists a function $f \in W^{m,p}(B)$ such that $X^j f_k \to X^j f$ in $L^{q_{mj}}(B)$ for all j with $0 \leqslant j < m$ and $X^m f_k \to X^m f$ in $L^p(\mathbb{G})$. Since B can be arbitrarily large, we thus have shown $f \in W^{m,p}_{\text{loc}}(\mathbb{G})$ and $|X^m f| \in L^p(\mathbb{G})$. Letting $l \to \infty$ we get $f_k \to f$ in $B^{m,p}(\mathbb{G})$. $\square$

We now prove a density theorem similar to Theorem 3.3. We will first prove the following theorem.

Theorem 4.12. *Let $q_{mj} = \frac{Qp}{Q-(m-j)p}$ for $0 \leqslant j \leqslant m$. If w satisfies*

$$\sum_{j=0}^{m} \left\| X^j w \right\|_{q_{mj}} < \infty,$$

then for any $\epsilon > 0$ there exists a function $\phi \in C_0^\infty(\mathbb{G})$ such that

$$\sum_{j=0}^{m-1} \left\| X^j (w - \phi) \right\|_{q_{mj}} + \left\| X^m (w - \phi) \right\|_p < \epsilon,$$

where $q_{mj} = \frac{pQ}{Q-(m-j)p}$.

Proof. Let $R > 0$ and let $\psi_R(x)$ be a $C_0^\infty(\mathbb{G})$ function satisfying

$$\begin{cases} \psi_R(x) = 1 & \text{if } |x| \leqslant R, \\ \psi_R(x) = 0 & \text{if } |x| \geqslant 2R, \\ \left|\psi_R(x)\right| \leqslant 1 & \text{for all } x, \\ \left|X^j \psi_R(x)\right| \leqslant C/R^j & \text{for all } x \text{ and } j \geqslant 1. \end{cases}$$

Then for any multi-index I, J, K with $J = I + K$, $j = d(J) = d(I) + d(K) = i + k$, and $0 \leqslant j \leqslant m$ and $k > 0$ we have

$$\begin{aligned} &\left(\int_{\mathbb{G}} \left| X^I f X^K \psi_R \right|^{q_{mj}} dx \right)^{\frac{1}{q_{mj}}} \\ &\quad \leqslant \frac{C}{R^k} \left(\int_{B_{2R} \setminus B_R} \left| X^i f \right|^{q_{mj}} dx \right)^{\frac{1}{q_{mj}}} \\ &\quad = C(p, Q) R^{-k + \frac{Q}{q_{mj}}} \left(\frac{1}{|B_{2R} \setminus B_R|} \int_{B_{2R} \setminus B_R} \left| X^i f \right|^{q_{mj}} dx \right)^{\frac{1}{q_{mi}}} \\ &\quad \leqslant C(p, Q) R^{k + \frac{Q}{q_{mj}}} \left(\frac{1}{|B_{2R} \setminus B_R|} \int_{B_{2R} \setminus B_R} \left| X^i f \right|^{q_{mi}} dx \right)^{\frac{1}{q_{mi}}} \to 0 \end{aligned}$$

$$= C\left(\int\limits_{B_{2R}\setminus B_R} \left|X^i f\right|^{q_{mi}} dx\right)^{\frac{1}{q_{mi}}} \to 0$$

as $R \to \infty$, where in the last equality we have used the fact that $q_{mi} = \frac{pQ}{Q-(m-i)p}$, $q_{mj} = \frac{pQ}{Q-(m-j)p}$ and $j = i + k$. On the other hand, $J = I$ when $k = 0$. So we have

$$\left(\int\limits_{\mathbb{G}} \left|\left(X^J f\right)(1-\psi_R)\right|^{q_{mj}} dx\right)^{\frac{1}{q_{mj}}} \leqslant \left(\int\limits_{\mathbb{G}\setminus B_R} \left|X^j f\right|^{q_{mj}} dx\right)^{\frac{1}{q_{mj}}} \to 0$$

as $R \to \infty$. Thus for any J with $0 \leqslant d(J) \leqslant m$ we have

$$\begin{aligned}
&\left(\int\limits_{\mathbb{G}} \left|X^J\left(f(1-\psi)\right)\right|^{q_{mj}} dx\right)^{\frac{1}{q_{mj}}} \\
&\quad \leqslant \sum_{I,K:\ J=I+K} \left(\int\limits_{\mathbb{G}} \left|X^I f X^K \psi_R\right|^{q_{mj}} dx\right)^{\frac{1}{q_{mj}}} \\
&\quad \leqslant C\left(\int\limits_{B_{2R}\setminus B_R} \left|X^j f\right|^{q_{mj}} dx\right)^{\frac{1}{q_{mj}}} + C\left(\int\limits_{\mathbb{G}\setminus B_R} \left|X^j f\right|^{q_{mj}} dx\right)^{\frac{1}{q_{mj}}} \to 0
\end{aligned}$$

as $R \to \infty$. Therefore, as long as we choose R large enough we have that

$$\sum_{j=0}^{m}\left(\int\limits_{\mathbb{G}} \left|X^j\left(f(1-\psi)\right)\right|^{q_{mj}} dx\right)^{\frac{1}{q_{mj}}} < \epsilon.$$

Next, since $f\psi_R$ has a compact support, we can find a $C_0^\infty(\mathbb{G})$ function $\phi(x)$ such that

$$\sum_{j=0}^{m}\left(\int\limits_{\mathbb{G}} \left|X^j \psi - X^j \phi\right|^{q_{mj}} dx\right)^{\frac{1}{q_{mj}}} < \epsilon.$$

Then this ϕ satisfies (3.4). The theorem is proved. □

As an application of Theorems 4.10–4.12, we get

Theorem 4.13. *If we define $B_0^{m,p}(\mathbb{G})$ as the closed subspace of $B^{m,p}(\mathbb{G})$ which contains the $C_0^\infty(\mathbb{G})$ functions as a dense subset. Namely, given any function $f \in B^{m,p}(\mathbb{G})$ and for any $\epsilon > 0$, there exists a polynomial P of degree no more than $m-1$, and a function $\phi \in C_0^\infty(\mathbb{G})$ such that*

$$\sum_{j=0}^{m-1} \left\|X^j(f-P-\phi)\right\|_{q_{mj}} + \left\|X^m(f-P-\phi)\right\|_p < \epsilon,$$

where $q_{mj} = \frac{pQ}{Q-(m-j)p}$. *Consequently, the co-dimension of this subspace equals the dimension of the linear space consisting of all polynomials of order less than* m.

5. Weighted Poincaré inequalities of higher order

We first recall weighted simultaneous Poincaré inequalities for high order vector field gradients on metric balls in a stratified group $\mathbb{G}$ derived in [19], assuming a balance condition similar to the one in [3]. Such results were obtained by using the simultaneous representation formulas given in [19] together with weighted results for integral operators of potential type derived in [24,25]. We note here that the higher order simultaneous representation formulas given in [19] was motivated by the first order representation formula in [8].

If $w(x) \in L^1_{\mathrm{loc}}(\mathbb{G})$ and $w(x) \geqslant 0$, we say that w is a weight and use the notation $w(E) = \int_E w(x)\,dx$ for any measurable set E. If w is a weight, we say that $w \in A_p$, $1 < p < \infty$, if there is a constant C such that for all metric balls B,

$$\left(\frac{1}{|B|}\int_B w(x)\,dx\right)^{\frac{1}{p}}\left(\frac{1}{|B|}\int_B w(x)^{-\frac{p'}{p}}\,dx\right)^{\frac{1}{p'}} \leqslant C,$$

where $p' = p/(p-1)$. A Borel measure μ on $\mathbb{G}$ is said to be doubling of order N if there is a constant $C > 0$ such that for any balls B_1 and B_2 with $B_1 \subset B_2$,

$$\mu(B_2) \leqslant C\left(\frac{r(B_2)}{r(B_1)}\right)^N \mu(B_1).$$

Clearly, Lebesgue measure is doubling of order Q. In case the measure $d\mu = w\,dx$ is a doubling measure, we will say that w is doubling. It is not hard to see that w is doubling if $w \in A_p$ for some p.

We now recall the following weighted local Poincaré inequality proved in [19].

Theorem 5.1. *Let* B_0 *be a ball in a stratified Lie group* $\mathbb{G}$, *and let* m, j *be integers with* $0 \leqslant j < m$. *Suppose that* w_1, w_2 *are weights satisfying the following balance conditions for some* p, q_j *with* $1 < p < q_j < \infty$:

$$\left(\frac{r(B)}{r(B_0)}\right)^{m-j}\left(\frac{w_2(B)}{w_2(B_0)}\right)^{\frac{1}{q_j}} \leqslant C\left(\frac{w_1(B)}{w_1(B_0)}\right)^{\frac{1}{p}} \tag{5.1}$$

for all metric balls B *with* $B \subset cB_0$, *where* c *is a suitably large geometric constant. Suppose also that* $w_1 \in A_p$ *and* w_2 *is doubling. If* $f \in W^{m,p}(B_0)$, *then for either of the polynomials* $P = P_m(B_0, f)$ *or* $P = \pi_m(B_0, f)$,

$$\left(\frac{1}{w_2(B_0)}\int_{B_0}\left|X^j(f-P)\right|^{q_j} w_2\,dx\right)^{\frac{1}{q_j}} \leqslant Cr(B_0)^{m-j}\left(\frac{1}{w_1(B_0)}\int_{B_0}\left|X^m f\right|^p w_1\,dx\right)^{\frac{1}{p}}.$$

Remark. The balance condition (5.1) leads to the restrictions on the indices (p, q_j).

We also have the following simultaneous weighted Poincaré inequality.

Theorem 5.2. *Let B_0 be a ball in a stratified Lie group $\mathbb{G}$, and let m, i and j be integers with $0 \leqslant j < i \leqslant m$. Suppose that w_1, w_2 are weights satisfying the following balance conditions for some p, q_{ij} with $1 < p < q_{ij} < \infty$:*

$$\left(\frac{r(B)}{r(B_0)}\right)^{i-j}\left(\frac{w_2(B)}{w_2(B_0)}\right)^{\frac{1}{q_{ij}}} \leqslant C\left(\frac{w_1(B)}{w_1(B_0)}\right)^{\frac{1}{p}}$$

for all metric balls B with $B \subset cB_0$, where c is a suitably large geometric constant. Suppose also that $w_1 \in A_p$ and w_2 is doubling. If $f \in W^{m,p}(B_0)$, then

$$\left(\frac{1}{w_2(B_0)}\int_{B_0}\left|X^j\left(f-\pi_m(B_0,f)\right)\right|^{q_{ij}} w_2\,dx\right)^{\frac{1}{q_{ij}}}$$
$$\leqslant Cr(B_0)^{i-j}\left(\frac{1}{w_1(B_0)}\int_{B_0}\left|X^i f\right|^p w_1\,dx\right)^{\frac{1}{p}}.$$

With the same techniques used in non-weighted global Poincaré inequalities and using the localized weighted Poincaré inequalities in Theorems 5.1 and 5.2, we are able to show that

Theorem 5.3. *Let m, j be integers with $0 \leqslant j < m$. Suppose that w_1, w_2 are weights satisfying the following balance conditions for some p, q_j with $1 < p < q_j < \infty$:*

$$\left(\frac{r(B)}{r(B_0)}\right)^{m-j}\left(\frac{w_2(B)}{w_2(B_0)}\right)^{\frac{1}{q_j}} \leqslant C\left(\frac{w_1(B)}{w_1(B_0)}\right)^{\frac{1}{p}}$$

for all metric balls B and B_0 with $B \subset cB_0$, where c is a suitably large geometric constant. Suppose also that $w_1 \in A_p$ and w_2 is doubling and

$$r(B)^{m-j} w_2(B)^{\frac{1}{q}} w_1(B)^{-\frac{1}{p}} \leqslant C \tag{5.2}$$

for all sufficiently large balls $B \subset \mathbb{G}$. If $f \in W^{m,p}_{\mathrm{loc}}(\mathbb{G})$ and $|X^m f| \in L^p(\mathbb{G})$, then there exists a polynomial P of degree no more than $m-1$ such that

$$\left(\int_{\mathbb{G}}\left|X^j(f-P)\right|^{q_j} w_2\,dx\right)^{\frac{1}{q_j}} \leqslant C\left(\int_{\mathbb{G}}\left|X^m f\right|^p w_1\,dx\right)^{\frac{1}{p}}.$$

Remark 1. The assumption (5.2) that

$$r(B)^{m-j} w_2(B)^{\frac{1}{q}} w_1(B)^{-\frac{1}{p}} \leqslant C$$

for all sufficiently large balls $B \subset \mathbb{G}$ is reasonable. As a matter of fact, this assumption also implies that

$$r(B)^{m-j} w_2(B)^{\frac{1}{q}} w_1(B)^{-\frac{1}{p}} \leqslant C$$

for all balls $B \subset \mathbb{G}$ by using the balance condition. This follows from the fact that

$$r(B)^{m-j} w_2(B)^{\frac{1}{q}} w_1(B)^{-\frac{1}{p}} \leqslant C r(B^*)^{m-j} w_2\big(B^*\big)^{\frac{1}{q}} w_1\big(B^*\big)^{-\frac{1}{p}}$$

for all balls $B \subset B^* \subset \mathbb{G}$. By fixing any large ball B^* first, then we conclude that for all small balls B, (5.2) also holds. Thus, (5.2) holds for all balls.

Remark 2. Assumption (5.2) actually restricts the values of the pairs (p, q) as in the case for non-weighted case $w_1 = w_2 = 1$, in which case $q = \frac{Qp}{Q-(m-j)p}$. Therefore, such restriction is natural.

Furthermore, we can show the simultaneous weighted Poincaré inequality as follows.

Theorem 5.4. *Let m, i and j be integers with $0 \leqslant j < i \leqslant m$. Suppose that w_1, w_2 are weights satisfying the following balance conditions for some p, q_{ij} with $1 < p < q_{ij} < \infty$:*

$$\left(\frac{r(B)}{r(B_0)}\right)^{i-j} \left(\frac{w_2(B)}{w_2(B_0)}\right)^{\frac{1}{q_{ij}}} \leqslant C \left(\frac{w_1(B)}{w_1(B_0)}\right)^{\frac{1}{p}}$$

for all metric balls B and B_0 with $B \subset cB_0$, where c is a suitably large geometric constant. Suppose also that $w_1 \in A_p$ and w_2 is doubling and

$$r(B)^{i-j} w_2(B)^{\frac{1}{q_{ij}}} w_1(B)^{-\frac{1}{p}} \leqslant C$$

for all balls $B \subset \mathbb{G}$. If $f \in W^{m,p}_{\mathrm{loc}}(\mathbb{G})$ and $|X^i f| \in L^p(\mathbb{G})$, then there exists a polynomial P of degree no more than $m-1$ such that

$$\left(\int_{\mathbb{G}} \big|X^j(f-P)\big|^{q_{ij}} w_2\,dx\right)^{\frac{1}{q_{ij}}} \leqslant C \left(\int_{\mathbb{G}} \big|X^i f\big|^p w_1\,dx\right)^{\frac{1}{p}}.$$

Proof of Theorem 5.3. Since we assume that $f \in W^{m,p}_{\mathrm{loc}}(\mathbb{G})$ and the weights w_1 and w_2 satisfy the balance condition, then by Theorem 5.1 for every ball $B \subset \mathbb{G}$ there exists a polynomial $P_m(B, f)$ of degree no more than $m-1$ such that

$$\left(\frac{1}{w_2(B)} \int_B \big|X^j\big(f - P_m(B, f)\big)\big|^{q_j} w_2\,dx\right)^{\frac{1}{q_j}} \leqslant C r(B)^{m-j} \left(\frac{1}{w_1(B)} \int_B \big|X^m f\big|^p w_1\,dx\right)^{\frac{1}{p}}.$$

This implies for every ball $B \subset \mathbb{G}$ we have

$$\left(\int_B \left|X^j\big(f - P_m(B, f)\big)\right|^{q_j} w_2\, dx\right)^{\frac{1}{q_j}}$$

$$\leqslant Cr(B)^{m-j} w_2(B)^{\frac{1}{q_j}} w_1(B)^{-\frac{1}{p}} \left(\int_B \left|X^m f\right|^p w_1\, dx\right)^{\frac{1}{p}}.$$

The right-hand side above is bounded above by

$$C\left(\int_B \left|X^m f\right|^p w_1\, dx\right)^{\frac{1}{p}} \leqslant C\left(\int_{\mathbb{G}} \left|X^m f\right|^p w_1\, dx\right)^{\frac{1}{p}}.$$

Given any ball $B^* = B(0, R_0)$, we take balls $B_k = B(0, 2^k)$ such that $B^* \subset B_k$ for k large enough. Thus we have

$$\left(\int_{B^*} \left|X^j\big(f - P_m(B_k, f)\big)\right|^{q_j} w_2\, dx\right)^{\frac{1}{q_j}} \leqslant C\left(\int_{\mathbb{G}} \left|X^m f\right|^p w_1\, dx\right)^{\frac{1}{p}}.$$

Since we assume $|X^m f| \in L^p(\mathbb{G})$, we have that $P_m(B_k, f)$ converges to some polynomial P by Proposition 4.8. By taking $k \to \infty$ we get

$$\left(\int_{B_0} \left|X^j(f - P)\right|^{q_j} w_2\, dx\right)^{\frac{1}{q_j}} \leqslant C\left(\int_{\mathbb{G}} \left|X^m f\right|^p w_1\, dx\right)^{\frac{1}{p}}.$$

Finally, we let $R_0 \to \infty$ and thus conclude that

$$\left(\int_{\mathbb{G}} \left|X^j(f - P)\right|^{q_j} w_2\, dx\right)^{\frac{1}{q_j}} \leqslant C\left(\int_{\mathbb{G}} \left|X^m f\right|^p w_1\, dx\right)^{\frac{1}{p}}. \qquad \square$$

The proof of Theorem 5.4 is similar and we will omit that.

6. Poincaré inequalities on unbounded extension domains

For $1 \leqslant p \leqslant \infty$ and positive integer m, and any weight w, $W_w^{m,p}(\Omega)$ and $E_w^{m,p}(\Omega)$ are the spaces of functions having weak derivatives $X^I f$ with $d(I) \leqslant m$, and satisfying

$$\|f\|_{W_w^{m,p}(\Omega)} = \sum_{0 \leqslant d(I) \leqslant m} \left\|X^I f\right\|_{L_w^p(\Omega)} = \sum_{0 \leqslant d(I) \leqslant m} \left(\int_\Omega \left|X^I f\right|^p dw\right)^{\frac{1}{p}} < \infty \quad \text{if } 1 \leqslant p < \infty,$$

and

$$\|f\|_{E_w^{m,p}(\Omega)} = \sum_{d(I) = m} \left\|X^I f\right\|_{L_w^p(\Omega)} < \infty,$$

respectively. Moreover, in the case when $w = 1$, we will denote $W_w^{m,p}(\Omega)$ and $E_w^{m,p}(\Omega)$ by $W^{m,p}(\Omega)$ and $E^{m,p}(\Omega)$, respectively.

A bounded extension operator on $W_w^{m,p}(\Omega)$ is by definition a bounded linear operator $\Lambda : W_w^{m,p}(\Omega) \to W_w^{m,p}(\mathbb{G})$ such that $\Lambda f|_\Omega = f$, $\forall f \in W_w^{m,p}(\Omega)$. Similarly, we can define a bounded extension operator on $E_w^{m,p}(\Omega)$. Moreover, we use the notation

$$\|\Lambda\| = \sup_{\|f\|_{W_w^{m,p}(\Omega)}=1} \|\Lambda f\|_{W_w^{m,p}(\mathbb{G})} \quad \Big(\text{or} \sup_{\|f\|_{E_w^{m,p}(\Omega)}=1} \|\Lambda f\|_{E_w^{m,p}(\mathbb{G})} \text{ in the case for } E_w^{m,p}\Big).$$

We now give the definition of (ϵ, δ) domains in $\mathbb{G}$.

Definition 6.1. An open set Ω is an (ϵ, δ) domain if for all $x, y \in \Omega$, $\varrho(x, y) < \delta$, there exists a rectifiable curve γ connecting x, y such that γ lies in Ω and

$$l(\gamma) < \frac{\varrho(x, y)}{\varepsilon},$$

$$d(z) > \frac{\epsilon\varrho(x, z)\varrho(y, z)}{\varrho(x, y)} \quad \forall z \in \gamma.$$

Here $l(\gamma)$ is the length of γ and $d(z)$ is the distance between z and the boundary of Ω. Let us decompose $\Omega = \bigcup \Omega_i$ into connected components and define

$$r = \mathrm{rad}(\Omega) = \inf_i \inf_{x \in \Omega_i} \sup_{y \in \Omega_i} \varrho(x, y).$$

The main purpose of this section is to prove Poincaré inequalities on both non-weighted and weighted Folland–Stein Sobolev spaces on unbounded extension domain Ω, where the weight satisfies Muckenhoupt's A_p condition on $\mathbb{G}$ (see Section 5 for definition).

Before we go to state the main theorems of this section, we first recall some known results on extension theorems on weighted Folland–Stein Sobolev spaces on extension domains proved in [16,17].

First, we state the extension theorem on the full weighted Sobolev space $W_w^{m,p}(\Omega)$.

Theorem 6.2. *Let Ω be an (ϵ, δ) domain with* $\mathrm{rad}(\Omega) > 0$ *and let k be a positive integer. If $1 \leqslant p \leqslant \infty$ and $w \in A_p$ when $1 \leqslant p < \infty$, then there exists an extension operator Λ on Ω (i.e., $\Lambda f = f$ a.e. on Ω) such that*

$$\|\Lambda f\|_{W_w^{k,p}(L^p(\mathbb{G}))} \leqslant C\|f\|_{W_w^{k,p}(\Omega)}$$

for all $f \in W_w^{k,p}(\Omega)$ where C depends only on $\epsilon, \delta, k, w, p, Q$ and $\mathrm{rad}(\Omega)$. *Moreover, $\|\Lambda\| \to \infty$ as* $\mathrm{rad}(\Omega) \to 0$ *or as $\epsilon \to 0$ or as $\delta \to 0$.*

Remark. When $p = \infty$, we understand $w = 1$.

Next we also state the extension theorem proved in [17] on the single derivative Sobolev space $E_w^{k,p}(\Omega)$ which is crucial in proving the global Poincaré inequalities on unbounded extension domains.

Theorem 6.3. *Let Ω be an (ϵ, ∞) domain and let k be a positive integer. If $1 \leqslant p \leqslant \infty$ and $w \in A_p$ when $1 \leqslant p < \infty$, then there exists an extension operator Λ on Ω (i.e., $\Lambda f = f$ a.e. on Ω) such that*

$$\|\Lambda f\|_{E_w^{k,p}(\mathbb{G})} \leqslant C\|f\|_{E_w^{k,p}(\Omega)} \quad \textit{for all } f \in E_w^{k,p}(\Omega),$$

where C is independent of $\operatorname{rad}(\Omega)$.

Moreover, we also have the simultaneous extension theorem on different weighted Sobolev spaces (see [16,17]).

Theorem 6.4. *Let $w_i \in A_{p_i}$, $1 \leqslant p_i < \infty$, for $i = 1, \ldots, N$. If Ω is an unbounded (ϵ, ∞) domain, then there exists an extension operator on Ω such that*

$$\|X^{k_i}(\Lambda f)\|_{L^{p_i}_{w_i}(\mathbb{G})} \leqslant C_i \|X^{k_i} f\|_{L^{p_i}_{w_i}(\Omega)}$$

for all i and $f \in \bigcap_{i=1}^N E_{w_i}^{k_i,p_i}(\Omega)$. Here C_i depends only on ϵ, $\operatorname{rad}(\Omega)$, w_i, p_i, k_i, Q and $\max_i k_i$.

The following remarks are in order. First of all, we like to point out that Theorems 6.3 and 6.4 are different from Theorem 6.2 in the sense that the $E_w^{k,p}(\mathbb{G})$ norm of the function Λf is bounded by the $E_w^{k,p}(\Omega)$ norm of f alone, without using the $L_w^p(\Omega)$ norm of lower order derivatives $X^I f$ for $d(I) < k$. This is particularly important in deriving the global Poincaré inequalities on unbounded extension domains. This enables us to control the higher order sub-elliptic derivatives of the extended functions by the same order of sub-elliptic derivatives of the original function. Consequently, we are able to get the Poincaré inequalities on unbounded extension domains by using extension theorems. Secondly, Theorem 6.4 indicate that we can actually construct a single extension operator which is bounded on different weighted Sobolev spaces (e.g., different weights or exponents) simultaneously. In particular, a special case of Theorem 6.4 when $p_i = p$, $w_i = w$ and $0 \leqslant k_i = i \leqslant k$ says that there is an extension operator Λ on $W_w^{m,p}(\Omega)$ such that

$$\|X^i(\Lambda f)\|_{L^p(\mathbb{G})} \leqslant \|X^i f\|_{L^p(\Omega)}$$

for each $0 \leqslant i \leqslant k$. This is exactly Theorem 6.3.

The proof of all extension theorems in [16,17] closely follows from the original ideas of Jones for the non-weighted case in Euclidean spaces. However, we adapted the projection polynomial rather than the polynomial constructed in Theorem 4.2 to define the extension operators. This choice of polynomials is necessary for the proof of Theorems 6.3 and 6.4. The advantage of using the projection polynomials $\pi_k(B, f)$ rather than the approximation $P_k(B, f)$ in Theorem 4.2 to construct the extension operators is that we will be able to get estimates as

$$\|X^I \Lambda f\|_{L^p(\mathbb{G})} \leqslant C\|X^{d(I)} f\|_{L^p(\Omega)}$$

rather than

$$\|X^I \Lambda f\|_{L^p(\mathbb{G})} \leqslant C\sum_{i=0}^{k} \|X^i f\|_{L^p(\Omega)}.$$

This is thanks to the validity of Theorem 4.5. Thirdly, we point out that Theorem 6.2 was also proved in [21] in the case of $w = 1$. However, this theorem is not sufficient to prove the global Poincaré inequalities on unbounded extension domains as we explained above.

We are now ready to state and prove the Poincaré inequalities on unbounded extension domains.

Theorem 6.5. *Assume that $\Omega \subset \mathbb{G}$ is an unbounded (ϵ, ∞) extension domain. Let m be a positive integer, $p \geqslant 1$, and $f \in W^{m,p}_{\mathrm{loc}}(\Omega)$ and $|X^m f| \in L^p(\Omega)$. Then there exists a polynomial $P \in \mathcal{P}_m$ such that for any integer j with $0 \leqslant j < m$,*

$$\left(\int_\Omega \left|X^j(f-P)(x)\right|^{q_{mj}}\,dx\right)^{\frac{1}{q_{mj}}} \leqslant C\left(\int_\Omega \left|X^m f(x)\right|^p\,dx\right)^{\frac{1}{p}}$$

for all $1 \leqslant p < \frac{Q}{m-j}$ and $q_{mj} = \frac{pQ}{Q-(m-j)p}$, where C is independent of f.

Proof. By the extension Theorem 6.3, there exists an extension operator $\Lambda : E^{m,p}(\Omega) \to E^{m,p}(\mathbb{G})$ such that $\Lambda f = f$ on Ω and $\|X^m(\Lambda f)\|_{L^p(\mathbb{G})} \leqslant C\|X^m f\|_{L^p(\Omega)} < \infty$.

By Sobolev embedding theorem, we can conclude that $f \in W^{k,p}_{\mathrm{loc}}(\mathbb{G})$ for all $0 \leqslant k < m$. Therefore, we have $\Lambda f \in W^{m,p}_{\mathrm{loc}}(\mathbb{G})$ and $|X^m(\Lambda f)| \in L^p(\mathbb{G})$. Thus, by Theorem 4.3, there exists a polynomial $P \in \mathcal{P}_m$ of degree not greater than $m-1$ such that for any integer j with $0 \leqslant j < m$,

$$\left(\int_{\mathbb{G}} \left|X^j(\Lambda f-P)(x)\right|^{q_{mj}}\,dx\right)^{\frac{1}{q_{mj}}} \leqslant C\left(\int_{\mathbb{G}} \left|X^m(\Lambda f)(x)\right|^p\,dx\right)^{\frac{1}{p}}$$

for all $1 \leqslant p < \frac{Q}{m-j}$ and $q_{mj} = \frac{pQ}{Q-(m-j)p}$, where C is independent of f. This immediately gives the desired inequality using the properties of the extension operator, namely,

$$\Lambda f = f, \quad \text{a.e. on } \Omega \quad \text{and} \quad \left\|X^m(\Lambda f)\right\|_{L^p(\mathbb{G})} \leqslant C\left\|X^m f\right\|_{L^p(\Omega)}. \qquad \square$$

Similarly, we can prove

Theorem 6.6. *Assume that $\Omega \subset \mathbb{G}$ is an unbounded (ϵ, ∞) extension domain. Let m be a positive integer, $p \geqslant 1$ and $f \in W^{m,p}_{\mathrm{loc}}(\Omega)$ and $|X^i f| \in L^p(\Omega)$. Then there exists a polynomial $P \in \mathcal{P}_m$ such that for any integers i, j with $0 \leqslant j < i \leqslant m$,*

$$\left(\int_\Omega \left|X^j(f-P)(x)\right|^{q_{ij}}\,dx\right)^{\frac{1}{q_{ij}}} \leqslant C\left(\int_\Omega \left|X^i f(x)\right|^p\,dx\right)^{\frac{1}{p}}$$

for all $1 \leqslant p < \frac{Q}{i-j}$ and $q_{ij} = \frac{pQ}{Q-(i-j)p}$, where C is independent of f.

Next we extend Theorem 3.2.

Theorem 6.7. *Let $B^{m,p}(\Omega)$ be the function space consisting of all functions satisfying*

$$f \in W^{m,p}_{\mathrm{loc}}(\Omega) \quad \text{and} \quad \|X^m f\|_{L^p(\Omega)} < \infty.$$

For each element f, let $P \in \mathcal{P}_m$ be the polynomial of order $m-1$ associated to f in Theorem 4.3 *and let $P(x) = \sum_{I:\, d(I)\leqslant m-1} a_I \eta^I(x)$. Then $B^{m,p}(\Omega)$ is a complete Banach space with the norm*

$$\|f\| = \left(\int_\Omega |X^m f|^p\,dx + \sum_{d(I)\leqslant m-1} |a_I|^p\right)^{\frac{1}{p}}.$$

Proof. Let f_k be a Cauchy sequence in $B^{m,p}(\Omega)$ and let $P_k(x) = \sum_{d(I)\leqslant m-1} a_I^k \eta^I(x)$ be the polynomial in $\mathcal{P}_m$ associated to f_k. Then

$$\left(\sum_{d(I)\leqslant m-1} |a_I^k - a_I^l|^p\right)^{\frac{1}{p}} \to 0$$

as $k, l \to \infty$. Therefore, there exist $\{a_I\}_{I:\, d(I)\leqslant m-1}$ and a subsequence of $\{a_I^k\}$ (still denoted by the same notation) such that for $d(I) \leqslant m-1$ we have $a_I^k \to a_I$ as $k \to \infty$.

Since $f_k \in B^{m,p}(\Omega)$, by Theorem 6.5 we have for any integer j with $0 \leqslant j < m$,

$$\left(\int_\Omega |X^j(f_k - P_k)(x)|^{q_{mj}}\,dx\right)^{\frac{1}{q_{mj}}} \leqslant C\left(\int_\Omega |X^m f_k(x)|^p\,dx\right)^{\frac{1}{p}}$$

for all $1 \leqslant p < \frac{Q}{m-j}$ and $q_{mj} = \frac{pQ}{Q-(m-j)p}$, where C is independent of f.

Given any compact subset $K \subset \Omega$, we thus have

$$\begin{aligned}
&\left(\int_K |X^j(f_k - f_l)(x)|^{q_{mj}}\,dx\right)^{\frac{1}{q_{mj}}} \\
&\quad \leqslant C\left(\int_K \left|X^m\big(f_k(x) - f_l(x)\big)\right|^p\,dx\right)^{\frac{1}{p}} + |K|\left(\sum_{d(I)\leqslant m-1} |a_I^k - a_I^l|^p\right)^{\frac{1}{p}}.
\end{aligned}$$

Thus,

$$\left(\int_K |X^j(f_k - f_l)(x)|^{q_{mj}}\,dx\right)^{\frac{1}{q_{mj}}} \to 0$$

as $k, l \to \infty$. It is thus a standard argument that there exists a function $f \in W^{m,p}(K)$ such that $X^j f_k \to X^j f$ in $L^p(K)$ for all j with $0 \leqslant j < m$ and $X^m f_k \to X^m f$ in $L^p(\mathbb{G})$. Since K can be arbitrarily large, we thus have shown $f \in W^{m,p}_{\mathrm{loc}}(\Omega)$ and $|X^m f| \in L^p(\Omega)$. Letting $l \to \infty$ we get $f_k \to f$ in $B^{m,p}(\Omega)$. $\square$

Using the density theorem Theorem 6.8, we are able to derive the following theorem on unbounded extension theorem.

Theorem 6.8. *If we define $B_0^{m,p}(\Omega)$ as the closed subspace of $B^{m,p}(\Omega)$ which contains functions in $C_0^\infty(\Omega)$ as a dense subset, the codimension of $B_0^{m,p}(\Omega)$ in $B^{m,p}(\Omega)$ is finite. Namely, given any function $f \in B^{m,p}(\Omega)$ and for any $\epsilon > 0$, there exists a polynomial P of degree no more than $m-1$, and a function $\phi \in C_0^\infty(\mathbb{G})$ such that*

$$\sum_{j=0}^{m-1} \left\| X^j(f-P-\phi) \right\|_{q_{mj},\Omega} + \left\| X^m(f-P-\phi) \right\|_{p,\Omega} < \epsilon,$$

where $q_{mj} = \frac{pQ}{Q-(m-j)p}$.

For the weighted higher order Poincaré inequalities on unbounded extension domains, we have the following theorem.

Theorem 6.9. *Assume that $\Omega \subset \mathbb{G}$ is an unbounded (ϵ,∞) extension domain. Let m, j be integers with $0 \leqslant j < m$. Suppose that w_1, w_2 are weights satisfying the following balance conditions for some p, q_j with $1 < p < q_j < \infty$:*

$$\left(\frac{r(B)}{r(B_0)}\right)^{m-j} \left(\frac{w_2(B)}{w_2(B_0)}\right)^{\frac{1}{q_j}} \leqslant C \left(\frac{w_1(B)}{w_1(B_0)}\right)^{\frac{1}{p}}$$

for all metric balls B and B_0 with $B \subset cB_0$, where c is a suitably large geometric constant. Suppose also that $w_1 \in A_p$ and w_2 is doubling and

$$r(B)^{m-j} w_2(B)^{\frac{1}{q}} w_1(B)^{-\frac{1}{p}} \leqslant C$$

for all balls $B \subset \mathbb{G}$. If $f \in W^{m,p}(\Omega)$ and $|X^m f| \in L^p(\Omega)$, then there exists a polynomial P of degree no more than $m-1$ such that

$$\left(\int_\Omega \left| X^j(f-P) \right|^{q_j} w_2 \, dx \right)^{\frac{1}{q_j}} \leqslant C \left(\int_\Omega \left| X^m f \right|^p w_1 \, dx \right)^{\frac{1}{p}}.$$

Furthermore, we can show that

Theorem 6.10. *Assume that $\Omega \subset \mathbb{G}$ is an unbounded (ϵ,∞) extension domain. Let m, i and j be integers with $0 \leqslant j < i \leqslant m$. Suppose that w_1, w_2 are weights satisfying the following balance conditions for some p, q_{ij} with $1 < p < q_{ij} < \infty$:*

$$\left(\frac{r(B)}{r(B_0)}\right)^{i-j} \left(\frac{w_2(B)}{w_2(B_0)}\right)^{\frac{1}{q_{ij}}} \leqslant C \left(\frac{w_1(B)}{w_1(B_0)}\right)^{\frac{1}{p}}$$

for all metric balls B and B_0 in $\mathbb{G}$ with $B \subset cB_0$, where c is a suitably large geometric constant. Suppose also that $w_1 \in A_p$ and w_2 is doubling and

$$r(B)^{i-j} w_2(B)^{\frac{1}{q_{ij}}} w_1(B)^{-\frac{1}{p}} \leqslant C$$

for all balls $B \subset \mathbb{G}$. If $f \in W^{m,p}_{\mathrm{loc}}(\Omega)$ and $|X^i f| \in L^p(\Omega)$, then there exists a polynomial P of degree no more than $m-1$ such that

$$\left(\int_\Omega \left|X^j(f-P)\right|^{q_{ij}} w_2\,dx\right)^{\frac{1}{q_{ij}}} \leqslant C\left(\int_\Omega \left|X^i f\right|^p w_1\,dx\right)^{\frac{1}{p}}.$$

Proof. We first note that this theorem does not follow directly from the weighted inequalities on the entire group $\mathbb{G}$ proved in Theorem 5.4. This is because that we only assume w_2 being doubling and thus the extension theorem is not known to hold for weighted Sobolev space with weight w_2. By Theorem 6.4, there exists an extension operator Λ on Ω such that

$$\left\|X^i \Lambda f\right\|_{L^p_{w_1}(\mathbb{G})} \leqslant C\left\|X^i f\right\|_{L^p_{w_1}(\Omega)}$$

and

$$\left\|X^i \Lambda f\right\|_{L^p(\mathbb{G})} \leqslant C\left\|X^i f\right\|_{L^p(\Omega)},$$

where C depends only on $\epsilon, w_1, p, i, m, Q$. From the constructions of the extension operator Λ in [17], we can easily see that $|X^j(\Lambda f)| \in L^p_{\mathrm{loc}}(\mathbb{G})$ since $|X^j f| \in L^p_{\mathrm{loc}}(\mathbb{G})$ for all $0 \leqslant j \leqslant m$.

By the global Poincaré inequalities on $\mathbb{G}$ (see Theorem 5.4), there exists a polynomial $P \in \mathcal{P}_m$ of degree $m-1$ such that

$$\left(\int_{\mathbb{G}} \left|X^j(\Lambda f-P)\right|^{q_{ij}} w_2\,dx\right)^{\frac{1}{q_{ij}}} \leqslant \left(\int_{\mathbb{G}} \left|X^i(\Lambda f)\right|^p w_1\,dx\right)^{\frac{1}{p}}.$$

Using the properties of the extension operator Λ that $\Lambda f = f$ a.e. on Ω and $\|X^i(\Lambda f)\|_{L^p_{w_1}(\mathbb{G})} \leqslant C\|X^i f\|_{L^p_{w_1}(\mathbb{G})}$, we have

$$\left(\int_\Omega \left|X^j(f-P)\right|^{q_{ij}} w_2\,dx\right)^{\frac{1}{q_{ij}}} \leqslant \left(\int_{\mathbb{G}} \left|X^i f\right|^p w_1\,dx\right)^{\frac{1}{p}} \leqslant \left(\int_\Omega \left|X^i f\right|^p w_1\,dx\right)^{\frac{1}{p}}. \qquad \square$$

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