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Higher order Poincaré inequalities associated with linear operators on stratified groups and applications

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Abstract. This paper considers the dual of anisotropic Sobolev spaces on any stratified groups $\mathbb{G}$. For $0 \leq k < m$ and every linear bounded functional T on anisotropic Sobolev space $W^{m-k,p}(\Omega)$ on $\Omega \subset \mathbb{G}$, we derive a projection operator L from $W^{m,p}(\Omega)$ to the collection $\mathcal{P}_{k+1}$ of polynomials of degree less than $k+1$ such that $T(X^I(Lu)) = T(X^I u)$ for all $u \in W^{m,p}(\Omega)$ and multi-index I with $d(I) \leq k$. We then prove a general Poincaré inequality involving this operator L and the linear functional T. As applications, we often choose a linear functional T such that the associated L is zero and consequently we can prove Poincaré inequalities of special interests. In particular, we obtain Poincaré inequalities for functions vanishing on tiny sets of positive Bessel capacity on stratified groups. Finally, we derive a Hedberg-Wolff type characterization of measures belonging to the dual of the fractional anisotropic Sobolev spaces $W^{\alpha,p}(\mathbb{G})$.

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1. Known results on higher order Poincaré inequalities

First order Poincaré inequalities associated with anisotropic vector fields have received extensive attention in recent years. There is a very long list of references and we shall not review them here. However, higher order Poincaré inequalities in such a setting are only known on the stratified groups (see e.g., [L1], [L2]). This paper will focus on the study of some fairly general higher order Poincaré inequalities on stratified groups which improve those of [L1-2] substantially. In particular, we obtain Poincaré inequalities for functions vanishing on sets of positive Lebesgue

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measure or positive Bessel capacity on stratified groups. The main machinery of proving such Poincaré inequalities is to first establish a Poincaré type inequality associated with any given bounded linear functional on the anisotropic Sobolev spaces. More precisely, for $0 \le k < m$ and every linear bounded functional T on $W^{m-k,p}(\Omega)$, we derive a projection operator L from $W^{m,p}(\Omega)$ to the collection $\mathcal{P}_{k+1}$ of polynomials of degree less than $k+1$ such that $T(X^I(Lu)) = T(X^I u)$ for all $u \in W^{m,p}(\Omega)$ and multi-index I with $d(I) \le k$. We then prove a general Poincaré inequality involving this operator L and the linear functional T. By choosing a linear functional T such that the associated L is zero, we can prove Poincaré inequalities of special interests. Theorems proved in this paper even when $m = 1$ (e.g., in the first order anisotropic Sobolev spaces) yield some new Poincaré type inequalities of first order in the setting of stratified groups.

We begin with some preliminaries concerning stratified Lie groups (or so-called Carnot groups). We refer the reader to the books [FS] and [VSCC] for analysis on stratified groups. Let $\mathcal{G}$ be a finite-dimensional, stratified, nilpotent Lie algebra. Assume that

$$\mathcal{G} = \oplus_{i=1}^{s} V_i \ ,$$

with $[V_i, V_j] \subset V_{i+j}$ for $i+j \le s$ and $[V_i, V_j] = 0$ for $i+j > s$. Let $X_1, \cdots, X_l$ be a basis for V_1 and suppose that $X_1, \cdots, X_l$ generate $\mathcal{G}$ as a Lie algebra. Then for $2 \le j \le s$, we can choose a basis $\{X_{ij}\}$, $1 \le i \le k_j$, for V_j consisting of commutators of length j. We set $X_{i1} = X_i, \ i = 1, \cdots, l$ and $k_1 = l$, and we call X_{i1} a commutator of length 1.

If $\mathbb{G}$ is the simply connected Lie group associated with $\mathcal{G}$, then the exponential mapping is a global diffeomorphism from $\mathcal{G}$ to $\mathbb{G}$. Thus, for each $g \in \mathbb{G}$, there is $x = (x_{ij}) \in \mathbb{R}^N$, $1 \le i \le k_j, \ 1 \le j \le s, \ N = \sum_{j=1}^{s} k_j$, such that

$$g = \exp(\sum x_{ij} X_{ij}).$$

A homogeneous norm function $|\cdot|$ on $\mathbb{G}$ is defined by

$$|g| = (\sum |x_{ij}|^{2s!/j})^{1/2s!},$$

and $Q = \sum_{j=1}^{s} j k_j$ is said to be the **homogeneous dimension** of $\mathbb{G}$. The dilation δ_r on $\mathbb{G}$ is defined by

$$\delta_r(g) = \exp(\sum r^j x_{ij} X_{ij}) \ \text{ if } \ g = \exp(\sum x_{ij} X_{ij}).$$

The convolution operation on $\mathbb{G}$ is defined by

$$f * h(x) = \int_{\mathbb{G}} f(xy^{-1}) h(y) dy = \int_{\mathbb{G}} f(y) h(y^{-1}x) dy,$$

where y^{-1} is the inverse of y and xy^{-1} denotes group multiplication of x by y^{-1}. It is known that for any left invariant vector field X on $\mathbb{G}$,

$$X(f * h) = f * (Xh).$$

We call a curve $\gamma : [a, b] \to \mathbb{G}$ "a horizontal curve" connecting two points $x, y \in \mathbb{G}$ if $\gamma(a) = x$, $\gamma(b) = y$ and $\gamma^{'}(t) \in V_1$ for all t. Then the Carnot-Caratheodory distance between x, y is defined as

$$d_{cc}(x, y) = \inf_{\gamma} \int_a^b < \gamma^{'}(t), \gamma^{'}(t) >^{\frac{1}{2}} dt,$$

where the infimum is taken over all horizontal curves γ connecting x and y. It is known that any two points x, y on $\mathbb{G}$ can be joined by a horizontal curve of finite length and then d_{cc} is a left invariant metric on $\mathbb{G}$. We can define the metric ball centered at x and with radius r associated with this metric by

$$B_{cc}(x, r) = \{y : d_{cc}(x, y) < r\}.$$

We must notice that this metric d_{cc} is equivalent to the pseudo-metric $\rho(x, y) = |x^{-1}y|$ defined by the homogeneous norm $|\cdot|$ in the following sense (see [FS])

$$C\rho(x, y) \le d_{cc}(x, y) \le C\rho(x, y).$$

We denote the metric ball associated with ρ as $D(x, r) = \{y \in \mathbb{G} : \rho(x, y) < r\}$. An important feature of both of these distance functions is that these distances and thus the associated metric balls are left invariant, namely,

$$d_{cc}(zx, zy) = d(x, y), B_{cc}(x, r) = xB_{cc}(0, r)$$

and

$$\rho(zx, zy) = \rho(x, y), D(x, r) = xD(0, r).$$

From now on, we will always use the metric d_{cc} and drop the subscript from d_{cc}. Similarly, we will use $B(x, r)$ to denote $B_{cc}(x, r)$.

We now recall the definition of the class of polynomials on $\mathbb{G}$ given by Folland and Stein in [FS]. Let $X_1, \cdots, X_l$ in V_1 be the generators of the Lie algebra $\mathcal{G}$, and let $X_1, \cdots, X_l, \cdots, X_N$ be a basis of $\mathcal{G}$. We denote $d(X_j) = d_j$ to be the length of X_j as a commutator, and we arrange the order so that $1 \le d_1 \le \cdots \le d_N$. Then it is easy to see that $d_j = 1$ for $j = 1, \cdots, l$. Let $\xi_1, \cdots, \xi_N$ be the dual basis for $\mathcal{G}^*$, and let $\eta_i = \xi_i \circ \exp^{-1}$. Each η_i is a real-valued function on $\mathbb{G}$, and $\eta_1, \cdots, \eta_N$ gives a system of global coordinates on $\mathbb{G}$. A function P on $\mathbb{G}$ is said to be a polynomial on $\mathbb{G}$ if $P \circ \exp$ is a polynomial on $\mathcal{G}$. Every polynomial on $\mathbb{G}$ can be written uniquely as

$$P(x) = \sum_I a_I \eta^I(x), \quad \eta^I = \eta_1^{i_1} \cdots \eta_N^{i_N}, \quad a_I \in \mathbb{R},$$

where all but finitely many of the coefficients a_I vanish. Clearly η^I is homogeneous of degree $d(I) = \sum_{j=1}^N i_j d_j$, i.e., $\eta^I(\delta_r x) = r^{d(I)}\eta_i(x)$. If $P = \sum_I a_I \eta^I$, then we define the homogeneous degree (or order) of P to be $\max\{d(I) : a_I \neq 0\}$.

Throughout this paper, we use $\mathcal{P}_k$ to denote polynomials of homogeneous degree less than k for each positive integer k.

We also adopt the following multi-index notation for higher order derivatives. If $I = (i_1, \cdots, i_N) \in \mathbb{N}^N$, we set

$$X^I = X_1^{i_1} \cdot X_2^{i_2} \cdots X_N^{i_N}.$$

By the Poincaré–Birkhoff–Witt theorem (cf. Bourbaki [B], I.3.7), the differential operators X^I form a basis for the algebra of left-invariant differential operators in $\mathbb{G}$. Furthermore, we set

$$|I| = i_1 + i_2 + \cdots + i_N, \quad d(I) = d_1 i_1 + d_2 i_2 + \cdots + d_N i_N.$$

Thus, $|I|$ is the order of the differential operator X^I, and $d(I)$ is its degree of homogeneity; $d(I)$ is called the homogeneous degree of X^I. We will also use the notation

$$|X^m f| = \Big(\sum_{I:d(I)=m} |X^I f|^2 \Big)^{1/2}$$

for any positive integer m.

Let m be a positive integer, $1 \leq p < \infty$, and Ω be an open set in $\mathbb{G}$. The Folland–Stein Sobolev space $W^{m,p}(\Omega)$ associated with the vector fields $X_1, \cdots, X_l$ is defined to consist of all functions $f \in L^p(\Omega)$ with distributional derivatives $X^I f \in L^p(\Omega)$ for every X^I defined above with $d(I) \leq m$. Here, we say that the distributional derivative $X^I f$ exists and equals a locally integrable function g_I if for every $\phi \in C_0^\infty(\Omega)$,

$$\int_\Omega f X^I \phi \, dx = (-1)^{d(I)} \int_\Omega g_I \phi dx.$$

$W^{m,p}(\Omega)$ is equipped with the norm

$$||f||_{W^{m,p}(\Omega)} = ||f||_{L^p(\Omega)} + \sum_{1 \leq d(I) \leq m} ||X^I f||_{L^p(\Omega)}. \tag{1.1}$$

When $\Omega = \mathbb{G}$, we sometimes use $||f||_{k,p}$ to denote $||f||_{W^{m,p}(\mathbb{G})}$. We also sometimes use $||f||_{k,p;\Omega}$ to denote $||f||_{W^{m,p}(\Omega)}$.

To explain the motivation of this paper, we need to review some known results concerning high order Poincaré inequalities on stratified groups. The following is Theorem 3.7 in [L2] (see also [N1]).

Theorem 1.2. *Let $\Omega \subset \mathbb{G}$ be an open set of finite Lebesgue measure. Then given any positive integer k and $f \in W^{k,1}(\Omega)$, there exists a unique polynomial $P = P(f, \Omega)$ on $\mathbb{G}$ of degree less than k such that*

$$\int_\Omega X^I (f - P) = 0, \ \ for\ any\ 0 \leq d(I) < k. \tag{1.3}$$

Moreover, $P(f, \Omega)$ linearly depends on f.

In [L2], we also showed a different type of polynomials associated with the Sobolev functions which we call "projection polynomials" in [L2].

Definition 1.4. *For each $k \in \mathbb{N}$ and ball $B \subset \Omega \subset \mathbb{G}$, a projection of order k associated with the ball B is defined to be a linear map*

$$\pi_k(B) : W^{k,1}(\Omega) \longrightarrow \mathcal{P}_k$$

such that

$$\sup_{x \in B} |\pi_k(B) f(x)| \leq C r(B)^{-Q} \| f \|_{L^1(B)} \tag{1.5}$$

with C independent of f and B and

$$\pi_k(B) P = P \text{ for all } P \in \mathcal{P}_k. \tag{1.6}$$

Clearly, polynomials constructed in Theorem 1.2 satisfy (1.6) but not necessarily (1.5). The existence of projection polynomials was proved in Theorem 3.6 in [L2]. It is also shown in [L2] (see Theorem 3.8 in [L2]) that the following theorem holds.

Theorem 1.7. *For each $k \in \mathbb{N}$ and ball $B \subset \Omega$, then for any projection $\pi_k(B) : W^{k,1}(\Omega) \longrightarrow \mathcal{P}_k$ of order k, the following holds: for any $1 \leq q \leq \infty$, and any multiple index I with $d(I) = l \geq 0$*

$$\| X^I \pi_k(B) f \|_{L^q(B)} \leq C \| X^l f \|_{L^q(B)} \tag{1.8}$$

with C independent of f and B (noticing that when $l \geq k$ the left side is zero).

This shows that a certain order of subelliptic derivative of $\pi_k(B) f$ is controlled by the same order of subelliptic derivative of f.

We now recall some results concerning higher order Poincaré inequalities proved in [L1], [L2].

Theorem 1.9. *Let $f \in W^{k,p}(\Omega)$. Given any ball $B \subset \Omega$, there exists $P_k(f, B) \in \mathcal{P}_k$ such that we have for any $0 \leq j < i \leq k$*

$$\begin{aligned} &\left(\frac{1}{|B|} \int_B |X^j\, (f(x) - P_k(f, B)(x))\,|^{q_{ij}} dx \right)^{\frac{1}{q_{ij}}} \\ &\qquad \leq C r(B)^{i-j} \left(\frac{1}{|B|} \int_B |X^i\, (f(x) - P_k(f, B)(x))\,|^p dx \right)^{\frac{1}{p}} \end{aligned}$$

for all $1 \leq p < \frac{Q}{i-j}$ and $q_{ij} \leq \frac{pQ}{Q-(i-j)p}$, where C is independent of B and f.

We remark that the existence of the polynomial $P_m(f, B)$ is guaranteed by Theorem 1.2. The proof of theorem (1.9) follows from the repeated use of the standard sharp Poincaré inequality of first order proved in [L4] for $p > 1$ and for all $p \geq 1$ in [MS] and [FLW1] (see also [CDG]). In the proof we have used the property of the polynomial $P_k(f, B)$, i.e., the vanishing integral property (1.3).

As a simple corollary of this theorem we get the special Poincaré inequality when $i = k$ (see Corollary 6.2 in [L2]) by dropping the polynomial on the right hand side (see also [N1]). This follows from theorem (1.9) by taking $i = k$ and noting that $X^k P_k(f, B) = 0$.

If we choose the projection polynomial $\pi_k(B)f \in \mathcal{P}_k$, then Theorem 1.9 can be improved. Namely, we can drop the polynomial $P_k(f, B)$ on the right hand side of the Poincaré inequality even for $1 \le i < k$. Theorem 6.3 in [L2] states as follows:

Theorem 1.10. *Let $f \in W^{m,p}(\Omega)$ and let $B \subset \Omega$ be any ball. Then we have for any* $0 \le j < i \le k$

$$\left(\frac{1}{|B|}\int_B |X^j\,(f(x) - \pi_k(B)f(x))\,|^{q_{ij}}dx\right)^{\frac{1}{q_{ij}}} \le Cr(B)^{i-j}\left(\frac{1}{|B|}\int_B |X^i f(x)|^p dx\right)^{\frac{1}{p}}$$

for all $1 \le p < \frac{Q}{i-j}$ *and* $q_{ij} \le \frac{pQ}{Q-(i-j)p}$, *where C is independent of B and f.*

Since we do not necessarily have the vanishing integral property for the projection polynomial $\pi_k(B)f$, the proof of Theorem (1.10) does not follow immediately from the Poincaré inequality of first order. The interesting feature of the theorem is that even for $i < k$ (thus the degree of the polynomial $\pi_k(B)f$ is bigger than i), the left hand side is still controlled by the $i-$th order derivatives of f alone. Similar Poincaré inequalities on domains satisfying the Boman chain condition are also proved (see Theorem 6.4 in [L2]).

Polynomials constructed in Theorems 1.2, 1.7 have applications to proving extension theorems on high order Sobolev spaces on stratified groups as given in [L2], [L3] and Sobolev interpolation inequalities of any order (see [L1], [L2]). In particular, polynomials satisfying (1.5) enable us to construct a bounded linear extension operator on extension domains such that the derivatives of the extended functions can be controlled by the same order of derivatives of the original Sobolev functions (see [L2], [L3]). We should mention the existence of polynomials satisfying (1.3) was proved earlier on the Heisenberg group with application to Sobolev extension theorems on the Heisenberg group in [N2].

We mention in passing that polynomials in metric spaces were introduced in [LW] and [LLW]. The authors showed that the existence of polynomials satisfying L^1 to L^1 Poincaré inequalities implies higher order representation formulas. This is motivated by the first order result of [FLW2]. In particular, on the stratified groups, we showed in [LW1] that such representation formulas of higher order do hold (simultaneous representation formulas for the derivatives of functions were obtained in [LW2] recently). We also defined the notion of higher order Sobolev spaces in metric spaces in [LLW] and proved that several definitions are equivalent if functions of polynomial type exist. In the case of stratified groups, where polynomials do exist, we showed in [LLW] that our spaces are equivalent to the Sobolev spaces defined by Folland and Stein in [FS]. Our results in [LLW] extended the notion of first order Sobolev spaces in metric spaces of [Ha] and also give some alternate definitions of higher order Sobolev spaces in the classical Euclidean case and on stratified groups.

We now restate the result in Theorem 1.2 in a form which is consistent with what we will show more generally in this paper. If we consider the linear operator

$$T : L^p(\Omega) \to \mathbb{R},\ T(u) = \int_\Omega u(x)dx,\ \text{for}\ p > 1$$

then Theorem 1.2 actually says that there exists a map $L : W^{m,p}(\Omega) \to \mathcal{P}_m$ such that for each $u \in W^{m,p}(\Omega)$ and multi-index I with $d(I) \leq m-1$,

$$T(X^I u) = T(X^I P) \tag{1.13}$$

where $P = L(u)$.

Inspired by this simple result on the stratified group $\mathbb{G}$, and the results of Maz'ya [Ma1]-[Ma3], Meyers [Me1], Meyers-Ziemer [MZ], Hedberg [He] (see also the books by Adams and Hedberg [AH], Ziemer [Z], and Maz'ya [Ma1]) in Euclidean space, we will prove a more general theorem on a stratified group $\mathbb{G}$ which shows such a result holds for fairly general linear operators T. Results of this paper generalize those in Euclidean space by Meyers [Me1], Meyers-Ziemer [MZ] to this subelliptic setting and improve those in [L1,L2] substantially in several ways. First of all, our results show that for any element T in the dual space of $W^{m-k,p}(\Omega)$, where $0 < k < m$, we can associate with T a linear projection operator $L : W^{m,p}(\Omega) \to \mathcal{P}_{k+1}$ such that the norm of L is well controlled by T. Second, we can prove a Poincaré inequality such that $||u - L(u)||_{W^{p,k}(\Omega)}$ is controlled by $||X^{k+1}u||_{W^{m-(k+1),p}(\Omega)}$ and the norm $||T||$ of T. If we choose appropriately the linear functional T, we can show the corresponding linear projection operator L is zero. Thus we can derive considerably more general theorems than those in [L1,L2]. In particular, we can derive Poincaré inequalities for functions vanishing on sets of positive Lebesgue measure or furthermore on sets of merely positive Bessel capacity (see Theorems 2.4, 2.5 and 2.6 in §2). Finally, we obtain an analogue of Hedberg-Wolff's characterization concerning Radon measures being in the dual space of the fractional Sobolev space $W^{\alpha,p}(\mathbb{G})$ (see Adams [Ad1] and Hedberg-Wolff [HW]). In order to derive these results mentioned above, we must proceed with caution because of the noncommutative group multiplication. The noncommutativity of the convolutions on the group $\mathbb{G}$ also results in difficulties in dealing with the measures in the dual space of $W^{m,p}(\Omega)$ in conjunction with the Bessel potentials of the measures.

The organization of the paper is as follows. Main theorems are stated in Section §2. In Section §3, we consider the dual of anisotropic Folland-Stein Sobolev space $W^{m,p}(\Omega)$ on stratified groups. For every linear bounded functional T on $W^{m,p}(\Omega)$, we derive in Section §3 a projection operator L from $W^{m,p}(\Omega)$ to the collection $\mathcal{P}_k$ of polynomials of degree less than k such that $T(X^I(Lu)) = T(X^I u)$ for all $u \in W^{m,p}(\Omega)$ and multi-index I with $d(I) < k$ (see Theorem 2.1). We then prove in Section §4 a general Poincaré inequality involving this operator L and the linear functional T (see Theorems 2.2 and 2.3). As applications, in Section §5 we choose a linear functional T such that the associated L is zero and consequently we prove Poincaré inequalities for functions vanishing on sets of positive Lebesgue measures (see Theorem 2.4). In particular, in Section §6 we obtain Poincaré inequalities for functions vanishing on tiny sets of positive Bessel capacity on stratified groups (see Theorems 2.5 and 2.6). Finally, we derive in Section §7 a Hedberg-Wolff type characterization of measures in the dual of the fractional anisotropic Folland-Stein Sobolev spaces $W^{\alpha,p}(\Omega)$ (see Theorem 2.7).

2. Statements of Main Theorems

To state our main theorems, we let $(W^{m,p}(\Omega))^*$ denote the dual space of the Sobolev space $W^{m,p}(\Omega)$.

Theorem 2.1. *Let k and m be two integers with $0 \le k < m$ and $p \ge 1$. Let $\Omega \subset \mathbb{G}$ be an open set of finite Lebesgue measure and χ_Ω be the characteristic function of Ω. Suppose that $T \in \left(W^{m-k,p}(\Omega)\right)^*$ has the property that $T(\chi_\Omega) \neq 0$. Then there is a projection $L : W^{m,p}(\Omega) \to \mathcal{P}_{k+1}$ such that for each $u \in W^{m,p}(\Omega)$ and multi-index I with $d(I) \le k$,*

$$T(X^I u) = T(X^I P)$$

where $P = L(u)$. Moreover, L has the form

$$L(u) = \sum_{I:d(I)\le k} a_I \eta^I(x)$$

where $a_I = \sum_{J:d(J)\ge d(I)} b_J X^J u$, and

$$||L|| \le C \left(\frac{||T||}{T(\chi_\Omega)}\right)^{k+1},$$

where $C = C(k, p, |\Omega|)$ and $||T||$ is the norm of the linear functional T and $||L||$ is the operator norm of the map

$$L : W^{m,p}(\Omega) \to \mathcal{P}_k(\mathbb{G}) \subset W^{m,p}(\Omega).$$

Before we state the next theorem, we now recall the notion of extension domains on $\mathbb{G}$. A domain $\Omega \subset \mathbb{G}$ is said to be an extension domain if there is a bounded extension operator on $W^{k,p}(\Omega)$. A bounded extension operator on $W^{k,p}(\Omega)$ is a bounded linear operator $\Lambda : W^{k,p}(\Omega) \to W^{k,p}(\mathbb{G})$ such that $\Lambda f \mid_\Omega = f \ \forall f \in W^{k,p}(\Omega)$. Moreover, we use the notation

$$\|\Lambda\| = \sup_{\|f\|_{W^{k,p}(\Omega)}=1} \|\Lambda f\|_{W^{k,p}(\mathbb{G})}.$$

Theorem 2.2. *Let k and m be two integers with $0 \le k < m$ and $p \ge 1$. Let $\Omega \subset \mathbb{G}$ be an open, bounded extension domain and χ_Ω be the characteristic function of Ω. Suppose that $T \in \left(W^{m-k,p}(\Omega)\right)^*$ has the property that $T(\chi_\Omega) \neq 0$. Then for the projection operator $L : W^{m,p}(\Omega) \to \mathcal{P}_{k+1}$ associated with T whose existence is guaranteed by Theorem 2.1, we have*

$$||u - L(u)||_{W^{k,p}(\Omega)} \le C \left(\frac{||T||}{T(\chi_\Omega)}\right)^{k+1} ||X^{k+1}u||_{W^{m-(k+1),p}(\Omega)},$$

with $C = C(k, p, \Omega)$.

Theorem 2.3. *With the same hypotheses as in Theorem 2.2, we have*

$$||u - L(u)||_{L^{p*}(\Omega)} \leq C \left(\frac{||T||}{T(\chi_\Omega)} \right)^{k+1} ||X^{k+1}u||_{W^{m-(k+1),p}(\Omega)}$$

where $p^* = \frac{pQ}{Q-kp}$ *for* $1 \leq p < \frac{Q}{k}$, $p^* < \infty$ *for* $pk = Q$ *and* $p^* = \infty$ *when* $pk > Q$.

Theorem 2.4. *Let* $\Omega \subset \mathbb{G}$ *be a bounded extension domain. Let* $0 \leq k < m$ *be integers and* $p \geq 1$. *Suppose that* $u \in W^{m,p}(\Omega)$ *has the property that*

$$\int_E X^I u dx = 0, \quad for \ \ 0 \leq d(I) \leq k,$$

where $E \subset \Omega$ *is a measurable set of positive Lebesgue measure. Then, there is a constant* $C = C(k, m, p, |E|)$ *such that for* $u \in W^{m,p}(\Omega)$.

$$||u||_{W^{k,p}(\Omega)} \leq C||X^{k+1}u||_{W^{m-(k+1),p}(\Omega)}$$

and

$$||u||_{L^{p*}(\Omega)} \leq C||X^{k+1}u||_{W^{m-(k+1),p}(\Omega)}$$

where $p^* = \frac{pQ}{Q-kp}$ *for* $1 \leq p < \frac{Q}{k}$, $p^* < \infty$ *for* $pk = Q$ *and* $p^* = \infty$ *when* $pk > Q$.

Next two theorems are concerning Poincaré inequalities in terms of Bessel capacities on $\mathbb{G}$. We refer the reader to Section §6 for notations and definitions of Bessel capacity.

Theorem 2.5. *Let* $\Omega \subset \mathbb{G}$ *be a bounded extension domain, and let* $A \subset \Omega$ *be a* $B_{m-k,p}-$*capacitable set with* $B_{m-k,p}(A) > 0$ *where* $0 \leq k < m$ *are integers and* $p \geq 1$. *Then there exists a projection* $L : W^{m,p}(\Omega) \to \mathcal{P}_{k+1}(\mathbb{G})$ *such that*

$$||u - L(u)||_{W^{k,p}} \leq C \left(B_{m-k,p}(A)\right)^{-1/p} ||X^{k+1}u||_{W^{m-(k+1),p}(\Omega)},$$

and

$$||u - L(u)||_{L^{p^*}(\Omega)} \leq C \left(B_{m-k,p}(N)\right)^{-1/p} ||X^{k+1}u||_{W^{m-(k+1),p}(\Omega)},$$

with $C = C(k, p, m, \Omega)$ *and where* $p^* = \frac{pQ}{Q-kp}$ *for* $1 \leq p < \frac{Q}{k}$, $p^* < \infty$ *for* $pk = Q$ *and* $p^* = \infty$ *when* $pk > Q$.

Theorem 2.6. *Let* $\Omega \subset \mathbb{G}$ *be a bounded extension domain,* $u \in W^{m,p}(\Omega)$ *and let* $N \subset \Omega$ *be a set defined by*

$$N = \overline{\Omega} \bigcap \{x : X^I u(x) = 0 \, for \ all \ \ 0 \leq d(I) \leq k\}.$$

If $B_{m-k,p}(N) > 0$ *where* $0 \leq k < m$ *are integers and* $p \geq 1$. *Then*

$$||u||_{W^{k,p}} \leq C \left(B_{m-k,p}(N)\right)^{-1/p} ||X^{k+1}u||_{W^{m-(k+1),p}(\Omega)},$$

and

$$||u||_{L^{p^*}(\Omega)} \leq C \left(B_{m-k,p}(N)\right)^{-1/p} ||X^{k+1}u||_{W^{m-(k+1),p}(\Omega)},$$

with $C = C(k, p, m, \Omega)$ *and where* $p^* = \frac{pQ}{Q-kp}$ *for* $1 \leq p < \frac{Q}{k}$, $p^* < \infty$ *for* $pk = Q$ *and* $p^* = \infty$ *when* $pk > Q$.

The case of $m = 1, k = 0$ is of particular interest which gives rise to first order Poincaré inequalities for functions vanishing on sets of positive capacity. The difference between Theorems 2.4 and 2.6 is that functions vanishing on sets of positive capacity may not vanish on sets of positive Lebesgue measure. Thus, Theorem 2.6 is a stronger result than Theorem 2.4. Finally, we prove in Section §7 the following Hedberg-Wolff type characterization of Radon measures being in the dual of the fractional Sobolev spaces $W^{\alpha,p}(\mathbb{G})$ (see Section §6 for definition of such Sobolev spaces).

Theorem 2.7. *Let $p > 1$ and $0 < \alpha p \le Q$. If μ is a Radon measure, then $\mu \in (W^{\alpha,p}(\mathbb{G}))^*$ if and only if*

$$\int_{\mathbb{G}} \int_0^1 \left(\frac{\mu[B(y,r)]}{r^{Q-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} d\mu(y) < \infty.$$

3. Theorem 2.1: Projection operators associated with linear functionals

Definition 3.1. *If X is a Banach space and $Y \subset X$ a subspace, then a bounded linear map $L : X \to Y$ onto Y is called a projection if $L \cdot L = L$*

Note that $L(y) = y$ for $y \in Y$, for there exists $x \in X$ such that $L(x) = y$ and $y = L(x) = L(L(x)) = L(y)$.

We now are ready to prove Theorem 2.1. The proof of this theorem is similar to the case when $T(u) = \int_\Omega u$ for $u \in W^{m,p}(\Omega)$ given in [L1,2], and combining the ideas from [Me1] in Euclidean spaces. The proof also gives more precise expressions of $L(u)$, and the norm of L is given here. We must mention that we shall proceed with caution and use the Poincaréé-Birkhoff-Witt theorem on the stratified group $\mathbb{G}$ to express polynomials uniquely.

Proof of Theorem 2.1. Recall $X^I = X_1^{i_1} \cdots X_N^{i_N}$ with $I = (i_1, \cdots, i_N)$ and $d(I) < k$. The proof is given by examining closely the definition of polynomials on stratified groups. We first observe that for each multi-index I and J with the same length $d(I) = d(J)$ we have

$$X^I(\eta^J) = I!\delta_{IJ},\ where\ I! = i_1! \cdots i_N!\ \delta_{IJ} = 1\ if\ I = J;\ and\ = 0\ if\ I \ne J. \tag{3.2}$$

This follows from the definition of the dual basis $\eta_1, \cdots, \eta_N$ on $\mathcal{G}^*$, which indicates the inner product defining the dual basis satisfies

$$< X^J, \eta^I >= I!\delta_{IJ}.$$

We also note that for any polynomial P of degree less than or equal to k. i.e., $P \in \mathcal{P}_{k+1}$ we have $X^I P = 0$ for any $d(I) > k$. Since by the Poincaré-Birkhoff-Witt theorem any polynomial P on $\mathbb{G}$ of degree less than or equal to k can be uniquely written as

$$P(x) = \sum_{I:d(I)\le k} a_I \eta^I(x),\ \eta^I = \eta_1^{i_1} \cdots \eta_N^{i_N},$$

we thus only need to determine the coefficients a_I.

If $P \in \mathcal{P}_{k+1}$ and P has the form $P(x) = \sum_{I:d(I)\le k} a_I \eta^I(x)$, then $X^I P(x) = a_I I!$ if $d(I) = k$. In particular, in order for the following identity

$$T(X^I u) = T(X^I P) \tag{3.3}$$

to hold, the coefficients a_I of the polynomial must satisfy

$$a_I = \frac{T(X^I u)}{I!T(\chi_\Omega)},$$

if $d(I) = k$. Similarly, if $d(I) = k - 1$ then

$$X^I P(x) = a_I I! + \sum_{J:d(J)=1} a_{I+J} \frac{(I+J)!}{J!} \eta^J(x) \tag{3.4}$$

where $I + J = (i_1 + j_1, \cdots, i_N + j_N)$ if $I = (i_1, \cdots, i_N)$ and $J = (j_1, \cdots, j_N)$. In view of (3.3), (3.4) will hold if a_I satisfies the identity

$$a_I = \frac{T(X^I u)}{I!T(\chi_\Omega)} - \sum_{J:d(J)=1} a_{I+J} \frac{(I+J)!}{I!J!} \frac{T(\eta^J)}{T(\chi_\Omega)},$$

where $d(I) = k - 1$. Continue recursively, for all I with $d(I) \le k$ we thus have

$$a_I = \frac{T(X^I u)}{I!T(\chi_\Omega)} - \sum_{J:d(J)=1}^{k-d(I)} a_{I+J} \frac{(I+J)!}{I!J!} \frac{T(\eta^J)}{T(\chi_\Omega)}. \tag{3.5}$$

Since $L(u) = P$ implies $X^I[L(u)] = X^I P$ for any multi-index I, then for $d(I) \le k$,

$$T(X^I u) = T(X^I P) = T[X^I(Lu)].$$

This immediately leads to $L \cdot L = L$ by the definition of a_I given in (3.5). Therefore, L is a projection operator.

We next estimate the norm of the operator L. Note that

$$\begin{aligned} ||L|| &= \sup_{u:||u||_{W^{m,p}(\Omega)}\le 1} ||L(u)||_{W^{m,p}(\Omega)} \\ &= \sup_{u:P=L(u):||u||_{W^{m,p}(\Omega)}\le 1} ||P||_{W^{m,p}(\Omega)}. \end{aligned}$$

Let $u \in W^{m,p}(\Omega)$ with $||u||_{W^{m,p}(\Omega)} \le 1$ and $P = L(u)$, where

$$P(x) = \sum_{I:d(I)\le k} a_I \eta^I(x).$$

To estimate the $W^{m,p}$ norm of P, we need to have the upper bound of the coefficients a_I. To this end, we first consider a_I for $d(I) = k$. Note that for $d(I) = k$ and any nonnegative integer l,

$$\frac{||T||}{I!T(\chi_\Omega)} = \frac{||T|| \cdot T(\chi_\Omega)^l}{I!T(\chi_\Omega)^{1+l}} \le C(l, p, |\Omega|) \left(\frac{||T||}{T(\chi_\Omega)} \right)^{l+1}$$

because $|T(\chi_\Omega)| \le |\Omega|^{1/p} ||T||$.

In particular, this holds for $l = k$. Thus,

$$|a_I| \leq \frac{||T||}{I!T(\chi_\Omega)} \leq C(k, p, |\Omega|) \left(\frac{||T||}{T(\chi_\Omega)}\right)^{k+1}.$$

If $d(I) = k - 1$, $k \geq 1$, then from the estimate for a_I when $d(I) = k$ and the fact that

$$\frac{||T||}{T(\chi_\Omega)} \geq |\Omega|^{-1/p},$$

we get

$$\begin{aligned}|a_I| &\leq \frac{||T||}{I!T(\chi_\Omega)} + C(k, |\Omega|) \sum_{J:d(J)=1} |a_{I+J}| \frac{||T||}{T(\chi_\Omega)} \\ &\leq C(k, p, |\Omega|) \left(\frac{||T||}{T(\chi_\Omega)}\right)^2 \\ &\leq C'(k, p, |\Omega|) \left(\frac{||T||}{T(\chi_\Omega)}\right)^{k+1}.\end{aligned}$$

Continuing in this fashion, we get if $d(I) = k - i$, $k \geq i$, then

$$\begin{aligned}|a_I| &\leq C(k, p, |\Omega|) \left(\frac{||T||}{T(\chi_\Omega)}\right)^{i+1} \\ &\leq C(k, p, |\Omega|) \left(\frac{||T||}{T(\chi_\Omega)}\right)^{k+1}.\end{aligned}$$

Therefore, we have shown that

$$||L|| \leq C(k, p, |\Omega|) \left(\frac{||T||}{T(\chi_\Omega)}\right)^{k+1}.$$

This completes the proof of the theorem. □

4. Theorem 2.2: Poincaré inequalities associated with linear functionals

The main purpose of this section is to show Theorem 2.2. We first need the following lemma due to N. Meyers [Me1].

Lemma 4.1. *Let X_0 be a normed linear space with norm $||\cdot||_0$ and let $X \subset X_0$ be a Banach space with norm $||\cdot||$. Suppose $||\cdot|| = ||\cdot||_0 + ||\cdot||_1$, where $||\cdot||_1$ is a semi-norm and assume that bounded sets in X are precompact in X_0. Let $Y = X \bigcap \{x : ||x||_1 = 0\}$. If $L : X \to Y$ is a projection, then there is a constant C independent of L such that*

$$||x - L(x)||_0 \leq C||L|| \cdot ||x||_1,$$

for all $x \in X$.

Proof of Theorem 2.2. Set $X = W^{m,p}(\Omega)$, $X_0 = W^{k,p}(\Omega)$ and $Y = \mathcal{P}_{k+1}$. From the Rellich compact embedding theorem on extension domain Ω, the bounded sets in $W^{m,p}(\Omega)$ are precompact in $W^{k,p}(\Omega)$. Set $||u||_0 = ||u||_{W^{k,p}(\Omega)}$ and $||u||_1 = ||X^{k+1}u||_{W^{m-(k+1),p}(\Omega)}$. Clearly, $||u|| = ||u||_0 + ||u||_1$ is an equivalent norm on $W^{m,p}(\Omega)$. Moreover, $||u||_1 = 0$ if and only if $u \in \mathcal{P}_{k+1}$. Thus, if $T \in \left(W^{m-k,p}(\Omega)\right)^*$ with $T(\chi_\Omega) \neq 0$, and L is the associated projection operator

$$L : W^{m,p}(\Omega) \to \mathcal{P}_{k+1}$$

whose existence and estimate of upper bound of the norm $||L||$ are guaranteed by Theorem 2.1, we will have

$$\begin{aligned}
||u - L(u)||_{W^{k,p}(\Omega)} &\leq C||L|| \cdot ||X^{k+1}u||_{W^{m-(k+1),p}(\Omega)} \\
&\leq C\left(\frac{||T||}{T(\chi_\Omega)}\right)^{k+1} ||X^{k+1}u||_{W^{m-(k+1),p}(\Omega)}.
\end{aligned}$$

□

We now show the norm on the left hand side above can be replaced by the L^{p*} norm of $u - L(u)$ where $p^* = \frac{Qp}{Q-mp}$, namely, Theorem 2.3. To this end, we need the Sobolev interpolation inequality which can be deduced from those proved in [L1,2].

Lemma 4.2. *Suppose $m > 1$ is an integer and $p \geq 1$. Let $\Omega \subset \mathbb{G}$ be a bounded extension domain. Then for each integer k with $1 \leq k \leq m-1$, and $\epsilon > 0$ there is a constant $C = C(Q, m, p, k, \epsilon, \Omega)$ such that*

$$||X^k u||_{L^p(\Omega)} \leq C||u||_{L^p(\Omega)} + \epsilon||X^m u||_{L^p(\Omega)},$$

whenever $u \in W^{m,p}(\Omega)$.

To prove this lemma, we need to use the following result from [L2] (Theorem 10.1 in [L2])

Lemma 4.3. *Let Ω be an extension domain in $\mathbb{G}$. Let $1 \leq p \leq q < \infty$ be such that the first order L^p to L^q Poincaré inequality holds, namely, $q = \frac{pQ}{Q-p}$ for $1 \leq p < Q$ and $1 \leq q < \infty$ for $p \geq Q$. Let i, k be positive integers such that $1 \leq i < k$ and $1 \leq r \leq q$. Then*

$$||X^i f||_{L^q(\Omega)} \leq C||f||_{L^r(\Omega)}^{\frac{\frac{Q}{q}+k-\frac{Q}{p}-i}{k+\frac{Q}{r}-\frac{Q}{p}}} \cdot ||X^k f||_{L^p(\Omega)}^{\frac{\frac{Q}{r}+i-\frac{Q}{q}}{k+\frac{Q}{r}-\frac{Q}{p}}}$$

for all f such that $X^k f \in L^p(\Omega)$ with $||X^k f||_{L^p(\Omega)} \neq 0$.

Proof of Lemma 4.2. Taking $p = q = r$ and $k = m$ in Lemma 4.3, we get the inequality

$$\begin{aligned}
||X^i f||_{L^p(\Omega)} &\leq C||f||_{L^p(\Omega)}^{1-\frac{i}{m}} \cdot ||X^m f||_{L^p(\Omega)}^{\frac{i}{m}} \\
&= C\left(\epsilon^{-\frac{m}{i}}||f||_{L^p(\Omega)}^{1-\frac{i}{m}}\right) \cdot \left(\epsilon^{\frac{m}{i}}||X^m f||_{L^p(\Omega)}^{\frac{i}{m}}\right).
\end{aligned}$$

Using Holder's inequality on the right hand side with two conjugate exponents $\frac{m}{m-i}$ and $\frac{m}{i}$, we get immediately Lemma 4.2. □

We now prove Theorem 2.3 stated in Section §2.

Proof of Theorem 2.3. If $v \in W^{m,p}(\mathbb{G})$ has compact support, then

$$||v||_{L^{p*}(\mathbb{G})} \leq C||v||_{W^{m,p}(\mathbb{G})}.$$

Since $\Omega \subset \mathbb{G}$ is an extension domain, $u \in W^{m,p}(\Omega)$ has an extension to $v \in W^{m,p}(\mathbb{G})$ with compact support such that $||v||_{W^{m,p}(\mathbb{G})} \leq C||u||_{W^{m,p}(\Omega)}$. Therefore,

$$\begin{aligned}
||u||_{L^{p^*}(\Omega)} &\leq C||v||_{L^{p^*}(\mathbb{G})} \\
&\leq C||v||_{W^{m,p}(\mathbb{G})} \\
&\leq C||u||_{W^{m,p}(\Omega)} \\
&\leq C\left[||u||_{L^p(\Omega)} + ||X^m u||_{L^p(\Omega)}\right].
\end{aligned}$$

In the last inequality above we have used the Sobolev interpolation inequality from Lemma 4.2. Thus,

$$||u||_{L^{p^*}(\Omega)} \leq C\left[||u||_{L^p(\Omega)} + ||X^m u||_{L^p(\Omega)}\right]. \tag{4.4}$$

Since $X^I(L(u)) = 0$ for I with $d(I) = m$, we obtain by replacing u with $u - L(u)$ and using Theorem 2.2 to control $||u - L(u)||_{L^p(\Omega)}$

$$\begin{aligned}
||u - L(u)||_{L^{p*}(\Omega)} &\leq C\left[||u - L(u)||_{L^p(\Omega)} + ||X^m u||_{L^p(\Omega)}\right] \\
&\leq C\left(\frac{||T||}{T(\chi_\Omega)}\right)^{k+1} ||X^{k+1}u||_{W^{m-(k+1),p}(\Omega)}.
\end{aligned}$$

This completes the proof of Theorem 2.3. □

5. Poincaré inequalities for functions vanishing on sets of positive Lebesgue measure

In this section, we will use the general Poincaré inequalities associated with the given operator T to derive some basic Poincaré estimates involving high order derivatives. By considering Lebesgue measure and its variants as elements of $(W^{m,p}(\Omega))^*$ and introducing some appropriate linear functionals T such that the associated projection operator L is zero, we will prove some interesting Poincaré inequalities for functions vanishing on sets of positive Lebesgue measure, namely, Theorem 2.4 stated in Section §2.

Theorem 5.1. *Let $\Omega \subset \mathbb{G}$ be a bounded set. Let $0 \leq k \leq m$ be integers and $p \geq 1$. Then, there is a constant $C = C(k, m, p, diam(\Omega))$ such that*

$$||X^k u||_{L^p(\Omega)} \leq C||X^m u||_{L^p(\Omega)}$$

for $u \in W_0^{m,p}(\Omega)$. *Moreover, for* $q = \frac{Qp}{Q-(m-k)p}$ *and* $1 \leq p < \frac{Q}{m-k}$

$$||X^k u||_{L^q(\Omega)} \leq C||X^m u||_{L^p(\Omega)}$$

for $u \in W_0^{m,p}(\Omega)$ *with* $C = C(k, m, p)$ *independent of* Ω.

Proof. We first note, by a scaling argument, that the L^p to L^q inequality holds with the constant C independent of the domain Ω. We thus may assume with no loss of generality that Ω is a ball $\Omega = B(0, 2r)$ centered at the origin 0 and with radius $2r$. We may further assume that all the functions $u \in W^{m,p}(\Omega)$ are supported in $B(0, r)$.

Define $T \in \left(W^{m-(k+1),p}(\Omega)\right)^*$ by

$$T(w) = \int_\Omega vw dx$$

for $w \in W^{m-(k+1),p}(\Omega)$, where $v = \chi_{B(0,2r)} - \chi_{B(0,r)}$. Since u is supported in $B(0, r)$,

$$T(X^I u) = 0, 0 \leq d(I) \leq k.$$

Therefore, by the construction of the associated operator L given in Theorem 2.1, we derive that all the coefficients of L are zero. Therefore, $L(u) = 0$. Note that $X^k u \in W^{m-(k+1),p}(\Omega)$, thus the theorem follows from Theorems 2.2 and 2.3. This completes the proof of the desired result. □

We are now ready to prove the main theorem of this section, namely Theorem 2.4 in §2. Again, we set $p^* = \frac{pQ}{Q-kp}$ for $1 \leq p < \frac{Q}{k}$, $p^* < \infty$ for $pk = Q$ and $p^* = \infty$ when $pk > Q$.

Proof of Theorem 2.4. Define $T \in \left(W^{m-k,p}(\Omega)\right)^*$ by

$$T(w) = \int_E w dx, \;\; w \in W^{m-k,p}(\Omega).$$

Then, $T(\chi_\Omega) \neq 0$, and

$$T(X^I u) = 0, \text{ for } 0 \leq d(I) \leq k.$$

Then the associated functional (projection) L, constructed in Theorem 2.1, has the property that $L(u) = 0$. The theorem then follows again from Theorem 2.2 and 2.3. □

Corollary 5.2. *Let* $\Omega \subset \mathbb{G}$ *be a bounded extension domain. Let* $0 \leq k < m$ *be integers and* $p \geq 1$. *Suppose that* $u \in W^{m,p}(\Omega)$ *has the property that*

$$X^I u = 0, \text{ a.e. } x \in E \text{ for } 0 \leq d(I) \leq k,$$

where $E \subset \Omega$ *is a measurable set of positive Lebesgue measure. Then, there is a constant* $C = C(k, m, p, |E|)$ *such that*

$$||u||_{W^{k,p}(\Omega)} \leq C||X^{k+1}u||_{W^{m-(k+1),p}(\Omega)}$$

for $u \in W^{m,p}(\Omega)$. *In particular,*

$$||u||_{L^{p*}(\Omega)} \leq C||X^{k+1}u||_{W^{m-(k+1),p}(\Omega)}$$

The case $m = 1$ has its special interest.

Theorem 5.3. *Let $\Omega \subset \mathbb{G}$ be a bounded extension domain. Let $p \geq 1$. Suppose that $u \in W^{1,p}(\Omega)$ has the property that $u > 0$ on A and $u < 0$ on B, where A and B are measurable subsets of Ω of positive Lebesgue measures. Then, there is a constant $C = C(k, m, p, |A|, |B|)$ such that*

$$||u||_{W^{m,p}(\Omega)} \leq C||Xu||_{L^p(\Omega)}$$

and

$$||u||_{L^{p^*}(\Omega)} \leq C||Xu||_{L^p(\Omega)}$$

for $p^* = \frac{pQ}{Q-p}$ when $1 \leq p < Q$ and $p^* < \infty$ when $p = Q$, and $p^* = \infty$ when $p > Q$.

Proof. Given the above u we let $\alpha = \int_A udx$ and $\beta = \int_B udx$ and define $T \in (W^{1,p}(\Omega))^*$ by

$$T(w) = \int_\Omega vwdx, w \in W^{1,p}(\Omega)$$

with $v = \frac{1}{\alpha}\chi_A - \frac{1}{\beta}\chi_B$. Then $T(u) = 0$. Therefore, the projection operator L associated to T, as constructed in Theorem 2.1, satisfies $L(u) = 0$, and the results again follow from Theorem 2.2 and 2.3. □

6. Poincaré's inequalities for functions vanishing on sets of positive capacities

In this section we develop further the results obtained in section §3, §4 and §5. to derive Poincaré inequalities for which the term $L(u)$ is zero in the inequality. We will show that this term vanishes when the set $\{x : u(x) \neq 0\}$ is non-zero when measured by an appropriate capacity. While results in §5 require the sets where Sobolev functions vanish to have positive Lebesgue measure, these results are stronger than those in section §5 because sets of positive capacities can still have zero Lebesgue measure. The main purpose here is to prove Theorems 2.5 and 2.6 stated in Section §2.

To state and prove our main theorems, we need to recall some results concerning Riesz and Bessel capacities on a stratified group $\mathbb{G}$ given in [L5].

Definition 6.1. *The Riesz kernel I_α, $0 < \alpha < Q$, is defined by*

$$I_\alpha(x) = d(x, 0)^{\alpha - Q}$$

The Riesz potential of a function f defined as the convolution

$$f * I_\alpha(x) = \int_{\mathbb{G}} \frac{f(y)\,dy}{d(x, y)^{Q-\alpha}}.$$

We now let $h_t(x)$ be the heat kernel associated with the sub-Laplacian on $\mathbb{G}$, namely, if we set $H_t f(x) = f * h_t(x)$, then

$$\frac{\partial}{\partial t} H_t f(x) + \mathcal{L} H_t f(x) = 0$$

on $\mathbb{G} \times (0, \infty)$, where $\mathcal{L}$ is the sub-Laplacian on $\mathbb{G}$. Many properties of heat kernels can be found in [F] and [VSCC].

Definition 6.2. *For each $\alpha > 0$ we define*

$$G_\alpha(x) = \int_0^\infty t^{\frac{\alpha}{2}-1} e^{-t} h_t(x) dt,$$

where $h_t(x)$ is the heat kernel associated with the sub-Laplacian on $\mathbb{G}$. We call G_α the Bessel kernel and $f * G_\alpha$ the Bessel potential of f.

This integral has the following properties (see [F]):

1) For each $\alpha > 0$, $G_\alpha \in L^1$.
2) For $\alpha > 0$, $\beta > 0$, $G_\alpha * G_\beta = G_{\alpha+\beta}$.
3) For each $\alpha > 0$ and a multi-index I, and any $x \neq 0$

$$X^I G_\alpha(x) = \int_0^\infty t^{\frac{\alpha}{2}-1} e^{-t} X^I h_t(x) dt.$$

Let $1 \leq p \leq \infty$. We use $L^{\alpha,p}(\mathbb{G})$, $\alpha > 0$, $1 \leq p \leq \infty$ to denote all functions u such that $u = f * G_\alpha$ for some $f \in L^p(\mathbb{G})$.

The following theorem is due to Folland [F].

Theorem 6.3. *If k is a positive integer and $1 < p < \infty$, then*

$$L^{k,p}(\mathbb{G}) = W^{k,p}(\mathbb{G}).$$

Moreover, if $u \in L^{k,p}(\mathbb{G})$ with $u = f * G_\alpha$, then

$$C^{-1} \|f\|_p \leq \|u\|_{k,p} \leq C \|f\|_p$$

where $C = C(\alpha, p, Q)$.

Remark. The equivalence of the spaces $L^{k,p}$ and $W^{k,p}$ fails when $p = 1$ or $p = \infty$.

Definition 6.4. *For $\alpha > 0$ and $p > 1$, the Bessel capacity is defined as*

$$B_{\alpha,p}(E) = \inf\{||f||_p^p : f * G_\alpha \geq 1 \text{ on } E,\ f \geq 0\}$$

whenever $E \subset \mathbb{G}$. In case $\alpha = 0$, we take $B_{\alpha,p}$ as Lebesgue measure. The Riesz capacity for $0 < \alpha < Q$ is defined as

$$R_{\alpha,p}(E) = \inf\{||f||_p^p : f * I_\alpha \geq 1 \text{ on } E,\ f \geq 0\}$$

whenever $E \subset \mathbb{G}$.

Our definition of Bessel kernel G_α on stratified groups is motivated by above Theorem 6.3 of Folland. Namely, the anisotropic Sobolev spaces $L^{\alpha,p}$ defined by the Bessel potential G_α coincides, when $\alpha = m$, with the Folland-Stein space $W^{m,p}(\mathbb{G})(1 < p < \infty)$ given earlier in the introduction. The benefit of our definition of G_α is as we mentioned earlier that $L^{\alpha,p}$ coincides with $W^{m,p}$ when $1 < p < \infty$ and $\alpha = m$ following Folland's work. This important result in the classical Euclidean space was established by A.P. Calderon [Ca] (see also Stein [St]). Moreover, our definition of the Bessel kernel G_α allows us to show that it satisfies the following important property of its lower and upper bound near the origin and infinity. Indeed, this is the key observation on which the Hedberg-Wolff estimates in Section §7 rely. Using this property of the Bessel kernel, we will derive in §7 a Hedberg-Wolff type characterization of Radon measures belonging to the dual of anisotropic Sobolev space $L^{\alpha,p}(\mathbb{G})$ on stratified groups.

Theorem 6.5. *The Bessel kernel G_α is a positive, integrable function satisfying $G_\alpha(x) = G_\alpha(x^{-1})$. There exist some $C_1 > 0, C_2 > 0, M > 1$ such that when $d(x,0) \to \infty$ we have*

$$C_1 d(x,0)^{(1/2)(\alpha-Q-1)} e^{-Md(x,0)} \leq G_\alpha(x) \leq C_2 d(x,0)^{(1/2)(\alpha-Q-1)} e^{-\frac{d(x,0)}{M}}.$$

Moreover, there exist C_3 and C_4 such that as $d(x,0) \to 0$ we have

$$C_3 d(x,0)^{\alpha-Q} + o\left(d(x,0)^{\alpha-Q}\right) \leq G_\alpha(x) \leq C_4 d(x,0)^{\alpha-Q} + o\left(d(x,0)^{\alpha-Q}\right)$$

and there exist C_5, C_6, C_7 and C_8 such that for all $x \in \mathbb{G}$ and any multi-index I we have

$$\frac{C_5}{d(x,0)^{Q-\alpha+d(I)}} e^{-C_6 d(x,0)}. \leq \left|X^I G_\alpha(x)\right| \leq \frac{C_7}{d(x,0)^{Q-\alpha+d(I)}} e^{-C_8 d(x,0)}.$$

We mention in passing that the Riesz potential and Riesz capacity on stratified groups were already given in [V1][1]. Many potential theoretical properties for Riesz capacity similar to those in Euclidean space were also developed in [V1] and [V2].

The following observations are then easy to see: Since $G_\alpha(x) \leq C I_\alpha(x)$ for all $x \in \mathbb{G}$, it follows from the definition that for $0 < \alpha < Q$, $1 < p < \infty$, there exists a constant $C = C(\alpha, p, Q)$ such that

$$R_{\alpha,p}(E) \leq C B_{\alpha,p}(E), \text{ whenever } E \subset \mathbb{G}.$$

Moreover, it is also true (see [L5]) that for $\alpha p < Q$

$$R_{\alpha,p}(E) = 0 \text{ if and only if } B_{\alpha,p}(E) = 0.$$

We also note that for the Riesz capacity, we can easily get by an argument of dilation

$$R_{\alpha,p}(B(x,r)) = C_{\alpha,p} r^{Q-\alpha p} R_{\alpha,p}(B(x,1)) = C_{\alpha,p} r^{Q-\alpha p} R_{\alpha,p}(B(0,1)).$$

This is easily seen because the Riesz kernel I_α is homogeneous of degree $\alpha - Q$. However, it is considerably harder to give the estimates of Bessel capacity for metric balls because of the complicated kernel G_α. Fortunately, we were able to succeed in [L5] thanks to Theorem 6.5. Indeed, we derive in [L5] the estimates of Bessel capacities on metric balls on stratified groups, which extends N. Meyers' theorem in Euclidean space ([Me2]). We also establish in [L5] the relationship between Riesz and Bessel capacities, which extends results by Adams [Ad3] in Euclidean space. The relationship between Hausdorff measures and capacities on Carnot groups was established in [L6].

We now give the following

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[1] The second author wishes to thank T. Coulhon for bringing the work of [V1] to his attention and for sending him a copy of [V1].

Definition 6.6. *A set $E \subset \mathbb{G}$ is called a $B_{\alpha,p}-$capacitable set if*

$$B_{\alpha,p}(E) = \inf\{B_{\alpha,p}(U) : E \subset U,\ U \text{ open}\} = \sup\{B_{\alpha,p}(K) : K \subset E,\ K \text{ compact}\}.$$

It is shown in [L5] that any Borel sets in $\mathbb{G}$ are $B_{\alpha,p}-$capacitable. More general sets in $\mathbb{G}$ can be $B_{\alpha,p}-$capacitable. The notion of so-called **analytical sets** in Euclidean space can be introduced without difficulty (see [CH], [Me2]). However, we shall not discuss this here further.

We now introduce an equivalent formulation of Bessel capacity.

Definition 6.7. *For $1 < p < \infty$, and $E \subset \mathbb{G}$ be a $B_{\alpha,p}$-capacitable set, let $\mathcal{M}(E)$ denote the class of Radon measures μ on $\mathbb{G}$ such that $\mu(\mathbb{G} - E) = 0$. We define*

$$b_{\alpha,p}(E) = \sup\{\mu(\mathbb{G})\}$$

where the supremum is taken over all $\mu \in \mathcal{M}(E)$ such that

$$\|\mu * G_\alpha\|_{p'} \le 1.$$

The following result is shown in [L5].

Proposition 6.8. *For any $B_{\alpha,p}-$capacitable set E we have*

$$b_{\alpha,p}(E) = B_{\alpha,p}(E)^{\frac{1}{p}}.$$

We will need the following lemma in the rest of this section.

Lemma 6.9. *Let G_α be the Bessel kernel and I_α be the Riesz kernel, we have*

$$\int_{\mathbb{G}} (f * G_\alpha)(x) h(x) dx = \int_{\mathbb{G}} (h * G_\alpha)(x) f(x) dx$$

and

$$\int_{\mathbb{G}} (f * I_\alpha)(x) h(x) dx = \int_{\mathbb{G}} (h * I_\alpha)(x) f(x) dx.$$

This follows from the important fact that $G_\alpha(x) = G_\alpha(x^{-1})$ and $I_\alpha(x) = I_\alpha(x^{-1})$.

We now prove

Lemma 6.10. *Let Ω be a bounded extension domain. Let us assume that μ is a non-negative measure with the properties that $spt\,\mu \subset \overline{\Omega}$ and $\mu * G_{m-k} \in L^{p'}(\mathbb{G})$, where k is an integer, $0 \le k < m$. Then μ can be viewed as an element of $\left(W^{m-k,p}(\Omega)\right)^*$ if we define $T : W^{m-k,p}(\Omega) \to \mathbb{R}$ by $T(u) = \int u d\mu$. Moreover,*

$$||T|| \le ||\mu * G_{m-k}||_{p'}.$$

Proof. Let $u \in W^{m-k,p}(\Omega)$ where $\Omega \subset \mathbb{G}$ is a bounded extension domain. Then there is an extension operator Λ such that

$$||\Lambda u||_{W^{m-k,p}(\mathbb{G})} \leq C||u||_{W^{m-k,p}(\Omega)}.$$

With no loss of generality we assume that Λu has compact support. Then Λu has the representation

$$\Lambda u = f * G_{m-k}$$

where $f \in L^p(\mathbb{G})$ and $||f||_{L^p(\mathbb{G})} \approx ||\Lambda u||_{W^{m-k,p}(\mathbb{G})}$. Using Lemma 6.9

$$\begin{aligned}\int u d\mu &= \int \Lambda u d\mu \\ &= \int f * G_{m-k} d\mu \\ &= \int (\mu * G_{m-k}) \cdot f dx \\ &\leq ||\mu * G_{m-k}||_{p'}||f||_p \\ &\leq C||\mu * G_{m-k}||_{p'}||\Lambda u||_{m-k,p} \\ &\leq C||\mu * G_{m-k}||_{p'}||u||_{W^{m-k,p}(\Omega)}.\end{aligned}$$

Thus, $\mu \in \left(W^{m-k,p}(\Omega)\right)^*$ and $||T|| \leq C||\mu * G_{m-k}||_{p'}$. □

We are now ready to prove the following main theorem of this section (namely, Theorem 2.5 stated in Section §2).

Proof of Theorem 2.5. By the alternative formulation of capacity, i.e., Proposition 6.8, there exists a nonnegative measure μ supported in A and

$$||\mu * G_{m-k}||_{p'} \leq 1$$

and $\mu(\mathbb{G}) \geq \frac{1}{2}(B_{m-k,p}(A))^{1/p}$. If we set $T = \mu$ in Theorem 2.1, we have $T(\chi_\Omega) = \mu(\mathbb{G}) \geq \frac{1}{2}(B_{m-k,p}(A))^{1/p} > 0$ and from the above Lemma 6.10 that

$$||T|| \leq C||\mu * G_{m-k}||_{p'} \leq C$$

and thus it follows from Theorem 2.1

$$||L|| \leq C\left(B_{m-k,p}(A)\right)^{-1/p}$$

which yields the first part of Theorem 2.5. The L^{p^*}-norm estimate follows from the same argument as in the proof of Theorem 2.3 in §4. □

Using Theorem 2.5, we can give the following

Proof of Theorem 2.6. Clearly, N is $B_{m-k,p}-$capacitable. Recall from Definition 6.7 and Proposition 6.8, if $B_{m-k,p}(N) > 0$ where

$$N = \overline{\Omega} \cap \{x : X^I u(x) = 0, d(I) \leq k\},$$

there exists a nonnegative measure μ supported in N such that

$$||\mu * G_{m-k}||_{p'} \leq 1$$

and $\mu(\mathbb{G}) \geq \frac{1}{2}(B_{m-k,p}(A))^{1/p}$. We thus have $T(\chi_\Omega) = \mu(\mathbb{G}) \geq \frac{1}{2}(B_{m-k,p}(A))^{1/p}$.

Define $T(u) = \int_N u d\mu$. Then

$$T(X^I u) = \int X^I u d\mu$$

for all $0 \leq d(I) \leq k$ and thus are all zeros because μ is supported on N. Since the coefficients of the polynomial $L(u)$ depend on $T(X^I u) = \int_N X^I u d\mu$ and thus are all zeros. This completes the proof of the first inequality of the theorem. The second inequality follows from the Sobolev interpolation inequality.This completes the proof of Theorem 2.5 as we did in the proofs of Theorems 2.2 and 2.3. □

Because of the particular importance of the case $m = 1, k = 0$, we state the Poincaré inequality separately.

Corollary 6.11. *Let $\Omega \subset \mathbb{G}$ be a bounded extension domain, $u \in W^{m,p}(\Omega)$ and let $N \subset \Omega$ be a set defined by*

$$N = \overline{\Omega} \bigcap \{x : u(x) = 0\}.$$

If $B_{1,p}(N) > 0$, then

$$||u||_{L^{p^*}(\Omega)} \leq C \left(B_{1,p}(N)\right)^{-1/p} ||Xu||_{L^p(\Omega)},$$

with $C = C(k, p, m, \Omega)$.

As a generalization, we obtain

Theorem 6.12. *Let $p > 1$ and suppose $0 \leq k \leq m$ are integers. Let $\Omega \subset \mathbb{G}$ be a bounded extension domain. If there is some μ which is a nonnegative measure on $\mathbb{G}$ such that $\mu \in (W^{m-k,p}(\Omega))^*$, $\mu(\mathbb{G}) \neq 0$ and*

$$\int X^I u d\mu = 0, \quad \textit{for all} \quad d(I) \leq k,$$

then

$$||u||_{p^*;\Omega} \leq C||X^{k+1}u||_{m-(k+1),p;\Omega}$$

where $C = C(k, p, m, \mu, \Omega)$.

7. Hedberg-Wolff's characterization of the dual of anisotropic Sobolev spaces

If μ is a Radon measure, we define the fractional maximal operator

$$\mathcal{M}_\alpha \mu(x) = \sup\{r^{\alpha-Q}\mu(B(x,r)) : r > 0\}.$$

It is easy to see $\mathcal{M}_\alpha \mu(x) \le C\mu * I_\alpha(x)$. The following is much less trivial and is due to Muckenhoupt and Wheeden [MW] in Euclidean space. For a generalization to homogeneous space by adapting the method in [MW], we refer the reader to the independent work of [V1] and the second author's thesis in 1991 [L7] in which it is shown:

Lemma 7.1. *Let $1 < p < \infty$ and $0 < k < Q$, then there exists $C = C(\alpha, p, Q)$ such that*

$$||\mu * I_\alpha||_p \le C||\mathcal{M}_\alpha \mu||_p.$$

The (α, p) energy of μ is defined as

$$\mathcal{E}_{\alpha,p}(\mu) = \int (\mu * G_\alpha)^{p'} dx.$$

Clearly, by Lemma 6.9

$$\mathcal{E}_{\alpha,p}(\mu) = \int (\mu * G_\alpha)^{\frac{1}{p-1}} * G_\alpha d\mu.$$

Since the Bessel kernel is dominated by the Riesz kernel, we have

$$\begin{aligned}
\mathcal{E}_{\alpha,p}(\mu) &\le C \int (\mu * I_\alpha)^{p'} dx \\
&= C \int (\mu * I_\alpha) \cdot (\mu * I_\alpha)^{\frac{1}{p-1}} dx \\
&= C \int (\mu * I_\alpha)^{\frac{1}{p-1}} * I_\alpha d\mu
\end{aligned}$$

where we have also used Lemma 6.9 in the last equality above. The expression $(\mu * I_\alpha)^{\frac{1}{p-1}} * I_\alpha$ is called the nonlinear potential of μ.

To prove the main theorem of this section (stated as Theorem 2.7 in Section §2), we need the following lemma which gives another characterization of Radon measures belonging to the dual of the fractional Sobolev space $W^{\alpha,p}(\Omega)$.

Lemma 7.2. *A Radon measure μ is in $(W^{\alpha,p}(\mathbb{G}))^*$ $(1 < p < \infty)$ if and only if $||\mu * G_\alpha||_{p'} < \infty$, namely, if and only if the (α, p) energy of μ is finite.*

Proof. For $u \in W^{\alpha,p}(\mathbb{G})$, write $u = f * G_\alpha$. Thus

$$\begin{aligned}\int u d\mu &= \int f * G_\alpha d\mu \\ &= \int \mu * G_\alpha \cdot f dx \\ &\le ||\mu * G_\alpha||_{p'} \cdot ||f||_p \\ &\le C||\mu * G_\alpha||_{p'} \cdot ||u||_{\alpha,p},\end{aligned}$$

which implies that $\mu \in (W^{\alpha,p}(\mathbb{G}))^*$.

Conversely, if $\mu \in (W^{\alpha,p}(\mathbb{G}))^*$, then

$$||\mu * G_\alpha||_{p'} = \sup_{f\in L^p} \int (\mu * G_\alpha) f dx = \sup_{f\in L^p} \int (f * G_\alpha) d\mu < \infty.$$

Therefore, $||\mu * G_\alpha||_{p'} < \infty$, namely, the (α, p) energy of μ is finite. □

Proof of Theorem 2.7. Observe that

$$\begin{aligned}\frac{\mu\,(B(x,r))}{r^{Q-\alpha}} &\le \left(\int_r^{2r} \left[\frac{\mu\,(B(x,t))}{t^{Q-\alpha}}\right]^{p'} \frac{dt}{t}\right)^{\frac{1}{p'}} \\ &\le \left(\int_0^{\infty} \left[\frac{\mu\,(B(x,t))}{t^{Q-\alpha}}\right]^{p'} \frac{dt}{t}\right)^{\frac{1}{p'}}\end{aligned}$$

Thus, the fractional maximal function is bounded by

$$\mathcal{M}_\alpha \mu(x) \le \left(\int_0^{\infty} \left[\frac{\mu\,(B(x,t))}{t^{Q-\alpha}}\right]^{p'} \frac{dt}{t}\right)^{\frac{1}{p'}}.$$

Hence, using the $L^{p'}$ boundedness of

$$||\mu * I_\alpha||_{p'} \le C||\mathcal{M}_\alpha \mu||_{p'}$$

we get

$$\begin{aligned}\mathcal{E}_{\alpha,p}(\mu) &\le C \int_{\mathbb{G}} (\mu * I_\alpha)^{p'} dx \\ &\le C \int_{\mathbb{G}} (\mathcal{M}_\alpha \mu)^{p'} dx \\ &\le C \int_{\mathbb{G}} \left(\int_0^{\infty} \left[\frac{\mu\,(B(x,t))}{t^{Q-\alpha}}\right]^{p'} \frac{dt}{t}\right) dx.\end{aligned}$$

To estimate the last term, we notice that

$$\begin{aligned}\int_{\mathbb{G}} \mu(B(x,t))^{p'}dx &= \int_{\mathbb{G}} \mu(B(x,t))^{\frac{1}{p-1}} \mu(B(x,t))dx \\ &= \int_{\mathbb{G}} \mu(B(x,t))^{\frac{1}{p-1}} \int_{B(x,t)} d\mu(y)dx \\ &\leq \int_{\mathbb{G}} \int_{B(x,t)} \mu(B(y,2t))^{\frac{1}{p-1}} d\mu(y)dx \\ &\leq C \int_{\mathbb{G}} \mu[B(y,2t)]^{\frac{1}{p-1}} |B(y,t)| d\mu(y) \\ &= Ct^Q \int_{\mathbb{G}} \mu[B(y,2t)]^{\frac{1}{p-1}} d\mu(y).\end{aligned}$$

In next to the last equality we have used the Fubini's theorem and we used $|B(y,t)| = Ct^Q$. Thus,

$$\begin{aligned}\mathcal{E}_{\alpha,p}(\mu) &\leq C \int_0^\infty (t^{\alpha-Q})^{p'} t^Q \int_{\mathbb{G}} \mu[B(y,2t)]^{\frac{1}{p-1}} d\mu(y) \frac{dt}{t} \\ &= C \int_{\mathbb{G}} \int_0^\infty \left(\frac{\mu[B(y,t)]}{t^{Q-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} d\mu(y).\end{aligned}$$

Since μ has compact support and $\mu(\mathbb{G}) < \infty$, thus it is easy to see that the expression on the right hand side of the above inequality is finite if

$$\int_{\mathbb{G}} \int_0^1 \left(\frac{\mu[B(y,t)]}{t^{Q-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} d\mu(y) < \infty$$

and consequently $\mathcal{E}_{\alpha,p}(\mu)$ is finite and then by Lemma 7.2 we get $\mu \in (W^{\alpha,p}(\mathbb{G}))^*$. This proves the sufficiency of Theorem 2.7.

To prove the necessity, we need to use the lower bound of the Bessel kernel (see Theorem 6.5)

$$G_\alpha(x) \geq C|x|^{\alpha-Q} e^{-M|x|} \quad \text{for} \quad x \in \mathbb{G}, \quad x \neq 0.$$

The (α, p) energy of μ can be written as

$$\mathcal{E}_{\alpha,p}(\mu) = \int_{\mathbb{G}} (\mu * G_\alpha)^{\frac{1}{p-1}} * G_\alpha d\mu.$$

If write $f = (\mu * G_\alpha)^{\frac{1}{p-1}}$, then

$$\begin{aligned}f * G_\alpha(x) &= \int_{\mathbb{G}} f(xy^{-1}) G_\alpha(y) dy \\ &\geq C \int_0^\infty \left(\int_{B(x,r)} f(y)dy \right) r^{\alpha-Q} e^{-Mr} \frac{dr}{r}.\end{aligned}$$

But for $y \in B(x, r)$

$$
\begin{aligned}
f(y) &\geq \left(\int_{B(x,r)} G_\alpha(z^{-1}y)d\mu(z)\right)^{\frac{1}{p-1}} \\
&\geq C\left(\int_{B(x,r)} |z^{-1}y|^{\alpha-Q}e^{-M|z^{-1}y|}d\mu(z)\right)^{\frac{1}{p-1}} \\
&\geq C\left(\int_{B(x,r)} r^{\alpha-Q}e^{-2Mr}d\mu(z)\right)^{\frac{1}{p-1}} \\
&= C\left(\mu(B(x,r))r^{\alpha-Q}e^{-2Mr}\right)^{\frac{1}{p-1}}
\end{aligned}
$$

where we have used the fact $|z^{-1}y| \leq 2r$ for $y, z \in B(x, r)$. Thus,

$$
\begin{aligned}
f * G_\alpha(x) &\geq C\int_0^\infty \left(\int_{B(x,r)} f(y)dy\right) r^{\alpha-Q}e^{-Mr}\frac{dr}{r} \\
&\geq C\int_0^\infty \left(\int_{B(x,r)} \left(\mu(B(x,r))r^{\alpha-Q}e^{-2Mr}\right)^{\frac{1}{p-1}} dy\right) r^{\alpha-Q}e^{-Mr}\frac{dr}{r} \\
&\geq C\int_0^\infty \left(\frac{\mu[B(x,r)]}{r^{Q-\alpha p}}\right)^{\frac{1}{p-1}} e^{-2p'Mr}\frac{dr}{r}.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
\mathcal{E}_{\alpha,p}(\mu) &\geq C\int_{\mathbb{G}}\int_0^\infty \left(\frac{\mu[B(x,r)]}{r^{Q-\alpha p}}\right)^{\frac{1}{p-1}} e^{-2p'Mr}\frac{dr}{r}d\mu(x) \\
&\geq C\int_{\mathbb{G}}\int_0^1 \left(\frac{\mu[B(y,r)]}{r^{Q-\alpha p}}\right)^{\frac{1}{p-1}} e^{-2p'Mr}\frac{dr}{r}d\mu(x) \\
&\geq Ce^{-2Mp'}\int_{\mathbb{G}}\int_0^1 \left(\frac{\mu[B(x,r)]}{r^{Q-\alpha p}}\right)^{\frac{1}{p-1}} \frac{dr}{r}d\mu(x)
\end{aligned}
$$

This establishes the necessity of Theorem 2.7 by using Lemma 7.2 again. □

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