Sharp Constants for Moser-Trudinger Inequalities on Spheres in Complex Space $\mathbb{C}^n$

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Abstract

The main results of this paper concern sharp constants for the Moser-Trudinger inequalities on spheres in complex space $\mathbb{C}^n$. We derive Moser-Trudinger inequalities for smooth functions and holomorphic functions with different sharp constants (see Theorem 1.1). The sharp Moser-Trudinger inequalities under consideration involve the complex tangential gradients for the functions and thus we have shown here such inequalities in the CR setting. Though there is a close connection in spirit between inequalities proven here on complex spheres and those on the Heisenberg group for functions with compact support in any finite domain proven earlier by the same authors [17], derivation of the sharp constants for Moser-Trudinger inequalities on complex spheres are more complicated and difficult to obtain than on the Heisenberg group. Variants of Moser-Onofri-type inequalities are also given on complex spheres as applications of our sharp inequalities (see Theorems 1.2 and 1.3). One of the key ingredients in deriving the main theorems is a sharp representation formula for functions on the complex spheres in terms of complex tangential gradients (see Theorem 1.4).

1 Introduction and Statement of Main Theorems

In 1971 J. Moser [28] found the largest positive constant β_0 (which sharpened the result of Trudinger [39]) such that if Ω is an open subset of euclidean space $\mathbb{R}^n$ $(n \geq 2)$ with finite Lebesgue measure, then there is a constant C_0 depending only on n such that

$$\frac{1}{|\Omega|} \int_{\Omega} \exp\left(\beta |f(x)|^{\frac{n}{n-1}}\right) dx \leq C_0 \tag{1.1}$$

for any $\beta \leq \beta_0$ and any f in the Sobolev space $W_0^{1,n}(\Omega)$ provided $\|\nabla f\|_{L^n(\Omega)} \leq 1$.

In fact, Moser showed $\beta_0 = n\omega_{n-1}^{1/(n-1)}$, where ω_{n-1} is the area of the surface of the unit n-ball. He also proved that if β exceeds β_0, then the above inequality cannot hold with uniform C_0 independent of f. Later, D. Adams found sharp constants for the Moser inequality in higher-order Sobolev spaces [1], and these higher-order Moser-Trudinger inequalities were proven for Riemannian manifolds by Fontana [22] and Branson, Chang, and Yang [7].

Along with the Moser-Trudinger inequality, the question of whether the supremum

$$\sup\left\{\frac{1}{|\Omega|}\int_{\Omega}\exp\left(n\omega_{n-1}^{1/(n-1)}|f(x)|^{\frac{n}{n-1}}\right)dx : f \in W_0^{1,n}(\Omega),\ \|\nabla f\|_{L^n(\Omega)} \leq 1\right\} \tag{1.2}$$

is attained has also been considered. In 1986 Carleson and Chang [10] proved that the above supremum has extremals for the case where Ω is a ball in $\mathbb{R}^n$ for $n \geq 2$. Their result came as a surprise since it was already known that Sobolev inequality has no extremals supported in balls for $p > 1$ (see Talenti [38] and Aubin [2]). Carleson and Chang proved the existence of extremals by reduction to a one-dimensional problem using a symmetrization argument. We note that the Carleson-Chang result was extended to arbitrary bounded smooth domains by Flucher when $n = 2$ [21] and by Lin for the case $n > 2$ [27]. Furthermore, Soong extended the result to some domains contained in n-spheres [36].

With a modification of Moser's proof of his inequality (1.1), Moser also proved in [29] that there exists a constant C_0 such that 4π is the best constant for the inequality

$$\int_{\mathbb{S}^2}\exp(4\pi|f(x)|^2)d\sigma \leq C_0 \tag{1.3}$$

for any smooth f on $\mathbb{S}^2$ with $\int_{\mathbb{S}^2}|\nabla f|^2\,d\sigma \leq 1$ and $\int_{\mathbb{S}^2} f\,d\sigma = 0$, where $d\sigma$ is the surface measure and ∇ is the gradient on $\mathbb{S}^2$.

Moser further proved that when the function f is in the class of even functions on $\mathbb{S}^2$, i.e., $f(\xi) = f(-\xi)$, the best constant for Moser's inequality is 8π. Namely, there exists a constant C_0 such that 8π is the best constant for the inequality

$$\int_{\mathbb{S}^2}\exp(8\pi|f(x)|^2)d\sigma \leq C_0 \tag{1.4}$$

for any smooth f on $\mathbb{S}^2$ with $\int_{\mathbb{S}^2}|\nabla f|^2\,d\sigma \leq 1$, $\int_{\mathbb{S}^2} f\,d\sigma = 0$, and $f(\xi) = f(-\xi)$.

Inequality (1.3) is motivated by the Nirenberg problem: characterize all Gauss curvature functions $K(x)$ belonging to metrics ds^2 that are conformally related to the standard metric ds_0^2 so that $ds^2 = e^f\,ds_0^2$ for a function f on $\mathbb{S}^2$. Let $\triangle$ be the Laplace-Beltrami operator on $\mathbb{S}^2$ with respect to the standard metric ds_0^2. Thus K and f are related by the equation

$$\triangle f + Ke^{2f} - 1 = 0\,. \tag{1.5}$$

Moser applied inequality (1.3) to the problem of prescribing Gaussian curvature on $\mathbb{S}^2$. He considered the functional

$$G(f) = \log\left\{\frac{1}{4\pi}\int_{\mathbb{S}^2} Ke^{2f}\,d\sigma\right\} - \frac{1}{4\pi}\int_{\mathbb{S}^2} |\nabla f|^2\,d\sigma - \frac{1}{2\pi}\int_{\mathbb{S}^2} f\,d\sigma\,. \tag{1.6}$$

Using inequality (1.3) Moser showed that there is a positive constant C such that

$$\frac{1}{4\pi}\int_{\mathbb{S}^2} e^{2f}\,d\sigma \le C\exp\left\{\frac{1}{4\pi}\int_{\mathbb{S}^2} |\nabla f|^2\,d\sigma + \frac{1}{2\pi}\int_{\mathbb{S}^2} f\,d\sigma\right\}. \tag{1.7}$$

It follows from (1.7) that the functional $G(f)$ is bounded from above. Under the further assumption that $K(\xi) = K(-\xi)$ on $\mathbb{S}^2$, Moser showed that equation (1.5) has a solution f such that $f(\xi) = f(-\xi)$ provided that $\max_{\mathbb{S}^2} K(\xi) > 0$ using the boundedness from above for the functional $G(f)$ and inequality (1.4).

We now recall that the Moser-Onofri inequality on $\mathbb{S}^2$ states that

$$\frac{1}{4\pi}\int_{\mathbb{S}^2} e^{2f}\,d\sigma \le \exp\left(\frac{1}{4\pi}\int_{\mathbb{S}^2} (2f + |\nabla f|^2)d\sigma\right) \tag{1.8}$$

with equality if and only if $e^{2f}g_0$ is isometric to g_0. The contribution of Onofri [31] lies in the fact that (1.8) is an improvement of (1.7) in the sense that the sharp constant C in (1.7) is 1. Onofri's inequality was also independently proven by Hong [23] (see also another proof in [32]). It was applied to geometry by Osgood, Phillips, and Sarnak [32, 33], who used it to compute extremals for determinants of Laplacians on two-dimensional manifolds. As a corollary of Onofri's inequality, it follows that among all the metrics on $\mathbb{S}^2$, $\log\det\triangle_0$ is the maximum. Moreover, it is shown by Osgood, Phillips, and Sarnak that the isospectral family of metrics on compact surfaces without boundary forms a compact set in the C^∞ topology.

We now turn to the generalization of Onofri's inequality to the higher-dimensional case. Let $\triangle_{\mathbb{S}}$ denote the Laplacian on $\mathbb{S}^n$. We note that the inequality involves the Paneitz operator on the sphere. The Paneitz operator on 4-manifolds was discovered by Paneitz [34]. It was extended for all dimensions $n \ne 2$ by Branson [6] and Beckner [5].

On the n-dimensional sphere, the Paneitz operator is defined as the pullback of $(-\triangle)^{n/2}$ under the stereographic projection. More precisely, define the operator A_n on $L^2(\mathbb{S}^n)$ by

$$A_n = \prod_{j=0}^{\frac{n-2}{2}}(-\triangle_{\mathbb{S}} + j(n-1-j)) \quad \text{if } n \text{ is even}$$

and

$$A_n = \prod_{j=0}^{\frac{n-3}{2}}(-\triangle_{\mathbb{S}} + j(n-1-j))\left(-\triangle_{\mathbb{S}} + \left(\frac{n-1}{2}\right)^2\right)^{\frac{1}{2}} \quad \text{when } n \text{ is odd.}$$

The eigenvalues of A_n are $l(l+1)\cdot(l+n-1)$, $l=0,1,2,\ldots$, and the eigenfunctions are spherical harmonics. It is easy to check that

$$A_1 = (-\triangle_{\mathbb{S}})^{\frac{1}{2}}\,, \quad A_2 = -\triangle_{\mathbb{S}}\,, \quad \text{and} \quad A_4 = (-\triangle_{\mathbb{S}})^2 + 2(-\triangle_{\mathbb{S}})\,.$$

This operator A_n is a natural generalization of the conformal Laplacian on $\mathbb{S}^2$ and the Paneitz operator on $\mathbb{S}^4$ [7, 34]. We note that A_n is given in the stereographic coordinates by $h^{-1}(-\triangle)^{\frac{n}{2}}$, where

$$h = |\mathbb{S}^n|^{-1}\left(\frac{2}{1+|x|^2}\right)^n$$

is the Jacobian of the stereographic projection. It is also easy to see that A_n is covariant in the sense that for each conformal map

$$\tau : \mathbb{S}^n \to \mathbb{S}^n$$

and for $f : \mathbb{S}^n \to \mathbb{R}$,

$$A_n(f\circ\tau) = |\mathcal{J}_\tau|(A_n f)\circ\tau\,.$$

Thus the higher-dimensional Onofri's inequality of Beckner (see also Carlen and Loss [9]) states that

$$\frac{1}{2n!}\int_{\mathbb{S}^n} \left|A_n^{1/2} f\right|^2 d\mu \ge \log\left(\int_{\mathbb{S}^n} e^f\, d\mu\right) - \int_{\mathbb{S}^n} f\, d\mu \tag{1.9}$$

holds for all $f \in L^2(\mathbb{S}^n)$ for which the left side of the above inequality is finite (here we use $d\mu$ to denote the uniform normalized surface measure on $\mathbb{S}^n$). The equality holds if and only if there is a conformal τ whose Jacobian is $\mathcal{J}_\tau$ such that $f = \text{const} + \log|\mathcal{J}_\tau|$.

The important relationship between the higher-dimensional Moser-Onofri inequality and the problem of prescribing Q-curvature on high-dimensional Riemannian manifolds has been explored extensively in the work of Paneitz [34], Branson [6], Branson, Chang, and Yang [7], Branson and Ørsted [8], Chang and Yang [13], and Chang, Gursky, and Yang [12]. We refer to Chang [11] and Chang and Yang [14, 15] for extensive recent accounts and many references in this direction.

Although there has been substantial development of the theory of conformal geometry on spheres, euclidean space, and more general Riemannian manifolds, much less is known in the CR setting.

In the 1980s Jerison and Lee successfully completed the program of prescribing the scalar curvature problem in the CR setting (see [24, 25, 26]). In particular, they found the best constant and extremals for the L^2 to $L^{2Q/(Q-2)}$ Sobolev inequality on the Heisenberg group $\mathbb{H}^n$ (where $Q = 2n+2$ is the homogeneous dimension), and solved the CR Yamabe problem of conformally changing the contact form to one with constant Webster curvature in the compact setting. The extensive exploration of conformal geometry and Moser-Trudinger and Moser-Onfri inequalities on Riemannian manifolds as mentioned above, and the work in the CR setting by

Jerison and Lee [24, 25, 26], Fefferman [19], Fefferman and Graham [20], and Bailey, Eastwood, and Graham [3] suggest that a satisfactory theory in the CR setting should be developed. To this end, Cohn and Lu obtained the sharp constant for the Moser-Trudinger inequality on the Heisenberg group in [17]. The method of [17] is to derive a sharp pointwise estimate for any function with compact support on $\mathbb{H}^n$ by the convolution of an integral kernel with the subelliptic gradient of the function. The rearrangement of the convolution (rather than considering the rearrangement of the function itself) is then used to derive the sharp constant for the Moser-Trudinger inequality on $\mathbb{H}^n$. The ideas and methods developed in [17] have also been used to generalize to the Heisenberg-type group in [18] and more general groups in [4].

Motivated by [17], the prominent role of the Heisenberg group $\mathbb{H}^n$ and the sphere in $\mathbb{C}^n$ in CR geometry, and the intimate relationship between them (see expositions for such a relationship in [16, 37]), in this paper we investigate the sharp Moser-Trudinger and Moser-Onofri inequalities on the sphere in complex space $\mathbb{C}^n$.

Our main result is a sharp Moser-Trudinger-type inequality with the complex tangential gradient on spheres of odd dimension. A sphere of odd dimension can be viewed as a sphere in an n-dimensional complex space; therefore there is a complex tangential gradient on such a sphere. This complex tangential gradient is bounded above pointwise by the usual gradient.

To state our main theorems, we must first introduce some necessary notation and definitions. A more precise and detailed account will be given in Section 2. Let $\mathbb{C}$ denote the complex numbers and $\mathbb{C}^n$ be the usual n-dimensional complex vector space equipped with the Hermitian inner product

$$\langle z, w\rangle = \sum_{j=1}^{n} z_j \bar{w}_j$$

where $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n) \in \mathbb{C}^n$.

We define the standard vector fields

$$D_k = \frac{1}{2}\left(\frac{\partial}{\partial x_k} - i\frac{\partial}{\partial y_k}\right) \quad \text{and} \quad \overline{D}_k = \frac{1}{2}\left(\frac{\partial}{\partial x_k} + i\frac{\partial}{\partial y_k}\right), \qquad k = 1, \dots, n,$$

and recall the complex structure map on vector fields determined by the equations $JD_k = iD_k$ and $J\overline{D}_k = -i\overline{D}_k$.

Let $\mathbb{S}$ $(= \mathbb{S}^{2n-1})$ be the (real) $(2n-1)$–dimensional sphere $\{z : \langle z, z\rangle = 1\}$ contained in $\mathbb{C}^n$, and $R(z) = z \cdot D + \overline{z} \cdot \overline{D} = \sum_{j=1}^n z_j \cdot D_j + \overline{z}_j \cdot \overline{D}_j$ denote the vector field normal to the sphere.

If f is a smooth function defined on a neighborhood of a point $z \in \mathbb{C}^n$, then the ordinary gradient in $\mathbb{C}^n$ is

$$\nabla f(z) = \sum_j 2\overline{D_j f(z)} D_j + 2\overline{D_j \overline{f}(z)}\,\overline{D}_j\,.$$

Let $\nabla_{\mathrm{t}} f$ be the projection of ∇f onto the tangent space at $\mathbb{S}$ normal to R, and let $\nabla_{\mathbb{C}} f$ be the projection of ∇f onto the complex tangent space at $\mathbb{S}$ normal to both R and JR, respectively. Thus

$$\nabla_{\mathrm{t}} f(z) = \nabla f(z) - \langle \nabla f(z), R(z) \rangle R(z)$$

and

$$\nabla_{\mathbb{C}} f(z) = \nabla f(z) - \langle \nabla f(z), R(z) \rangle R(z) - \langle \nabla f(z), JR(z) \rangle JR(z) .$$

From this it is clear that

$$|\nabla_{\mathbb{C}} f(z)|^2 = |\nabla_t f(z)|^2 - |JRf(z)|^2 .$$

Therefore,

$$|\nabla_{\mathbb{C}} f(z)| \leq |\nabla_t f(z)| ;$$

that is, the complex tangential gradient is pointwise bounded above in the norm by the euclidean tangential.

Let

$$M_{jk}(z) = \overline{z_j} D_k - \overline{z_k} D_j , \quad \overline{M}_{jk}(z) = z_j \overline{D}_k - z_k \overline{D}_j , \tag{1.10}$$

and

$$E_k = \sqrt{2} \sum_j z_j M_{j,k} , \quad \overline{E}_k = \sqrt{2} \sum_j \overline{z}_j \overline{M}_{j,k} . \tag{1.11}$$

Then we get the formulas

$$\nabla_{\mathbb{C}} f(z) = \sum_j \overline{E_j f(z)} E_j + E_j \overline{f}(z) \overline{E}_j \tag{1.12}$$

and

$$|\nabla_{\mathbb{C}} f(z)|^2 = \sum_j |E_j f(z)|^2 + |\overline{E}_j f(z)|^2 . \tag{1.13}$$

Let $d\sigma$ denote the normalized volume element on $\mathbb{S}$, and let

$$B(p, q) = \int_0^1 x^{p-1} (1 - x)^{q-1} \, dx$$

be the beta function for $p > 0$, $q > 0$. We also denote by $\|g\|_{L^{2n}(\mathbb{S})}$ or $\|g\|_{2n}$ the L^{2n} norm of a function g on $\mathbb{S}$ with respect to the measure $d\sigma$. One of the main theorems of this paper is the following result:

THEOREM 1.1 *Let*

$$B = n \left((n-1) \pi^{-1} 2^{2n-2} \cdot B\left(\frac{2n-1}{2}, \frac{1}{2} \right) \right)^{\frac{1}{2n-1}} .$$

There is a constant C_0 *such that, for all smooth functions* f *with* $\int_{\mathbb{S}} f\,d\sigma = 0$ *and* $\|\nabla_{\mathbb{C}} f\|_{L^{2n}(\mathbb{S})} \le 1$*, the inequality*

$$\int_{\mathbb{S}} \exp\big(B|f|^{\frac{2n}{2n-1}}\big) d\sigma \le C_0 \tag{1.14}$$

is verified. No larger value of B *verifies the inequality. If* f *is taken from the class of all holomorphic functions with* $f(0) = 0$*, then the sharp constant can be enlarged to* $B_H = 2^{n/(2n-1)} \cdot B$.

Recall that the Moser-Trudinger inequality on the real sphere $\mathbb{S} = \mathbb{S}^{2n-1}$ as stated in the work of Moser [28] and Fontana [22] states that

$$\int_{\mathbb{S}} \exp\big(\beta_0|f|^{\frac{2n-1}{2n-2}}\big) d\sigma \le C_0 \tag{1.15}$$

with a different sharp constant $\beta_0 > 0$ for f with $\int_{\mathbb{S}} f\,d\sigma = 0$ and $\|\nabla_{\mathrm{t}} f\|_{2n-1} \le 1$.

There are several differences between inequalities (1.14) and (1.15):

- The smaller gradient $|\nabla_{\mathbb{C}} f|$ occurs in place of the usual gradient $|\nabla_{\mathrm{t}} f|$.
- The assumptions on the exponents of integrability for the gradients are different. In (1.14) the exponent is $2n$ as opposed to the exponent $2n-1$, which occurs in (1.15).

Using Theorem 1.1 we can derive the following Moser-Onofri-type inequality. (We do not know, however, the exact values of the sharp constants C_0 and C_1.)

THEOREM 1.2 *Let* B *be the same constant as in Theorem* 1.1. *Then there is a constant* C_0 *such that*

$$\log \int_{\mathbb{S}} e^{2nf}\,d\sigma \le \log C_0 + 2n \int_{\mathbb{S}} f\,d\sigma + (2n)^{2n-1}\left(\frac{2nB}{2n-1}\right)^{-(2n-1)} \int_{\mathbb{S}} |\nabla_{\mathbb{C}} f|^{2n}\,d\sigma\,.$$

If f *is taken from the class of all holomorphic functions, then there is a constant* C_1 *such that*

$$\log \int_{\mathbb{S}} e^{2nf}\,d\sigma \le \log C_1 + 2n \int_{\mathbb{S}} f\,d\sigma + (2n)^{2n-1}\left(\frac{2nB_H}{2n-1}\right)^{-(2n-1)} \int_{\mathbb{S}} |\nabla_{\mathbb{C}} f|^{2n}\,d\sigma$$

where $B_H = 2^{\frac{n}{2n-1}} \cdot B$.

We now consider the functionals

$$I(f) = \log \int_{\mathbb{S}} e^{2nf}\, d\sigma - 2n \int_{\mathbb{S}} f\, d\sigma - (2n)^{2n-1}\left(\frac{2nB}{2n-1}\right)^{-(2n-1)} \int_{\mathbb{S}} |\nabla_{\mathbb{C}} f|^{2n}\, d\sigma$$

for all smooth f and

$$J(f) = \log \int_{\mathbb{S}} e^{2nf}\, d\sigma - 2n \int_{\mathbb{S}} f\, d\sigma - (2n)^{2n-1}\left(\frac{2nB_H}{2n-1}\right)^{-(2n-1)} \int_{\mathbb{S}} |\nabla_{\mathbb{C}} f|^{2n}\, d\sigma$$

for holomorphic f. Then Theorem 1.2 implies the following:

THEOREM 1.3 *The functionals $I(f)$ and $J(f)$ are bounded from above. Namely,*

$$\sup I(f) < \infty \quad \textit{and} \quad \sup J(f) < \infty\,.$$

In the proof of Theorem 1.1 above, the main tool is a sharp representation formula. For $\lambda \in \mathbb{C}$, let

$$K_\beta(\lambda) = \left(2^{n+\beta-1} B\left(\frac{n+\beta}{2}, \frac{1}{2}\right)\right)^{-1} (n+\beta-1) \cdot \frac{(1-|\lambda|^2)^\beta}{|1-\lambda|^{n+\beta-1}}\,,$$

and let the convolution on the sphere $\mathbb{S}$ be defined as

$$f * K_\beta(\xi) = \int_{\mathbb{S}} f(z) K_\beta(\langle z, \xi\rangle) dz\,.$$

Then we get a one-parameter representation formula, which leads to the following pointwise inequality, where, from now on, $c_n = (n-1)\pi^{-1}$:

THEOREM 1.4 *Suppose $\beta > -n+1$ and*

$$B_\beta = \left(c_n 2^{n+\beta-1} B\left(\frac{n+\beta}{2}, \frac{1}{2}\right)\right)^{-1}.$$

Then

$$|f(\zeta) - f * K_\beta(\zeta)| \le B_\beta \int_{\mathbb{S}} |\nabla_{\mathbb{C}} f(z)| \frac{(1-|\langle z, \zeta\rangle|^2)^{\beta+1/2}}{|1-\langle z, \zeta\rangle|^{n+\beta}}\, d\sigma(z)\,. \tag{1.16}$$

If f is holomorphic, then the constant B_β on the right-hand side can be replaced by the smaller number $\sqrt{2}/2 \cdot B_\beta$.

By choosing $\beta = n - 1$ in Theorem 1.4, we have

$$(1.17)\quad |f(\zeta) - f * K_{n-1}(\zeta)| \le B_{n-1} \int_{\mathbb{S}} |\nabla_{\mathbb{C}} f(z)| \frac{(1 - |\langle z, \zeta\rangle|^2)^{n-1/2}}{|1 - \langle z, \zeta\rangle|^{2n-1}} \, d\sigma(z)\,.$$

Using (1.17) we get the sharp Moser-Trudinger inequality

$$(1.18)\qquad \int_{\mathbb{S}} \exp\left(B \left(\frac{|f - f * K_{n-1}|}{\|\nabla_{\mathbb{C}} f\|_{2n}} \right)^{\frac{2n}{2n-1}} \right) d\sigma \le C_0$$

with the same sharp constant B as in Theorem 1.1. The passage from this to Theorem 1.1 requires the use of an embedding theorem relating the complex gradient $\nabla_{\mathbb{C}} f$ and the ordinary gradient $\nabla_t f$ (see Section 7 for more details).

To conclude this introduction, we make some remarks concerning the relationship between the sharp Moser-Trudinger inequalities on the Heisenberg group and those on complex spheres. One might suppose that the sharp Moser-Trudinger inequalities on the sphere $\mathbb{S}$ in $\mathbb{C}^n$ should follow from those on the Heisenberg group by using the well-known Cayley transform. As far as we understand, this is not the case. Indeed, none of the theorems in this paper on the complex sphere seem to follow in such a way.

To be more precise, we recall what we have proven in [17] concerning the Moser-Trudinger inequality on the Heisenberg group $\mathbb{H}^n$. Let $\mathbb{H}^n$ be the n-dimensional Heisenberg group

$$\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$$

whose group structure is given by

$$(z, t) \cdot (z', t') = (z + z', t + t' + 2\,\mathrm{Im}(z \cdot \overline{z'}))$$

for any two points (z, t) and (z', t') in $\mathbb{H}^n$.

The Lie algebra of $\mathbb{H}^n$ is generated by the left-invariant vector fields

$$T = \frac{\partial}{\partial t}\,, \qquad X_i = \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t}\,, \qquad Y_i = \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t}\,,$$

for $i = 1, \dots, n$. These generators satisfy the noncommutative relationship

$$[X_i, Y_j] = -4\delta_{ij} T\,.$$

Moreover, all the commutators of length greater than 2 vanish, and thus this is a nilpotent, graded, and stratified group of step 2.

We now use $|\nabla_{\mathbb{H}^n} f|$ to express the (euclidean) norm of the subelliptic gradient of f:

$$|\nabla_{\mathbb{H}^n} f| = \sum_{i=1}^{n} \big((X_i f)^2 + (Y_i f)^2 \big)^{\frac{1}{2}}\,.$$

We also use $W_0^{1,p}(\Omega)$ for the open set $\Omega \subset \mathbb{H}^n$ to denote the completion of $C_0^\infty(\Omega)$ under the norm

$$\|f\|_{L^p(\Omega)} + \|\nabla_{\mathbb{H}^n} f\|_{L^p(\Omega)}\,.$$

The following sharp Moser-Trudinger inequality on the n-dimensional Heisenberg group $\mathbb{H}^n$ was proven by Cohn and Lu in [17].

THEOREM A *Let* $A_Q = Q(c_Q)^{Q'/Q}$, *where*

$$Q = 2n+2\,, \quad Q' = \frac{Q}{Q-1}\,, \quad \textit{and} \quad c_Q = \frac{\omega_{2n-1}\Gamma(\frac{1}{2})\Gamma(\frac{Q-1}{2})}{\Gamma(\frac{Q}{2})}\,.$$

There exists a constant C_0 *such that for all* $\Omega \subset \mathbb{H}^n$, $|\Omega| < \infty$, *and* $f \in W_0^{1,Q}(\Omega)$,

$$\frac{1}{|\Omega|}\int_\Omega \exp\big(A_Q|f(u)|^{Q'}\big)du \le C_0\,,$$

provided $\|\nabla_{\mathbb{H}^n} f\|_{L^Q} \le 1$. *Furthermore, if* A_Q *is replaced by any number greater than* A_Q, *then the statement is false.*

We now let $\mathbb{B}^n = \{w \in \mathbb{C}^n : |w| < 1\}$ be the unit ball in $\mathbb{C}^n$ whose boundary is $\mathbb{S}$. Let

$$\mathbb{D} = \{z = (z', z_n) : \operatorname{Im} z_n > |z'|^2\} \subset \mathbb{C}^n\,.$$

The Cayley transform $C : \mathbb{B}^n \to \mathbb{D}$ is thus defined as

$$C(w) = (w', w_n) = \left(\frac{w'}{1+w_n}, i\frac{1-w_n}{1+w_n}\right).$$

This transform explores the biholomorphic equivalence between $\mathbb{D}$ and $\mathbb{B}^n$. The $(n-1)$–dimensional Heisenberg group $\mathbb{H}^{n-1}$ arises as the group of translations of $\mathbb{D}$, and this leads to its identification with the boundary of $\partial\mathbb{D}$; see Stein's book [37, p. 530].

This Cayley transform also maps the sphere $\mathbb{S}$ minus the south pole to the boundary of $\mathbb{D}$, the $(n-1)$–dimensional Heisenberg group $\mathbb{H}^{n-1}$. It is natural to see if there is any way to derive the Moser-Trudinger inequality on $\mathbb{S}$ from the one on $\mathbb{H}^{n-1}$ obtained from Theorem A. (Note that we need to replace n by $n-1$ in Theorem A). This does not seem to be a straightforward matter, as is explained below.

Theorem A gives only a sharp Moser-Trudinger inequality on the Heisenberg group for functions with compact support for any given bounded open set Ω. Under the Cayley transform, such functions are transformed to those functions with compact support away from the south pole $(0, \dots, 0, -1)$ in $\mathbb{S}$. More precisely, we can derive through Theorem A an inequality of the following type on $\mathbb{S}$:

$$\int_E \exp\big(A_{2n}|f|^{\frac{2n}{2n-1}}\big)\frac{d\sigma}{|1+w_n|^{2n}} \le C\int_E \frac{d\sigma}{|1+w_n|^{2n}}$$

provided $\|\nabla_{\mathbb{C}} f\|_{2n} \le 1$, where E is an open subset of $\mathbb{S}$ and $E \cap (0, \dots, 0, -1) = \varnothing$ and f are functions supported in E.

This inequality is unsatisfactory for two reasons. First, the inequality only holds for functions supported in an open subset away from the south pole. The Moser-Trudinger inequality derived in this paper (see Theorem 1.1) involves arbitrary smooth functions on the complex sphere $\mathbb{S}$ without restriction on their support. To achieve this we must have a term $\int_{\mathbb{S}} f d\sigma$ subtracted from f in the inequality of Theorem 1.1. Second, inequalities derived from Theorem A will involve an extra weight $\frac{d\sigma}{|1+w_n|^{2n}}$ in the integrals on both sides. Such an inequality is clearly different from the sharp Moser-Trudinger inequality given by Theorem 1.1.

In summary, although the Cayley transform certainly can be used to reformulate results on the Heisenberg group to results on the complex sphere, Theorem 1.1 is not simply a reformulation of Theorem A. This is why we have taken the direct approach used in this paper.

The organization of the paper is as follows: In Section 2, we introduce in detail the necessary notation and consider the divergence of vector fields in $\mathbb{C}^n$. The purpose of using this divergence is to make it easier to realize the differential operator $\mathcal{L}_{\beta,\zeta}$ under consideration in Section 3 as a divergence operator. In Section 3, we prove a representation formula in terms of this differential operator $\mathcal{L}_{\beta,\zeta}$ (see Theorem 3.3) and thus prove the sharp pointwise estimates Theorem 3.4 (namely, Theorem 1.4 stated above). Section 4 deals with the estimates for distribution functions for the kernel arising in the sharp pointwise estimates in Theorem 1.4. In Section 5, we use the distribution estimates in Section 4 and the sharp pointwise estimates to prove the Moser-Trudinger inequality (1.12) for $f - f * K_{n-1}$. We then prove in Section 6 that such constants derived in Section 5 are optimal. Section 7 provides the proof of the main theorem, Theorem 1.1 stated above. In Section 8, we prove Theorems 1.2 and 1.3 and give some remarks concerning the functionals $I(f)$ and $J(f)$.

2 Preliminaries

Recall from Section 1 that we let $\mathbb{C}$ denote the complex numbers and $\mathbb{C}^n$ be the usual n-dimensional complex vector space equipped with the Hermitian inner product

$$\langle z, w\rangle = \sum_{j=1}^{n} z_j \bar{w}_j$$

where $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n) \in \mathbb{C}^n$. If z and w are identified with vectors in $\mathbb{R}^{2n}$, then $z \cdot w = \mathrm{Re}\langle z, w\rangle$ is the usual real inner product.

At each point in $\mathbb{C}^n$ we also use the notation $\langle \cdot, \cdot \rangle$ to denote the usual Hermitian Riemannian metric on complex-valued tangent vectors. Defining the standard vector fields

$$D_k = \frac{1}{2}\left(\frac{\partial}{\partial x_k} - i\frac{\partial}{\partial y_k}\right) \quad \text{and} \quad \overline{D}_k = \frac{1}{2}\left(\frac{\partial}{\partial x_k} + i\frac{\partial}{\partial y_k}\right), \quad k = 1, \ldots, n,$$

it follows that the vector fields $\{\sqrt{2}\,D_k, \sqrt{2}\,\overline{D}_k\}$ form an orthonormal system. If $a \in \mathbb{C}^n$ it will also be convenient to use the notation $a \cdot D = \sum_{j=1}^n a_j D_j$ and $\overline{a} \cdot \overline{D} = \sum_{j=1}^n \overline{a}_j \overline{D}_j$. It follows that any vector field X is of the form $X(z) = a \cdot D + \overline{b} \cdot \overline{D}$ where $a : \mathbb{C}^n \to \mathbb{C}^n$ and $b : \mathbb{C}^n \to \mathbb{C}^n$ are $\mathbb{C}^n$-valued functions. Note that $\langle a \cdot D + \overline{b} \cdot \overline{D}, c \cdot D + \overline{d} \cdot \overline{D} \rangle = \frac{1}{2}(\langle a, c\rangle + \langle d, b\rangle)$. In addition, we recall the complex structure map on vector fields determined by the equations $JD_k = iD_k$ and $J\overline{D}_k = -i\overline{D}_k$. Thus $J(a \cdot D + \overline{b} \cdot \overline{D}) = i(a \cdot D - \overline{b} \cdot \overline{D})$. It is easy to see that $J^2 = -I$, $\langle JX, JY\rangle = \langle X, Y\rangle$ and $\langle JX, Y\rangle = -\langle X, JY\rangle$ for all vector fields X and Y.

Let $\mathbb{S}$ be the (real) $(2n-1)$–dimensional sphere $\{z : \langle z, z\rangle = 1\}$ contained in $\mathbb{C}^n$. We will be working with functions defined on the sphere $\mathbb{S}$ and vector fields that are tangent to the sphere. Let $R(z) = z \cdot D + \overline{z} \cdot \overline{D}$ denote the vector field normal to the sphere. Let $X = X(z)$ be a vector field defined on a neighborhood of $\mathbb{S}$. Then X is called *tangential* if $\langle X(z), R(z)\rangle = 0$ for all $z \in \mathbb{S}$, i.e., $X(|z|^2) = 0$ for all $z \in \mathbb{S}$. It follows that the vector field $X = a \cdot D + \overline{b} \cdot \overline{D}$ is tangential if and only if $\langle a(z), z\rangle + \langle z, b(z)\rangle = 0$ for all $z \in \mathbb{S}$.

A vector field X is called *complex tangential* if both X and JX are tangential. Let $N = -iJR = (z \cdot D - \overline{z} \cdot \overline{D})$. Equivalently, X is complex tangential if and only if both $\langle X, R\rangle = 0$ and $\langle X, N\rangle = 0$ at all points of the sphere. It is not hard to verify that $X = a \cdot D + \overline{b} \cdot \overline{D}$ is complex tangential if and only if both $\langle a(z), z\rangle = 0$ and $\langle b(z), z\rangle = 0$ for all $z \in \mathbb{S}$. Obviously, X is complex tangential if and only if both $a(z) \in \mathbb{C}^n \ominus z\mathbb{C}$ and $b(z) \in \mathbb{C}^n \ominus z\mathbb{C}$ for all $z \in \mathbb{S}$.

If f is a smooth function defined on a neighborhood of a point $z \in \mathbb{C}^n$, then $\nabla f(z)$ denotes the unique tangent vector with the property that, for all vectors $X(z)$ in the tangent space at z, $Xf(z) = \langle X(z), (\nabla f)(z)\rangle$. It is easy to see that

$$\nabla f(z) = \sum_j 2\overline{D_j f(z)} D_j + 2D_j \overline{f}(z) \overline{D}_j \,.$$

If f is a smooth function defined on $\mathbb{S}$, then $\nabla_{\mathrm{t}} f$ denotes the unique vector field that is tangential to $\mathbb{S}$ with the property that $Xf(z) = \langle X(z), (\nabla_{\mathrm{t}} f)(z)\rangle$ for all tangential vector fields X.

If f is a smooth function, then there is also a unique complex tangential vector field that we denote by $\nabla_{\mathbb{C}} f$ with the property that $Xf(z) = \langle X(z), (\nabla_{\mathbb{C}} f)(z)\rangle$ for all complex tangential vector fields X. We will need formulas for $\nabla_{\mathrm{t}} f$ and $\nabla_{\mathbb{C}} f$.

To get $\nabla_{\mathrm{t}} f$ and $\nabla_{\mathbb{C}} f$, we simply take the projection of ∇f onto the tangent space and complex tangent space of $\mathbb{S}$. Thus

$$\nabla_{\mathrm{t}} f(z) = \nabla f(z) - \langle \nabla f(z), R(z)\rangle R(z)$$

and

$$\begin{aligned}\nabla_{\mathbb{C}} f(z) &= \nabla f(z) - \langle \nabla f(z), R(z)\rangle R(z) - \langle \nabla f(z), JR(z)\rangle JR(z) \\ &= \nabla_{\mathrm{t}} f(z) - \overline{JRf(z)}\, JR(z)\,.\end{aligned}$$

From this it is clear that

$$|\nabla_{\mathbb{C}} f(z)|^2 = |\nabla_t f(z)|^2 - |JRf(z)|^2 .$$

It is easy to check that if E_j and $\overline{E}_j$ are the projections defined by

$$E_j(z) = \sqrt{2}\, D_j - \langle \sqrt{2}\, D_j, R(z)\rangle R(z) - \langle \sqrt{2}\, D_j, JR(z)\rangle JR(z)$$

and

$$\overline{E}_j(z) = \sqrt{2}\, \overline{D}_j - \langle \sqrt{2}\, \overline{D}_j, R(z)\rangle R(z) - \langle \sqrt{2}\, \overline{D}_j, JR(z)\rangle JR(z) ,$$

then

$$E_j(z) = \sqrt{2}\,(D_j - \overline{z}_j(z \cdot D)) \quad \text{and} \quad \overline{E}_j(z) = \sqrt{2}\,(\overline{D}_j - z_j(\overline{z} \cdot \overline{D})) .$$

Since

$$\nabla_{\mathbb{C}} f = \sum_j \langle \nabla_{\mathbb{C}} f, E_j\rangle E_j + \langle \nabla_{\mathbb{C}} f, \overline{E}_j\rangle \overline{E}_j ,$$

we get the formulas

$$\nabla_{\mathbb{C}} f(z) = \sum_j \overline{E_j f(z)} E_j + E_j \overline{f}(z) \overline{E}_j$$

and

$$\begin{aligned} |\nabla_{\mathbb{C}} f(z)|^2 &= \langle \nabla_{\mathbb{C}} f(z), \nabla_{\mathbb{C}} f(z)\rangle \\ &= \nabla_{\mathbb{C}} f(f) \\ &= \sum_j |E_j f(z)|^2 + |\overline{E}_j f(z)|^2 . \end{aligned}$$

Let $V : \mathbb{C}^n \to \mathbb{C}^n$ be a complex linear map. Then the map defined on tangent vectors by

$$V(a \cdot D + \overline{b} \cdot \overline{D}) = V(a) \cdot D + \overline{V(b)} \cdot \overline{D}$$

is also complex linear, and if V^* is the Hermitian adjoint of V with respect to the inner product on $\mathbb{C}^n$, then it is also true that $\langle VX, Y\rangle = \langle X, V^*Y\rangle$. We have the following proposition:

PROPOSITION 2.1 *Let V be a unitary map. Then*

$$\nabla_{\mathbb{C}}(f \circ V)(z) = V^*[(\nabla_{\mathbb{C}} f)(V(z))] .$$

PROOF: Notice that if $X(z) = a(z) \cdot D + \overline{b}(z) \cdot \overline{D}$ is complex tangential, then the vector field defined by

$$Y(z) = V^*(X(V(z)) = V^*(a(Vz)) \cdot D + \overline{V^*(b(Vz))} \cdot \overline{D}$$

is also complex tangential. By the chain rule,

$$(a \cdot D + \overline{b} \cdot \overline{D})(f \circ V)(z) = (V(a(z)) \cdot D + \overline{V(b(z))} \cdot \overline{D})(f)(Vz) .$$

From this it follows that if X is complex tangential, then

$$\begin{aligned} X(f \circ V)(z) &= \langle VX(z), (\nabla_{\mathbb{C}} f)(Vz) \rangle \\ &= \langle X(z), V^*(\nabla_{\mathbb{C}} f)(Vz) \rangle . \end{aligned}$$

Since $Y(z) = V^*(\nabla_{\mathbb{C}} f)(Vz)$ defines a complex tangential vector field, the proposition is established. □

Let

$$M_{jk}(z) = \overline{z_j} D_k - \overline{z_k} D_j , \quad \overline{M}_{jk}(z) = z_j \overline{D}_k - z_k \overline{D}_j ,$$

and

$$N(z) = z \cdot D - \overline{z} \cdot \overline{D} , \quad \overline{N}(z) = \overline{z} \cdot \overline{D} - z \cdot D = -N .$$

It is easy to see that these are tangential vector fields and that $M_{j,k}$ and $\overline{M}_{j,k}$ are complex tangential. Furthermore,

$$\langle N, M_{j,k} \rangle = \langle N, \overline{M}_{j,k} \rangle = \langle M_{j,k}, \overline{M}_{r,s} \rangle = 0 .$$

LEMMA 2.2 *The following formulas are true*:

$$E_k = \sqrt{2} \sum_j z_j M_{j,k} \quad \textit{and} \quad \overline{E}_k = \sqrt{2} \sum_j \overline{z}_j \overline{M}_{j,k} .$$

The advantage of the vector fields $M_{j,k}$ and $\overline{M}_{j,k}$ is that they satisfy the following formula, which is easy to check:

LEMMA 2.3 *Let f and g be smooth functions on $\mathbb{S}$. Then*

$$\int_{\mathbb{S}} g M_{j,k} f \, dS = \int_{\mathbb{S}} f M_{j,k} g \, dS \quad \textit{and} \quad \int_{\mathbb{S}} g \overline{M}_{j,k} f \, dS = \int_{\mathbb{S}} f \overline{M}_{j,k} g \, dS .$$

We now turn to the discussion of divergence of tangential vector fields. The purpose of defining such a divergence is for the convenience of presentation in Section 3 in proving the sharp representation formulas using the differential operator $\mathcal{L}_{\beta,\zeta}$ in the form of divergence. It is convenient to adapt this notion to do integration by parts.

It follows from Lemma 2.3 that if X is a complex tangential vector field and X^* its formal adjoint, then

$$X = \sum_{j,k} \langle X, E_k \rangle z_j M_{j,k} + \langle X, \overline{E}_k \rangle \overline{z}_j \overline{M}_{j,k}$$

and

$$\begin{aligned} X^* + X &= -\sum_k E_k(\langle X, E_k\rangle) + \overline{E}_k(\langle X, \overline{E}_k\rangle) \\ &\quad - \sum_{j,k} \langle X, E_k\rangle M_{j,k}(z_j) + \langle X, \overline{E}_k\rangle \overline{M}_{j,k}(\overline{z}_j) \\ &= -\sum_k E_k(\langle X, E_k\rangle) + \overline{E}_k(\langle X, \overline{E}_k\rangle) \\ &\quad - \sum_{j,k} \langle X, E_k\rangle (\overline{z}_j \delta_{jk} - \overline{z}_k) + \langle X, \overline{E}_k\rangle (z_j \delta_{jk} - z_k) \\ &= -\sum_k E_k(\langle X, E_k\rangle) + \overline{E}_k(\langle X, \overline{E}_k\rangle) - (n-1)\langle X, R(z)\rangle \\ &= -\sum_k E_k(\langle X, E_k\rangle) + \overline{E}_k(\langle X, \overline{E}_k\rangle) . \end{aligned}$$

If f is a smooth function on $\mathbb{S}$ and X is a tangential vector field on $\mathbb{S}$, then the tangential divergence div_t is defined by the equation

$$\int_{\mathbb{S}} f \ \mathrm{div}_\mathrm{t} X \, d\sigma = \int_{\mathbb{S}} X f d\sigma = \int_{\mathbb{S}} \langle X, \nabla_\mathrm{t} f\rangle d\sigma ,$$

where $d\sigma$ is the volume element on the sphere. Thus, div_t is the formal adjoint of ∇_t. It follows from integration by parts, i.e., Stokes theorem (see Rudin's book [35]) that if

$$X = \langle X, N\rangle N + \sum_j \langle X, E_j\rangle E_j + \langle X, \overline{E}_j\rangle \overline{E}_j$$

is tangential, then

$$\mathrm{div}_\mathrm{t} X = -N(\langle X, N\rangle) - \sum_j E_j(\langle X, E_j\rangle - \overline{E}_j(\langle X, \overline{E}_j\rangle) .$$

3 Sharp Representation Formulas on Complex Spheres

The goal of this section is to prove the sharp representation formulas (Theorem 1.4), which are some of the key ingredients in deriving the sharp Moser-Trudinger inequalities (Theorem 1.1).

We will use the following notation. Let $\lambda \in \mathbb{C}$ and $\beta > -n+1$. Define

$$H_\beta(\lambda) = (1 - |\lambda|^2)^\beta \quad \text{and} \quad C_\beta(\lambda) = |1 - \lambda|^{1-n-\beta} ,$$

and for $\zeta \in \mathbb{S}$ define

$$H^{\beta,\zeta}(z) = H_\beta(\langle z, \zeta\rangle) .$$

Define the operator

$$\begin{aligned}\mathcal{L}_{\beta,\zeta} f(z) &= -\operatorname{div}_{\mathrm{t}}\big(H^{\beta,\zeta}\nabla_{\mathbb{C}} f\big)(z)\\ &= \sum_j E_j\big(H^{\beta,\zeta}\overline{E}_j\overline{f}\big)(z) + \overline{E}_j\big(H^{\beta,\zeta}E_j\overline{f}\big)(z)\,,\end{aligned}$$

and let $\check{\mathcal{L}}_{\beta,\zeta} f = \mathcal{L}_{\beta,\zeta}\overline{f}$.

LEMMA 3.1 *Let $Q_\zeta(z) = z - \langle z,\zeta\rangle\zeta$ be the projection of z onto the orthogonal complement of the subspace of $\mathbb{C}^n$ spanned by ζ. Then*

$$\begin{aligned}\check{\mathcal{L}}_{\beta,\zeta} = H^{\beta,\zeta}&\Big(\sum_j 4(D_j\overline{D}_j - \overline{z}\cdot\overline{D}\, z\cdot D) - 2(n+\beta-1)R(z)\Big)\\ &- 2\beta H^{\beta-1,\zeta}R(Q_\zeta(z))\,.\end{aligned}$$

Note that

$$\check{\mathcal{L}}_0 = \sum_j 4(D_j\overline{D}_j - \overline{z}\cdot\overline{D}\, z\cdot D) - 2(n-1)R(z)$$

is the usual sub-Laplacian in $\mathbb{C}^n$ and therefore

$$\check{\mathcal{L}}_{\beta,\zeta} = H^{\beta,\zeta}\big(\check{\mathcal{L}}_0 - 2\beta R(z)\big) - 2\beta H^{\beta-1,\zeta}R(Q_\zeta(z))\,.$$

For each fixed $\zeta\in\mathbb{S}$, there is a parametrization of the sphere $\mathbb{S}$ using the variables $\lambda\in\mathbb{C}$ and $\xi\in\mathbb{C}^n$ given by the equation

$$z = \lambda\zeta + \sqrt{1-|\lambda|^2}\,\xi\,,$$

where $|\lambda| < 1$ and ξ belongs to the lower-dimensional sphere orthogonal to ζ $\{\xi : |\xi| = 1,\ \langle\xi,\zeta\rangle = 0\}$.

We will use the symbol $d\sigma$ to denote the normalized volume element on $\mathbb{S}$.

Then it follows from Rudin's book [35] that

$$d\sigma(z) = c_n(1-|\lambda|^2)^{n-2}\, dA(\lambda)d\tilde{S}(\xi)$$

where $d\tilde{S}$ is the normalized volume element on the lower-dimensional sphere

$$\{\xi : |\xi| = 1,\ \langle\xi,\zeta\rangle = 0\}\,,$$

$dA(\lambda)$ is the two-dimensional Lebesgue measure, and $c_n = (n-1)\pi^{-1}$. It will be convenient to let $d\tilde{A}(\lambda) = c_n dA(\lambda)$.

Let

$$a_\beta = (n+\beta-1)^2\int\limits_{|\lambda|<1}\frac{(1-|\lambda|^2)^{\beta+n-2}}{|1-\lambda|^{n+\beta-1}}\, d\tilde{A}(\lambda)\,.$$

From Lemma 3.2 below we conclude that

$$\begin{aligned}a_\beta &= (n+\beta-1)c_n\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(2\cos\theta)^{n+\beta-1}\, d\theta\\ &= (n+\beta-1)c_n 2^{(n+\beta-1)}\cdot B\Big(\frac{n+\beta}{2},\frac{1}{2}\Big)\,.\end{aligned}$$

LEMMA 3.2

$$\int\limits_{|\lambda|<1} \frac{(1-|\lambda|^2)^{\beta+n-2}}{|1-\lambda|^{n+\beta-1}}\, d\tilde{A}(\lambda) = \frac{1}{n+\beta-1}\, c_n \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (2\cos\theta)^{n+\beta-1}\, d\theta\,.$$

PROOF: Let $z = re^{i\theta} = \frac{1}{1-\lambda}$. Then

$$1-|\lambda|^2 = \frac{2\cos\theta - \frac{1}{r}}{r}\,.$$

Therefore,

$$\begin{aligned}\int\limits_{|\lambda|<1} \frac{(1-|\lambda|^2)^{\beta+n-2}}{|1-\lambda|^{n+\beta-1}}\, dA(\lambda) &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int\limits_{r>\frac{1}{2\cos\theta}} \left(2\cos\theta - \frac{1}{r}\right)^{n+\beta-2} r^{-2}\, dr\, d\theta \\ &= \frac{1}{n+\beta-1} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (2\cos\theta)^{n+\beta-1}\, d\theta\,.\end{aligned}$$

Note that $d\tilde{A}(\lambda) = c_n dA(\lambda)$, so the lemma follows. □

The main theorem of this section is the following sharp representation formula:

THEOREM 3.3 *For f sufficiently smooth, we have*

$$a_\beta f(\zeta) = \int\limits_{\mathbb{S}} \left\{-\check{\mathcal{L}}_{\beta,\zeta} f(z) + (n+\beta-1)^2 H^{\beta,\zeta}(z) f(z)\right\} C_\beta(\langle z,\zeta\rangle)\, d\sigma(z)\,.$$

Before we prove this theorem, we will use it to derive the following pointwise estimates, which are crucial to derive the sharp constants for the Moser-Trudinger inequalities. Recall $c_n = (n-1)\pi^{-1}$.

THEOREM 3.4 *Let $\beta > -n+1$ and $B_\beta = (c_n 2^{n+\beta-1} B(\frac{n+\beta}{2}, \frac{1}{2}))^{-1}$. Then*

$$|f(\zeta) - f * K_\beta(\zeta)| \le B_\beta \int\limits_{\mathbb{S}} |\nabla_{\mathbb{C}} f(z)| \frac{(1-|\langle z,\zeta\rangle|^2)^{\beta+\frac{1}{2}}}{|1-\langle z,\zeta\rangle|^{n+\beta}}\, d\sigma(z)\,.$$

If f is holomorphic, then

$$|f(\zeta) - f * K_\beta(\zeta)| \le \frac{\sqrt{2}}{2} B_\beta \int\limits_{\mathbb{S}} |\nabla_{\mathbb{C}} f(z)| \frac{(1-|\langle z,\zeta\rangle|^2)^{\beta+\frac{1}{2}}}{|1-\langle z,\zeta\rangle|^{n+\beta}}\, d\sigma(z)\,.$$

PROOF: Let g be a complex-valued function of a single complex variable λ. If f is a function defined on the sphere, we have a substitute for the operation of convolution given by

$$f * g(\zeta) = \int\limits_{\mathbb{S}} f(z) g(\langle \zeta, z\rangle)\, d\sigma(z)\,.$$

Notice that if V denotes a unitary transformation of $\mathbb{C}^n$, then

$$(f \circ V) * g = (f * g) \circ V .$$

Let

$$K_\beta(\lambda) = a_\beta^{-1}(n+\beta-1)^2 \cdot \frac{(1-|\lambda|^2)^\beta}{|1-\lambda|^{n+\beta-1}} . \tag{3.1}$$

Then the representation formula above (Theorem 3.3) may be restated as

$$a_\beta f(\zeta) = -\check{\mathcal{L}}_{\beta,\zeta} f * C_\beta(\zeta) + a_\beta f * K_\beta(\zeta) .$$

Thus, it follows that

$$\begin{aligned} a_\beta(f(\zeta) - f * K_\beta(\zeta)) &= \int_{\mathbb{S}} \operatorname{div}_{\mathrm{t}}\big(H^{\beta,\zeta}\nabla_{\mathbb{C}}\overline{f}\big) C_\beta(\langle z,\zeta\rangle) d\sigma(z) \\ &= \int_{\mathbb{S}} \big\langle H^{\beta,\zeta}\nabla_{\mathbb{C}}\overline{f}, \nabla_{\mathrm{t}} C_\beta(\langle z,\zeta\rangle)\big\rangle d\sigma(z) \\ &= \int_{\mathbb{S}} \langle \nabla_{\mathbb{C}}\overline{f}, \nabla_{\mathbb{C}} C_\beta(\langle z,\zeta\rangle)\rangle H^{\beta,\zeta}(z) d\sigma(z) . \end{aligned}$$

Therefore,

$$\begin{aligned} &a_\beta|f(\zeta) - f * K_\beta(\zeta)| \\ &\quad\le \int_{\mathbb{S}} |\nabla_{\mathbb{C}} f(z)| H^{\beta,\zeta}(z) \left|\nabla_{\mathbb{C}} \frac{1}{|1-\langle z,\zeta\rangle|^{n+\beta-1}}\right| d\sigma(z) \\ &\quad= (n+\beta-1)\int_{\mathbb{S}} |\nabla_{\mathbb{C}} f(z)| H^{\beta,\zeta}(z) \left|\frac{(1-|\langle z,\zeta\rangle|^2)^{\frac12}}{|1-\langle z,\zeta\rangle|^{n+\beta}}\right| d\sigma(z) . \end{aligned}$$

Thus, we have derived

$$|f(\zeta) - f * K_\beta(\zeta)| \le B_\beta \int_{\mathbb{S}} |\nabla_{\mathbb{C}} f(z)| \frac{(1-|\langle z,\zeta\rangle|^2)^{\beta+\frac12}}{|1-\langle z,\zeta\rangle|^{n+\beta}} \, d\sigma(z) .$$

In the case where f is holomorphic, we can say a little more. Decomposing the complex tangential gradient into holomorphic and antiholomorphic parts, we have

$$\nabla_{\mathbb{C}} = \nabla_{1,0} + \nabla_{0,1}$$

where

$$\nabla_{1,0} f = \sum_j (\overline{E}_j \overline{f}) E_j \quad \text{and} \quad \nabla_{0,1} f = \sum_j (E_j \overline{f}) \overline{E}_j .$$

If f is holomorphic, i.e., $\nabla_{\mathbb{C}}\overline{f} = \nabla_{0,1}\overline{f}$, then it follows that

$$\begin{aligned} a_\beta(f(\zeta) - f * K_\beta(\zeta)) &= \int_{\mathbb{S}} \operatorname{div}_{\mathrm{t}}\big(H^{\beta,\zeta}\nabla_{0,1}\overline{f}\big)C_\beta(\langle z,\zeta\rangle)d\sigma(z) \\ &= \int_{\mathbb{S}} \big\langle H^{\beta,\zeta}\nabla_{0,1}\overline{f}, \nabla_{\mathrm{t}}C_\beta(\langle z,\zeta\rangle)\big\rangle d\sigma(z) \\ &= \int_{\mathbb{S}} \big\langle \nabla_{0,1}\overline{f}, \nabla_{0,1}C_\beta(\langle z,\zeta\rangle)\big\rangle H^{\beta,\zeta}(z)d\sigma(z)\,. \end{aligned}$$

Notice that if G is a real-valued function defined on $\mathbb{S}$, then $|\nabla_{\mathbb{C}}G|^2 = 2|\nabla_{1,0}G|^2$.

Therefore, we have proven in the case where f is holomorphic that

$$|f(\zeta) - f * K_\beta(\zeta)| \le \frac{\sqrt{2}}{2}B_\beta \int_{\mathbb{S}} |\nabla_{\mathbb{C}}f(z)|\,\frac{(1-|\langle z,\zeta\rangle|^2)^{\beta+\frac{1}{2}}}{|1-\langle z,\zeta\rangle|^{n+\beta}}\,d\sigma(z)\,.$$

$\square$

We now turn to the proof of Theorem 3.3.

PROOF OF THE REPRESENTATION FORMULA (THEOREM 3.3): Let $A < 1$. Let $\zeta = e_1 = (1,0,\dots,0)$. We show first that for any $f \in C^\infty(\mathbb{S})$

$$\begin{aligned} a_\beta f(e_1) = \lim_{A\to 1}\int_{\mathbb{S}} -\check{\mathcal{L}}_{\beta,e_1}f(z)\,\frac{d\sigma(z)}{|1-Az_1|^{n+\beta-1}} \\ + (n+\beta-1)^2\int_{\mathbb{S}} f(z)H^{\beta,e_1}(z)\,\frac{d\sigma(z)}{|1-z_1|^{n+\beta-1}}\,. \end{aligned}$$

Let

$$\Phi_{\beta,A}(z) = |1-Az_1|^{-(n+\beta-1)}\,.$$

We integrate by parts (see [35, 18.2.2, eq. (4)]) and see that this is equivalent to showing that

$$\begin{aligned} a_\beta f(e_1) = \lim_{A\to 1}\bigg\{\int_{\mathbb{S}} -f(z)\mathcal{L}_{\beta,e_1}\Phi_{\beta,A}(z)d\sigma(z) \\ + (n+\beta-1)^2\int_{\mathbb{S}} f(z)H^{\beta,e_1}(z)\Phi_{\beta,A}(z)d\sigma(z)\bigg\}\,. \end{aligned}$$

In other words,

$$a_\beta\delta_{e_1} = \lim_{A\to 1}\big\{-\mathcal{L}_{\beta,e_1}\Phi_{\beta,A}(z) + (n+\beta-1)^2H^{\beta,e_1}(z)\Phi_{\beta,A}(z)\big\}\,.$$

A straightforward calculation based on Lemma 3.1 shows that

$$\mathcal{L}_{\beta,e_1}\Phi_{\beta,A}(z) = H^{\beta,e_1}(z)\big(4(1-|z_1|^2)D_1\overline{D}_1 - 2(n+\beta-1)(z_1D_1+\overline{z}_1\overline{D}_1)\big)\Phi_{\beta,A}(z)\,.$$

More computation gives

$$\mathcal{L}_{\beta,e_1}\Phi_{\beta,A}(z) = (n+\beta-1)^2\big[H^{\beta,e_1}(z)\Phi_{\beta,A}(z) + (A^2-1)H^{\beta,e_1}(z)\Phi_{\beta+2,A}(z)\big]\,.$$

Integration by parts shows that

$$\int_{\mathbb{S}} \mathcal{L}_{\beta,e_1}\Phi_{\beta,A}(z)d\sigma(z) = 0$$

for all $A < 1$. From this it follows that

$$-(n+\beta-1)^2\int_{\mathbb{S}} H^{\beta,e_1}(z)\Phi_{\beta,A}(z)d\sigma(z) = (A^2-1)\int_{\mathbb{S}} H^{\beta,e_1}(z)\Phi_{\beta+2,A}(z)d\sigma(z)$$

for all $A < 1$. It follows that

$$(A^2-1)H^{\beta,e_1}(z)\Phi_{\beta+2,A}(z)d\sigma(z)$$

converges in the weak-star topology on $C(\mathbb{S})$ to $c\cdot\delta_{e_1}$ where

$$c = -(n+\beta-1)^2\int_{\mathbb{S}} H^{\beta,e_1}(z)\Phi_{\beta,1}(z)d\sigma(z) = -a_\beta\,.$$

This gives the formula for $\zeta = e_1$.

Now let $\zeta \in \mathbb{S}$ and let V be a unitary transformation such that $Ve_1 = \zeta$. Let $C_{\beta,\zeta}(z) = C(\langle z,\zeta\rangle)$ so $C_{\beta,\zeta}(Vz) = C_\beta(\langle z,e_1\rangle)$. Since $(f\circ V)*K(e_1) = f*K(Ve_1) = f*K(\zeta)$, it follows from the formula for e_1 that

$$\begin{aligned}
a_\beta(f(\zeta) - f*K_\beta(\zeta)) &= \int_{\mathbb{S}} \mathrm{div}_{\mathrm{t}}\big(H^{\beta,e_1}\nabla_{\mathbb{C}}(\overline{f}\circ V)\big)C_\beta(\langle z,e_1\rangle)d\sigma(z)\\
&= \int_{\mathbb{S}} \big\langle H^{\beta,\zeta}(Vz)\nabla_{\mathbb{C}}(\overline{f}\circ V)(z),\, \nabla_{\mathrm{t}}(C_{\beta,\zeta}\circ V)(z)\big\rangle d\sigma(z)\\
&= \int_{\mathbb{S}} \big\langle(\nabla_{\mathbb{C}}\overline{f})(Vz),\, (\nabla_{\mathbb{C}}C_{\beta,\zeta})(Vz)\big\rangle H^{\beta,\zeta}(Vz)d\sigma(z)\\
&= \int_{\mathbb{S}} \big\langle(\nabla_{\mathbb{C}}\overline{f})(z),\, \nabla_{\mathbb{C}}C_{\beta,\zeta}(z)\big\rangle H^{\beta,\zeta}(z)d\sigma(z)\,.
\end{aligned}$$

This completes the argument and then Theorem 3.3 follows. □

Remark. By considering a class of quasi-linear subelliptic differential operators involving the complex tangential gradients on $\mathbb{S}$, we can find an alternative way to derive Theorem 3.3. This is not hard to do since we already know a priori the form of the fundamental solutions to such quasi-linear subelliptic equations from this representation formula (Theorem 3.3).

4 The Distribution Function of the Kernel

We will need to use the rearrangement of functions on $\mathbb{S}$.

Suppose F is a nonnegative function defined on $\mathbb{S}$. Define the nonincreasing rearrangement of F by

$$F^*(t) = \inf\{s > 0 : \lambda_F(s) \leq t\},$$

where $\lambda_F(s) = \sigma(\{u \in \mathbb{S} : F(u) > s\})$. In addition, define

$$F^{**}(t) = t^{-1} \int_0^t F^*(s)ds .$$

Suppose g is a nonnegative function of one complex variable. For each $\zeta \in \mathbb{S}$ let $g_\zeta(z) = g(\langle z, \zeta\rangle)$ for $z \in \mathbb{S}$. Since the measure σ is invariant under unitary transformations, the nonincreasing rearrangement $(g_\zeta)^*$ is the same for all $\zeta \in \mathbb{S}$. In the sequel, we will omit the subscript ζ and simply write g^*.

Now let $U = f * g$ be the convolution on $\mathbb{S}$. It is easy to check that the following version of O'Neil's lemma [30] regarding rearrangement of convolution of two functions holds on $\mathbb{S}$:

LEMMA 4.1 *Suppose $U = f * g$ on $\mathbb{S}$. Then*

$$U^*(t) \leq U^{**}(t) \leq tf^{**}(t)g^{**}(t) + \int_t^\infty f^*(s)g^*(s)ds .$$

In order to use O'Neil's lemma, we will need to estimate the distribution function for various kernels.

Define the measure dm on the unit disk $D = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$ by

$$dm(\lambda) = (1 - |\lambda|^2)^{n-2}\, d\tilde{A}(\lambda)$$

where $d\tilde{A}(\lambda) = c_n dA(\lambda)$ and $c_n = (n-1)\pi^{-1}$. Let

$$K_{\beta,\alpha}(\lambda) = \frac{(1 - |\lambda|^2)^\beta}{|1 - \lambda|^{n+\beta-\alpha}} \tag{4.1}$$

and

$$E_t = \{|\lambda| < 1 \text{ and } K_{\beta,\alpha}(\lambda) > t\}.$$

Set

$$m(E_t) = \int_{E_t} (1 - |\lambda|^2)^{n-2}\, d\tilde{A}(\lambda) .$$

Observe that for $\zeta \in \mathbb{S}$

$$\sigma(\{z \in \mathbb{S} : K_{\beta,\alpha}(\langle z, \zeta \rangle) > t\}) = m(E_t)\,.$$

LEMMA 4.2 *Let*

$$\begin{aligned} A_{\beta,\alpha} &= \frac{n}{c_n \int_{-\pi/2}^{\pi/2} (2\cos\theta)^{\frac{\beta n}{n-\alpha}+n-2}\, d\theta} \\ &= n\left(c_n 2^{\frac{\beta n}{n-2}+n-2} B\left(\frac{n^2+(\beta-\alpha-1)n+\alpha}{2(n-\alpha)}, \frac{1}{2}\right)\right)^{-1}. \end{aligned} \tag{4.2}$$

The following estimate holds:

$$m(E_t) \le (A_{\beta,\alpha})^{-1} t^{-\frac{n}{n-\alpha}}\,.$$

PROOF: For $0 < \alpha < n$, $\beta > 0$, and $t > 0$, the set E_t is

$$E_t = \{|\lambda| < 1,\ K_{\beta,\alpha}(\lambda) > t\}\,.$$

Use the change of variables $w = re^{i\theta} = (1-\lambda)^{-1}$. Then the disk D is mapped to the half-plane

$$H = \{re^{i\theta} : r\cos\theta > \tfrac{1}{2}\}$$

and

$$\begin{aligned} dA(\lambda) &= r^{-3}\, dr\, d\theta\,, \\ (1-|\lambda|^2) &= r^{-2}(2\cos\theta - 1)\,, \end{aligned}$$

and therefore

$$dm(\lambda) = c_n r^{-2n+1}(2r\cos\theta - 1)^{n-2}\, dr\, d\theta\,.$$

Furthermore, the set E_t is mapped to the region

$$H_t = \left\{re^{i\theta} : \left(2\cos\theta - \frac{1}{r}\right)^{\beta} r^{n-\alpha} > t\right\} \cap H\,.$$

Since

$$dm(\lambda) \le c_n r^{-n-1}(2\cos\theta)^{n-2}\, dr\, d\theta$$

and

$$\begin{aligned} H_t &\subset \{re^{i\theta} : (2\cos\theta)^{\beta} r^{n-\alpha} > t\} \cap H \\ &\subset \{re^{i\theta} : (2\cos\theta)^{\beta} r^{n-\alpha} > t,\ -\tfrac{\pi}{2} < \theta < \tfrac{\pi}{2}\}\,, \end{aligned}$$

it follows that

$$\begin{aligned} m(E_t) &\leq c_n \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{[\frac{t}{(2\cos\theta)^\beta}]^{\frac{1}{n-\alpha}}}^{\infty} r^{-n-1}\, dr \cdot (2\cos\theta)^{n-2}\, d\theta \\ &= \frac{c_n}{n} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (2\cos\theta)^{n-2} (t2^{-\beta} \cos^{-\beta}\theta)^{-\frac{n}{n-\alpha}}\, d\theta \\ &= c_n n^{-1} t^{-\frac{n}{n-\alpha}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (2\cos\theta)^{\frac{\beta n}{n-\alpha}+n-2}\, d\theta \\ &= c_n n^{-1} t^{-\frac{n}{n-\alpha}} 2^{\frac{\beta n}{n-\alpha}+n-2} B\left(\frac{n^2 + (\beta-\alpha-1)n + \alpha}{2(n-\alpha)}, \frac{1}{2}\right). \end{aligned}$$

□

5 Moser-Trudinger Inequalities for Functions Minus a Potential

In this section, we will first use Lemmas 4.1 and 4.2, and follow the argument as done in the work of authors [17] in the nonisotropic setting to conclude the following:

THEOREM 5.1 *If $u(z) = f * K_{\beta,\alpha}(z)$, then*

$$u^*(t) \leq A_{\beta,\alpha}^{-\frac{n-\alpha}{n}} \left(\frac{n}{\alpha} t^{-\frac{n-\alpha}{n}} \int_0^t f^*(s) ds + \int_t^\infty f^*(s) s^{-\frac{n-\alpha}{n}}\, ds \right).$$

Moreover, there exists a constant C_0 such that with $n = \alpha p$

$$\int_{\mathbb{S}} \exp\left(A_{\beta,\alpha} \left(\frac{f * K_{\beta,\alpha}}{\|f\|_p} \right)^{p'} \right) d\sigma \leq C_0 .$$

We shall not repeat the proof since it is very similar to that for the case on the Heisenberg group [17] as long as we have Lemma 4.2 available (see also [1] for an earlier argument of this type in an isotropic euclidean setting).

Combining Theorem 5.1 and Theorem 3.4, we will derive the following theorems:

THEOREM 5.2 *Let*

$$B = n\left((n-1)\pi^{-1} 2^{2n-2} \cdot B\left(\frac{2n-1}{2}, \frac{1}{2} \right) \right)^{\frac{1}{2n-1}} .$$

Then there is a constant C_0 such that the inequality

$$\int_{\mathbb{S}} \exp\left(B \left(\frac{|f - f * K_{n-1}|}{\|\nabla_{\mathbb{C}} f\|_{2n}} \right)^{\frac{2n}{2n-1}} \right) d\sigma \leq C_0$$

is verified for all smooth functions f.

THEOREM 5.3 *Let*

$$B_H = 2^{\frac{n}{2n-1}} B = 2^{\frac{n}{2n-1}} \cdot n\left((n-1)\pi^{-1}2^{2n-2} \cdot B\left(\frac{2n-1}{2}, \frac{1}{2}\right)\right)^{\frac{1}{2n-1}} .$$

Then there is a constant C_0 such that the inequality

$$\int\limits_{\mathbb{S}} \exp\left(B_H\left(\frac{|f - f * K_{n-1}|}{\|\nabla_{\mathbb{C}} f\|_{2n}}\right)^{\frac{2n}{2n-1}}\right) d\sigma \leq C_0$$

holds for all holomorphic functions f on $\mathbb{S}$.

To prove Theorems 5.2 and 5.3, we first restate Theorem 3.4 using the kernel $K_{\beta,\alpha}$. We recall that $K_{\beta,\alpha}$ is defined in (4.1), K_β is defined in (3.1), and $A_{\beta,\alpha}$ is defined in (4.2).

PROPOSITION 5.4 *Let $\beta > -n+1$ and $B_\beta = (c_n 2^{n+\beta-1} B(\frac{n+\beta}{2}, \frac{1}{2}))^{-1}$. Then for any smooth f*

$$|f(\zeta) - f * K_\beta(\zeta)| \leq B_\beta\big(|\nabla_{\mathbb{C}} f| * K_{\beta+\frac{1}{2},\frac{1}{2}}\big)(\zeta) .$$

If f is holomorphic, then

$$|f(\zeta) - f * K_\beta(\zeta)| \leq \frac{\sqrt{2}}{2} B_\beta\big(|\nabla_{\mathbb{C}} f| * K_{\beta+\frac{1}{2},\frac{1}{2}}\big)(\zeta) .$$

PROOF OF THEOREMS 5.2 AND 5.3: We apply Proposition 5.4 with $\beta = n-1$ and $\alpha = \frac{1}{2}$ and get

$$|f(\zeta) - f * K_{n-1}(\zeta)| \leq B_{n-1}|\nabla_{\mathbb{C}} f| * K_{n-\frac{1}{2},\frac{1}{2}}(\zeta) .$$

Therefore, using Theorem 5.1 we have

$$\int\limits_{\mathbb{S}} \exp\left(B\left(\frac{|f - f * K_{n-1}|}{\|\nabla_{\mathbb{C}} f\|_{2n}}\right)^{\frac{2n}{2n-1}}\right) d\sigma \leq C_0$$

provided

$$B \cdot B_{n-1}^{2n/(2n-1)} \leq A_{n-\frac{1}{2},\frac{1}{2}} ,$$

i.e.,

$$\begin{aligned} B &\leq B_{n-1}^{-2n/(2n-1)} \cdot A_{n-\frac{1}{2},\frac{1}{2}} \\ &= n B_{n-1}^{-1/(2n-1)} \\ &= n\left((n-1)\pi^{-1}2^{2n-2} \cdot B\left(\frac{2n-1}{2}, \frac{1}{2}\right)\right)^{\frac{1}{2n-1}} . \end{aligned}$$

If f is holomorphic, we use the following:

$$|f(\zeta) - f * K_{n-1}(\zeta)| \le \frac{\sqrt{2}}{2} B_{n-1} |\nabla_{\mathbb{C}} f| * K_{n-\frac{1}{2},\frac{1}{2}}(\zeta)\,,$$

and the same argument shows

$$\int\limits_{\mathbb{S}} \exp\left(B_H \left(\frac{|f - f * K_{n-1}|}{\|\nabla_{\mathbb{C}} f\|_{2n}} \right)^{\frac{2n}{2n-1}} \right) d\sigma \le C_0$$

provided

$$B_H \cdot \left(\sqrt{2}\, c_n B_{n-1}^{-1}\right)^{-\frac{2n}{2n-1}} \le A_{n-\frac{1}{2},\frac{1}{2}}\,,$$

i.e.,

$$B_H \le 2^{\frac{n}{2n-1}} B\,.$$

In summary, the above argument has already shown Theorems 5.2 and 5.3. □

6 Sharpness of Moser-Trudinger Inequalities for Functions Minus a Potential

To show that the constants B and B_H in Theorems 5.2 and 5.3 are optimal, we need several propositions.

PROPOSITION 6.1 *Let $R < 1$ and let $o(1-R)$ denote that* $\lim_{R\to 1^-} o(1-R) = 0$. *Then*

$$I_R = \int\limits_{|\lambda|<1} \frac{(1-|\lambda|^2)^{2n-2}}{|1-R\lambda|^{2n}}\, dA(\lambda)$$

$$= \log \frac{1}{1-R} \cdot J_n \cdot (1 + o(1-R))$$

where the number J_n is defined by

$$J_n = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (2\cos\theta)^{2n-2}\, d\theta \tag{6.1}$$

$$= 2^{2n-2} B\left(\frac{2n-1}{2}, \frac{1}{2} \right).$$

PROOF: Make the change of variables $w = (1-R)(1-\lambda)^{-1}$. Let $w = u + iv$. Then an easy calculation shows that

$$I_R = \int\limits_{2u > 1-R} (2u - (1-R))^{2n-2} |w|^{-2n} |w + R|^{-2n} r\, dr\, d\theta$$

where $w = re^{i\theta}$, which can be written as

$$\int_{\frac{1-R}{2}}^{\infty} \left[\int\limits_{\{\cos\theta > \frac{1-R}{2r}\}} \frac{(2\cos\theta - \frac{1-R}{r})^{2n-2}}{|re^{i\theta} + R|^{2n}}\, d\theta \right] \frac{dr}{r} = \mathrm{I} + \mathrm{II}\,,$$

where

$$\mathrm{I} = \int_{\frac{1-R}{2}}^{1} \left[\int_{-\cos^{-1}(\frac{1-R}{2r})}^{\cos^{-1}(\frac{1-R}{2r})} \frac{(2\cos\theta - \frac{1-R}{r})^{2n-2}}{|re^{i\theta} + R|^{2n}} \, d\theta \right] \frac{dr}{r}$$

and

$$\begin{aligned} \mathrm{II} &= \int_{1}^{\infty} \left[\int_{\{\cos\theta > \frac{1-R}{2r}\}} \frac{(2\cos\theta - \frac{1-R}{r})^{2n-2}}{|re^{i\theta} + R|^{2n}} \, d\theta \right] \frac{dr}{r} \\ &\le \int_{1}^{\infty} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(2\cos\theta)^{2n-2}}{|re^{i\theta} + R|^{2n}} \, d\theta \, \frac{dr}{r} \\ &= C < \infty \, . \end{aligned}$$

To calculate I, we make the change of variable $r^{-1} = e^t$ and write

$$\mathrm{I} = \int_{0}^{\log \frac{2}{1-R}} \left[\int_{-\cos^{-1}(\frac{e^t(1-R)}{2})}^{\cos^{-1}(\frac{e^t(1-R)}{2})} \frac{(2\cos\theta - (1-R)e^t))^{2n-2}}{|e^{-t+i\theta} + R|^{2n}} \, d\theta \right] dt \, .$$

Now make the change of variable

$$x = \frac{t}{\log \frac{2}{1-R}} \quad \text{so} \quad t = \log\left(\frac{2}{1-R}\right)^{x}$$

and get

$$\mathrm{I} = \log\left(\frac{2}{1-R}\right) \int_{0}^{1} \int_{-\cos^{-1}((\frac{1-R}{2})^{1-x})}^{\cos^{-1}((\frac{1-R}{2})^{1-x})} \frac{(2\cos\theta - (1-R)^{1-x})^{2n-2}}{|(\frac{1-R}{2})^{x} e^{i\theta} + R|^{2n}} \, d\theta \, dx \, .$$

By the dominated convergence theorem

$$\lim_{R\to 1} \left(\log\left(\frac{1}{1-R}\right) \right)^{-1} I_R = J_n$$

where we recall J_n is given in (6.1). This proves the proposition. $\square$

We now define

$$f_R(z) = \left(\log \frac{1}{1-R}\right)^{-1} \cdot \log \frac{1}{|1-Rz_1|} \,.$$

Then an easy calculation shows that we have the following:

PROPOSITION 6.2

$$|\nabla_{\mathbb{C}} f_R(z)| = R\left(\log \frac{1}{1-R}\right)^{-1} \frac{\sqrt{1-|z_1|^2}}{|1-Rz_1|}\,,$$

$$\begin{aligned}
&(\|\nabla_{\mathbb{C}} f_R(z)\|_{2n})^{\frac{2n}{2n-1}} \\
&= \left(R\left(\log \frac{1}{1-R}\right)^{-1}\right)^{\frac{2n}{2n-1}} \left(\int\limits_{|\lambda|<1} \frac{(1-|\lambda|^2)^{2n-2}}{|1-R\lambda|^{2n}}\, d\tilde{A}(\lambda)\right)^{\frac{1}{2n-1}} \\
&= \left(R\left(\log \frac{1}{1-R}\right)^{-1}\right)^{\frac{2n}{2n-1}} \left(\log \frac{1}{1-R} \cdot J_n \cdot \Big(1+o(1-R)\Big)\right)^{\frac{1}{2n-1}} .
\end{aligned}$$

Furthermore, we have the following:

PROPOSITION 6.3

(i) $f_R * K_{n-1}(z) \to 0$ *uniformly in* z *as* $R \to 1$.
(ii) *If* $|z_1 - 1| < 1 - R$, *then*

$$f_R(z) \geq 1 + \frac{\frac{1}{2}}{\log \frac{1}{1-R}} = 1 + o(1-R)\,.$$

(iii) $\sigma\{z : |z_1 - 1| < 1 - R\} = c(1 + o(1-R))(1-R)^n \geq c(1-R)^n$.

The proofs of Propositions 6.2 and 6.3 are not hard and we omit the details.

With these preparations, we will be able to show the sharpness of the constant B.

PROOF OF THE SHARPNESS OF B IN THEOREM 5.2: Suppose now that

$$\int\limits_{\mathbb{S}} \exp\left(\beta\left(\frac{|f - f * K_{n-1}|}{\|\nabla_{\mathbb{C}} f\|_{2n}}\right)^{\frac{2n}{2n-1}}\right) d\sigma \leq C_0$$

for all smooth f. Let $\Omega_R = \{z : |1 - z_1| < 1 - R\}$ and $\sigma(\Omega_R) = \int_{\Omega_R} d\sigma$. Then

$$\sigma(\Omega_R) \exp\left(\beta\left(\frac{1 - o(1-R)}{\|\nabla_{\mathbb{C}} f_R\|_{2n}}\right)^{\frac{2n}{2n-1}}\right) \leq C_0\,.$$

Therefore,

$$\begin{aligned}\beta(1-o(1-R))^{\frac{2n}{2n-1}} &\leq \log\big(C_0\cdot\sigma(\Omega_R)^{-1}\big)(\|\nabla_{\mathbb{C}} f_R\|_{2n})^{\frac{2n}{2n-1}}\\ &\leq \left(\log C_0 + n\log\frac{1}{1-R}\right)\cdot\left(\log\frac{1}{1-R}\right)^{-1}\\ &\quad\cdot (c_n J_n\cdot(1+o(1-R)))^{\frac{1}{2n-1}}\cdot R^{\frac{2n}{2n-1}}\\ &= [n+o(1-R)]\cdot(c_n\cdot J_n(1+o(1-R)))^{\frac{1}{2n-1}}\cdot R^{\frac{2n}{2n-1}}.\end{aligned}$$

Let $R\to 1$ and we get that

$$\beta \leq n(c_n J_n)^{\frac{1}{2n-1}} = B.$$

□

To show that the constant B_H is sharp, we abuse the notation f_R again and define

$$f_R(z) = \left(\log\frac{1}{1-R}\right)^{-1}\cdot\log\frac{1}{(1-Rz_1)}.$$

Then we have the following:

PROPOSITION 6.4

$$|\nabla_{\mathbb{C}} f_R(z)| = R\sqrt{2}\left(\log\frac{1}{1-R}\right)^{-1}\frac{\sqrt{1-|z_1|^2}}{|1-Rz_1|},$$

$$\begin{aligned}&(\|\nabla_{\mathbb{C}} f_R\|_{2n})^{\frac{2n}{2n-1}}\\ &= \left(\sqrt{2}\,R\left(\log\frac{1}{1-R}\right)^{-1}\right)^{\frac{2n}{2n-1}}\left(\int_{|\lambda|<1}\frac{(1-|\lambda|^2)^{2n-2}}{|1-R\lambda|^{2n}}\,d\tilde{A}(\lambda)\right)^{\frac{1}{2n-1}}\\ &= \left(\sqrt{2}\,R\left(\log\frac{1}{1-R}\right)^{-1}\right)^{\frac{2n}{2n-1}}\left(\log\frac{1}{1-R}\cdot J_n\cdot\Big(1+o(1-R)\Big)\right)^{\frac{1}{2n-1}}.\end{aligned}$$

Moreover, we have the following estimates:

PROPOSITION 6.5

(i) $f_R * K_{n-1}(z)\to 0$ *uniformly in* z *as* $R\to 1$.
(ii) *If* $|z_1-1|<1-R$, *then*

$$|f_R(z)| \geq 1+\frac{\frac{1}{2}}{\log\frac{1}{1-R}} = 1+o(1-R)$$

on $|z_1-1|<1-R$.

PROOF OF THE SHARPNESS OF B_H IN THEOREM 5.3: Suppose now that

$$\int_{\mathbb{S}} \exp\left(\beta\left(\frac{|f - f * K_{n-1}|}{\|\nabla_{\mathbb{C}} f\|_{2n}}\right)^{\frac{2n}{2n-1}}\right) d\sigma \leq C_0$$

for all holomorphic f. Repeating the previous argument using the test functions f_R gives the upper bound $\beta \leq n 2^{\frac{n}{2n-1}} (c_n J_n)^{\frac{1}{2n-1}} = B_H$. This completes the proof. $\square$

7 Moser-Trudinger Inequalities for Functions Minus the Average: Theorem 1.1

In Section 5, we derived the constants for the Moser-Trudinger inequalities stated in Theorems 5.2 and 5.3. In Section 6, we proved the sharpness of those constants. However, we note that the inequalities involve $|f - f * K_{n-1}|$ rather than $|f - \int_{\mathbb{S}} f\, d\sigma|$. The goal of this section is to show Theorem 1.1, stated in the introduction. This will follow from Theorems 5.2 and 5.3 combined with Proposition 7.3 below.

We first show the following:

PROPOSITION 7.1

$$M_{jk}(f * K_{n-1})(z) = (M_{jk} f) * \tilde{K}_{n-1}(z)$$

and

$$\overline{M}_{jk}(f * K_{n-1})(z) = (\overline{M}_{jk} f) * \overline{\tilde{K}}_{n-1}(z)$$

where

$$\tilde{K}_{n-1}(\lambda) = \frac{n-1}{n-3} \cdot (1-\lambda)^{-\frac{n+1}{2}} \cdot (1-\overline{\lambda})^{-\frac{n-3}{2}} \quad \textit{when } n \neq 3$$

and

$$\tilde{K}_{n-1}(\lambda) = \frac{n-1}{2} \cdot \frac{\log(1-\overline{\lambda})}{(1-\lambda)^{\frac{n+1}{2}}} \quad \textit{when } n = 3\,.$$

Moreover,

$$|M_{jk}(f * K_{n-1})(z)| \leq C(f * J)(z)$$

and

$$|\overline{M}_{jk}(f * K_{n-1})(z)| \leq C(f * J)(z)$$

where

$$J(\lambda) = \frac{1}{|1-\lambda|^{n-\frac{1}{2}}}\,.$$

PROOF: We first note

$$f * K_{n-1}(z) = \int_{\mathbb{S}} f(\zeta) K_{n-1}(\langle z, \zeta\rangle) d\sigma(\zeta)$$

and $K_{n-1}(\langle z, \zeta\rangle) = K_{n-1}(\langle \zeta, z\rangle)$.

Then it follows from this identity that

$$M^z_{jk}K_{n-1}(\langle z,\xi\rangle) = -M^\xi_{jk}\tilde{K}_{n-1}(\langle z,\xi\rangle)$$

where

$$M^z_{jk} = \overline{z_j}\frac{\partial}{\partial z_k} - \overline{z_k}\frac{\partial}{\partial z_j} \quad \text{and} \quad M^\xi_{jk} = \overline{\xi_j}\frac{\partial}{\partial \xi_k} - \overline{\xi_k}\frac{\partial}{\partial \xi_j}\,.$$

Using integration by parts, we get

$$\begin{aligned} M_{jk}(f * K_{n-1})(z) &= \int_{\mathbb{S}} f(\xi)\left[M^z_{jk}\frac{1}{(1-\langle z,\xi\rangle)^{\frac{n-1}{2}}}\cdot\frac{1}{(1-\langle \xi,z\rangle)^{\frac{n-1}{2}}}\right]d\sigma(\xi) \\ &= \int_{\mathbb{S}} f(\xi)\left[\frac{n-1}{2}\cdot\frac{\overline{z_j}\overline{\xi_k}-\overline{z_k}\overline{\xi_j}}{(1-\langle z,\xi\rangle)^{\frac{n+1}{2}}}\cdot\frac{1}{(1-\langle \xi,z\rangle)^{\frac{n-1}{2}}}\right]d\sigma(\xi) \\ &= (M_{jk}f) * \tilde{K}_{n-1}(z)\,. \end{aligned}$$

Using the second equality above and noticing that

$$\begin{aligned} \sum_{j,k}|\overline{z_j}\overline{\xi_k}-\overline{z_k}\overline{\xi_j}|^2 &= \sum_{j,k}\left(|\overline{z_j}|^2|\overline{\xi_k}|^2+|\overline{z_k}|^2|\overline{\xi_j}|^2\right) - \sum_{j,k}(\overline{z_j}z_k\overline{\xi_k}\xi_j - \overline{z_k}z_j\overline{\xi_j}\xi_k) \\ &= 2-2|\langle z,\xi\rangle|^2 \\ &= 2(1+|\langle z,\xi\rangle|)(1-|\langle z,\xi\rangle|) \\ &\le 4|1-|\langle z,\xi\rangle||\,, \end{aligned}$$

we have

$$|\overline{z_j}\overline{\xi_k}-\overline{z_k}\overline{\xi_j}| \le C|1-\lambda|^{\frac{1}{2}}\,.$$

Combining this with the second equality above gives

$$|M_{jk}(f * K_{n-1})(z)| \le C(f * J)(z)$$

where

$$J(\lambda) = \frac{1}{|1-\lambda|^{n-\frac{1}{2}}}\,.$$

Similarly, we can show

$$\overline{M}_{jk}(f * K_{n-1})(z) = (\overline{M}_{jk}f) * \overline{\tilde{K}}_{n-1}(z)$$

and

$$|\overline{M}_{jk}(f * K_{n-1})(z)| \le C|(f * J)(z)|\,.$$

$\square$

PROPOSITION 7.2

$$|\overline{M}_{jk}(M_{jk}(f * K_{n-1}))(z)| \le C|M_{jk}f| * J(z)$$

and

$$|M_{jk}((\overline{M}_{jk}(f * K_{n-1}))(z)| \le C|\overline{M}_{jk}f| * J(z)\,.$$

PROOF: We note that

$$\begin{aligned}
&\overline{M}_{jk}(M_{jk}(f * K_{n-1}))(z)\\
&\quad = \overline{M}_{jk}((M_{jk}f) * \tilde{K}_{n-1})(z)\\
&\quad = \int_{\mathbb{S}} M_{jk}f(\zeta)\overline{M}^{z}_{jk}\tilde{K}_{n-1}(\langle z,\zeta\rangle)d\sigma(\zeta)\\
&\quad = \frac{n-1}{2}\int_{\mathbb{S}} M_{jk}f(\zeta)\left[\frac{\overline{z_j}\overline{\zeta_k} - \overline{z_k}\overline{\zeta_j}}{(1-\langle\zeta,z\rangle)^{\frac{n-1}{2}}}\cdot\frac{1}{(1-\langle z,\zeta\rangle)^{\frac{n+1}{2}}}\right]d\sigma(\zeta)\,.
\end{aligned}$$

So the estimates follow. □

PROPOSITION 7.3 *Let* $\int_{\mathbb{S}} |\nabla_{\mathbb{C}} f|^{2n}\, d\sigma = 1$ *and assume* $\int_{\mathbb{S}} f d\sigma = 0$. *Let*

$$F = f * K_{n-1}\,.$$

Then

$$\|F\|_\infty \le C\,.$$

PROOF: Suppose now that $F = f * K_{n-1}$ where

$$\nabla_{\mathbb{C}} f \in L^{2n} \quad \text{and} \quad \int_{\mathbb{S}} |\nabla_{\mathbb{C}} f|^{2n}\, d\sigma = 1\,.$$

We will show that if $q < \infty$ then

$$\left(\int_{\mathbb{S}} |\nabla_{\mathrm{t}} F|^q d\sigma\right)^{\frac{1}{q}} \le C_q \|\nabla_{\mathbb{C}} f\|_{2n}\,.$$

We observe that if $NF = \sum_j (z_j D_j - \overline{z}_j \overline{D}_j)F$, then

$$|\nabla_{\mathrm{t}} F|^2 = |\nabla_{\mathbb{C}} F|^2 + |NF|^2\,.$$

But

$$(n-1)NF = -\sum_{j,k}[M_{jk}, \overline{M}_{jk}]F\,,$$

and each commutator is of the form

$$M_{jk}\overline{M}_{jk}F - \overline{M}_{jk}M_{jk}F = M_{jk}(\overline{M}_{jk}(f * K_{n-1})) - \overline{M}_{jk}(M_{jk}(f * K_{n-1}))\,,$$

and it follows from Proposition 7.2 that

$$|(NF)(z)| \leq C(|\nabla_{\mathbb{C}} f| * J)(z) = \int_{\mathbb{S}} |\nabla_{\mathbb{C}} f|(\zeta) J(\langle z, \zeta\rangle) d\sigma(\zeta)$$

where

$$J(\lambda) = \frac{1}{|1-\lambda|^{n-\frac{1}{2}}} .$$

We also note that

$$M_{jk} F(z) = M_{jk}(f * K_{n-1})(z) = ((M_{jk} f) * \tilde{K}_{n-1})(z)$$

and

$$|\tilde{K}_{n-1}(\lambda)| \leq \frac{C}{|1-\lambda|^{n-1}} \leq \frac{C}{|1-\lambda|^{n-\frac{1}{2}}} .$$

Thus, the estimate

$$|\nabla_t F(z)| \leq C(|\nabla_{\mathbb{C}} f| * J)(z)$$

holds.

Now if $K(z, \zeta) = J(\langle z, \zeta\rangle)$, it follows that

$$\int_{\mathbb{S}} |K(z, \zeta)|^s d\sigma \leq M_s < \infty \quad \text{for } s < \frac{n}{n-\frac{1}{2}} = \frac{2n}{2n-1} .$$

Let $\epsilon = q^{-1}$. Then there is a constant C_q such that

$$\begin{aligned} \left(\int_{\mathbb{S}} |\nabla_{\mathrm{t}} F|^q d\sigma\right)^{\frac{1}{q}} &\leq C_q \left(\int_{\mathbb{S}} (|\nabla_{\mathbb{C}} f| * K)^q d\sigma\right)^{\frac{1}{q}} \\ &\leq C_q \left(\int_{\mathbb{S}} |\nabla_{\mathbb{C}} f|^{2n}\, d\sigma\right)^{\frac{1}{q}} = C_q \end{aligned}$$

because with $s^{-1} = \frac{2n-1}{2n} + \epsilon$,

$$q^{-1} = (2n)^{-1} + s^{-1} - 1 = \epsilon .$$

We have now shown that $|\nabla_{\mathrm{t}} F|$ is in $L^q(d\sigma)$ (with norm less than a constant depending on q) for any $q < \infty$. Let $\zeta \cdot z = \mathrm{Re}\langle \zeta, z\rangle$ denote the real inner product on $\mathbb{R}^{2n}$. Using the well-known integral representation formula for the real sphere ($\mathbb{S}$ is considered to be a $(2n-1)$–dimensional real sphere) in terms of the tangential gradient $\nabla_{\mathrm{t}} F$, we have

$$|F(z) - \int_{\mathbb{S}} F d\sigma| \leq \int_{\mathbb{S}} |\nabla_{\mathrm{t}} F(\zeta)| \cdot G(\zeta \cdot z) d\sigma(\zeta) ,$$

where

$$G(\zeta \cdot z) \leq \frac{C}{|1 - \zeta \cdot z|^{n-\frac{1}{2}}} .$$

If $\int_{\mathbb{S}} f\, d\sigma = 0$, then $\int_{\mathbb{S}} f * K\, d\sigma = 0$, i.e., $\int_{\mathbb{S}} F\, d\sigma = 0$. We also note that

$$2|1 - \zeta \cdot z| = |\zeta - z|^2 \quad \text{on } \mathbb{S}\,.$$

By choosing q sufficiently large (and thus its conjugate $q^{'}$ is very close to 1), it follows that

$$\begin{aligned} |F(z)| &\leq \int_{\mathbb{S}} |\nabla_{\mathrm{t}} F(\zeta)| \cdot G(\zeta \cdot z) d\sigma(\zeta) \\ &\leq \left(\int_{\mathbb{S}} |\nabla_{\mathrm{t}} F(\zeta)|^q) d\sigma \right)^{\frac{1}{q}} \cdot \left(\int_{\mathbb{S}} |G(\zeta \cdot z)|^{q^{'}}\, d\sigma(\zeta) \right)^{\frac{1}{q^{'}}} \\ &\leq M \end{aligned}$$

where M is independent of F or z. □

Using Proposition 7.3, we can immediately deduce the sharp Moser-Trudinger inequalities, Theorem 1.1 stated in the introduction, from Theorems 5.2 and 5.3. Indeed, given $\int_{\mathbb{S}} f\, d\sigma = 0$ and $\int_{\mathbb{S}} |\nabla_{\mathbb{C}} f|^{2n}\, d\sigma = 1$, we have $\|F\|_\infty \leq C$ for $F = f * K_{n-1}$. Therefore, combining Theorems 5.2 and 5.3 with this, we can conclude the Moser-Trudinger inequalities in Theorem 1.1 with the same constants B and B_H as in Theorems 5.2 and 5.3. The sharpness of these constants for such inequalities as in Theorem 1.1 follow the same argument as in Section 6.

8 Proof of Theorems 1.2 and 1.3

We now give the proof of Theorem 1.2 using Theorem 1.1. Write

$$2n(f - f_{\mathbb{S}}) = \left(\epsilon \cdot \frac{f - f_{\mathbb{S}}}{\|\nabla_{\mathbb{C}} f\|_{2n}} \right) \cdot \left(\epsilon^{-1}(2n) \cdot \|\nabla_{\mathbb{C}} f\|_{2n} \right).$$

Using Hölder's inequality with the exponents $p = \frac{2n}{2n-1}$ and $p' = 2n$, we get

$$2n(f - f_{\mathbb{S}}) \leq \frac{2n-1}{2n} \left(\epsilon \cdot \frac{f - f_{\mathbb{S}}}{\|\nabla_{\mathbb{C}} f\|_{2n}} \right)^p + (2n)^{-1} \left(\epsilon^{-1}(2n) \cdot \|\nabla_{\mathbb{C}} f\|_{2n} \right)^{2n}.$$

Taking $\epsilon = (Bp)^{(2n-1)/2n}$ and using the Moser-Trudinger inequality in Theorem 1.1, we get

$$\int_{\mathbb{S}} e^{2n(f - f_{\mathbb{S}})}\, d\sigma \leq C_0 \cdot \exp\!\left((2n)^{-1} \left(\epsilon^{-1}(2n) \cdot \|\nabla_{\mathbb{C}} f\|_{2n} \right)^{2n} \right).$$

Inserting the value of ϵ, we conclude the first part of Theorem 1.2.

The proof of the second part of Theorem 1.2 for holomorphic functions is the same by using the second part of Theorem 1.1.

Theorem 1.3 is an immediate consequence of Theorem 1.2.

We now end this section with the following remark. The interest of the functionals $I(f)$ and $J(f)$ lie in the fact that the term $\|\nabla_{\mathbb{C}} f\|_{2n}$ is conformally invariant. This can be seen in the following way: Let $\nabla^0 f = \sum_k (E_k \overline{f}) E_k$; thus $\langle E_k, E_j \rangle_0 = \delta_{kj}$ under the inner product defined by the standard Hermitian metric $\langle \cdot, \cdot \rangle_0$ on $\mathbb{S}$. Let $\langle \cdot, \cdot \rangle_1 = g \cdot \langle \cdot, \cdot \rangle_0$ be the inner product defined by the conformal metric where g is a function. Thus, $\langle g^{-1/2} E_k, g^{-1/2} E_j \rangle_1 = \delta_{kj}$, and the subelliptic gradient associated with the new metric is

$$\nabla^1 f = \sum_k g^{-\frac{1}{2}} E_k f \cdot g^{-\frac{1}{2}} E_k = g^{-1} \cdot \nabla^0 f \,,$$

and under the inner product associated with the new metric we have

$$\langle \nabla^1 f, \nabla^1 f \rangle_1 = g^{-1} \langle \nabla^0 f, \nabla^0 f \rangle_0 \,.$$

Note that $d\sigma_1 = g^n \, d\sigma_0$ on $\mathbb{S}$, where $d\sigma_1$ and $d\sigma_0$ are the volume elements associated with the new and old metrics. Thus, we can see that $\|\nabla_{\mathbb{C}} f\|_{2n}$ is conformally invariant. We will further investigate this conformal invariant in the future.

Acknowledgments. Results of this paper were presented at invited talks of the International Conference in Harmonic Analysis and Applications in August 2002 in Hangzhou, China, at the 51st Midwest PDE Seminar at UIC in April 2003, and at the Workshop on Second Order Subelliptic Equations and Applications in Cortona, Italy, in June 2003. Research was partly supported by an NSF grant.

We would like to thank Prof. Sun-Yung A. Chang for her comments on an early draft of this paper. In particular, we thank her for encouraging us to include Theorems 1.2 and 1.3 as an application of Theorem 1.1 and for helpful discussions in this direction. We would also like to thank Prof. David Jerison for discussions concerning the relationship between Moser-Trudinger inequalities on the Heisenberg group and complex spheres.

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Received July 2003.