Best Constants for Moser-Trudinger Inequalities on the Heisenberg Group

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ABSTRACT. A Moser-Trudinger inequality (with sharp constant) is proven for convolution type potentials defined on stratified groups. When combined with a new representation formula for functions defined on the Heisenberg group (which is based on explicit fundamental solutions to a class of singular or degenerate subelliptic differential operators) we obtain the following Moser-Trudinger inequality with sharp constant on the Heisenberg group $\mathbb{H}^n$. Let $\Omega \subset \mathbb{H}^n$ be an open subset of $\mathbb{H}^n$ with finite volume. Then

$$\sup_{F \in W_0^{1,Q}(\Omega)} \left\{ \frac{1}{|\Omega|} \int_\Omega \exp\left(A \left(\frac{F(u)}{\|\nabla_{\mathbb{H}^n} F\|_Q} \right)^{Q'} \right) du \right\} < \infty,$$

with

$$A = Q\left(2\pi^n \Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{Q-1}{2}\right)\Gamma\left(\frac{Q}{2}\right)^{-1}\Gamma(n)^{-1}\right)^{Q'-1}.$$

Here, $Q = 2n+2$, $Q' = Q/(Q-1)$, $\nabla_{\mathbb{H}^n}$ denotes the subelliptic gradient, and $W_0^{1,Q}(\Omega)$ is the nonisotropic Sobolev space on $\mathbb{H}^n$. Furthermore, A can not be replaced by any greater number.

1. INTRODUCTION

Let $\mathbb{H}^n$ be the n-dimensional Heisenberg group $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$ whose group structure is given by

$$(z,t) \cdot (z',t') = (z + z', t + t' + 2\,\mathrm{Im}(z \cdot \overline{z'})),$$

for any two points (z,t) and (z',t') in $\mathbb{H}^n$.

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The Lie algebra of $\mathbb{H}^n$ is generated by the left invariant vector fields

$$T = \frac{\partial}{\partial t}, \quad X_i = \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t}, \quad Y_i = \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t}$$

for $i = 1, \dots, n$. These generators satisfy the non-commutative relationship

$$[X_i, Y_j] = -4\delta_{ij} T.$$

Moreover, all the commutators of length greater than two vanish, and thus this is a nilpotent, graded, and stratified group of step two.

For each real number $r \in \mathbb{R}$, there is a dilation naturally associated with the Heisenberg group structure which is usually denoted as

$$\delta_r u = \delta_r(z,t) = (rz, r^2 t).$$

However, for simplicity we will write ru to denote $\delta_r u$. The Jacobian determinant of δ_r is r^Q, where $Q = 2n + 2$ is the homogeneous dimension of $\mathbb{H}^n$.

The anisotropic dilation structure on $\mathbb{H}^n$ introduces a homogeneous norm $|u| = |(z,t)| = (|z|^4 + t^2)^{1/4}$. With this norm, we can define the Heisenberg ball centered at $u = (z,t)$ with radius R $B(u,R) = \{v \in \mathbb{H}^n : |u^{-1} \cdot v| < R\}$. The volume of such a ball is $C_Q R^Q$ for some constant depending on Q.

The Heisenberg group is a prominent example of a non-abelian nilpotent Lie group whose anisotropic structure plays an essential role in the development of analysis based on the model of Euclidean space. In particular, it is now well known that the central objects to consider when studying Sobolev spaces on $\mathbb{H}^n$ are the Kohn sub-Laplacian and the subelliptic gradient. Recall that the sub-Laplacian on $\mathbb{H}^n$ equals

$$\Delta_{\mathbb{H}^n} = -\frac{1}{4} \sum_{i=1}^{n} (X_i^2 + Y_i^2),$$

and the subgradient is the $2n$ dimensional vector given by

$$\nabla_{\mathbb{H}^n} f(z,t) = (X_1 f, Y_1 f, \dots, X_n f, Y_n f).$$

The operators $-\frac{1}{4}\sum_{i=1}^{n}(X_i^2 + Y_i^2)$ and $\nabla_{\mathbb{H}^n}$ are left invariant differential operators of degrees two and one respectively. It has been known for years that the following Sobolev inequality holds for $f \in C_0^\infty(\mathbb{H}^n)$:

$$\left(\int_{\mathbb{H}^n} |f(z,t)|^q \, dz\, dt\right)^{1/q} \le C_{p,q} \left(\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} f(z,t)|^p \, dz\, dt\right)^{1/p} \tag{1.1}$$

provided that $1 \le p < Q = 2n+2$ and $1/p - 1/q = 1/Q$. In the above inequality, we have used $|\nabla_{\mathbb{H}^n} f|$ to express the (Euclidean) norm of the subelliptic gradient of f:

$$|\nabla_{\mathbb{H}^n} f| = \sum_{i=1}^{n} ((X_i f)^2 + (Y_i f)^2)^{1/2}.$$

It is clear that the above inequality is also true for functions in the anisotropic Sobolev space $W_0^{1,p}(\mathbb{H}^n)$ $(p \geq 1)$, where $W_0^{1,p}(\Omega)$ for open set $\Omega \subset \mathbb{H}^n$ is the completion of $C_0^\infty(\Omega)$ under the norm $\|f\|_{L^p(\Omega)} + \|\nabla_{\mathbb{H}^n} f\|_{L^p(\Omega)}$.

Of course the Sobolev inequality above is directly analogous to the well known Sobolev inequality for Euclidean space. Furthermore, it is also true that, as in the Euclidean case, the Sobolev inequality no longer holds if $p = Q$. One expects, however, that there will be a substitute, namely a Moser-Trudinger inequality for the Heisenberg group which states that there are absolute constants A_Q and C_0 such that if $Q' = Q/(Q-1)$ then

$$\frac{1}{|\Omega|} \int_\Omega \exp\left(A_Q |f(u)|^{Q'}\right) du \leq C_0,$$

for any $f \in W_0^{1,Q}(\Omega)$, $|\Omega| < \infty$, provided $\|\nabla_{\mathbb{H}^n} f\|_{L^Q} \leq 1$ (see, e.g., [S-C] for a proof).

Once these inequalities are established for the Heisenberg group, there arises, just as in the Euclidean case, the question of computing the best constants for the inequalities. In fact, sharp constants for Sobolev and Moser-Trudinger inequalities have been studied by many authors in the various contexts of Euclidean space $\mathbb{R}^N$, spheres $\mathbb{S}^N$ and manifolds. They play a fundamental role in solving geometric problems; see, for example, [Ad], [Au], [Be1], [Be2], [Be3], [C], [CC], [CL1], [CL2], [Fl], [Fon], [Lie], [Lin], [O], [R], [S], [M1], [M2], [T].

To put the results of this paper in context, we now recall briefly some well-known results on the Moser-Trudinger inequalities in Euclidean space $\mathbb{R}^n$.

In 1971, J. Moser [M1] found the largest positive constant β_0 (which sharpened the result of Trudinger [Tr]) such that if Ω is an open subset of Euclidean space $\mathbb{R}^n$, $n \geq 2$, with finite Lebesgue measure, then there is a constant C_0 depending only on n such that

$$\frac{1}{|\Omega|} \int_\Omega \exp(\beta |f(u)|^{n/(n-1)})\, dx \leq C_0,$$

for any $\beta \leq \beta_0$, any $f \in W_0^{1,Q}(\Omega)$, $|\Omega| < \infty$, provided $\|\nabla f\|_{L^n} \leq 1$.

In fact, Moser showed $\beta_0 = n\omega_{n-1}^{1/(n-1)}$, where ω_{n-1} is the area of the surface of the unit $n-$ball. He also proved that if β exceeds β_0, then the above inequality can not hold with uniform C_0 independent of u. Moser's result was extended to higher order Sobolev spaces in $\mathbb{R}^n$, by a quite different method, by D. Adams [Ad].

After Moser established his result, then the question of whether the following supremum

$$\sup\left\{\frac{1}{|\Omega|}\int_\Omega \exp\left(n\omega_{n-1}^{1/(n-1)}u^{n/(n-1)}\right)dx : u\in W_0^{1,n}(\Omega),\ \|\nabla u\|_n\le 1\right\}$$

is attained arises. In 1986, Carleson and Chang [CC] proved that the above supremum indeed has extremals for balls in $\mathbb{R}^n$ for $n \ge 2$. Their result came as a surprise since it was already known that Sobolev inequality has no extremals on balls for $p > 1$ (see [T], [Ad]). Carleson and Chang proved the existence of extremals by reduction to a one-dimensional problem. Fairly recently, Flucher extended their result to arbitrary bounded smooth domains when $n = 2$ [Fl] , and Lin established the existence of extremals for any bounded smooth domains for $n > 2$.

There has also been substantial progress for the Moser-Trudinger inequality on spheres $\mathbb{S}^n$, and on Riemannian manifolds. We shall not discuss here but refer the reader to the works by Beckner [Be1], [Be2], [Be3], Carlen-Loss [CL2] and the article by S.Y. Chang [C], and references therein.

Much less is known about sharp constants for Sobolev inequalities for the Heisenberg group than for Euclidean space. In fact, the first major breakthrough came after the works by D. Jerison and J. Lee [JL1-3] on the sharp constants for the Sobolev inequality and extremal functions on the Heisenberg group in conjunction with the solution to the CR Yamabe problem (we should note the well-known results of Talenti [T] and Aubin [Au] for sharp constants and extremal functions in the isotropic case). More precisely, in a series of papers [JL1]-[JL3], the Yamabe problem on CR manifolds was first studied. In particular, Jerison and Lee study the problem of conformally changing the contact form to one with constant Webster curvature in the compact setting.

In [JL3], the best constant $C_{p,q}$ for the Sobolev inequality (1.1) on $\mathbb{H}^n$ for $p = 2$ was found to be $C_{2,(2n+2)/n} = (4\pi)^{-1}n^{-2}[\Gamma(n+1)]^{1/(n+1)}$ and it is also shown that all the extremals of (1.1) are obtained by dilations and left translations of the function $K|(t+i(|z|^2+1))|^{-n}$. Furthermore, Jerison and Lee prove that the extremals in (1.1) are constant multiples of images under the Cayley transform of extremals for the Yamabe functional on the sphere $\mathbb{S}^{2n+1}$ in $\mathbb{C}^{n+1}$.

We should mention in passing that Beckner [Be4] derived the sharp constant for the inequality of Stein-Weiss integral on the Heisenberg group by using $SL(2,\mathbb{R})$ symmetry. He showed that for any $f, g \in S(\mathbb{H}^n)$, the Schwartz class, the following inequality holds with

$$C_\alpha = 2^{\alpha/2}(2\pi)^{n+1}\Gamma\left(\frac{\alpha}{2}\right)\left[\frac{\Gamma[(2n-\alpha)/4]}{\Gamma[(2n-\alpha)/4+1/2]\Gamma[(2n+\alpha)/4]}\right]^2 :$$

$$\left| \int_{\mathbb{H}^n \times H^n} f(z,t)|z|^{-\alpha/2} \left| \, |z-w|^2 + i(t-s-2\operatorname{Im} z\bar{w}) \right|^{-(n+1)+\alpha/2} \right.$$
$$\left. \times |w|^{-\alpha/2} g(w,s)\, dm\, dm \right| \le C_\alpha \|f\|_{L^2(\mathbb{H}^n)} \|g\|_{L^2(\mathbb{H}^n)},$$

where $dm = 4^n\, dz\, dt$.

The work of Jerison and Lee [JL3] raised two natural questions.

Question 1. What is the best constant $C_{p,q}$ for the L^p to L^q Sobolev inequality for $1 \le p < Q$ and $q = Qp/(Q-p)$ when $p \ne 2$?

Question 2. What is the best constant for the Moser-Trudinger inequality for $p = Q$? Namely, what is the best constant A_Q such that

$$\frac{1}{|\Omega|} \int_\Omega \exp(A_Q |f(u)|^{Q'})\, du \le C_0,$$

for any $f \in W_0^{1,Q}(\Omega)$, $|\Omega| < \infty$, provided $\|\nabla_{\mathbb{H}^n} f\|_{L^Q} \le 1$, where $C_0 > 0$ is an absolute constant?

The main result of this paper is to provide an answer to Question 2.

Now, by the result of Folland [F], the the Kohn sub-Laplacian on $\mathbb{H}^n$ is similar to the classical Laplacian in Euclidean space in that $-\frac{1}{4}\sum_{i=1}^n (X_i^2 + Y_i^2)$ has a fundamental solution with a pole at the origin given by $\Gamma(u) = B_Q|u|^{2-Q}$, where

$$\begin{aligned} B_Q &= 4\left((Q-2)(Q+2) \int_{\mathbb{H}^n} \frac{|z|^2}{(|u|^4+1)^{(Q+6/4)}}\, du \right)^{-1} \\ &= 2^{n-2}\pi^{-n-1}\Gamma\left(\frac{n}{2}\right)^2; \end{aligned}$$

Since the fundamental solution is given so explictly, it might be expected that, analogous to the Euclidean case, it could be used to determine the best constant for the Moser-Trudinger inequality on the Heisenberg group. It is interesting that this approach fails. What is needed, we find, is a fundamental solution to a more general operator than the sub-Laplacian. This is the substance of Theorem 1.2 below.

We actually obtain sharp constants for a number of Moser-Trudinger type inequalities on any stratified groups $\mathbb{G}$. The most general result is a Moser-Trudinger inequality on $\mathbb{G}$ (with sharp constant) for the convolution of a function with a kernel which is possibly non-radial with respect to the homogeneous norm. We then obtain the answer to Question 2 above by choosing a particular non-radial kernel and combining the result described above, for that kernel, with a representation formula for a function in terms of its subgradient. We also obtain Moser-Trudinger inequalities for certain nonradial kernels in Euclidean space.

To state the main theorems, we need to introduce some notation. Given any stratified group $\mathbb{G}$ (see Folland-Stein [FS] for definitions), let $|u|$ be a homogeneous norm on $\mathbb{G}$ and let Q be the homogeneous dimension on $\mathbb{G}$, and let u^* denote a point on the unit sphere $\Sigma = \{u \in \mathbb{G} : |u| = 1\}$ in $\mathbb{G}$.

The following result from [FS] concerns polar coordinates on $\mathbb{G}$.

Proposition. *Let $\Sigma = \{u \in \mathbb{G} : |u| = 1\}$ be the unit sphere in a stratified group $\mathbb{G}$. Then there is a unique Radon measure $d\mu$ on Σ such that for all $f \in L^1(\mathbb{G})$,*

$$\int_{\mathbb{G}} f(u)\,du = \int_0^\infty \int_\Sigma f(ru^*) r^{Q-1}\,d\mu(u^*)\,dr.$$

On the Heisenberg group $\mathbb{G} = \mathbb{H}^n$, we will also use the following notation. Let $ru = (rz, r^2 t)$ denote the dilation on $\mathbb{H}^n$ and given any $u = (z,t)$ set $z^* = z/|u|$, $t^* = t/|u|^2$ and $u^* = (z^*, t^*)$. Thus, for any $u \in \mathbb{H}^n$ and $u \neq 0$ we have $u^* \in \Sigma = \{u \in \mathbb{H}^n : |u| = 1\}$, the Heisenberg sphere.

We will make frequent use of this proposition. In particular, let $\omega_{2n-1} = 2\pi^n/\Gamma(n)$ be the surface area of the unit sphere in $\mathbb{C}^n$ and for $\beta > -2n$ let

$$c_\beta = \int_\Sigma |z^*|^\beta\,d\mu.$$

The best constants we find for Moser-Trudinger inequalities are naturally expressed in terms of c_β which we now write in terms of the Gamma function. Compute that, for $\beta > -Q$,

$$\begin{aligned}\int_\Sigma |z^*|^\beta\,d\mu &= (Q+\beta)\int_0^1 r^{\beta+Q-1}\,dr \int_\Sigma |z^*|^\beta\,d\mu \\ &= (Q+\beta)\int_\Sigma\int_0^1 |rz^*|^\beta r^{Q-1}\,dr\,d\mu \\ &= (Q+\beta)\int_{|u|<1} |z|^\beta\,du.\end{aligned}$$

Next, if $\beta > -2n$,

$$\begin{aligned}\int_{|u|<1} |z|^\beta\,du &= 2\int_0^1 \int_{|z|<(1-t^2)^{1/4}} |z|^\beta\,dz\,dt \\ &= 2\omega_{2n-1}\int_0^1\int_0^{(1-t^2)^{1/4}} r^{\beta+2n-1}\,dr\,dt \\ &= \frac{2\omega_{2n-1}}{2n+\beta}\int_0^1 (1-t^2)^{(2n+\beta)/4}\,dt \\ &= \frac{\omega_{2n-1}}{2n+\beta}\int_0^1 (1-t)^{(2n+\beta)/4} t^{-1/2}\,dt \\ &= \frac{\omega_{2n-1}}{2n+\beta}\frac{\Gamma[(2n+4+\beta)/4]\Gamma(1/2)}{\Gamma[(2n+6+\beta)/4]}.\end{aligned}$$

Thus, if $\beta > -2n$,

$$
\begin{aligned}
c_\beta &= \frac{(Q+\beta)\omega_{2n-1}}{2n+\beta} \frac{\Gamma[(2n+\beta)/4+1]\Gamma(1/2)}{\Gamma[(Q+\beta)/4+1]} \\
&= \frac{\omega_{2n-1}\Gamma(1/2)\Gamma[(2n+\beta)/4]}{\Gamma[(Q+\beta)/4])} \\
&= \frac{\omega_{2n-1}\Gamma(1/2)\Gamma[(Q-2+\beta)/4]}{\Gamma[(Q+\beta)4]}.
\end{aligned}
$$

We can now state our main result.

Theorem 1.1. *Let $A_Q = Q(c_Q)^{Q'/Q}$, where $Q' = Q/(Q-1)$. There exists a constant C_0 such that for all $\Omega \subset \mathbb{H}^n$, $|\Omega| < \infty$, and for all $f \in W_0^{1,Q}(\Omega)$,*

$$
\frac{1}{|\Omega|} \int_\Omega \exp(A_Q |f(u)|^{Q'})\, du \le C_0,
$$

provided $\|\nabla_{\mathbb{H}^n} f\|_{L^Q} \le 1$. Furthermore, if A_Q is replaced by any number greater than A_Q, then the statement is false.

The proof of Theorem 1.1 relies on the next theorem, Theorem 1.2, which provides various ways to represent and estimate the value of a function in terms of its subelliptic gradient.

In what follows, $\langle A, B\rangle$ will denote the usual inner product of vectors A and B in $\mathbb{R}^{2n}$.

Theorem 1.2. *Suppose that $\beta > -2n+1$ and $f \in C_0^\infty(\mathbb{H}^n)$. Then*

$$
f(v) = \frac{-(c_\beta)^{-1}}{4} \int_{\mathbb{H}^n} \frac{|z|^{\beta-2}}{|u|^{Q+\beta}} \langle \nabla_{\mathbb{H}^n} f(vu^{-1}), \nabla_{\mathbb{H}^n}(|u|^4)\rangle\, du
$$

and

$$
|f(v)| \le (c_\beta)^{-1} \int_{\mathbb{H}^n} \frac{|z|^{\beta-1}}{|u|^{Q+\beta-2}} |\nabla_{\mathbb{H}^n} f(vu^{-1})|\, du
$$

where $u = (z,t)$, and $\nabla_{\mathbb{H}^n} f| = \sum_{i=1}^n ((X_i f)^2 + (Y_i f)^2)^{1/2}$ is the norm of the subgradient of f.

As we mentioned earlier, the above representation formula is equivalent to the fundamental solutions to a one-parameter degenerate or singular subelliptic differential operators. We state this as follows.

Theorem 1.2*. *The functions $-[1/(Q+\beta-4)c_\beta]|u|^{4-Q-\beta}$ are fundamental solutions to the differential operators $\nabla_{\mathbb{H}^n} \circ |z|^{\beta-2}\nabla_{\mathbb{H}^n}$ in the sense that*

$$
\nabla_{\mathbb{H}^n} \circ |z|^{\beta-2} \nabla_{\mathbb{H}^n} \left(-\frac{1}{(Q+\beta-4)c_\beta} (|u|^{4-Q-\beta}) \right) = \delta_0,
$$

where $\beta > -2n + 1$.

There is an alternative proof of Theorem 1.2 in the special case $\beta = 2$ by using the fundamental solution due to Folland [F]. We discuss this in Section 2. However, as we mentioned earlier, the special case $\beta = 2$ does not lead to the optimal constant for the Moser-Trudinger inequality.

Theorem 1.1 will follow from Theorem 1.2 above and Theorem 1.3 below which is a sharp Moser-Trudinger inequality for potentials on general stratified groups $\mathbb{G}$. To state Theorem 1.3 we need the following definitions. First, define the operation of convolution on $\mathbb{G}$ by setting

$$f * h(u) = \int_{\mathbb{G}} f(uv^{-1})h(v)\,dv = \int_{\mathbb{G}} f(v)h(v^{-1}u)\,dv.$$

Next, let Q denote the homogeneous dimension on $\mathbb{G}$ and let $0 < \alpha < Q$. We will say that a non-negative function g defined on $\mathbb{G} - \{0\}$ is a kernel of order α if there is a function (also denoted by g) defined on the unit sphere $\Sigma = \{u \in \mathbb{G} : |u| = 1\}$ such that for $u \neq 0$, $g(u) = |u|^{\alpha - Q} g(u^*)$, where $u^* = u/|u|$.

We will also make the following technical assumption about the function g. For $\delta > 0$ let Σ_δ be the subset of the sphere given by

$$\Sigma_\delta = \{u^* \in \Sigma : \delta \le g(u^*) \le \delta^{-1}\}.$$

We will need to assume that for every $\delta > 0$ and $0 < M < \infty$ there are constants $C(\delta, M)$ such that

$$\int_{\Sigma_\delta} \int_0^M |g(u^*(sv^*)^{-1}) - g(u^*)| \frac{ds}{s}\, d\mu(u^*) \le C(\delta, M)$$

for all $v^* \in \Sigma$.

If a kernel of order α satisfies this assumption, we will say that g is "allowed". It is easy to verify that functions defined by $g(u^*) = |z^*|^\beta$ on the Heisenberg group $\mathbb{H}^n$ are allowed.

Theorem 1.3. *Let* $0 < \alpha < Q$. *Suppose* g *is an allowed kernel of order* α *on the stratified group* $\mathbb{G}$ *and* $Q - \alpha p = 0$ (*i.e.,* $\alpha = Q/p$). *Let* $p' = p/(p-1)$ *and*

$$A(g,p) = A_\alpha(g) = \frac{Q}{\int_\Sigma |g(u^*)|^{p'}\,d\mu}.$$

There exists a constant C_0 *such that for all* $\Omega \subset \mathbb{G}$, $|\Omega| < \infty$, *and for all* $f \in L^p(\mathbb{G})$ *with support contained in* Ω,

$$\frac{1}{|\Omega|} \int_\Omega \exp\left(A(g,p) \left(\frac{f * g(u)}{\|f\|_p} \right)^{p'} \right) du \le C_0.$$

Furthermore, if $A(g, p)$ is replaced by a greater number, the resulting statement is false.

If we choose the kernel function $g(u) = |z|^{Q-1}/|u|^{2Q-2}$ on the Heisenberg group $\mathbb{H}^n$ then the first statement of Theorem 1.1 follows as a consequence of Theorem 1.3 and Theorem 1.2.

By selecting $g(u) = g_\alpha(u) = |u|^{\alpha-Q}$ on $\mathbb{G}$ for $0 < \alpha < Q$ which is the analogue on the stratified group $\mathbb{G}$ of the standard fractional integral kernel in Euclidean space, we obtain the following result.

Theorem 1.4. *Let $0 < \alpha < Q$, $Q - \alpha p = 0$, $p' = Q/(Q - \alpha)$ and let $I_\alpha f(u) = \int_{\mathbb{G}} |u \cdot v^{-1}|^{\alpha-Q} f(v)\, dv$. There exists a constant C_0 such that for all $\Omega \subset \mathbb{G}$, $|\Omega| < \infty$, and for all $f \in L^p(\mathbb{G})$ with support in Ω,*

$$\frac{1}{|\Omega|} \int_\Omega \exp\left(\frac{Q}{c_0^{p'}} \left| \frac{I_\alpha f}{\|f\|_{L^p(\mathbb{G})}} \right|^{p'} \right) \le C_0,$$

where $c_0 = \int_\Sigma d\mu$ (i.e., the same as c_β when $\beta = 0$). Furthermore, if $Q/c_0^{p'}$ is replaced by a greater number, then the statement is false.

We remark here that the statement and proof of Theorem 1.3 also provide a Moser-Trudinger inequality for nonradial kernels in Euclidean space. We are not aware of any such results even in the Euclidean space, and thus we include it below. Note that we denote in Theorem 1.5 dH^{N-1} the $(N-1)$-dimensional Hausdorff measure on the sphere $\{x \in \mathbb{R}^N : |x| = 1\}$.

Theorem 1.5. *Let $0 < \alpha < N$, $g(x) = |x|^{\alpha-N} g(x')$ (where $x = |x|x'$) and let $N - \alpha p = 0$ and $p' = N/(N - \alpha)$. There exists a constant C_0 such that for all $|\Omega| < \infty$ and for all if $f \in L^N(\mathbb{R}^N)$ with support in Ω,*

$$\frac{1}{|\Omega|} \int_\Omega \exp\left(\left(N/\int_{|x|=1} g^{p'}\, dH^{N-1} \right) \left(\frac{f * g(x)}{\|f\|_p} \right)^{p'} \right) dx \le C_0,$$

Furthermore, if $N/\int_{|x|=1} g^{p'}\, dH^{N-1}$ is replaced by a greater number, then the statement is false.

The following remarks are in order. First of all, as has been the case in most proofs of sharp constants we shall use the radial non-increasing rearrangement u^* of functions u (in terms of the homogeneous norm). However, it is not known whether or not the L^p norm of the subelliptic gradient of the rearrangement of a function is dominated by the L^p norm of the subelliptic gradient of the function. In other words, an inequality like $\|\nabla_{\mathbb{H}^n} u^*\|_{L^p} \le \|\nabla_{\mathbb{H}^n} u\|_{L^p}$ is not available as a tool. In fact, the work of Jerison-Lee on the best constant and extremals [JL3] indicates that this inequality fails to hold for the case $p = 2$. [1]

[1] We thank S. Chanillo for pointing out this to us after we finished this paper.

Thus, we will adapt D. Adams' idea in deriving the Moser-Trudinger inequality for higher order derivatives in Euclidean space [Ad], which requires, roughly speaking, a sharp bound on the size of a function in terms of the potential of its gradient, namely a sharp representation formula. This is given by Theorem 1.2. By using this one parameter representation formula, we are able to avoid considering the subelliptic gradient of the rearrangement function. Instead, we will consider the rearrangement of the convolution of the subelliptic gradient with an optimal kernel of order 1.

More precisely, the representation formula for Sobolev functions on $\mathbb{H}^n$ in terms of their subelliptic gradients holds for a wide range of choices of the parameter β (e.g., $\beta > -2n + 1$). As we pointed out earlier, only the case $\beta = 2$ follows from Folland's fundamental solution to the sub-Laplacian on $\mathbb{H}^n$. However, what is needed to derive the sharp constants is the case when $\beta = Q$. Moreover, the derivation of our representation formula is completely elementary and has its own interest. It also provides an alternative way of seeking the fundamental solution to sub-Laplacian (see the remark made after the end of the proof of Theorem 1.2). However, our representation formula gives rise to a non-radial kernel in terms of the homogeneous norm on $\mathbb{H}^n$. This presents more complexity and difficulties in deriving the sharp constants for the Moser-Trudinger inequalities on $\mathbb{H}^n$ than we have in Euclidean space. In particular, the proof of the sharpness of the desired constant in Theorem 1.3 for the Moser-Trudinger inequality on general stratified groups $\mathbb{G}$ becomes rather technical, and we must choose the test functions very carefully (see Section 3). We remark in passing that we hope that we will find useful the one-parameter representation formula in answering Question 1 raised at the earlier part of the introduction, namely, the sharp constants for the Sobolev inequality (1.1) for $p \neq 2$.

The plan of the paper is as follows: Section 2 shows the one-parameter representation formula (Theorem 1.2) and the fundamental solutions to a one-parameter degenerate or singular subelliptic differential operators (Theorem 1.2*) on the Heisenberg group $\mathbb{H}^n$; In Section 3, we prove the sharp constant for Moser-Trudinger inequality on general stratified groups $\mathbb{G}$ for potentials with possibly non-radial kernels motivated by Theorem 1.2. Section 4 gives the proof of Theorem 1.1, i.e., the Moser-Trudinger inequality for Sobolev functions on $\mathbb{H}^n$.

2. PROOF OF THE REPRESENTATION FORMULA: THEOREM 1.2

In this section, we prove Theorem 1.2, which, for each $\beta > -2n + 1$ gives a formula expressing a function in terms of its subelliptic gradient. In the case $\beta = 2$, the formula follows from the fundamental solution to the sub-Laplacian by integration by parts; however for the application we have in mind, we will need the case $\beta = Q$.

We will use the following simple lemma whose proof will be included for the sake of completeness.

Lemma 2.1. *Let $\rho = |u|$ denote the homogeneous norm of the element $u = (z,t) \in \mathbb{H}^n$. Let $f(u) = f(\rho)$ be a C^1 radial function on $\mathbb{H}^n$. Then*

$$|\nabla_{\mathbb{H}^n} f(u)| = \frac{|f'(\rho)|}{\rho}|z|.$$

Proof. It is easy to calculate that

$$X_j f(\rho) = \frac{f'(\rho)}{\rho^3}(|z|^2 x_j + y_j t) \quad \text{and} \quad Y_j f(\rho) = \frac{f'(\rho)}{\rho^3}(|z|^2 y_j - x_j t).$$

Thus

$$\begin{aligned}
|\nabla_{\mathbb{H}^n} f(\rho)|^2 &= \sum_{j=1}^n (|X_j f(\rho)|^2 + |Y_j f(\rho)|^2)\\
&= \Big[\sum_{j=1}^n (x_j^2|z|^4 + y_j^2 t^2) + \sum_{j=1}^n (|z|^4 y_j^2 + x_j^2 t^2)\Big] \cdot \left(\frac{f'(\rho)}{\rho^3}\right)^2\\
&= \Big[\sum_{j=1}^n (x_j^2 + y_j^2)|z|^4 + \sum_{j=1}^n (x_j^2 + y_j^2)t^2\Big] \cdot \left(\frac{f'(\rho)}{\rho^3}\right)^2\\
&= (|z|^4 + t^2)|z|^2 \cdot \left(\frac{f'(\rho)}{\rho^3}\right)^2 = |z|^2 \cdot \left(\frac{f'(\rho)}{\rho}\right)^2 = \left(\frac{f'(\rho)}{\rho}|z|\right)^2.
\end{aligned}$$

This leads to the conclusion of the lemma. □

Proof of Theorem 1.2. Let u^* be a point on the Heisenberg sphere, that is $u^* = (z^*, t^*)$, where $|z^*|^4 + (t^*)^2 = 1$. We first establish the first formula of Theorem 1.2 in the case $v = 0$. Since f has compact support

$$\begin{aligned}
-f(0) &= \int_0^\infty \frac{d}{dr} f(ru^*)\, dr\\
&= \int_0^\infty \sum_{j=1}^n \left(\frac{x_j}{r}\frac{\partial f}{\partial x_j}(ru^*) + \frac{y_j}{r}\frac{\partial f}{\partial y_j}(ru^*)\right) + \frac{2t}{r}\frac{\partial f}{\partial t}(ru^*)\, dr\\
&= \int_0^\infty \sum_{j=1}^n \left(\frac{x_j}{r}\frac{\partial f}{\partial x_j}(ru^*) + \frac{y_j}{r}\frac{\partial f}{\partial y_j}(ru^*) + \frac{y_j^2 + x_j^2}{|z|^2}\frac{2t}{r}\frac{\partial f}{\partial t}(ru^*)\right) dr,
\end{aligned}$$

where $u = ru^* = (x_1 + iy_1, \ldots, x_n + iy_n, t) = (rz^*, r^2 t^*)$.

Multiplying both sides of the last equation by $|z^*|^\beta$ and integrating over the Heisenberg sphere with respect to $d\mu(u^*)$ yields the equation

$$
\begin{aligned}
&-\left(\int_\Sigma |z^*|^\beta\, d\mu\right) f(0)\\
&= \int_{\mathbb{H}^n} \frac{|z|^\beta}{|u|^{Q+\beta}} \sum_{j=1}^n \left(x_j \frac{\partial f}{\partial x_j}(u) + y_j \frac{\partial f}{\partial y_j}(u) + 2t\left(\frac{y_j^2 + x_j^2}{|z|^2}\right)\frac{\partial f}{\partial t}(u)\right) dz\, dt\\
&= \int_{\mathbb{H}^n} \frac{|z|^{\beta-2}}{|u|^{Q+\beta}} \sum_{j=1}^n \left(\frac{\partial f}{\partial x_j} + 2y_j\frac{\partial f}{\partial t}\right)(|z|^2 x_j + y_j t)\\
&\qquad\qquad + \left(\frac{\partial f}{\partial y_j} - 2x_j \frac{\partial f}{\partial t}\right)(|z|^2 y_j - x_j t))\, du\\
&\qquad\qquad\qquad - \int_{\mathbb{H}^n} \frac{t|z|^{\beta-2}}{|u|^{Q+\beta}} \sum_{j=1}^n \left(y_j \frac{\partial f}{\partial x_j} - x_j \frac{\partial f}{\partial y_j}\right) du\\
&= \int_{\mathbb{H}^n} \frac{|z|^{\beta-2}}{|u|^{Q+\beta}} \sum_{j=1}^n ((X_j f)(|z|^2 x_j + y_j t) + (Y_j f)(|z|^2 y_j - x_j t))\, du\\
&\qquad\qquad\qquad - \int_{\mathbb{H}^n} \frac{t|z|^{\beta-2}}{|u|^{Q+\beta}} \sum_{j=1}^n \left(y_j \frac{\partial f}{\partial x_j} - x_j \frac{\partial f}{\partial y_j}\right) du.
\end{aligned}
$$

By the proof of Lemma 2.1,

$$X_j(|u|^4) = 4|z|^2 x_j + 4y_j t \quad \text{and} \quad Y_j(|u|^4) = 4|z|^2 y_j - 4x_j t.$$

Therefore the last equation can be rewritten as

$$
\begin{aligned}
-(c_\beta) f(0) = &\int_{\mathbb{H}^n} \frac{|z|^{\beta-2}}{|u|^{Q+\beta}} \left\langle \nabla_{\mathbb{H}^n} f(u), \nabla_{\mathbb{H}^n}\left(\frac{|u|^4}{4}\right)\right\rangle du\\
&- \int_{\mathbb{H}^n} \frac{t|z|^{\beta-2}}{|u|^{Q+\beta}} \sum_{j=1}^n \left(y_j \frac{\partial f}{\partial x_j} - x_j \frac{\partial f}{\partial y_j}\right) du.
\end{aligned}
$$

Thus, the first formula of Theorem 1.2 will follow (for the case $v = 0$) if we prove that the assertion that the second integral on the right hand side in the last equation vanishes. To see this, for each j, let

$$T_j f = y_j \frac{\partial f}{\partial x_j} - x_j \frac{\partial f}{\partial y_j}.$$

Notice that the integrand in the second integral is absolutely integrable, provided $\beta > -2n + 1$ and that the dominated convergence theorem shows that it is equal to the limit as $\varepsilon \to 0$ of the integrals

$$\int_{\mathbb{H}^n} \frac{t(|z|^2+\varepsilon)^{(\beta-2)/2}}{(|u|^4+\varepsilon)^{(Q+\beta)/4}} T_j f(u)\, dz\, dt.$$

Since T_j annihilates functions of $|z|$, integration by parts shows that for any $\varepsilon > 0$

$$\int_{\mathbb{H}^n} \frac{t(|z|^2+\varepsilon)^{(\beta-2)/2}}{(|u|^4+\varepsilon)^{(Q+\beta)/4}} T_j f(u)\, dz\, dt = 0,$$

for each j, and this proves the assertion.

It follows now that

$$f(0) = -\frac{(c_\beta)^{-1}}{4} \int_{\mathbb{H}^n} \frac{|z|^{\beta-2}}{|u|^{Q+\beta}} \langle \nabla_{\mathbb{H}^n} f(u), \nabla_{\mathbb{H}^n}(|u|^4)\rangle\, du$$

which is the first assertion of Theorem 1.2 with $v = 0$. Translation shows that the first assertion holds for arbitrary v. For the second statement, by Lemma 2.1, $|\nabla_{\mathbb{H}^n}(|u|^4)| = 4|z|\,|u|^2$. The second statement in Theorem 1.2 follows now from the pointwise Schwartz inequality. This concludes the proof of Theorem 1.2. □

The proof of Theorem 1.2 above also leads to the proof of Theorem 1.2*, giving explicit formulas for fundamental solutions to a one-parameter family of degenerate or singular subelliptic differential operators.

Proof of Theorem 1.2*. If we observe that

$$-4(Q+\beta-4)^{-1}\nabla_{\mathbb{H}^n}(|u|^{4-Q-\beta}) = |u|^{-Q-\beta}\nabla_{\mathbb{H}^n}(|u|^4),$$

then we can integrate by parts in the representation formula above to get the additional fact that

$$\begin{aligned} \nabla_{\mathbb{H}^n} \circ |z|^{\beta-2}\nabla_{\mathbb{H}^n}(|u|^{4-Q-\beta}) &= \sum_{j=1}^{n} (X_j|z|^{\beta-2}X_j + Y_j(|z|^{\beta-2}Y_j)(|u|^{4-Q-\beta}) \\ &= -(Q+\beta-4)c_\beta\delta_0. \end{aligned}$$

This formula may also be verified by setting $\rho_\varepsilon = |u|^4 + \varepsilon^4$ and verifying that

$$\lim_{\varepsilon\to 0} -\frac{(c_\beta)^{-1}}{Q+\beta-4} \sum_{j=1}^{n} (X_j|z|^{\beta-2}X_j + Y_j|z|^{\beta-2}Y_j)(\rho_\varepsilon)^{(4-Q-\beta)/4} = \delta_0.$$

This completes the proof of Theorem 1.2*. □

Remark 2.1. If Theorem 1.2* is known a-priori, integration by parts will lead to another proof of Theorem 1.2.

Remark 2.2. Theorem 1.2* above has stated that $\Delta_{\mathbb{H}^n}$ has fundamental solution $B|u|^{-Q+2}$ where $B^{-1} = (Q-2)c_2/4$. We verify that this agrees with the constant B_Q given by Folland in [F] by computing that

$$\begin{aligned}
(B_Q)^{-1} &= \frac{(Q-2)(Q+2)}{4}\int_{\mathbb{H}^n}\frac{|z|^2}{(|u|^4+1)^{(Q+6)/4}}\,du \\
&= \frac{(Q-2)(Q+2)}{4}\int_{\Sigma}|z^*|^2\,d\mu\int_0^\infty\frac{\rho^{Q+1}}{(\rho^4+1)^{(Q+6)/4}}\,d\rho \\
&= \frac{c_2(Q-2)(Q+2)}{4}\int_0^\infty\frac{\rho^{Q+1}}{(\rho^4+1)^{(Q+6)/4}}\,d\rho.
\end{aligned}$$

We may rewrite the integral on the right as

$$\int_0^\infty\frac{\rho^{Q+1}}{(\rho^4+1)^{(Q+6)4}}\,d\rho = \int_0^\infty\frac{\rho^{-5}}{(1+\rho^{-4})^{(Q+6)/4}}\,d\rho$$

and make the substitution $t = 1+\rho^{-4}$ to arrive at

$$\int_0^\infty\frac{\rho^{-5}}{(1+\rho^{-4})^{(Q+6)/4}}\,d\rho = \frac{1}{4}\int_1^\infty t^{(-Q-6)/4}\,dt = \frac{1}{Q+2}.$$

Thus

$$(B_Q)^{-1} = \frac{c_2(Q-2)}{4},$$

as claimed.

3. Proof of the Moser-Trudinger inequality for general kernel: Theorem 1.3

Let $\mathbb{G}$ be a stratified group and Q be the homogeneous dimension of $\mathbb{G}$. Recall that for $0 < \alpha < Q$ we will say that a non-negative function g on $\mathbb{G}$ is a kernel of order α if g has the form $g(u) = |u|^{\alpha-Q}g(u^*)$, where $u^* = u/|u|$ is a point on the unit sphere. We are also assuming that for every $\delta > 0$ and $0 < M < \infty$ there are constants $C(\delta, M)$ such that

$$\int_{\Sigma_\delta}\int_0^M |g(u^*(sv^*)^{-1}) - g(u^*)|\frac{ds}{s}\,d\mu(u^*) \le C(\delta, M) \tag{3.1}$$

for all $v^* \in \Sigma$, where Σ_δ is the subset of the unit sphere given by

$$\Sigma_\delta = \{u^* \in \Sigma : \delta \le g(u^*) \le \delta^{-1}\}.$$

We now recall the rearrangement of functions on $\mathbb{G}$. This can certainly be done in more general homogeneous groups (see also [FS]).

Suppose F is a non-negative function defined on $\mathbb{G}$. Define the non-increasing rearrangement of F by $F^*(t) = \inf\{s > 0 : \lambda_F(s) \leq t\}$, where $\lambda_F(s) = |\{u \in \mathbb{G} : F(u) > s\}|$. In addition, define

$$F^{**}(t) = t^{-1} \int_0^t F^*(s)\, ds.$$

Let $U = f * g$ be the convolution on $\mathbb{G}$. It is easy to check that O'Neil's lemma regarding rearrangement of convolution of two functions remains true on $\mathbb{G}$ (and even on any homogeneous groups or more general "convolution operators" defined on metric measure spaces as done in O'Neil's paper [O'N]). We state it here as

Lemma 3.1. *Suppose* $U = f * g$ *on* $\mathbb{G}$. *Then*

$$U^*(t) \leq U^{**}(t) \leq t f^{**}(t) g^{**}(t) + \int_t^\infty f^*(s) g^*(s)\, ds.$$

The following lemma gives rise to a specific estimate for the rearrangement of convolution of two functions on $\mathbb{G}$ when one of the functions is the aforementioned g for any $0 < \alpha < Q$. We prove it in the most general form for it has its own interest. However, what is needed in the proof of the Moser-Trudinger inequality is the case when $p = Q/\alpha$. See Corollary 3.3.

Lemma 3.2. *Suppose that* g *is a kernel of order* α *on* $\mathbb{G}$ *and* $0 < \alpha < Q$. *Let* $U = f * g$ *and*

$$A_\alpha(g) = \frac{Q}{\int_\Sigma |g(u^*)|^{Q/(Q-\alpha)}\, d\mu}.$$

Then

$$U^*(t) \leq (A_\alpha(g))^{-(Q-\alpha)/Q} \left(\frac{Q}{\alpha} t^{-(Q-\alpha)/Q} \int_0^t f^*(s)\, ds + \int_t^\infty f^*(s) s^{-(Q-\alpha)/Q}\, ds \right).$$

Proof. Set $U = f * g$, $f \geq 0$. We intend to apply O'Neil's lemma and therefore proceed to compute g^* and g^{**}. We have

$$\lambda_g(s) = |\{u \in \mathbb{G} : g(u) > s\}| = |\{u \in \mathbb{G} : |u| < (s^{-1} g(u^*))^{1/(Q-\alpha)}\}|.$$

Using polar coordinates we find that the last quantity equals

$$\int_\Sigma \int_0^{(s^{-1} g(u^*))^{1/(Q-\alpha)}} r^{Q-1}\, dr\, d\mu = (A_\alpha(g))^{-1} s^{-Q/(Q-\alpha)}.$$

It follows from this that

$$g^*(t) = (A_\alpha(g)t)^{-(Q-\alpha)/Q} \quad \text{and} \quad g^{**}(t) = \frac{Q}{\alpha} g^*(t).$$

By O'Neil's lemma,

$$U^*(t) \le U^{**}(t) \le t f^{**}(t) g^{**}(t) + \int_t^\infty f^*(s) g^*(s)\, ds$$

which is

$$(A_\alpha(g))^{-(Q-\alpha)/Q} \left(\frac{Q}{\alpha} t^{-(Q-\alpha)/Q} \int_0^t f^*(s)\, ds + \int_t^\infty f^*(s) s^{-(Q-\alpha)/Q}\, ds \right).$$

□

Corollary 3.3. *Suppose that g is a kernel of order α, $0 < \alpha < Q$ and $\alpha = Q/p$. Let $U = f * g$, and $A(g,p) = A_\alpha(g)$. Then*

$$U^*(t) \le (A(g,p))^{-1/p'} \left(p t^{-1/p'} \int_0^t f^*(s)\, ds + \int_t^\infty f^*(s) s^{-1/p'}\, ds \right).$$

We are now ready to prove the first of our main theorems.

Theorem 3.4. *Suppose g is an allowed kernel of order α on a stratified group $\mathbb{G}$ and $Q - \alpha p = 0$ (i.e., $\alpha = Q/p$). Let*

$$A(g,p) = A_\alpha(g) = \frac{Q}{\int_\Sigma |g(u^*)|^{p'}\, d\mu}.$$

Then there exists a constant C_0 such that for any $f \in L^p(\mathbb{G})$ with support contained in $\Omega \subset \mathbb{G}$, $|\Omega| < \infty$, the following holds:

$$\frac{1}{|\Omega|} \int_\Omega \exp\left(A(g,p) \left(\frac{f * g(u)}{\|f\|_p} \right)^{p'} \right) du \le C_0,$$

Furthermore, if $A(g,p)$ is replaced by a greater number, the resulting statement is false.

We will separate the proof of the theorem into two parts. The first part is to show that the constant $A(g,p)$ does make the inequality hold for any aforementioned function f. The second part is to show that $A(g,p)$ is actually sharp. The proof of the second part is rather technical.

Before we start the proof, we state a lemma from Adams' paper [Ad].

Lemma 3.5. *Let $a(s,t)$ be a nonnegative measurable function on $(-\infty,\infty)\times[0,\infty)$ such that almost everywhere (in Lebesgue measure in the product space)*

$$a(s,t)\le 1,\quad \text{when } 0<s<t;$$
$$\sup_{t>0}\left(\int_{-\infty}^{0}+\int_{t}^{\infty} a(s,t)^{p'}\,ds\right)^{1/p'}=b<\infty.$$

Then there is a constant $c_0=c_0(p,b)$ such that if for $\varphi\ge 0$,

$$\int_{-\infty}^{\infty}\varphi(s)^p\,ds\le 1$$

then

$$\int_0^\infty e^{-F(t)}\,dt\le c_0,$$

where

$$F(t)=t-\left(\int_{-\infty}^{\infty}a(s,t)\varphi(s)\,ds\right)^{p'}.$$

Proof of the first part of Theorem 3.4. Following Adams, [Ad], set

$$\varphi(s)=|\Omega|^{1/p}f^*(|\Omega|e^{-s})e^{-s/p}.$$

Then

$$\int_0^\infty\varphi(s)^p\,ds=\int_0^{|\Omega|}(f^*)^p(t)\,dt=\int_\Omega|f(u)|^p\,du.$$

Similarly

$$\int_\Omega e^{\beta U(x)^{p'}}\,dx=\int_0^{|\Omega|}e^{\beta U^*(t)^{p'}}\,dt.$$

But $\int_0^\infty\varphi(s)^p\,ds\le 1$ implies that $\int_0^\infty e^{-F(t)}\,dt\le C_0$, where

$$F(t)=t-\left(\int_0^\infty a(s,t)\varphi(s)\,ds\right)^{Q'},$$
$$a(s,t)=\begin{cases}1, & s<t\\ pe^{(t-s)/p'}, & t<s<\infty\\ 0, & -\infty<s\le 0.\end{cases}$$

We note, using the change of variables $|\Omega|e^{-s} = \lambda$ in the third equality below, that

$$\begin{aligned}
&\int_0^\infty e^{-F(t)}\,dt \\
&\quad= \int_0^\infty e^{-t} e^{(\int_0^\infty a(s,t)\varphi(s)\,ds)^{Q'}}\,dt \\
&\quad= \int_0^\infty e^{-t} e^{(\int_0^t \varphi(s)\,ds + p\int_t^\infty e^{(t-s)/p'}\varphi(s)\,ds)^{Q'}}\,dt \\
&\quad= \int_0^\infty e^{-t} e^{(\int_{|\Omega|}^{|\Omega|e^{-t}} f^*(\lambda)\lambda^{-1/p'}\,d\lambda + p\int_{|\Omega|e^{-t}}^0 |\Omega|^{-1/p'} f^*(\lambda)e^{t/p'}\,d\lambda e^{(t-s)/p'})^{Q'}}\,dt.
\end{aligned}$$

Using the change of variables $|\Omega|e^{-t} = \mu$ in the last expression above and combining with the result of Corollary 3.3 will complete the proof of the first statement of the theorem. ◻

Proof of the second part of Theorem 3.4. To prove the second statement, for $r > 0$ and $\delta > 0$ define the regions

$$\Omega_{r,\delta} = \{u : |u| < rg(u^*)^{1/(Q-\alpha)} \text{ and } \delta < g(u^*) < \delta^{-1}\}$$

and

$$A_{r,\delta} = \{u : rg(u^*)^{1/(Q-\alpha)} < |u| < g(u^*)^{1/(Q-\alpha)} \text{ and } \delta < g(u^*) < \delta^{-1}\}.$$

In addition, let $\Sigma_\delta = \{u : |u| = 1 \text{ and } \delta < g(u^*) < \delta^{-1}\}$. Define test functions $f_{r,\delta}$ by setting $f_{r,\delta}(u) = h_{r,\delta}(u^{-1})$ where

$$h_{r,\delta}(u) = \left(\log\left(\frac{1}{r}\right)\int_{\Sigma_\delta} g(u^*)^{p'}\,d\mu\right)^{-1} g(u)^{p'-1}$$

if u is in the region $A_{r,\delta}$ and setting $f_{r,\delta} = 0$ otherwise. Thus $f_{r,\delta}$ is supported in the region $\Omega_{1,\delta}^{-1} = \{u : u^{-1} \in \Omega_{1,\delta}\}$.

Claim. *We claim that, given $\varepsilon > 0$, there is an $r_0 > 0$ such that for all $0 < r < r_0$ the potential $f_{r,\delta} * g(v) \geq 1 - \varepsilon$, for all v in $\Omega_{r,\delta}^{-1}$.*

It is the proof of this claim which requires the introduction of the parameter δ and we will delay proving the claim until later. Accepting the claim, we first make the change of variable from u to u^{-1} to see that

$$\|f_{r,\delta}\|_p = \left(\log\left(\frac{1}{r}\right)\int_{\Sigma_\delta} g(u^*)^{p'}\,d\mu\right)^{-1}\left(\int_{A_{r,\delta}} |g(u)|^{p'}\,du\right)^{1/p}$$

and use polar coordinates to evaluate the integral as

$$\begin{aligned}\int_{A_{r,\delta}} |g(u)|^{p'}\,du &= \int_{\Sigma_\delta}\int_{rg(u^*)^{1/(Q-\alpha)}}^{g(u^*)^{1/(Q-\alpha)}} g(u^*)^{p'}\rho^{p'(\alpha-Q)}\rho^{Q-1}\,d\rho\,d\mu\\ &= \int_{\Sigma_\delta}\int_{rg(u^*)^{1/(Q-\alpha)}}^{g(u^*)^{1/(Q-\alpha)}} g(u^*)^{p'}\rho^{-1}\,d\rho\,d\mu\\ &= \log\left(\frac{1}{r}\right)\int_{\Sigma_\delta} g(u^*)^{p'}\,d\mu.\end{aligned}$$

Thus

$$\|f_{r,\delta}\|_p^{p'} = \left(\log\left(\frac{1}{r}\right)\int_{\Sigma_\delta} g(u^*)^{p'}\,d\mu\right)^{-1}.$$

Polar coordinates also show that $|\Omega_{r,\delta}^{-1}| = r^Q|\Omega_{1,\delta}^{-1}|$. Suppose that constants C_0 and β exist such that for all regions Ω, $|\Omega| < \infty$, and functions f supported in Ω

$$\frac{1}{|\Omega|}\int_\Omega \exp\left(\beta\left(\frac{f*g}{\|f\|_p}\right)^{p'}\right)du \le C_0.$$

Let $\Omega = \Omega_{1,\delta}^{-1}$ and $f = f_{r,\delta}$. Assuming the claim above and noting that $f_{r,\delta} * g(v) \ge 1-\varepsilon$ on $\Omega_{r,\delta}^{-1}$ and

$$\frac{1}{|\Omega_{1,\delta}^{-1}|}\int_{\Omega_{r,\delta}^{-1}} \exp\left(\beta\left(\frac{f*g}{\|f\|_p}\right)^{p'}\right)du \le \frac{1}{|\Omega_{1,\delta}^{-1}|}\int_{\Omega_{1,\delta}^{-1}} \exp\left(\beta\left(\frac{f*g}{\|f\|_p}\right)^{p'}\right)du,$$

it follows that

$$\frac{|\Omega_{r,\delta}^{-1}|}{|\Omega_{1,\delta}^{-1}|}\exp\left(\beta(1-\varepsilon)^{p'}\log\left(\frac{1}{r}\right)\int_{\Sigma_\delta} g^{p'}\,d\mu\right) \le C_0,$$

that is

$$\exp\left(Q\log\left(\frac{1}{r}\right)\left(-1+\beta(1-\varepsilon)^{p'}\frac{1}{Q}\int_{\Sigma_\delta} g^{p'}\,d\mu\right)\right) \le C_0$$

for all $0 < r < r_0$. It follows that

$$-1+\beta(1-\varepsilon)^{p'}\frac{1}{Q}\int_{\Sigma_\delta} g^{p'}\,d\mu \le 0.$$

Therefore

$$\beta \le (1-\varepsilon)^{-p'}Q\left(\int_{\Sigma_\delta} g^{p'}\,d\mu\right)^{-1}.$$

If we first let ε go to 0 and then let δ go to 0 we obtain the desired inequality.

It remains to prove the claim.

For this, suppose that $v \in \Omega_{r,\delta}$. Making the change of variable from u to u^{-1}, we may write

$$f_{r,\delta} * g(v^{-1}) = \left(\log\left(\frac{1}{r}\right)\int_{\Sigma_\delta} g^{p'}\,d\mu\right)^{-1}\int_{A_{r,\delta}} g(u)^{p'-1}g(uv^{-1})\,du.$$

Note that $uv^{-1} = (\rho u^*)v^{-1} = \rho(u^*(\rho^{-1}v^{-1}))$ and by the homogeneity of g, $g(uv^{-1}) = \rho^{(\alpha-Q)}g(u^*(\rho^{-1}v^{-1}))$. Keeping this in mind, we may use polar coordinates to calculate that

$$\begin{aligned}\int_{A_{r,\delta}} & g(u)^{p'-1}g(uv^{-1})\,du \\ &= \int_{\Sigma_\delta}\int_{rg(u^*)^{1/(Q-\alpha)}}^{g(u^*)^{1/(Q-\alpha)}} \rho^{p'(\alpha-Q)}g(u^*)^{p'-1}g(u^*(\rho^{-1}v^{-1}))\rho^{Q-1}\,d\rho\,d\mu \\ &= \int_{\Sigma_\delta}\int_{rg(u^*)^{1/(Q-\alpha)}}^{g(u^*)^{1/(Q-\alpha)}} g(u^*)^{p'-1}g(u^*(\rho^{-1}v^{-1}))\rho^{-1}\,d\rho\,d\mu.\end{aligned}$$

We change variables on the inside integral by letting $s = \rho^{-1}|v|$ and obtain the formula

$$\begin{aligned}\int_{A_{r,\delta}} & g(u)^{p'-1}g(uv^{-1})\,du \\ &= \int_{\Sigma_\delta}\int_{|v|g(u^*)^{1/(\alpha-Q)}}^{|v|r^{-1}g(u^*)^{1/(\alpha-Q)}} g(u^*)^{p'-1}g(u^*(sv^*)^{-1}))s^{-1}\,ds\,d\mu.\end{aligned}$$

Observe that since v is restricted to the region $\Omega_{r,\delta}$ it follows that

$$|v|r^{-1} \le g(v^*)^{1/(Q-\alpha)} \le \delta^{-1/(Q-\alpha)},$$

and since u^* is restricted to the set Σ_δ it follows that the upper limit of integration in the inside integral is uniformly bounded by $\delta^{-2/(Q-\alpha)}$. Note that the bound is independent of u^* and r. Using this observation, we write $g(u^*(sv^*)^{-1}) = g(u^*) + (g(u^*(sv^*)^{-1}) - g(u^*))$ and conclude that, by condition (3.1), for all $v \in \Omega_{r,\delta}$

$$\int_{A_{r,\delta}} g(u)^{p'-1}g(uv^{-1})\,du = \left(\log\left(\frac{1}{r}\right)\right)\int_{\Sigma_\delta} g(u^*)^{p'}\,d\mu + B,$$

where B is uniformly bounded on $\Omega_{r,\delta}$ by a constant which is independent of r. It follows that, given $\varepsilon > 0$, we may choose r sufficiently close to 0 such that

$$\begin{aligned}f_{r,\delta} * g(v^{-1}) &= \left(\log\left(\frac{1}{r}\right)\int_{\Sigma_\delta} g^{p'}\,d\mu\right)^{-1}\int_{A_{r,\delta}} g(u)^{p'-1}g(uv^{-1})\,du \\ &\ge (1-\varepsilon)\end{aligned}$$

for all $v \in \Omega_{r,\delta}$. This concludes the proof. $\square$

4. Proof of the Moser-Trudinger inequality for Sobolev functions on $\mathbb{H}^n$: Theorem 1.1

Theorem 4.1. *There exists a constant C_0 depending only on Q such that for any $f \in W_0^{1,Q}(\Omega)$, $|\Omega| < \infty$, then the following holds:*

$$\frac{1}{|\Omega|} \int_\Omega \exp(A_Q |f(u)|^{Q'})\, du \le C_0,$$

provided $\|\nabla_{\mathbb{H}^n} f\|_{L^Q} \le 1$, where

$$A_Q = Q\Big(\int_{|u|=1} |z^*|^Q \, d\mu \Big)^{Q'-1}.$$

Furthermore, no greater number than A_Q can replace this.

Proof of the Moser-Trudinger inequality with the constant A_Q. It is enough to prove that the inequality holds with the constant A_Q for all functions $f \in C_0^\infty(\Omega)$. By Theorem 1.2 with $\beta = Q$, and $f \in C_0^\infty(\Omega)$,

$$|f(v)| \le (c_Q)^{-1} |\nabla_{\mathbb{H}^n} f| * g(v),$$

where $g(u) = |z|^{Q-1}/|u|^{2Q-2}$ and $c_Q = \int_\Sigma |z^*|^Q \, d\mu$. From Theorem 1.3 it follows that there is a constant C_0 such that if $\|\nabla_{\mathbb{H}^n} f\|_{L^Q} \le 1$ then

$$\frac{1}{|\Omega|} \int_\Omega \exp(A |f(u)|^{Q'})\, du \le C_0$$

provided

$$A(c_Q)^{-Q'} \le A(g, Q) = \frac{Q}{\int_\Sigma |z^*|^Q \, d\mu}.$$

This proves the first statement of the theorem. $\square$

Proof of the sharpness of the constant A_Q. In order to show this A_Q is sharp, we look at the following example. Let

$$f_r(u) = \begin{cases} (\log(r^{-1}))^{-1} \log(|u|^{-1}) & \text{for } r \le |u| \le 1, \\ f_r(u) = 1 & \text{for } 0 \le |u| \le r. \end{cases}$$

By Lemma 2.1

$$|\nabla_{\mathbb{H}^n} f_r|(z,t) = \begin{cases} (\log(r^{-1}))^{-1} |z|\, |(z,t)|^{-2} & \text{for } r \le |u| \le 1. \\ 0 & \text{for } 0 \le |u| \le r. \end{cases}$$

Thus,

$$||\nabla_{\mathbb{H}^n} f_r||_Q^{Q'} = \left(\int_{r \le |u| \le 1} \left(\left(\log\left(\frac{1}{r}\right) \right)^{-1} |z|\,|u|^{-2} \right)^Q \right)^{Q'-1} du$$

and using polar coordinates we compute that

$$\int_{r \le |u| \le 1} \left(|z|\,|u|^{-2} \right)^Q du = \int_\Sigma \int_r^1 |z^*|^Q \rho^{-1}\, d\rho\, d\mu = \log\left(\frac{1}{r}\right) \int_\Sigma |z^*|^Q\, d\mu.$$

Therefore

$$||\nabla_{\mathbb{H}^n} f_r||_Q^{Q'} = \left(\log\left(\frac{1}{r}\right) \right)^{-1} (c_Q)^{Q'-1}.$$

Now assume that

$$\frac{1}{|\Omega|} \int_\Omega \exp\left(\beta \left(\frac{|f|}{\|\nabla_{\mathbb{H}^n} f\|_Q} \right)^{Q'} \right) \le C_0$$

for some $\beta > 0$. Take $\Omega = B = B_1(0) = \{u : |u| < 1\}$ and $B_r = \{u : |u| < r\}$. Using the above selected function f_r, it follows that

$$\frac{|B_r|}{|B|} \exp\left(\frac{\beta}{\|\nabla_{\mathbb{H}^n} f_r\|_Q^{Q'}} \right) \le C_0$$

and thus

$$\beta \le \log\left(C_0 \frac{|B|}{|B_r|} \right) ||\nabla_{\mathbb{H}^n} f_r||_Q^{Q'}.$$

Therefore,

$$\beta \le \left(\log C_0 + \log\left(\frac{|B|}{|B_r|} \right) \right) ||\nabla_{\mathbb{H}^n} f_r||_Q^{Q'}.$$

Note that since $\|\nabla_{\mathbb{H}^n} f_r\|_Q \to 0$ as $r \to 0$ for the above selected f_r, it follows that

$$\beta \le \lim \left(Q \log\left(\frac{1}{r} \right) \right) ||\nabla_{\mathbb{H}^n} f_r||_Q^{Q'}.$$

Thus, $\beta \le Q(c_Q)^{Q'-1}$.

This concludes the argument. □

Remark 4.1. If we use the representation formula with $\beta = 2$, then in the part of *Proof of the existence of the constant* A_Q we will get the constant A in place of A_Q with $A = Q(c_2)^{Q'}(c_{Q'})^{-1}$. By Hölder's inequality, $c_2 < c_Q^{1/Q} c_{Q'}^{1/Q'}$, from which it follows that A is strictly less than A_Q. Thus, if we use the representation formula $\beta = 2$ we will not be able to get the sharp constant A_Q for the Moser-Trudinger inequality for Sobolev functions.

Added in July, 2001. Results of this paper have been generalized to groups of Heisenberg type introduced in [K] by Cohn and Lu [CoL]. In particular, we derived in [CoL] the explicit fundamental solutions for a class of degenerate (or singular) one-parameter subelliptic differential operators on groups of Heisenberg (H) type. This extends the result of Kaplan [K] for sub-Laplacian on H-type groups, which in turn generalizes Folland's result on the Heisenberg group. As an application, we obtain a one-parameter representation formula for Sobolev functions of compact support on H-type groups. By choosing the parameter equal to the homogeneous dimension Q and using the Moser-Trudinger inequality for convolutional type operator on stratified groups obtained in the current paper (namely, Theorem 1.3 here) we get in [CoL] the following theorem which gives the best constant for the Moser-Trudinger inequality for Sobolev functions on H-type groups:

Let $\mathbb{G}$ be any group of Heisenberg type whose Lie algebra is generated by m left invariant vector fields and with q-dimensional center. Let $Q = m + 2q$, $Q' = Q/(Q-1)$ and

$$A_Q = Q\left[\left(\frac{1}{4}\right)^{q-1/2} \frac{\pi^{(q+m)/2}\Gamma[(Q+m)/4]}{Q\Gamma(m/2)\Gamma(Q/2)}\right]^{1/(Q-1)}.$$

Then,

$$\sup_{F\in W_0^{1,Q}(\Omega)} \left\{\frac{1}{|\Omega|}\int_\Omega \exp\left(A_Q\left(\frac{F(u)}{\|\nabla_{\mathbb{G}}F\|_Q}\right)^{Q'}\right)du\right\} < \infty$$

with A_Q as the sharp constant, where $\nabla_{\mathbb{G}}$ denotes the subelliptic gradient on $\mathbb{G}$.

Motivated by a preprint of the present paper, Blogh, Manfredi and Tyson [BMT] have derived a representation formula corresponding to ours (Theorem 1.2) in the special case $\beta = Q$ in the setting of Carnot groups (i.e., stratified groups) by using a homogeneous norm defined by fundamental solutions to the Q-sub-Laplacian on such groups. Combining their representation formula with our sharp constant for the Moser-Trudinger inequality for operators of convolutional type on Carnot groups (i.e., our Theorem 1.3) as we did in [CoL] and also here, they are able to find the best constant for the Moser-Trudinger inequality for Sobolev functions on Carnot groups in terms of this fundamental solution. Their constant can be calculated explicitly for the Heisenberg group and groups of Heisenberg type and coincides with the constants obtained in this paper and also [CoL] (see also a calculation of the constant on the H-type groups in [BT]). It is, however, still not known how to calculate this constant explicitly for general Carnot groups.

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