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Asymptotic Behavior of Radial Solutions for a Class of Semilinear Elliptic Equations

Shaohua Chen

Department of Mathematics and Statistics, Simon Fraser University, Burnaby, British Columbia, Canada, V5A 1S6

and

Guozhen Lu*

Department of Mathematics and Statistics, Wright State University, Dayton, Ohio 45435

1. INTRODUCTION

The main purpose of this paper is to investigate the asymptotic behavior of the oscillatory solutions to the semilinear equation of the form

$$\Delta u + f(|x|, u) = 0 \qquad \text{in } R^n, \qquad n \geqslant 3. \tag{1}$$

Equation (1) has its origin from, e.g., the prescribed curvature problems in Riemannian geometry, and astrophysics (i.e., the Lane–Emden–Fowler equation and the Matukuma equation as special cases). The asymptotic behavior of the positive solutions to (1) has recently received much attention, see e.g., [L1, LN, Na].

However, it is well-known that the above equation (1) does not always have positive solutions or positive radial solutions. In other words, under suitable conditions, radial solutions to (1) must oscillate about the zero at infinite times (see e.g., [DCC, NY, N]). Thus it becomes very interesting to know as precisely as possible the asympototic behaviors of the oscillating periods, the amplitudes of the oscillatory solutions.

In this paper, we restrict our attention to the study of radial solutions to (1). More precisely, we shall discuss the asymptotic behavior of oscillatory

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radial solutions for the following singular initial value problem for the semilinear elliptic equation

$$u'' + \frac{n-1}{r} u' + (br^{\mu} |u|^{q-1} + r^{\nu} |u|^{p-1}) u = 0, \qquad u(0) = a \neq 0, \qquad u'(0) = 0, \tag{2}$$

where $b > 0$, $n \geqslant 3$, $\mu > -2$, $\nu > -2$, $1 < q < (n+2+2\mu)/(n-2)$ and

$$p = \frac{n+2+2\nu}{n-2} = 1 + \frac{4+2\nu}{n-2} > 1. \tag{3}$$

Ni and Yotsutani [NY] have considered the equation

$$u'' + \frac{n-1}{r} u' + \sum_{i=1}^{k} c_i r^{l_i} (u^+)^{p_i} = 0, \qquad u(0) = u_0 > 0, \qquad u'(0) = 0, \tag{4}$$

where $p_i > 1$, $l_i > -2$, $c_i > 0$, $(1 \leqslant i \leqslant k)$ and $u^+ = \max[u(r), 0]$. They concluded for (4) that:

(a) if $p_i \leqslant (n+2+2l_i)/(n-2)$, for $1 \leqslant i \leqslant k$, with at least one inequality being a strict one, then $u(r)$ has a finite zero for every u_0.

(b) if $p_i \geqslant (n+2+2l_i)/(n-2)$, for $1 \leqslant i \leqslant k$, then $u(r)$ is a positive solution for every u_0.

Nishihara [N] dealt with the second order differential equation

$$y''(s) + f(s) |y(s)|^{p-1} y(s) = 0,$$

which is equivalent to

$$u''(r) + \frac{n-1}{r} u'(r) + k(r) |u(r)|^{p-1} u(r) = 0,$$

by the transformation $u(r) = y(s)$ and $s = (n-2)^{n-2} r^{-(n-2)}$. Under suitable assumptions on $f(s)$, Nishihara proved that the solution $y(s)$ is nonoscillatory.

Derrick *et al.* [DCC] recently discussed the following equation when the restoring function $f(|x|, u)$ is independent of the radial variable $|x|$:

$$u'' + \frac{n-1}{r} u' + (b |u|^{q-1} + |u|^{p-1}) u = 0, \qquad u(0) \neq 0, \qquad u'(0) = 0, \tag{5}$$

$1<q<p=(n+2)/(n-2)$. They proved that

$$\lim_{j\to\infty} t_j^{(2n-2)/(q+3)} |u(t_j)| = c \qquad \text{and} \qquad \frac{r_{j+1}-r_j}{r_{j+1}^{((n-1)(q-1))/(q+3)}} \geqslant c,$$

where c is a positive constant, t_j and r_j are the jth local extremum and zero of the solution $u(r)$, respectively. This shows that the period between two consecutive zeros $r_{j+1}-r_j$ has a lower bound of the order $r_{j+1}^{((n-1)(q-1))/(q+3)}$ as r_j goes to ∞.

This paper generalizes and improves the results of [DCC] in several ways. First of all, we allow the restoring function $f(|x|, u)$ to depend on $|x|$ rather than to be a function of u only. Secondly, we provide an exact asymptotic behavior of the period between any two consecutive zeros r_j and r_{j+1}. Namely, we also show an upper bound of the period, which is of the same order as the lower bound. The precise statement is given in Theorem 2 below. We also derive in this paper the precise asymptotic estimate for the energy associated to the equation (2) defined by (8) which is of independent interest, see Theorem 1 for details.

The paper is organized as follows: In Section 2, we include a proof that shows the nonexistence of eventually positive or negative solutions (see Lemma 1). Section 3 is devoted to the statement and proofs of the main theorems. The proof of Theorem 1, which shows that any solution of (2) has the decay property ($|u(r)| \to 0$), together with Lemma 1, conclude that any solution of (2) must oscillate around the zero infinitely many times. This observation is considerably simpler than the proof given in [DCC].

2. PRELIMINARY

We will often use the following Pokhozhaev's second identity (see [Po] and also [KNY]):

$$\begin{aligned} &r^{k+1}\left[\frac{1}{2}u'^2(r)+F(r,u(r))\right]+\alpha r^k u'(r)\,u(r)+\frac{\alpha}{2}(n-1-k)\,r^{k-1}u^2(r)\\ &\quad+\left(n-1-\frac{k+1}{2}-\alpha\right)\int_{r_0}^{r} u'^2(s)\,s^k\,ds\\ &\quad-\alpha\frac{(n-1-k)(k-1)}{2}\int_{r_0}^{r} u^2(s)\,s^{k-2}\,ds\\ &\quad+\int_{r_0}^{r}[\alpha u(s)\,f(s,u(s))-(k+1)\,F(s,u(s))-sF_s(s,u(s))]\,s^k\,ds=c_0 \qquad (6)\end{aligned}$$

for any k and $r_0 > 0$, where

$$c_0 = r_0^{k+1} \left[\frac{1}{2} u'^2(r_0) + F(r_0, u(r_0)) \right] + \alpha r_0^k u'(r_0)\, u(r_0)$$

$$+ \frac{\alpha}{2} (n-1-k)\, r_0^{k-1} u^2(r_0)$$

and

$$F(r, u) = \int_0^u f(r, v)\, dv,$$

and $u(r)$ is a solution of

$$u''(r) + \frac{n-1}{r} u'(r) + f(r, u) = 0. \tag{7}$$

Let $u(r)$ be a solution of (2) and define the energy function

$$Q(r, u(r)) = \frac{1}{2} u'^2(r) + F(r, u(r)) = \frac{1}{2} u'^2(r) + \frac{br^{\mu}\, |u(r)|^{q+1}}{q+1} + \frac{r^{\nu}\, |u(r)|^{p+1}}{p+1}. \tag{8}$$

Suppose that the solution $u(r)$ of (2) oscillates about the zero a finite number of times and has a local maximum at r_1 for which $u(r) > 0$ for all $r > r_1$. We call such solutions eventually positive solutions.

LEMMA 1. *The problem* (2) *has no eventually positive or negative solutions.*

Remark 1. Ni and Yotsutani [NY] proved that problem (2) has a finite number of zeros for every $a > 0$. The techniques of [NY] can be adapted to prove our Lemma 1. However, we provide here an alternate proof for the sake of completeness.

Proof of Lemma 1. Since the case for eventually negative solutions follows trivially by setting $v = -u$, we only prove that there is no eventually positive solutions. Suppose that the conclusion were not true, then there would exist some $r_1 \geqslant 0$ such that $u'(r_1) = 0$ and $u(r) > 0$ for all $r \geqslant r_1$.

Note

$$u'(r) = -\frac{1}{r^{n-1}} \int_{r_1}^{r} [bs^{\mu} u^q(s) + s^{\nu} u^p(s)]\, s^{n-1}\, ds < 0 \qquad \text{for} \quad r > r_1, \tag{9}$$

which implies that $u(r)$ is decreasing for $r > r_1$. Then

$$u'(r) \leqslant -\frac{b}{r^{n-1}}\, u^q(r) \int_{r_1}^{r} s^{n-1+\mu}\, ds = -\frac{b}{r^{n-1}} \cdot \frac{u^q(r)}{n+\mu}\,(r^{n+\mu} - r_1^{n+\mu}),$$

or

$$\frac{du(r)}{u^q(r)} \leqslant -\frac{b}{n+\mu}\,(r^{\mu+1} - r_1^{n+\mu} r^{1-n})\, dr.$$

Integrating from r_1 to r we get

$$\frac{1}{q-1}\left(\frac{1}{u^{q-1}(r_1)} - \frac{1}{u^{q-1}(r)}\right)$$
$$\leqslant -\frac{b}{n+\mu}\left(\frac{1}{\mu+2}\, r^{\mu+2} + \frac{1}{n-2}\, r_1^{n+\mu} r^{2-n} - \frac{n+\mu}{(n-2)(\mu+2)}\, r_1^{\mu+2}\right),$$

or

$$\frac{1}{u^{q-1}(r_1)} + \frac{b(q-1)}{n+\mu}\left(\frac{1}{\mu+2}\, r^{\mu+2} + \frac{1}{n-2}\, r_1^{n+\mu} r^{2-n} - \frac{n+\mu}{(n-2)(\mu+2)}\, r_1^{\mu+2}\right)$$
$$\leqslant \frac{1}{u^{q+1}(r)}.$$

Thus

$$u(r) \leqslant c r^{-((\mu+2)/(q-1))} \tag{10}$$

for $r > r_2$ with sufficiently large $r_2 \geqslant r_1$.

Inserting (10) into (9) we obtain

$$|u'(r)| \leqslant \frac{c}{r^{n-1}} + \frac{c}{r^{n-1}} \int_{r_1}^{r} s^{n-1+\mu-(((\mu+2)\, q)/(q-1))}\, ds$$
$$+ \int_{r_1}^{r} s^{n-1+\nu-((\mu(\ +2)\, p/()q-1))}\, ds$$
$$\leqslant c \cdot \max(r^{-((\mu+2)/(q-1))-1},\ r^{\nu+1-(((\mu+2)\, p)/(q-1))},\ r^{-(n-1)}). \tag{11}$$

Applying (6) with $k = n-1$, $r_0 = 0$ and $\alpha = (n-2)/2$ we get

$$r^n\left(\frac{1}{2}\, u'^2(r) + F(r, u(r))\right) + \frac{n-2}{2}\, r^{n-1} u'(r)\, u(r)$$
$$= \frac{n+2+2\mu-(n-2)\, q}{2(q+1)} \int_0^r |u(s)|^{q+1}\, s^{n-1+\mu}\, ds \tag{12}$$

Using (10) and (11), we derive

$$r^n\left(\frac{1}{2}u'^2(r)+\frac{br^{\mu}\,|u(r)|^{q+1}}{q+1}\right)$$
$$\leqslant c\cdot\max\left(r^{-((n+2+2\mu-(n-2)\,q)/(q-1))},\ r^{n+2(\nu+1-(((\mu+2)\,p)/(q-1)))},\ r^{-(n-2)}\right)$$

and

$$r^{n+\nu}\frac{|u(r)|^{p+1}}{p+1}\leqslant cr^{-((n+2+2\mu-(n-2)\,q)/((q-1)(n-2)))\cdot(n+\nu)}.$$

Letting $r\to\infty$ in (12) we get a contradiction, since the left side of (12) is nonpositive, while the right side is a positive number. Thus Lemma 1 follows.

3. MAIN THEOREMS AND THEIR PROOFS

By Lemma 1 and the first part of the proof of Theorem 1 below, the solutions of (2) must oscillate about zero infinitely many times. This can be seen by the decay property (22), which is true for any solution of (2). Then we may let r_j be the zeros of $u(r)$ and t_j be the local extremum points of $u(r)$ with $r_j<t_j<r_{j+1}$. Now we will study the amplitudes and the periods of the oscillations.

THEOREM 1. *There exists a positive constant c_1 such that*

$$\lim_{r\to\infty} r^{(2(n-1)(q+1)-2\mu)/(q+3)}Q(r,u(r))=c_1. \tag{13}$$

Consequently,

$$\lim_{j\to\infty} t_j^{(2n-2+\mu)/(q+3)}\,|u(t_j)|=\left[\frac{(q+1)\,c_1}{b}\right]^{1/(q+1)}$$

and

$$\lim_{j\to\infty} r_j^{((n-1)(q+1)-\mu)/(q+3)}\,|u'(r_j)|=\sqrt{2c_1}. \tag{14}$$

Proof. Applying (6) with $\alpha=(2n-2+\mu)/(q+3)$ and

$$k=\frac{2(n-1)(q+1)-2\mu}{q+3}-1=n-1-\frac{n+2+2\mu-(n-2)\,q}{q+3}<n-1, \tag{15}$$

we have

$$r^{k+1}(\tfrac{1}{2}u'^2(r)+F(r,u(r)))+\alpha r^k u'(r)\,u(r)+c_2 r^{k-1}u^2(r)$$
$$+c_3\int_{r_0}^{r}|u(s)|^{p+1}\,s^{k+v}\,ds+c_3'\int_{r_0}^{r}s^{\mu}\,|u|^{q+1}\,ds$$
$$=c_4\int_{r_0}^{r}u^2(s)\,s^{k-2}\,ds+c_0, \tag{16}$$

or

$$\frac{1}{4}r^{k+1}u'^2(r)+r^{k+1}\left(\frac{b}{q+1}r^{\mu}\,|u(r)|^{q+1}+\frac{1}{p+1}r^{v}\,|u(r)|^{p+1}\right)$$
$$+\frac{1}{4}[ru'(r)+2\alpha u(r)]^2+c_3\int_{r_0}^{r}|u(s)|^{p+1}\,s^{k+v}\,ds$$
$$+c_3'\int_{r_0}^{r}s^{k+\mu}\,|u|^{q+1}\,ds$$
$$=c_4\int_{r_0}^{r}u^2(s)\,s^{k-2}\,ds+c_0+(\alpha^2-c_2)\,r^{k-1}u^2(r), \tag{17}$$

where c_0 is defined as in (6),

$$c_3'=b\left(\alpha-\frac{k+1}{q+1}-\frac{\mu}{q+1}\right)=0,$$
$$c_2=\frac{(2n-2+\mu)(n+2+2\mu-(n-2)\,q)}{2(q+3)^2},\qquad c_4=c_2(k-1),$$

and

$$c_3=\alpha-\frac{k+1}{p+1}-\frac{v}{p+1}=\frac{2(n-1)(p-q)+\mu(p+3)-v(q+3)}{(q+3)(p+1)}.$$

Since

$$2(n-1)(p-q)+\mu(p+3)-v(q+3)-2[n+2+2\mu-(n-2)\,q]$$
$$=[2(n-1)+\mu]\,p-2(n-1)\,q+3\mu-vq-3v-2(n+2)-4\mu+2q(n-2)$$
$$=[2(n-1)+\mu]\,\frac{n+2+2v}{n-2}-\mu-2q-vq-3v-2(n+2)$$
$$=(2+v)\,\frac{n+2+2\mu-(n-2)\,q}{n-2}>0, \tag{18}$$

we then get $c_3 > 0$. By Holder's inequality with two conjugate exponents $(p+1)/2$ and $(p+1)/(p-1)$ we have

$$|c_4|\, u^2(s)\, s^{k-2} = (u^2(s)\, s^{(2(k+v))/(p+1)})(|c_4|\, s^{k-2-((2(k+v))/(p+1))})$$
$$\leqslant c_3\, |u(s)|^{p+1}\, s^{k+v} + cs^{\beta}, \tag{19}$$

where

$$\beta = \left(k - 2 - \frac{2k+2v}{p+1}\right) \cdot \frac{p+1}{p-1} = k - 2 - \frac{4+2v}{p-1}$$
$$= k - n + \frac{(n-2)\, p - n - 2 - 2v}{p-1} = k - n < -1$$

by (15). Using (19), we also get

$$|\alpha^2 - c_2|\, r^{k-1} u^2(r) = |\alpha^2 - c_2|\, r \cdot [u^2(r)\, r^{k-2}]$$
$$\leqslant \frac{1}{2(p+1)}\, |u(r)|^{p+1}\, r^{k+1+v} + cr^{\beta+1}. \tag{20}$$

Substituting (19) and (20) into (17) yields

$$\frac{1}{4} r^{k+1} u'^2(r) + r^{k+1} \left[\frac{b}{q+1}\, r^{\mu}\, |u(r)|^{q+1} + \frac{1}{p+1}\, r^{v}\, |u(r)|^{p+1}\right]$$
$$+ c_3 \int_{r_0}^{r} |u(s)|^{p+1}\, s^{k+v}\, ds$$
$$\leqslant c_3 \int_{r_0}^{r} |u(s)|^{p+1}\, s^{k+v}\, ds + c \int_{r_0}^{r} s^{\beta}\, ds$$
$$+ \frac{1}{2(p+1)}\, r^{k+1+v}\, |u(r)|^{p+1} + cr^{\beta+1} + c_0. \tag{21}$$

Thus we have simultaneously proved that

$$r^{k+1} Q(r, u(r)) \leqslant c, \qquad |u(r)| \leqslant cr^{-((2n-2+\mu)/(q+3))}$$

and

$$|u'(r)| \leqslant cr^{-(((n-1)(q+1)-\mu)/(q+3))}. \tag{22}$$

Consequently we have from (22)

$$|\alpha r^k u(r)\, u'(r)| \leqslant c r^{((2(n-1)(q+1)-2\mu)/(q+3))-1} \cdot r^{-((2n-2+\mu)/(q+3))}$$
$$\times r^{-(((n-1)(q+1)-\mu)/(q+3))}$$
$$= c r^{-((n+2+2\mu-(n-2)\,q)/(q+3))}, \tag{23}$$
$$r^{k-1} u^2(r) \leqslant c r^{((2(n-1)(q+1)-2\mu)/(q+3))-2} \cdot r^{-((2(2n-2+\mu)/(q+3))}$$
$$= c r^{-((2[n+2+2\mu-(n-2)\,q])/(q+3))}, \tag{24}$$

Letting $r \to \infty$ in (16) and using (23) and (24) we derive

$$\lim_{r \to \infty} r^{k+1} Q(r, u(r)) = c_4 \int_{r_0}^{\infty} u^2(s)\, s^{k-2}\, ds - c_3 \int_{r_0}^{\infty} |u(s)|^{p+1}\, s^{k+v}\, ds + c_0. \tag{25}$$

Now we claim that

$$\lim_{r \to \infty} r^{k+1} Q(r, u(r)) = c_1 > 0. \tag{26}$$

Suppose otherwise, then $c_1 = 0$, since $Q(r, u(r)) > 0$ for all $r > 0$. Set

$$\lambda = \frac{n+2+2\mu-(n-2)\,q}{q+3} - \varepsilon_0 > 2(q+1)\,\varepsilon_0 > 0$$

for small ε_0. By L'Hospital's rule, (22)–(24) and (18) we will get

$$\left| \lim_{r \to \infty} \frac{c_4 \int_{r_0}^{r} u^2(s)\, s^{k-2}\, ds - c_3 \int_{r_0}^{r} |u(s)|^{p+1}\, s^{k+v}\, ds + c_0}{r^{-\lambda}} \right|$$
$$\leqslant \frac{c}{\lambda} \Big(\lim_{r \to \infty} u^2(r)\, r^{k-1+\lambda} + \lim_{r \to \infty} |u(r)|^{p+1}\, r^{k+1+\lambda+v} \Big)$$
$$\leqslant \frac{c}{\lambda} \Big(\lim_{r \to \infty} r^{\lambda - ((2(n+2+2\mu-(n-2)\,q)/(q+3))}$$
$$+ \lim_{r \to \infty} r^{\lambda - ((2(n-1)(p-q)+(p+3)\,\mu - v(q+3))/(q+3))} \Big)$$
$$= 0. \tag{27}$$

Thus, multiplying (16) by r^{λ} and using (23), (24) and (27) we get

$$\lim_{r \to \infty} r^{k+1+\lambda} Q(r, u(r)) = 0$$

or

$$|u(r)| \leqslant cr^{-((k+1+\lambda+\mu)/(q+1))} \leqslant cr^{-((2n-2+\mu)/(q+3))-(\lambda/(q+1))}. \tag{28}$$

Consequently, we have by (28), (22) and (18),

$$\begin{aligned} |\alpha r^k u(r)\, u'(r)| &\leqslant cr^{-((n+2+2\mu-(n-2)\,q)/(q+3))-(\lambda/(q+1))} \\ &\leqslant cr^{-((2(n+2+2\mu-(n-2)\,q))/(q+3))-2\varepsilon_0} \end{aligned} \tag{29}$$

and

$$\begin{aligned} |u(r)|^{p+1}\, r^{k+1+\nu} &\leqslant cr^{-((2(n-1)(p-q)+(p+3)\mu-\nu(q+3))/(q+3))-((\lambda(p+1))/(q+1))} \\ &\leqslant cr^{-((n+2+2\mu-(n-2)\,q)/(q+3))-2\varepsilon_0}. \end{aligned} \tag{30}$$

We can repeat the process in (27) by using

$$\lambda_1 = \frac{n+2+2\mu-(n-2)\,q}{q+3} + \varepsilon_0$$

to obtain

$$\lim_{r\to\infty} r^{k+1+\lambda_1} Q(r, u(r)) = 0. \tag{31}$$

On the other hand, using (12) with $r = r_j$ and $u(r_j) = 0$, we get

$$r_j^n\, Q(r_j, u(r_j)) = \frac{n+2+2\mu-(n-2)\,q}{2(q+1)} \int_0^{r_j} |u(s)|^{q+1}\, s^{n-1+\mu}\, ds > 0$$

which contradicts (31) since $k+1+\lambda_1 = n+\varepsilon_0$. Hence $c_1 > 0$ in (26) and the proof is complete.

THEOREM 2. *There exist two positive constants* $0 < c_2 < c_3$ *such that*

$$c_2 \leqslant \frac{r_{j+1} - r_j}{r_{j+1}^{((n+1)(q-1)-2\mu)/(q+3)}} \leqslant c_3. \tag{32}$$

Proof. Since $u'(t_j) = 0$ and $u(r)$ does not change sign in $[t_j, r_{j+1}]$, we have by (14), (9) and (10) for j large enough,

$$\begin{aligned}\sqrt{c_1} &\leqslant r_{j+1}^{((n-1)(q+1)-\mu)/(q+3)} |u'(r_{j+1})| \\ &= r_{j+1}^{-((2(n-1)+\mu)/(q+3))} \int_{t_j}^{r_{j+1}} (bs^{\mu} |u(s)|^q + s^{\nu} |u(s)|^p) s^{n-1} \, ds \\ &\leqslant c r_{j+1}^{-((2(n-1)+\mu)/(q+3))} \int_{t_j}^{r_{j+1}} s^{n-1+\mu-(((2n-1+\mu)q)/(q+3))} \, ds \\ &\leqslant c r_{j+1}^{-((2(n-1)+\mu)/(q+3))} \int_{t_j}^{r_{j+1}} s^{((n-1)(3-q)+3\mu)/(q+3)q+3))} \, ds. \qquad (33)\end{aligned}$$

If $(n-1)(3-q)+3\mu \geqslant 0$, then the integrand is monotone increasing so that

$$\begin{aligned}\sqrt{c_1} &\leqslant c r_{j+1}^{-(((n-1)(q-1)-2\mu)/(q+3))} \int_{t_j}^{r_{j+1}} ds \\ &\leqslant c r_{j+1}^{-(((n-1)(q-1)-2\mu)/(q+3))} (r_{j+1} - r_j).\end{aligned}$$

If $(n-1)(3-q)+3\mu < 0$, since

$$\frac{(n-1)(3-q)+3\mu}{q+3} + 1 = \frac{3n-(n-2)\,q+3\mu}{q+3} > 0$$

we can calculate (33) directly and get

$$\begin{aligned}\sqrt{c_1} &\leqslant \frac{c(q+3)}{3n-q(n-2)+3\mu} r_{j+1}^{-((2(n-1)+\mu)/(q+3))} \\ &\quad \times (r_{j+1}^{(((n-1)(3-q)+3\mu)/(q+3))+1} - t_j^{(((n-1)(3-q)+3\mu)/(q+3))+1}) \\ &= c r_{j+1}^{-(((n-1)(q-1)-2\mu)/(q+3))} (r_{j+1} - r_{j+1}^{-(((n-1)(3-q)+3\mu)/(q+3))} \\ &\quad \times t_j^{(((n-1)(3-q)+3\mu)/(q+3))+1}) \\ &\leqslant c r_{j+1}^{-(((n-1)(q-1)-2\mu)/(q+3))} (r_{j+1} - t_j) \\ &\leqslant c r_{j+1}^{-(((n-1)(q-1)-2\mu)/(q+3))} (r_{j+1} - r_j). \qquad (34)\end{aligned}$$

Thus we have proved the first inequality in (32).

Now let us prove the second inequality in (32). First suppose $u(t_{2j}) > 0$ for $j = 1, 2, \ldots$. Let $s_{2j} \in (t_{2j}, r_{2j+1})$ such that

$$u'^2(s_{2j}) = 2F(s_{2j}, u(s_{2j})) \qquad \text{and} \qquad u'^2(r) > 2F(r, u(r)), \qquad (35)$$

for $r \in (s_{2j}, r_{2j+1})$. Note that such s_{2j} must exist because if we set

$$g(r) = u'^2(r) - 2F(r, u(r)) = u'^2(r) - 2\left(br^{\mu} \frac{|u(r)|^{q+1}}{q+1} + r^{\nu} \frac{|u(r)|^{p+1}}{p+1}\right),$$

then $g(t_{2j}) = -2F(t_{2j}, u(t_{2j})) < 0$ and $g(r_{2j+1}) = u'^2(r_{2j+1}) > 0$. Thus

$$s_{2j} = \sup_{r \in (t_{2j},\, r_{2j+1})} \{r \mid u'^2(r) = 2F(r, u(r))\}.$$

Using (13) and (35) we have

$$s_{2j}^{k+1} Q((s_{2j}, u(s_{2j})) = 2s_{2j}^{k+1} F(s_{2j}, u(s_{2j})) \to c_1 \qquad \text{as} \quad j \to \infty,$$

which implies that

$$s_{2j}^{(2(n-1)+\mu)/(q+3)} u(s_{2j}) \to \left(\frac{(q+1)\, c_1}{2b}\right)^{1/(q+1)} = c_0 \qquad \text{as} \quad j \to \infty. \quad (36)$$

Using (13) and (35) again, we get

$$\frac{c_1}{2} < r^{k+1} Q(r, u(r)) \leqslant r^{k+1} u'^2(r),$$

for $r \in (s_{2j}, r_{2j+1})$ and j large enough, which implies that (notice $u'(r) < 0$ in (t_{2j}, r_{2j+1}))

$$r^{((n-1)(q+1)-\mu)/(q+3)}(-u'(r)) = r^{(k+1)/2}(-u'(r)) \geqslant \sqrt{\frac{c_1}{2}}. \quad (37)$$

Multiplying both sides of (37) by $r^{-((n-1)(q-1)-2\mu)/(q+3)}$ and adding a similar term to both sides, gives

$$-\frac{2(n-1)+\mu}{q+3} u(r)\, r^{((2(n-1)+\mu)/(q+3))-1} - u'(r)\, r^{(2(n-1)+\mu)/(q+3)}$$
$$\geqslant \sqrt{\frac{c_1}{2}}\, r^{-((n-1)(q-1)-2\mu)/(q+3)} - \frac{2(n-1)+\mu}{q+3} u(r)\, r^{((2(n-1)+\mu)/(q+3))-1},$$

or

$$-\frac{d}{dr}\left(u(r)\, r^{(2(n-1)+\mu)/(q+3)}\right)$$
$$\geqslant r^{-(((n-1)(q-1)-2\mu)/(q+3))}\left(\sqrt{\frac{c_1}{2}} - \frac{2(n-1)+\mu}{q+3} \right.$$
$$\left. \times u(r)\, r^{(2(n-1)+\mu)/(q+3)} \cdot r^{-((n+2-(n-2)\, q+2\mu)/(q+3))}\right). \quad (38)$$

Since $u(r)\, r^{(2(n-1)+\mu)/(q+3)}$ is bounded by (22) and $r^{-((n+2+2\mu-(n-2)\,q)/(q+3))} \to 0$ as $r \to \infty$, we can assume that

$$\left(\sqrt{\frac{c_1}{2}} - \frac{2(n-1)+\mu}{q+3}\, u(r)\, r^{(2(n-1)+\mu)/(q+3)} \times r^{-((n+2+2\mu-(n-2)\,q)/(q+3))}\right) \geqslant \frac{\sqrt{c_1}}{2},$$

for $r \in (s_{2j}, r_{2j+1})$ and j sufficiently large. Thus (38) becomes

$$-\frac{d}{dr}\left(u(r)\, r^{(2(n-1)+\mu)/(q+3)}\right) \geqslant \frac{\sqrt{c_1}}{2}\, r^{-(((n-1)(q-1)-2\mu)/(q+3))} \tag{39}$$

Integrating (39) from s_{2j} to r_{2j+1} yields, by (36),

$$2c_0 \geqslant s_{2j}^{(2(n-1)+\mu)/(q+3)}\, u(s_{2j}) \geqslant \begin{cases} \frac{\sqrt{c_1}}{2}\, r_{2j+1}^{-(((n-1)(q-1)-2\mu)/(q+3))}(r_{2j+1}-s_{2j}), & \text{if} \quad (n-1)(q-1) \geqslant 2\mu; \\ \frac{\sqrt{c_1}}{2}\, s_{2j}^{-(((n-1)(q-1)-2\mu)/(q+3))}(r_{2j+1}-s_{2j}), & \text{if} \quad (n-1)(q-1) < 2\mu. \end{cases} \tag{40}$$

Noticing that $u(r)$ is a decreasing function on (t_{2j}, r_{2j+1}), it follows that by (35) and (9),

$$\begin{aligned} 2\sqrt{b}\, s_{2j}^{\mu/2} u^{(q+1)/2}(s_{2j}) &\geqslant \sqrt{2F(s_{2j}, u(s_{2j}))} \\ &= -u'(s_{2j}) \geqslant s_{2j}^{-(n-1)} \int_{t_{2j}}^{s_{2j}} bs^{\mu} u^{q}(s)\, s^{n-1}\, ds \\ &\geqslant bs_{2j}^{-(n-1)} u^{q}(s_{2j}) \int_{t_{2j}}^{s_{2j}} s^{n-1+\mu}\, ds \\ &= bs_{2j}^{\mu} u^{q}(s_{2j}) \int_{t_{2j}}^{s_{2j}} ds = bs_{2j}^{\mu} u^{q}(s_{2j})(s_{2j}-t_{2j}), \end{aligned}$$

This shows that

$$2 \geqslant \sqrt{b}\, s_{2j}^{\mu/2} u^{(q-1)/2}(s_{2j})(s_{2j}-t_{2j}).$$

Then we get by (36)

$$
\begin{aligned}
2 &\geqslant \sqrt{b}\, s_{2j}^{\mu/2} u^{(q-1)/2}(s_{2j})(s_{2j}-t_{2j}) \\
&\geqslant \frac{\sqrt{b}}{2}\, c_0^{(q-1)/2} s_{2j}^{-((2(n-1)+\mu)/(q+3))\cdot((q-1)/2)+(\mu/2)}(s_{2j}-t_{2j}) \\
&\times \begin{cases} \geqslant \dfrac{\sqrt{b}}{2}\, c_0^{(q-1)/2} r_{2j+1}^{-(((n-1)(q-1)-2\mu)/(q+3))}(s_{2j}-t_{2j}) \\ \quad \text{if} \quad (n-1)(q-1) \geqslant 2\mu; \\ = \dfrac{\sqrt{b}}{2}\, c_0^{(q-1)/2} s_{2j}^{-(((n-1)(q-1)-2\mu)/(q+3))}(s_{2j}-t_{2j}) \\ \quad \text{if} \quad (n-1)(q-1) < 2\mu \end{cases}
\end{aligned} \tag{41}
$$

If $(n-1)(q-1)<2\mu$ then by (40) and (41), $r_{2j+1}-t_{2j}\to 0$ as $j\to\infty$, which implies

$$
s_{2j}^{-(((n-1)(q-1)-2\mu)/(q+3))} \geqslant \tfrac{1}{2} r_{2j+1}^{-(((n-1)(q-1)-2\mu)/(q+3))}, \tag{42}
$$

for j sufficiently large.

Adding (40) and (41) together, using (42) in the case $(n-1)(q-1)<2\mu$, we thus have proved the second inequality of (32) in the interval (t_{2j}, r_{2j+1}), i.e., replacing r_{2j} by t_{2j} in (32).

We still need to prove the similar inequality in the interval $[r_{2j}, t_{2j}]$. Notice that

$$
u'(t_{2j}) = -\frac{1}{t_{2j}^{n-1}} \int_{t_{2j-1}}^{t_{2j}} f(s, u(s))\, s^{n-1}\, ds = 0,
$$

we have

$$
u'(r) = -\frac{1}{r^{n-1}} \int_{t_{2j-1}}^{r} f(s, u(s))\, s^{n-1}\, ds = \frac{1}{r^{n-1}} \int_{r}^{t_{2j}} f(s, u(s))\, s^{n-1}\, ds,
$$

for $r\in(r_{2j}, t_{2j})$. Then we can obtain the second inequality of (32) in the interval (r_{2j}, t_{2j}) by similar arguments. By adding these two inequalities we thus have proved the second inequality of (32) when $u(t_{2j})>0$. Similarly we can prove the result when $u(t_{2j+1})<0$.

Remark 2. From the conclusion of Theorem 2, we notice the very interesting phenomena, that is, if $(n-1)(q-1)-2\mu>0$ then oscillatory period of the solution becomes longer and longer while if $(n-1)(q-1)-2\mu<0$ it becomes shorter and shorter. Especially, when $(n-1)(q-1)-2\mu=0$, it changes between two constants. For $n=3$, $\mu=q-1$, the behaviors of solutions of (2) are very similar to that of the following linear equation

$$u'' + \frac{2}{r} u' + Au = 0, \qquad u(0) = a \neq 0, \qquad u'(0) = 0.$$

The solution is $u = (a/\sqrt{A}\, r) \sin \sqrt{A}\, r$ because the order of the amplitude of two solutions is the same (all are $1/r$). We note the zeros of the solution are $r_j = (j\pi)/\sqrt{A}$ and thus $r_{j+1} - r_j = \pi/\sqrt{A}$, which is a constant for all j.

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