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An overdetermined problem in Riesz-potential and fractional Laplacian

Guozhen Lu *, Jiuyi Zhu

Department of Mathematics, Wayne State University, Detroit, MI 48202, USA

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ABSTRACT

The main purpose of this paper is to address two open questions raised by Reichel (2009) in [2] on characterizations of balls in terms of the Riesz potential and fractional Laplacian. For a bounded C^1 domain $\Omega \subset \mathbb{R}^N$, we consider the Riesz-potential

$$u(x) = \int_{\Omega} \frac{1}{|\, x-y \,|^{N-\alpha}}\, dy$$

for $2 \le \alpha \ne N$. We show that u = constant on $\partial\Omega$ if and only if Ω is a ball. In the case of $\alpha = N$, the similar characterization is established for the logarithmic potential $u(x) = \int_{\Omega} \log \frac{1}{|x-y|}\, dy$. We also prove that such a characterization holds for the logarithmic Riesz potential

$$u(x) = \int_{\Omega} |\, x-y \,|^{\alpha-N} \log \frac{1}{|\, x-y \,|}\, dy$$

when the diameter of the domain Ω is less than $e^{\frac{1}{N-\alpha}}$ in the case when $\alpha - N$ is a nonnegative even integer. This provides a characterization for the overdetermined problem of the fractional Laplacian. These results answer two open questions in Reichel (2009) [2] to some extent. Moreover, we also establish some nonexistence result of positive solutions to a class of integral equations in an exterior domain.

1. Introduction

It is well-known that the gravitational potential of a ball of constant mass density is constant on the surface of the ball. It is shown by Fraenkel [1] that this property indeed provides a characterization of balls. In fact, Fraenkel proves the following theorem.

Theorem A *([1]). Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and ω_N be the surface measure of the unit sphere in $\mathbb{R}^N$. Consider*

$$u(x) = \begin{cases} \dfrac{1}{2\pi} \displaystyle\int_{\Omega} \log \frac{1}{|x-y|}\, dy, & N = 2, \\ \dfrac{1}{(N-2)\omega_N} \displaystyle\int_{\Omega} \frac{1}{|x-y|^{N-2}}\, dy, & N \ge 3. \end{cases} \tag{1.1}$$

If $u(x)$ is constant on $\partial\Omega$, then Ω is a ball.

* Corresponding author.
E-mail addresses: gzlu@math.wayne.edu, gzlu2001@gmail.com (G. Lu), jiuyi.zhu@wayne.edu (J. Zhu).

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This result has been extended by Reichel [2] to more general Riesz potential, but under a more restrictive assumption on the domain Ω, i.e., Ω is assumed to be convex. In [2], Reichel considers the integral equation

$$u(x) = \begin{cases} \int_{\Omega} \log \frac{1}{|x-y|}\, dy, & N = \alpha, \\ \int_{\Omega} \frac{1}{|x-y|^{N-\alpha}}\, dy, & N \neq \alpha, \end{cases} \tag{1.2}$$

and proves the following theorem.

Theorem B *([2]). Let $\Omega \subset \mathbb{R}^N$ be a bounded convex domain and $\alpha > 2$, if $u(x)$ is constant on $\partial\Omega$, then Ω is a ball.*

This more general Riesz potential is actually closely related to the fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}}$ in $\mathbb{R}^N$. Let $\mathbb{N}_0$ be the collection of nonnegative integers. It is known that the fundamental solution $G(x, y)$ for pseudo-differential operator $(-\Delta)^{\frac{\alpha}{2}}$ in $\mathbb{R}^N$ has the following representation

$$G(x,y) = \begin{cases} \dfrac{\Gamma\left(\frac{N-\alpha}{2}\right)}{2^{\alpha}\pi^{\frac{N}{2}}\Gamma\left(\frac{\alpha}{2}\right)} |x-y|^{\alpha-N}, & \text{if } \dfrac{\alpha-N}{2} \notin \mathbb{N}_0, \\ \dfrac{(-1)^k}{2^{\alpha-1}\pi^{\frac{N}{2}}\Gamma\left(\frac{\alpha}{2}\right)} |x-y|^{\alpha-N} \log \dfrac{1}{|x-y|}, & \text{if } \dfrac{\alpha-N}{2} \in \mathbb{N}_0. \end{cases} \tag{1.3}$$

We note that for the case of $\alpha = 2$, Fraenkel's result is under weaker assumption on the domain Ω, namely, Ω only needs to be bounded and open in $\mathbb{R}^N$. The surprising part for $\alpha = 2$ is that there is neither regularity nor convexity requirement for Ω. Thus, two open problems were raised by Reichel in [2].

Question 1. Is Theorem B true if we remove the convexity assumption of Ω?

Question 2. Is there an analogous result as Theorem B for Riesz-potential of the form

$$u(x) = \int_{\Omega} |x-y|^{\alpha-N} \log \frac{1}{|x-y|}\, dy? \tag{1.4}$$

It is meaningful to study (1.4) because in the case of $\frac{\alpha-N}{2} \in \mathbb{N}_0$, up to some rescaling, the kernel function in above integral is the fundamental solution of the fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}}$.

Our first goal in this paper is to address the above two open questions.

The first result we establish does remove the convexity assumption in Theorem B.

Theorem 1. *Assume that $\alpha > 1$. Let Ω be a C^1 bounded domain. If u in (1.2) is constant on $\partial\Omega$, then Ω is a ball.*

As far as Question 2 is concerned, we partially solve it under some additional assumption on the diameter of the domain Ω. Since we are only interested in the case when $\alpha > N$, we will assume this when we address Question 2.

Theorem 2. *Assume $\alpha > N$. Let Ω be a C^1 bounded domain with* diam $\Omega < e^{\frac{1}{N-\alpha}}$. *Thus, Ω is a ball if $u(x)$ in (1.4) is constant on $\partial\Omega$.*

Remark 1.1. In the above two theorems, if the conclusion that Ω is a ball is verified, then we can easily deduce that $u(x)$ is radially symmetric with respect to the center of the ball.

Concerning our Theorem 2, a natural open question is raised here.

Question. Is the assumption that the diameter of Ω satisfies diam $\Omega < e^{\frac{1}{N-\alpha}}$ necessary to conclude our characterization?

Our second goal of this paper is to study a general integral equation with Riesz-potential over an exterior domain. Set $G = \mathbb{R}^N \setminus \Omega_1$, where Ω_1 is a bounded and connected C^1 domain. The integral equation to our interest is as follows:

$$u(x) = \int_G \frac{f(u)}{|x-y|^{N-\alpha}}\, dy. \tag{1.5}$$

We will actually show the nonexistence of any positive solution to Eq. (1.5). Indeed, we establish the following theorem.

Theorem 3. *Let $1 < \alpha < N$. Assume that the positive solution $u \in L^q(G)$ for some $q > \frac{N}{N-\alpha}$ and $f(u)$ satisfies*

(i) *$f(u)$ is continuous, increasing and $f(0) = 0$;*
(ii) *$f'(u)$ is non-increasing and $\frac{f(u)}{u} \in L^{r+1}(G) \cap L^p(G)$ for some $r > \frac{N}{\alpha}$ and some $1 < p < \frac{N}{\alpha}$.*

Then there does not exist any positive solution u to (1.5) such that u is constant on the boundary of Ω_1.

Remark 1.2. Based on the assumption of (i) and (ii), we can infer that $f'(u) \in L^{\frac{N}{\alpha}}(G)$.

Remark 1.3. We note that we do not assume any regularity on the function u. As a matter of fact, we will be able to show that $u \in C^1(\overline{G})$ under the assumptions of Theorem 3 by using the regularity lifting method.

Remark 1.4. Since it is impossible to have any positive solution u in (1.5) under the assumptions of Theorem 3, the only nonnegative solution is the trivial one.

Remark 1.5. It is interesting to note that the method we employed here to prove Theorem 3 is to prove first that a nonnegative solution to (1.5) must be symmetric using the moving plane method. Then we prove such a symmetric solution must be trivial.

There has been extensive study in the literature about overdetermined problems in elliptic differential equations and integral equations. In his seminal paper [3], Serrin showed that the overdetermined boundary value determines the geometry of the underlying set. This is, if Ω is a bounded C^2 domain and $u \in C^2(\bar{\Omega})$ satisfies the following

$$\begin{cases} \Delta u = -1 & \text{in } \Omega, \\ u = 0, \quad \dfrac{\partial u}{\partial n} = \text{constant} & \text{on } \Omega, \end{cases} \tag{1.6}$$

then Ω is a ball and u is radially symmetric with respect to its center of the ball. Serrin's proof is based on what is nowadays called the moving planes method relying on the maximum principle of solutions to the differential equations, which is originally due to Alexandrov, and has been later used to derive further symmetry results for more general elliptic equations. Important progress as for the moving plane methods since then are the works of Gidas et al. [4], Caffarelli et al. [5], to just name some of the early works in this direction.

Immediately after Serrin's paper, Weinberger [6] obtained a very short proof of the same result, using the maximum principle applied to an auxiliary function. However, compared to Serrin's approach, Weinberger's proof relies crucially on the linearity of the Laplace operator.

Since the work of [3], many results are obtained about overdetermined problems. The interested reader may refer to [7–30] and the references therein, for more general elliptic equations. See also [31] and the reference therein for overdetermined problems in an exterior domain or general domain. In [10], an alternative shorter proof of Serrin's result, not relying explicitly on the maximum principle has been given, where they deduce some global information concerning the geometry of the solution.

Overdetermined problems are important from the point of view of mathematical physics. Many models in fluid mechanics, solid mechanics, thermodynamics, and electrostatics are relevant to the overdetermined Dirichlet or Neumann boundary problems of elliptic partial differential equations. We refer the reader to article [14] for a nice introduction in that aspect.

Instead of a volume potential, single layer potential is also considered in overdetermined problems. A single layer potential is given by

$$u(x) = \begin{cases} A \displaystyle\int_{\partial\Omega} \frac{-1}{2\pi} \log \frac{1}{|x-y|}\, d\sigma_y, & N = 2, \\ A \displaystyle\int_{\partial\Omega} \frac{1}{(N-2)\omega_N} \frac{1}{|x-y|^{N-2}}\, d\sigma_y, & N \geq 3, \end{cases} \tag{1.7}$$

where $A > 0$ is the constant source density on the boundary of the domain Ω. If u is constant in $\bar{\Omega}$, then Ω can be proved to be a ball under different smoothness assumption on the domain Ω. See [23] for the case of $n = 2$ and [31] for the case of $n \geq 3$, and also some related works in [21,28]. We also refer the reader to the book of Kenig [32] on this subject of layer potential.

Generally speaking, two approaches are widely applied in dealing with overdetermined problems. One is the classical moving plane method. In [3], the moving plane method with a sophisticated version of Hopf boundary maximum principle plays a very important role in the proof. The other way is based on an equality of Rellich type, as well as an interior maximum principle; see [6]. Our approach is a new variant of moving plane method—moving plane in integral forms. It is much different from the traditional methods of moving planes used for partial differential equations. Instead of relying on the differentiability and maximum principles of the structure, a global integral norm is estimated. The method of moving planes in integral forms can be adapted to obtain symmetry and monotonicity for solutions. The method of moving planes on integral equations was developed in the work of Chen et al. [33]; see also [34] for the moving sphere method in integral forms, the book [35] and an exhaustive list of references therein, where the symmetry of solutions in the entire space was proved. The moving plane method in integral form over bounded domains has also been carried out in [36].

We end this introduction with the following remark concerning the characterization of balls by using the Bessel potential. The Bessel kernel g_α in $\mathbb{R}^N$ with $\alpha \geq 0$ is defined by

$$g_\alpha(x) = \frac{1}{r(\alpha)} \int_0^\infty \exp\left(-\frac{\pi}{\delta}|x|^2\right) \exp\left(-\frac{\delta}{4\pi}\right) \delta^{\frac{\alpha-N-2}{2}}\, d\delta, \tag{1.8}$$

where $r(\alpha) = (4\pi)^{\frac{\alpha}{2}} \Gamma\left(\frac{\alpha}{2}\right)$.

We consider the Bessel potential type equation:

$$u(x) = \int_{\Omega} g_{\alpha}(x-y)\,dy. \tag{1.9}$$

Overdetermined problems for Bessel potential over a bounded domain in $\mathbb{R}^N$ can be studied. For instance, among other results, the following theorem has been established in [37]:

Theorem 4. *Let Ω be a C^1 bounded domain in $\mathbb{R}^N$. If u in (1.9) is constant on $\partial\Omega$, then Ω is a ball.*

It is well-known that (1.9) is closely related to the following fractional equation

$$(I-\Delta)^{\frac{\alpha}{2}} u = \chi_{\Omega}.$$

In the case of $\alpha = 2$, it turns out to be the ground state of the Schrödinger equation.

It turns out that the Riesz and Bessel potentials are just examples of more general potentials which can be used to give characterizations of balls. In fact, we can establish characterizations of balls using more general potentials. We refer the interested reader to Appendix of this paper where a precise statement of such characterization is given together with assumptions on the more general potentials. Therefore, Theorem 4 can be viewed as a special case of the more general theorem (Theorem A.1) in Appendix. We have chosen to present our paper by placing the more general theorem in Appendix because our primary goal is to address the two open questions raised by Reichel [2]. Finally, we mention that this paper is a revised version of our earlier article with the same title posted in the arXiv (arXiv:1101.1649). In this version, we have reformulated the symmetry result of nonnegative solutions to the integral equation (1.5) in the earlier version into the nonexistence result Theorem 3.

The paper is organized as follows. In Section 2, we show Theorem 1. In Section 3, we carry out the proof of Theorem 2. Section 4 deals with the nonexistence of any positive solution to the integral equation (1.5) over any exterior domain. Namely, we give the proof of Theorem 3. Appendix provides a characterization of balls using a more general potential. Throughout this paper, the positive constant C is frequently used in the paper. It may differ from line to line, even within the same line. It may also depend on u in some cases.

2. Proof of Theorem 1

In this section, we will prove Theorem 1 by adapting the moving plane method in integral forms; see [33]. Since we are dealing with the case of bounded domains, we modify the method accordingly (see also [36,38]).

We first introduce some notations. Choose any direction and, rotate coordinate system if it is necessary such that x_1-axis is parallel to it. For any $\lambda \in \mathbb{R}$, define

$$T_\lambda = \{(x_1, \ldots, x_n) \in \Omega | x_1 = \lambda\}.$$

Since Ω is bounded, if λ is sufficiently negative, the intersection of T_λ and Ω is empty. Then, we move the plane T_λ all the way to the right until it intersects Ω. Let

$$\lambda_0 = \min\{\lambda : T_\lambda \cap \bar{\Omega} \neq \emptyset\}.$$

For $\lambda > \lambda_0$, T_λ cuts off Ω. We define

$$\Sigma_\lambda = \{x \in \Omega | x_1 < \lambda\}.$$

Set

$$x_\lambda = \{2\lambda - x_1, \ldots, x_n\}$$

and

$$\Sigma'_\lambda = \{x_\lambda \in \Omega | x \in \Sigma_\lambda\}.$$

At the beginning of $\lambda > \lambda_0$, Σ'_λ remains within Ω. As the plane keeps moving to the right, Σ'_λ will still stay in Ω until at least one of the following events occurs:

(i) Σ'_λ is internally tangent to the boundary of Ω at some point P_λ not on T_λ.
(ii) T_λ reaches a position where it is orthogonal to the boundary of Ω at some point Q.

Let $\bar{\lambda}$ be the first value such that at least one of the above positions is reached.

We assert that Ω must be symmetric about $T_{\bar{\lambda}}$, i.e.,

$$\Sigma_{\bar{\lambda}} \cup T_{\bar{\lambda}} \cup \Sigma'_{\bar{\lambda}} = \Omega. \tag{2.1}$$

If this assertion is verified, for any given direction in $\mathbb{R}^N$, there also exists a plane $T_{\bar{\lambda}}$ such that Ω is symmetric about $T_{\bar{\lambda}}$. Moreover, Ω is connected. Then the only domain with those properties is a ball; see [39].

In order to assert (2.1), we introduce

$$u_\lambda(x) = u(x_\lambda),$$
$$\Omega_\lambda = \Omega \setminus (\overline{\Sigma_\lambda \cup \Sigma'_\lambda}).$$

We first establish some lemmas. Throughout the paper, we assume $\alpha > 1$.

Lemma 2.1. *Let $l \in \mathbb{N}$ with $1 \le l < \alpha$. Then for any solution in (1.2), $u \in C^l(\mathbb{R}^N)$ and differentiation of order l can be taken under the integral.*

Proof. The proof is standard. We refer the reader to [2]. $\square$

Lemma 2.2. *For $\lambda_0 < \lambda < \bar{\lambda}$ and $u(x)$ satisfying (1.2), we have the following.*

(i) *If $N \ge \alpha$, $u_\lambda(x) > u(x)$ for any $x \in \Sigma_\lambda$.*
(ii) *If $N < \alpha$, $u_\lambda(x) < u(x)$ for any $x \in \Sigma_\lambda$.*

Proof. For $x \in \Sigma_\lambda$, in the case of $N = \alpha$, we rewrite $u(x)$ and $u_\lambda(x)$ as

$$u(x) = \int_{\Sigma_\lambda} \log \frac{1}{|x-y|}\, dy + \int_{\Sigma_\lambda} \log \frac{1}{|x_\lambda - y|}\, dy + \int_{\Omega_\lambda} \log \frac{1}{|x-y|}\, dy,$$

and

$$u_\lambda(x) = \int_{\Sigma_\lambda} \log \frac{1}{|x_\lambda-y|}\, dy + \int_{\Sigma_\lambda} \log \frac{1}{|x - y|}\, dy + \int_{\Omega_\lambda} \log \frac{1}{|x_\lambda-y|}\, dy.$$

Then

$$u_\lambda(x) - u(x) = \int_{\Omega_\lambda} \log \frac{|x-y|}{|x_\lambda - y|}\, dy. \tag{2.2}$$

Since $|x-y| > |x_\lambda - y|$ for $x \in \Sigma_\lambda$ and $y \in \Omega_\lambda$, then

$$u_\lambda(x) > u(x).$$

While in the case of $N \neq \alpha$, $u_\lambda(x)$ and $u(x)$ have the following representations respectively:

$$u(x) = \int_{\Sigma_\lambda} |x-y|^{\alpha-N}\, dy + \int_{\Sigma_\lambda} |x_\lambda - y|^{\alpha-N}\, dy + \int_{\Omega_\lambda} |x-y|^{\alpha-N}\, dy,$$

and

$$u_\lambda(x) = \int_{\Sigma_\lambda} |x_\lambda-y|^{\alpha-N}\, dy + \int_{\Sigma_\lambda} |x - y|^{\alpha-N}\, dy + \int_{\Omega_\lambda} |x_\lambda-y|^{\alpha-N}\, dy.$$

Thus,

$$u_\lambda(x) - u(x) = \int_{\Omega_\lambda} (|x_\lambda - y|^{\alpha-N} - |x-y|^{\alpha-N})\, dy, \tag{2.3}$$

Note that $|x-y| > |x_\lambda - y|$ for $x \in \Sigma_\lambda$ and $y \in \Omega_\lambda$. Thus, (i) and (ii) are concluded. $\square$

Lemma 2.3. *Assume that $u(x)$ satisfies (1.2) and suppose $\lambda = \bar{\lambda}$ in the first case, i.e. $\Sigma'_{\bar{\lambda}}$ is internally tangent to the boundary of Ω at some point $P_{\bar{\lambda}}$ not on $T_{\bar{\lambda}}$, then $\Sigma_{\bar{\lambda}} \cup T_{\bar{\lambda}} \cup \Sigma'_{\bar{\lambda}} = \Omega$.*

Proof. When $N \ge \alpha$, thanks to Lemma 2.1, $u_{\bar{\lambda}}(x) \ge u(x)$ for $x \in \Sigma_{\bar{\lambda}}$. While $N < \alpha$, $u_{\bar{\lambda}}(x) \le u(x)$ for $x \in \Sigma_{\bar{\lambda}}$. We argue by contradiction. Suppose $\Sigma_{\bar{\lambda}} \cup T_{\bar{\lambda}} \cup \Sigma'_{\bar{\lambda}} \subsetneqq \Omega$; that is, $\Omega_{\bar{\lambda}} \neq \emptyset$. At $P_{\bar{\lambda}}$, from (2.2) and (2.3), $u(P_{\bar{\lambda}}) > u(P)$ in the case of $N \ge \alpha$. It is a contradiction since $P_{\bar{\lambda}},\ P \in \partial\Omega$ and $u(P_{\bar{\lambda}}) = u(P) = \text{constant}$. From the same reason, $u(P_{\bar{\lambda}}) < u(P)$ when $N < \alpha$. It also contradicts the fact that u is constant on the boundary. Therefore, the lemma is completed. $\square$

Lemma 2.4. *Assume that $u(x)$ satisfies (1.2) and suppose that the second case occurs, i.e. $T_{\bar{\lambda}}$ reaches a position where is orthogonal to the boundary of Ω at some point Q, then, $\Sigma_{\bar{\lambda}} \cup T_{\bar{\lambda}} \cup \Sigma'_{\bar{\lambda}} = \Omega$.*

Proof. Since $u(x)$ is constant on the boundary and $\Omega \in C^1$, ∇u is parallel to the normal at Q. As implied in the second case, $\frac{\partial u}{\partial x_1}\Big|_Q = 0$. We denote the coordinate of Q by z. Suppose $\Omega_{\bar\lambda} \neq \emptyset$, there exists a ball $B \subset\subset \Omega_{\bar\lambda}$. Choose a sequence $\{x^i\}_1^\infty \in \Sigma_{\bar\lambda} \setminus T_{\bar\lambda}$ such that $x^i \to z$ as $i \to \infty$. It is easy to see that $x^i_{\bar\lambda} \to z$ as $i \to \infty$. Since $B \subset\subset \Omega_{\bar\lambda}$, we can also find a δ such that diam $\Omega > |x^i_{\bar\lambda} - y| > \delta$ for any $y \in B$ and any $x^i_{\bar\lambda}$.

If $N = \alpha$, by (2.2),

$$u(x^i_{\bar\lambda}) - u(x^i) = \int_{\Omega_{\bar\lambda}} \log \frac{|x^i - y|}{|x^i_{\bar\lambda} - y|}\, dy.$$

Let $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^N$, then $(x^i_{\bar\lambda} - x^i) \cdot e_1$ is the first component of $(x^i_{\bar\lambda} - x^i)$. By the Mean Value Theorem,

$$\begin{aligned}\frac{u(x^i_{\bar\lambda}) - u(x)}{(x^i_{\bar\lambda} - x^i)\cdot e_1} &= \int_{\Omega_{\bar\lambda}} \frac{\log|x^i - y| - \log|x^i_{\bar\lambda} - y|}{(x^i_{\bar\lambda} - x^i)\cdot e_1}\, dy \\ &= \int_{\Omega_{\bar\lambda}} \frac{(y - \bar{x}^i_{\bar\lambda})\cdot e_1}{|y - \bar{x}^i_{\bar\lambda}|^2}\, dy \\ &> C\int_B \frac{1}{|\text{diam }\Omega|^2}\, dy \\ &> C, \end{aligned} \tag{2.4}$$

where $\bar{x}^i_{\bar\lambda}$ is some point between $x^i_{\bar\lambda}$ and x^i. Nevertheless,

$$\lim_{i\to\infty} \frac{u(x^i_{\bar\lambda}) - u(x^i)}{(x^i_{\bar\lambda} - x^i)\cdot e_1} = \frac{\partial u}{\partial x_1}\Big|_Q = 0,$$

which contradicts (2.4). Therefore, $\Omega_{\bar\lambda} = \emptyset$.

In the case of $N > \alpha$, similarly we have

$$\begin{aligned}\frac{u(x^i_{\bar\lambda}) - u(x^i)}{(x^i_{\bar\lambda} - x^i)\cdot e_1} &= \int_{\Omega_{\bar\lambda}} \frac{|x^i_{\bar\lambda} - y|^{\alpha-N} - |x^i - y|^{\alpha-N}}{(x^i_{\bar\lambda} - x^i)\cdot e_1}\, dy \\ &= \int_{\Omega_{\bar\lambda}} (\alpha - N)|\bar{x}^i_{\bar\lambda} - y|^{\alpha-N-2}((x^i_{\bar\lambda} - y)\cdot e_1)\, dy \\ &> \int_B (\alpha - N)|\bar{x}^i_{\bar\lambda} - y|^{\alpha-N-2}((x^i_{\bar\lambda} - y)\cdot e_1)\, dy \\ &> C. \end{aligned} \tag{2.5}$$

It also contradicts $\frac{\partial u}{\partial x_1}\Big|_Q = 0$, thus $\Omega_{\bar\lambda} = \emptyset$.

The same idea can be applied to the case of $N < \alpha$ with minor modification. In conclusion, $\Sigma_{\bar\lambda} \cup T_{\bar\lambda} \cup \Sigma'_{\bar\lambda} = \Omega$ when the second case occurs. □

Combining Lemmas 2.3 and 2.4, Theorem 1 is implied.

3. Proof of Theorem 2

In this section, we will prove Theorem 2 under some restriction on the diameter of Ω. Since we are mainly interested in the case of $\frac{\alpha-N}{2} \in \mathbb{N}_0$. This is the case when the fundamental solution of $(-\Delta)^{\frac{\alpha}{2}}$ has representation (1.3). Therefore, we will assume $\alpha > N$ in this section. Obviously, $u \in C^1(\mathbb{R}^N)$ in (1.4). We begin with establishing several lemmas.

Lemma 3.1. *For $\lambda_0 < \lambda < \bar\lambda$, assume $u(x)$ satisfies (1.4) with* diam $\Omega < e^{\frac{1}{N-\alpha}}$*; then $u_\lambda(x) < u(x)$ for any $x \in \Sigma_\lambda$.*

Proof. Since $|x_\lambda - y_\lambda| = |x - y|$, and $|x_\lambda - y| = |x - y_\lambda|$, we write $u(x)$ and $u_\lambda(x)$ in the following forms:

$$u(x) = \int_{\Sigma_\lambda} |x - y|^{\alpha-N} \log\frac{1}{|x - y|}\, dy + \int_{\Sigma_\lambda} |x_\lambda - y|^{\alpha-N} \log\frac{1}{|x_\lambda - y|}\, dy + \int_{\Omega_\lambda} |x - y|^{\alpha-N}\log\frac{1}{|x - y|}\, dy,$$

and

$$u_\lambda(x) = \int_{\Sigma_\lambda} |x_\lambda - y|^{\alpha-N} \log\frac{1}{|x_\lambda - y|}\, dy + \int_{\Sigma_\lambda} |x - y|^{\alpha-N} \log\frac{1}{|x - y|}\, dy + \int_{\Omega_\lambda} |x_\lambda - y|^{\alpha-N}\log\frac{1}{|x_\lambda - y|}\, dy.$$

Then,

$$u_\lambda(x) - u(x) = \int_{\Omega_\lambda} |x-y|^{\alpha-N}\log|x-y|\,dy - \int_{\Omega_\lambda} |x_\lambda - y|^{\alpha-N}\log|x_\lambda - y|\,dy. \tag{3.1}$$

We consider the function $s^{\alpha-N}\log s$. Note $\alpha > N$, thus

$$(s^{\alpha-N}\log s)' = s^{\alpha-N-1}[(\alpha-N)\log s + 1] < 0,$$

whenever $s < e^{\frac{1}{N-\alpha}}$. Since $|x-y| > |x_\lambda - y|$ for $x \in \Sigma_\lambda,\ y \in \Omega_\lambda$, and diam $\Omega < e^{\frac{1}{N-\alpha}}$, we easily infer that $u_\lambda(x) < u(x)$ for any $x \in \Sigma_\lambda$. □

Lemma 3.2. *$u(x)$ satisfies (1.4) and suppose $\lambda = \bar\lambda$ in the first case, i.e. $\Sigma'_{\bar\lambda}$ is internally tangent to the boundary of Ω at some point $P_{\bar\lambda}$ not on $T_{\bar\lambda}$, then $\Sigma_{\bar\lambda} \cup T_{\bar\lambda} \cup \Sigma'_{\bar\lambda} = \Omega$.*

Proof. The proof is essentially the same as that of Lemma 2.3. □

Lemma 3.3. *Suppose that $u(x)$ satisfies (1.4) with* diam $\Omega < e^{\frac{1}{N-\alpha}}$ *and that the second case occurs, i.e. $T_{\bar\lambda}$ reaches a position where is orthogonal to the boundary of Ω at some point Q, then, $\Sigma_{\bar\lambda} \cup T_{\bar\lambda} \cup \Sigma'_{\bar\lambda} = \Omega$.*

Proof. The argument follows that of the proof of Lemma 2.4. Since $u(x)$ is constant on $\partial\Omega$ and $\Omega \in C^1$, $\frac{\partial u}{\partial x_1}\Big|_Q = 0$. We denote the coordinate of Q by z. Suppose $\Omega_{\bar\lambda} \neq \emptyset$, there exists a ball $B \subset\subset \Omega_{\bar\lambda}$. Choosing a sequence $\{x^i\}_1^\infty \in \Sigma_{\bar\lambda} \setminus T_{\bar\lambda}$ such that $x^i \to z$ as $i \to \infty$, then $x^i_{\bar\lambda} \to z$ as $i \to \infty$. Since $B \subset\subset \Omega_{\bar\lambda}$, we find a δ such that diam $\Omega > |x^i_{\bar\lambda} - y| > \delta$ for any $y \in B$ and any $x^i_{\bar\lambda}$.

From (3.1), by Mean Value Theorem,

$$\begin{aligned}
\frac{u(x^i_{\bar\lambda}) - u(x^i)}{(x^i_{\bar\lambda} - x^i)\cdot e_1} &= \int_{\Omega_{\bar\lambda}} \frac{|x^i - y|^{\alpha-N}\log|x^i - y| - |x^i_{\bar\lambda} - y|^{\alpha-N}\log|x^i_{\bar\lambda} - y|}{(x^i_{\bar\lambda} - x^i)\cdot e_1}\,dy \\
&= \int_{\Omega_{\bar\lambda}} -|\bar x^i_{\bar\lambda} - y|^{\alpha-N-2}((x^i_{\bar\lambda} - y)\cdot e_1)((\alpha-N)\log|\bar x^i_{\bar\lambda} - y| + 1)\,dy \\
&< \int_B -|\bar x^i_{\bar\lambda} - y|^{\alpha-N-2}((x^i_{\bar\lambda} - y)\cdot e_1)((\alpha-N)\log|\bar x^i_{\bar\lambda} - y| + 1)\,dy \\
&< -C
\end{aligned} \tag{3.2}$$

where $\bar x^i_{\bar\lambda}$ is some point between $x^i_{\bar\lambda}$ and x^i. The assumption diam $\Omega < e^{\frac{1}{N-\alpha}}$ is applied in the last inequalities. Consequently, (3.2) contradicts $\frac{\partial u}{\partial x_1}\Big|_Q = 0$ as $i \to \infty$. Therefore, the lemma is verified. □

With the help of the above two lemmas, Theorem 2 is confirmed.

4. Proof of Theorem 3

We first show that the assumptions in Theorem 3 imply $u \in C^1(\bar G)$. To this end, we now introduce a regularity lifting lemma in [35] which is needed to show $u \in C^1(\bar G)$.

Lemma 4.1 (*Regularity Lifting*)**.** *Let V be a Hausdorff topological vector space. Suppose there are two extended norms (i.e. the norm of an element in V might be infinity) defined on V,*

$$\|\cdot\|_X, \|\cdot\|_Y : V \to [0,\infty].$$

Assume that the spaces

$$X := \{v \in V : \|v\|_X < \infty\} \quad \textit{and} \quad Y := \{v \in V : \|v\|_Y < \infty\}$$

are complete under the corresponding norms, and the convergence in X or in Y implies the convergence in V.

Let T be a contracting map from X into itself and from Y into itself. Assume that $f \in X$, and that there exists a function $g \in Z := X \cap Y$ such that $f = Tf + g$ in X. Then f also belongs to Z.

Then we can show the following.

Lemma 4.2. *If u and $f(u)$ satisfy the assumptions in Theorem 3, then $u \in C^1(\bar G)$.*

Proof. Define the linear operator

$$T_u v = \int_G \frac{\frac{f(u)}{u} v}{|x-y|^{N-\alpha}}\, dy.$$

For any real number $a > 0$, set

$$\begin{cases} u_a(x) = u(x), & |u(x)| > a \text{ or } |x| > a; \\ u_a(x) = 0, & \text{if otherwise.} \end{cases}$$

Let $u_b(x) = u(x) - u_a(x)$.

Since $u(x)$ satisfies (1.5), we can write it as

$$u_a(x) = T_{u_a} u_a + g(x) - u_b(x) \tag{4.1}$$

with $g(x) = \int_G \frac{f(u_b)}{|x-y|^{N-\alpha}}\, dy$.

Employing the Hardy–Littlewood–Sobolev inequality, then *Hölder*'s inequality to $g(x)$, for any $s > \frac{N}{N-\alpha}$,

$$\|g(x)\|_{L^s(G)} \le \left\| \frac{f(u_b)}{u_b} \right\|_{L^{\frac{N}{\alpha}}(G)} \|u_b\|_{L^s(G)}.$$

By the definition of u_b and the assumption of $f(u)$, we conclude that $g \in L^s(G)$ for any $s > \frac{N}{N-\alpha}$.

As for $T_{u_a} v$, applying the Hardy–Littlewood–Sobolev inequality, then *Hölder*'s inequality again, we have for any $t > \frac{N}{N-\alpha}$

$$\begin{aligned} \|T_{u_a} v\|_{L^t(G)} &\le \left\| \frac{f(u_a)}{u_a} v \right\|_{L^{\frac{Nt}{N+\alpha t}}(G)} \\ &\le \left\| \frac{f(u_a)}{u_a} \right\|_{L^{\frac{N}{\alpha}}(G)} \|v\|_{L^t(G)}. \end{aligned}$$

Choosing $a > 0$ sufficiently large, then

$$\|T_{u_a} v\|_{L^t(G)} \le \frac{1}{2} \|v\|_{L^t(G)}.$$

Therefore, T_{u_a} is a contracting map. By the Regularity lifting lemma above, $u_a \in L^t \cap L^q$ for any $t > \frac{N}{N-\alpha}$. This implies that $u \in L^t \cap L^q$ for any $t > \frac{N}{N-\alpha}$.

Next we show that $u \in L^\infty(G)$. For any $x \in G$, we choose a ball $\mathbb{B}_R(x)$ with fixed radius R, then

$$\begin{aligned} u(x) &= \int_{G \cap \mathbb{B}_R(x)} \frac{f(u)}{|x-y|^{N-\alpha}}\, dy + \int_{G \setminus \mathbb{B}_R(x)} \frac{f(u)}{|x-y|^{N-\alpha}}\, dy \\ &=: I_1 + I_2. \end{aligned}$$

We estimate I_1, I_2 respectively.

For I_1, from *Hölder*'s inequality,

$$\begin{aligned} |I_1| &\le \|\, |x-y|^{\alpha-N} \|_{L^{\frac{r}{r-1}}(\mathbb{B}_R(x))} \left\| \frac{f(u)}{u} \right\|_{L^{r+1}(G \cap \mathbb{B}_R(x))} \|u\|_{L^{r(r+1)}(G \cap \mathbb{B}_R(x))} \\ &\le C, \end{aligned} \tag{4.2}$$

by the fact that $r > \frac{N}{\alpha}$ implies that $-(N-\alpha)\frac{r}{r-1} + N > 0$ and $r(r+1) > \frac{N}{N-\alpha}$, the assumption of $f(u)$, and the fact that $u \in L^t$ for any $t > \frac{N}{N-\alpha}$.

For I_2,

$$\begin{aligned} |I_2| &\le \frac{1}{R^{N-\alpha}} \int_{G \setminus \mathbb{B}_R(x)} f(u)\, dy \\ &\le \frac{1}{R^{N-\alpha}} \left\| \frac{f(u)}{u} \right\|_{L^p(G \setminus \mathbb{B}_R(x))} \|u\|_{L^{\frac{p}{p-1}}(G \setminus \mathbb{B}_R(x))} \\ &\le C, \end{aligned} \tag{4.3}$$

due to the fact that $p < \frac{N}{\alpha}$ implies that $\frac{p}{p-1} > \frac{N}{N-\alpha}$ and the assumption of $f(u)$. Together with (4.2) and (4.3), we have shown $\|u\|_{L^\infty} < C$. Thanks to the continuity of $f(u)$, furthermore, we can infer that $f(u) < C$.

We next *claim* that $u \in C^1(\bar{G})$. Fix $\eta \in C_0^\infty(\mathbb{R}^n)$ satisfying $0 \le \eta \le 1$, and $\eta(t) = 0$ as $|t| \le 1$, and $\eta(t) = 1$ as $|t| \ge 2$. Define for any ϵ,

$$u_\epsilon = \int_G \frac{\eta_\epsilon f(u)}{|x-y|^{N-\alpha}}\, dy,$$

where $\eta_\epsilon = \eta(\frac{|x-y|}{\epsilon})$. We can easily deduce that

$$\begin{aligned} &u_\epsilon \to u; \\ &D_{x_i} u_\epsilon \to \int_G (\alpha - N)\frac{f(u)(x_i - y_i)}{|x-y|^{N-\alpha+2}}\, dy \end{aligned}$$

uniformly in G as $\epsilon \to 0$. Therefore, we have shown the claim holds. Consequently, the lemma follows. □

To prove Theorem 3, we also need to introduce some notations to avoid any confusion. If not specified, they are the same as those in above sections. Set

$$\begin{aligned} &T_\lambda = \{x \in G | x_1 = \lambda\}, \\ &\Sigma_\lambda = \{x \in G | x_1 < \lambda\}, \\ &H_\lambda = \{x \in \mathbb{R}^N | x_1 < \lambda\}, \\ &G_\lambda = \{x \in \Sigma_\lambda | x_\lambda \in \Omega_1\} \end{aligned}$$

and

$$\Omega_1^\lambda = \{x_\lambda \in \Omega_1 | x \in H_\lambda \cap \Omega_1\}.$$

Since we consider the exterior domain, the plane move from negative infinity towards Ω_1. Ω_1^λ will still stay in Ω_1 until at least one of the following events occurs:

(i) Ω_1^λ is internally tangent the boundary of Ω_1 at some point P not on T_λ.
(ii) T_λ reaches a position where it is orthogonal to the boundary of Ω_1 at some point Q.

Let $\bar{\lambda}$ be the first value such that at least one of above positions is reached.

We assert that G must be symmetric about $T_{\bar{\lambda}}$, i.e.,

$$\Sigma_{\bar{\lambda}} \cup T_{\bar{\lambda}} \cup \Sigma'_{\bar{\lambda}} = G. \tag{4.4}$$

If the assertion is true, Ω_1 is a ball as derived before.

For any solution u in (1.5), we have

$$u(x) = \int_{\Sigma_\lambda} \frac{f(u)}{|x-y|^{N-\alpha}}\, dy + \int_{\Sigma_\lambda \setminus G_\lambda} \frac{f(u_\lambda)}{|x_\lambda - y|^{N-\alpha}}\, dy$$

and

$$u_\lambda(x) = \int_{\Sigma_\lambda} \frac{f(u)}{|x_\lambda-y|^{N-\alpha}}\, dy + \int_{\Sigma_\lambda \setminus G_\lambda} \frac{f(u_\lambda)}{|x - y|^{N-\alpha}}\, dy.$$

Then

$$\begin{aligned} u_\lambda(x) - u(x) = &\int_{\Sigma_\lambda \setminus G_\lambda} [f(u_\lambda) - f(u)] \left[\frac{1}{|x-y|^{N-\alpha}} - \frac{1}{|x_\lambda - y|^{N-\alpha}}\right] dy \\ &- \int_{G_\lambda} f(u) \left[\frac{1}{|x-y|^{N-\alpha}} - \frac{1}{|x_\lambda - y|^{N-\alpha}}\right] dy. \end{aligned} \tag{4.5}$$

Since $|x-y| < |x_\lambda - y|$ for $x \in \Sigma_\lambda$ and $y \in G_\lambda$, furthermore, $f(u) > 0$ from the assumption (i) of $f(u)$, we have

$$u_\lambda(x) - u(x) \le \int_{\Sigma_\lambda \setminus G_\lambda} [f(u_\lambda) - f(u)] \left[\frac{1}{|x-y|^{N-\alpha}} - \frac{1}{|x_\lambda - y|^{N-\alpha}}\right] dy. \tag{4.6}$$

In order to carry out the moving plane method in integral form, we shall show that the plane can be started. Let

$$\Sigma_\lambda^- = \{x \in \Sigma_\lambda \setminus G_\lambda | u_\lambda(x) > u(x)\}$$

and

$$w_\lambda(x) = u_\lambda(x) - u(x).$$

Lemma 4.3. *If λ is close to negative infinity, then $u(x) \ge u_\lambda(x)$ for any $x \in \Sigma_\lambda \setminus G_\lambda$.*

Proof. Since $f(u)$ is increasing, from (4.6),

$$\begin{aligned} u_\lambda(x)-u(x) &\le \int_{\Sigma_\lambda^-}[f(u_\lambda)-f(u)]\left[\frac{1}{|x-y|^{N-\alpha}}-\frac{1}{|x_\lambda-y|^{N-\alpha}}\right]dy \\ &\le \int_{\Sigma_\lambda^-}[f(u_\lambda)-f(u)]\frac{1}{|x-y|^{N-\alpha}}\,dy \\ &= \int_{\Sigma_\lambda^-} f'(\theta u+(1-\theta)u_\lambda)(u_\lambda-u)\frac{1}{|x-y|^{N-\alpha}}\,dy, \end{aligned} \tag{4.7}$$

where $f'(\theta u+(1-\theta)u_\lambda)$ is deduced by Mean Value Theorem and $0<\theta<1$.

Applying the Hardy–Littlewood–Sobolev inequality, then *Hölder*'s inequality to (4.7), since $q>\frac{N}{N-\alpha}$, we get

$$\begin{aligned} \|w_\lambda\|_{L^q(\Sigma_\lambda^-)} &\le C\|f'(\theta u+(1-\theta)u_\lambda)w_\lambda\|_{L^{\frac{Nq}{N+\alpha q}}(\Sigma_\lambda^-)} \\ &\le C\|f'(\theta u+(1-\theta)u_\lambda)\|_{L^{\frac{N}{\alpha}}(\Sigma_\lambda^-)}\|w_\lambda\|_{L^q(\Sigma_\lambda^-)}. \end{aligned}$$

By the assumption (ii) of f, if λ is close to negative infinity, then,

$$C\|f'(\theta u+(1-\theta)u_\lambda)\|_{L^{\frac{N}{\alpha}}(\Sigma_\lambda^-)}\le\frac12,$$

which implies that

$$\|w_\lambda\|_{L^q(\Sigma_\lambda^-)}=0.$$

Hence Σ_λ^- measures 0, then $w_\lambda(x)\le 0$ for any $x\in\Sigma_\lambda\setminus G_\lambda$ if λ is sufficient negative. □

Next we show that the plane can continue to move all the way to the right.

Lemma 4.4. *Suppose $\lambda<\bar\lambda$ and $u(x)\ge u_\lambda(x)$ in $\Sigma_\lambda\setminus G_\lambda$, then there exists $\epsilon>0$ such that $u(x)>u_{\hat\lambda}(x)$ for any $x\in\Sigma_{\hat\lambda}\setminus G_{\hat\lambda}$, where $\bar\lambda>\hat\lambda:=\lambda+\epsilon$.*

Proof. Since $u(x)\ge u_\lambda(x)$, then $f(u)\ge f(u_\lambda)$ by the assumption of f. Suppose there exists some point x^0 in $\Sigma_\lambda\setminus G_\lambda$ such that $u(x_\lambda^0)-u(x^0)=0$; that is, from (4.5)

$$0=\int_{\Sigma_\lambda\setminus G_\lambda}[f(u_\lambda)-f(u)]\left[\frac{1}{|x^0-y|^{N-\alpha}}-\frac{1}{|x_\lambda^0-y|^{N-\alpha}}\right]dy-\int_{G_\lambda}f(u)\left[\frac{1}{|x^0-y|^{N-\alpha}}-\frac{1}{|x_\lambda^0-y|^{N-\alpha}}\right]dy.$$

Thus, $f(u)\equiv 0$ in G_λ, which is impossible since $f(u)>0$. Therefore, $u(x)>u_\lambda(x)$ in $\Sigma_\lambda\setminus G_\lambda$.

We next show that the plane T_λ can be moved a little further. Since $f'(u)\in L^{\frac{N}{\alpha}}(G)$, for any small μ, there exists large enough $\mathbb{B}_R$ such that

$$\|f'(u)\|_{L^{\frac{N}{\alpha}}(G\setminus\mathbb{B}_R)}\le\mu. \tag{4.8}$$

For such fixed $\mathbb{B}_R$, thanks to the integrability of $f'(u)$ again, we choose small enough ϵ such that

$$\|f'(u)\|_{L^{\frac{N}{\alpha}}((\Sigma_{\hat\lambda}\setminus\Sigma_{\lambda-\epsilon})\cap\mathbb{B}_R)}\le\mu. \tag{4.9}$$

Due to the continuity of u, $w_\lambda(x)<0$ in the compact set $\overline{\mathbb{B}_R\cap(\Sigma_{\lambda-\epsilon}\setminus G_{\lambda-\epsilon})}$. Thus the set $\Sigma_{\hat\lambda}^-$ only lies in $M:=\{((\Sigma_{\hat\lambda}\setminus\Sigma_{\lambda-\epsilon})\cap\mathbb{B}_R)\}\cup\{G\setminus\mathbb{B}_R\}$. From (4.7),

$$w_{\hat\lambda}\le\int_M f'(\theta u+(1-\theta)u_{\hat\lambda})w_{\hat\lambda}\frac{1}{|x-y|^{N-\alpha}}\,dy.$$

As before, we apply Hardy–Littlewood–Sobolev inequality, then *Hölder*'s inequality,

$$\|w_\lambda\|_{L^q(M)}\le C\|f'(\theta u+(1-\theta)u_\lambda)\|_{L^{\frac{N}{\alpha}}(M)}\|w_\lambda\|_{L^q(M)}.$$

By (4.8), (4.9) and above estimate, we have $\|w_\lambda\|_{L^q(M)}=0$. Therefore, $\Sigma_{\hat\lambda}^-$ is empty. Hence $u(x)\ge u_{\hat\lambda}(x)$. Using the same argument at the beginning of the lemma, we shall show that $u(x)>u_{\hat\lambda}(x)$ for any $x\in\Sigma_{\hat\lambda}\setminus G_{\hat\lambda}$. □

Lemma 4.5. *Suppose $u(x)$ satisfies (1.5) and $\lambda=\bar\lambda$ in the first case, i.e., $\Omega_1^{\bar\lambda}$ is internally tangent to the boundary of Ω_1 at some point $P_{\bar\lambda}$ not on $T_{\bar\lambda}$, then $\Sigma_{\bar\lambda}\cup T_{\bar\lambda}\cup\Sigma'_{\bar\lambda}=G$.*

Proof. If not, then $G_\lambda \neq \emptyset$. From (4.5), at P, $u(P) > u(P_{\bar{\lambda}})$ since $f(u_{\bar{\lambda}}) \leq f(u)$ in $\Sigma_{\bar{\lambda}} \setminus G_{\bar{\lambda}}$ and $f(u) > 0$ in G_λ. However, $u(P) = u(P_{\bar{\lambda}})$ by our assumption that u is constant on ∂G. Therefore, a contradiction is arrived. Hence $G_\lambda = \emptyset$, which implies that $\Sigma_{\bar{\lambda}} \cup T_{\bar{\lambda}} \cup \Sigma'_{\bar{\lambda}} = G$. □

Lemma 4.6. *$u(x)$ satisfies (1.5) and suppose that the second case occurs, i.e., $T_{\bar{\lambda}}$ reaches a position where is orthogonal to the boundary of Ω_1 at some point Q, then, $\Sigma_{\bar{\lambda}} \cup T_{\bar{\lambda}} \cup \Sigma'_{\bar{\lambda}} = G$.*

Proof. As deduced before, $\frac{\partial u}{\partial x_1}\Big|_Q = 0$. Denote the coordinate Q by z. Suppose $G_{\bar{\lambda}} \neq \emptyset$, then there exists a ball $\mathbb{B} \subset\subset G_{\bar{\lambda}}$. Choose a sequence $\{x^i\}_1^\infty \in \Sigma_{\bar{\lambda}} \setminus T_{\bar{\lambda}}$ such that $x^i \to z$ as $i \to \infty$. Correspondingly, $x^i_{\bar{\lambda}} \to z$ as $i \to \infty$. Since $\mathbb{B} \subset\subset G_{\bar{\lambda}}$, we can find a δ such that $|x^i_{\bar{\lambda}} - y| > \delta$ for any $y \in \mathbb{B}$ and any $x^i_{\bar{\lambda}}$. By (4.5),

$$\begin{aligned}
\frac{u(x^i_{\bar{\lambda}}) - u(x^i)}{(x^i_{\bar{\lambda}} - x^i)\cdot e_1} &\leq \int_{G_{\bar{\lambda}}} f(u)\frac{|x^i_{\bar{\lambda}} - y|^{\alpha-N} - |x^i - y|^{\alpha-N}}{(x^i_{\bar{\lambda}} - x^i)\cdot e_1}\,dy \\
&= \int_{G_{\bar{\lambda}}} (\alpha - N) f(u)|\bar{x}^i_{\bar{\lambda}} - y|^{\alpha-N-2}((x^i_{\bar{\lambda}} - y)\cdot e_1)\,dy \\
&< \int_{\mathbb{B}} (\alpha - N) f(u)|\bar{x}^i_{\bar{\lambda}} - y|^{\alpha-N-2}((x^i_{\bar{\lambda}} - y)\cdot e_1)\,dy \\
&< -C.
\end{aligned} \tag{4.10}$$

As before, $\bar{x}^i_{\bar{\lambda}}$ is some point between $x^i_{\bar{\lambda}}$ and x^i and Mean Value Theorem is applied above. However,

$$\lim_{i\to\infty} \frac{u(x^i_{\bar{\lambda}}) - u(x^i)}{(x^i_{\bar{\lambda}} - x^i)\cdot e_1} = \frac{\partial u}{\partial x_1}\Big|_Q = 0.$$

It apparently contradicts (4.10). In the end, the lemma holds. □

Through Lemmas 4.5 and 4.6, we infer that Ω_1 is a ball. Furthermore, Lemmas 4.3 and 4.4 lead to the radial symmetry and monotonicity of solution u if we regard x_1 as any given direction. Hence, u is radially symmetric with respect to the center of the ball and increasing in radial direction. Without loss of generality, let $u = a > 0$ on $\partial\Omega_1$ and $\Omega_1 = \mathbb{B}_1$. Then

$$\int_G u^q(x)\,dx = \int_{\mathbb{R}^N\setminus\mathbb{B}_1} u^q(|x|)\,dx > \int_{\mathbb{R}^N\setminus\mathbb{B}_1} a^q\,dx = \infty,$$

which obviously is a contradiction. Therefore, Theorem 3 is complete.

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Appendix

In this section, we extend our results to the more general integral equation for a bounded domain $\Omega \subset \mathbb{R}^N$, i.e.,

$$u(x) = \int_\Omega g(|x - y|)\,dy. \tag{A.1}$$

Assume that $g(r) \in C^1(\mathbb{R}_+)$ satisfies either

$$g'(r) < 0, \quad \forall 0 < r < \mathrm{diam}(\Omega), \tag{A.2}$$

or

$$g'(r) > 0, \quad \forall 0 < r < \mathrm{diam}(\Omega). \tag{A.3}$$

Moreover,

$$\epsilon^{-1}\int_0^\epsilon |g(r)|r^{N-1}\,dr \to 0 \tag{A.4}$$

and

$$\int_0^\epsilon |g'(r)|r^{N-1}\,dr \to 0, \tag{A.5}$$

as $\epsilon \to 0$. Since differentiability of u is applied in the second case of critical position, i.e., $T_{\bar{\lambda}}$ reaches a position where is orthogonal to the boundary of Ω at some point Q, we first prove that u is $C^1(\bar{\Omega})$. In fact, we will show the following lemma.

Lemma A.1. *If $u(x)$ satisfies (A.4) and (A.5) in (A.1), then $u(x) \in C^1(\mathbb{R}^N)$.*

Proof. Without loss of generality, we only show that $D_1 u = \frac{\partial u}{\partial x_1}$ is continuous in $\bar{\Omega}$. Since $g(r) \in C^1(\mathbb{R}_+)$, the possible singularity is $r = 0$. Let $\eta : [0, \infty) \to [0, 1]$ be a C^∞ function with $\eta \equiv 0$ on $[0, \frac{1}{2}]$ and $\eta \equiv 1$ on $[1, \infty)$. Let $\eta_\epsilon = \eta(\frac{t}{\epsilon})$ and define

$$u_\epsilon(x) := \int_\Omega g(|x-y|)\eta_\epsilon(|x-y|)\, dy$$

and

$$v_1(x) := \int_\Omega D_1 g(|x-y|)\, dy.$$

$v_1(x)$ exists because of (A.5). Furthermore, for any $x \in \bar{\Omega}$,

$$\begin{aligned} |D_1 u_\epsilon(x) - v_1(x)| &\le \int_\Omega |D_1((\eta_\epsilon(|x-y|) - 1) g(|x-y|))\, dy \\ &\le C\epsilon^{-1} \int_{\mathbb{B}_\epsilon(x)} |g(|x-y|)|\, dy + \int_{\mathbb{B}_\epsilon(x)} |g'(|x-y|)|\, dy \\ &= C\epsilon^{-1} \int_0^\epsilon |g(r)| r^{N-1}\, dr + \int_0^\epsilon |g'(r)| r^{N-1}\, dr \\ &\to 0 \end{aligned}$$

by the assumptions of (A.4) and (A.5). Thus, $D_1 u_\epsilon$ converges uniformly to v_1 on $\mathbb{R}^N$. Therefore $u(x) \in C^1(\mathbb{R}^N)$. □

Adapting the proofs of Theorems 1 and 2, we can similarly establish the following more general characterization of balls.

Theorem A.1. *Let Ω be a C^1 bounded domain. Then u in (A.1) is constant on $\partial\Omega$ if and only if Ω is a ball.*

We should point out that the monotonicity of the function g plays an essential role in the proofs of Theorems 1 and 2. The assumption on the function g assures that the argument in the proof of Theorem A.1 carries through without any substantial difficulty.

References

[1] L.E. Fraenkel, Introduction to Maximum Principles and Symmetry in Ellipitc Problems, in: Cambridge Tracts in Mathematics, vol. 128, Cambridge University Press, London, 2000.
[2] W. Reichel, Characterization of balls by Riesz-potentials, Ann. Mat. 188 (2009) 235–245.
[3] J. Serrin, A symmetry problem in potential theory, Arch. Ration. Mech. Anal. 43 (1971) 304–318.
[4] B. Gidas, W. Ni, L. Nirenberg, Symmetry of related properties via the maximum principle, Comm. Math. Phys. 68 (1979) 209–243.
[5] L. Caffarelli, B. Gidas, J. Spruck, Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth, Comm. Pure Appl. Math. XLII (1989) 271–297.
[6] H. Weinberger, Remark on the preceeding paper of Serrin, Arch. Ration. Mech. Anal. 43 (1971) 319–320.
[7] A. Aftalion, J. Busca, Radial symmetry of overdetermined boundary-value problems in exterior domains, Arch. Ration. Mech. Anal. 143 (2) (1998) 195–206.
[8] A. Bennett, Symmetry in an overdetermined four order ellipitc boundary value problem, SIAM J. Matrix Anal. Appl. 17 (1986) 1354–1358.
[9] M. Barkatou, S. Khatmi, Symmetry result for some overdetermined value problems, ANZIAM J. 49 (2008) 479–494.
[10] B. Brandolini, C. Nitsch, P. Salani, C. Trombetti, Serrin-type overdetermined problems: an alternative proof, Arch. Ration. Mech. Anal. 190 (2008) 267–280.
[11] B. Brandolini, C. Nitsch, P. Salani, C. Trombetti, On the stability of the Serrin problem, J. Differential Equations 245 (2008) 1566–1583.
[12] A. Cianchi, P. Salani, Overdetermined anisotropic elliptic problems, Math. Ann. 345 (4) (2009) 859–881.
[13] A. Enciso, D. Peralta-Salas, Symmetry for an overdetermined boundary problem in a punctured domain, Nonlinear Anal. 70 (2) (2009) 1080–1086.
[14] I. Fragala, F. Gazzola, Partially overdetermined elliptic boundary value problems, J. Differential Equations 245 (2008) 1299–1322.
[15] I. Fragala, F. Gazzola, B. Kawohl, Overdetermined problems with possibly degenerate ellipticity, a geometric approach, Math. Z. 254 (2006) 117–132.
[16] A. Farina, B. Kawohl, Remarks on an overdetermined boundary value problem, Calc. Var. Partial Differential Equations 31 (3) (2008) 351–357.
[17] A. Farina, E. Valdinoci, Flattening results for elliptic PDEs in unbounded domains with applications to overdetermined problems, Arch. Ration. Mech. Anal. 195 (2010) 1025–1058.
[18] A. Greco, Radial symmetry and uniqueness for an overdetermined problem, Math. Methods Appl. Sci. (24) (2001) 103–115.
[19] N. Garofalo, J.L. Lewis, A symmetry result related to some overdetermined boundary value problems, Amer. J. Math. 111 (1989) 9–33.
[20] A. Henrot, G. Philippin, H. Prebet, Overdetermined problems on ring shaped domains, Adv. Math. Sci. Appl. 9 (2) (1999) 737–747.
[21] M. Lim, Symmetry of a boundary integral operator and a characterization of a ball, Illinois J. Math. 45 (2001) 537–543.
[22] G. Liu, Symmetry theorems for the overdetermined eigenvalue problems, J. Differential Equations 233 (2007) 585–600.
[23] E. Martensen, Eine Integralgleichung fur die log. Gleichgewichtwverteilung und die Krummung der Randkurve eines Gebies, ZAMM Z. Angew. Math. Mech. 72 (1992) 596–599.
[24] O. Mendez, W. Reichel, Electrostatic characterization of spheres, Forum Math. 12 (1995) 223–245.
[25] L. Payne, G. Philippin, Some overdetermined boundary value problems for harmonic functions, Z. Angew. Math. Phys. 42 (6) (1991) 864–873.
[26] L. Payne, P. Schaefer, On overdetermined boundary value problems for the biharmonic operator, J. Math. Anal. Appl. 187 (2) (1994) 598–616.
[27] J. Prajapat, Serrin's result for domains with a corner or cusp, Duke Math. J. 91 (1998) 29–31.
[28] H. Shahgholian, A characterization of the sphere in terms of single-layer potentials, Proc. Amer. Math. Soc. 115 (1992) 1167–1168.
[29] B. Sirakov, Symmetry for exterior elliptic problems and two conjectures in potential theory, Ann. Inst. H. Poincaré Anal. Non Linéaire 18 (2001) 135–156.

[30] G. Wang, C. Xia, A characterization of the Wulff shape by an overdetermined anisotropic PDE, Arch. Ration. Mech. Anal. (2010).
[31] W. Reichel, Radial symmetry for elliptic boundary value problems on exterior domain, Arch. Ration. Mech. Anal. 137 (1997) 381–394.
[32] C. Kenig, Harmonic Analysis Techniques for Second Order Elliptic Boundary Value Problems, in: CBMS Series, vol. 83, Amer. Math. Soc., Providence, RI, 1994.
[33] W. Chen, C. Li, B. Ou, Classification of solutions for an integral equation, Commun. Pure Appl. Anal. 59 (2006) 330–343.
[34] Y.Y. Li, Remark on some conformally invariant integral equations: the method of moving spheres, J. Eur. Math. Soc. 6 (2004) 153–180.
[35] W. Chen, C. Li, Methods on Nonlinear Elliptic Equations, in: AIMS Book Series, vol. 4, 2010.
[36] D. Li, G. Strohmer, L. Wang, Symmetry of integral equations on bounded domain, Proc. Amer. Math. Soc. 137 (2009) 3695–3702.
[37] X. Han, G. Lu, J. Zhu, Characterization of balls in terms of Bessel-potential integral equation, J. Differential Equations 252 (2) (2012) 1589–1602.
[38] W. Chen, J. Zhu, Radial symmetry and regularity of solutions for poly-harmonic Dirichlet problems, J. Math. Anal. Appl. 377 (2011) 744–753.
[39] A.D. Alexandroff, A characteristic property of the sphere, Ann. Mat. Pura Appl. 58 (1962) 303–354.