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Bounds of Singular Integrals on Weighted Hardy Spaces and Discrete Littlewood–Paley Analysis

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Abstract We apply the discrete version of Calderón's reproducing formula and Littlewood–Paley theory with weights to establish the $H_w^p \to H_w^p$ $(0 < p < \infty)$ and $H_w^p \to L_w^p$ $(0 < p \leq 1)$ boundedness for singular integral operators and derive some explicit bounds for the operator norms of singular integrals acting on these weighted Hardy spaces when we only assume $w \in A_\infty$. The bounds will be expressed in terms of the A_q constant of w if $q > q_w = \inf\{s : w \in A_s\}$. Our results can be regarded as a natural extension of the results about the growth of the A_p constant of singular integral operators on classical weighted Lebesgue spaces L_w^p in Hytonen et al. (arXiv:1006.2530, 2010; arXiv:0911.0713, 2009), Lerner (Ill. J. Math. 52:653–666, 2008; Proc. Am. Math. Soc. 136(8):2829–2833, 2008), Lerner et al. (Int. Math. Res. Notes 2008:rnm 126, 2008; Math. Res. Lett. 16:149–156, 2009), Lacey et al. (arXiv:0905.3839v2, 2009; arXiv:0906.1941, 2009), Petermichl (Am. J. Math. 129(5):1355–1375, 2007; Proc. Am. Math. Soc. 136(4):1237–1249, 2008), and Petermichl and Volberg (Duke Math. J. 112(2):281–305, 2002). Our main result is stated in Theorem 1.1. Our method avoids the atomic decomposition which was usually used in proving boundedness of singular integral operators on Hardy spaces.

Keywords Discrete Littlewood–Paley theory · Discrete Calderón's identity · Singular integrals · Weighted Hardy spaces

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1 Introduction and Statement of Results

Weighted Hardy spaces have been studied extensively in the last fifty years (see, for example, Garcia-Guerva [7], Strömberg–Torchinsky [30, 31], and many other references therein), where the weighted Hardy space was defined by using the non-tangential maximal functions and atomic decompositions were derived. The relationship between H^p_w and L^p_w for $p > 1$ was considered in both one and multi-parameter cases (e.g., Strömberg and Wheeden in [32]).

In this paper, we consider the weighted Hardy space estimates for singular integrals using the discrete version of Calderón's identity and Littlewood–Paley theory developed in the work of Han with the first author [11]. In [11], the authors deal with the multiparameter Hardy spaces H^p $(0 < p \leq 1)$ associated with the flag singular integrals. The H^p to H^p and H^p to L^p boundedness are proved for flag singular integrals in [11] for all $0 < p \leq 1$ which extend the L^p theory for $1 < p < \infty$ developed in Nagel–Ricci–Stein [23]. The main purpose of the current paper is to derive some explicit bounds, in terms of the A_q constant $[w]_q$ of the Muckenhoupt weight $w \in A_q$ (see the definition of Muckenhoupt weight below) if $q > q_w = \inf\{s : w \in A_s\}$, for the H^p_w to L^p_w mapping norms for all $0 < p \leq 1$ and H^p_w to H^p_w mapping norms for all $0 < p < \infty$ on weighted Hardy spaces for a class of singular integral operators. In other words, we only assume that the weight w is in the class A_∞ (see definition below).

Let us first recall the definition of A_p weight. For $1 < p < \infty$, a locally integrable nonnegative function w on $\mathbf{R}^n$ is said to be in A_p if

$$[w]_{A_p} = \sup_I \left(\frac{1}{|I|} \int_I w(x)dx \right) \left(\frac{1}{|I|} \int_I w(x)^{-1/(p-1)} dx \right)^{p-1} < \infty,$$

where for every cube $I \in \mathbf{R}^n$, $|I|$ denotes its Lebesgue measure, and $[w]_{A_p}$ is called the A_p characteristic constant of w. For the case $p = 1$, w is said to be in A_1 if

$$Mw(x) \leq C_1 w(x) \quad \text{for almost all } x \in \mathbf{R}^n$$

and for some constant C_1. If $w \in A_1$, then the quantity

$$[w]_{A_1} = \sup_{I \subset \mathbf{R}^n} \left(\frac{1}{|I|} \int_I w(x)dx \right) \|w^{-1}\|_{L^\infty(I)}$$

is called the A_1 characteristic constant of w. Finally, we define

$$A_\infty = \bigcup_{1 \leq p < \infty} A_p.$$

For $w \in A_\infty$, we denote by $q_w = \inf\{q : w \in A_q\}$ the critical index of w.

It is well known that if $w \notin A_p$, then T may not be bounded on L^p_w. However, it does not contradict with our results since in general $H^p_w \neq L^p_w$ when $w \notin A_p$ for

$p > 1$. We refer the reader to the work of Strömberg and Wheeden [32] where the relations between L_u^p and H_u^p of the real line are studied in the case when $p > 1$ and $u(x) = |Q(x)|^p w(x)$, where $Q(x)$ is a polynomial and $w(x)$ satisfies the Muckenhoupt A_p condition. It turns out that H_u^p and L_u^p can be identified when all the zeros of Q are real, and that otherwise H_u^p can be identified with a certain proper subspace of L_u^p.

The growth of the A_p constants on classical weighted estimates in L^p spaces for $1 < p < \infty$ for the Hardy–Littlewood maximal function, singular integrals, and fractional integrals has been investigated extensively in recent years. We refer the reader to the work of Buckley [3], Petermichl and Volberg [27], Petermichl [25, 26], Lacey, Moen, Pérez, and Torres [16], Lerner [19, 20], Lerner, Ombrosi, and Pérez [21, 22], Lacey, Petermichl, and Reguera [15] and Hytonen, Lacey, Reguera, and Vagharshakyan [14], etc.

Buckley [3] showed that for $1 < p < \infty$, $w \in A_p$, the Hardy–Littlewood maximal operator M satisfies

$$\|M\|_{L^p(w)\to L^p(w)} \le c[w]_{A_p}^{1/(p-1)}, \qquad \|M\|_{L^p(w)\to L^{p,\infty}(w)} \le c[w]_{A_p}^{1/p}$$

and the exponent $1/(p-1)$ is the best possible. A new and rather simple proof of both Muckenhoupt's and Buckley's results were recently given by Lerner [20]. It is shown in [19] that the $L^p(w)$ $(1 < p < \infty)$ operator norms of Littlewood–Paley operators are bounded by a multiple of $[w]_{A_p}^{\gamma_p}$, where $\gamma_p = \max\{1, \frac{p}{2}\}\frac{1}{p-1}$.

For the singular integrals, Petermichl and Volberg [27] proved for the Ahlfors Beurling transform and Petermichl [25, 26] proved for the Hilbert transform and the Riesz transforms the following estimates:

$$\|T\|_{L_w^p} \le c_{p,n}[w]_p^{\max\{1, \frac{1}{p-1}\}}, \quad 1 < p < \infty,$$

when the operator T is any one of the aforementioned operators and the exponent $\max\{1, \frac{1}{p-1}\}$ is the best possible. Very recently, Lacey, Petermichl, and Reguera [15] and Hytonen, Lacey, Reguera, and Vagharshakyan [14] proved sharp bounds in terms of linear $[w]_{A_2}$ constant on weighted L^2 space and sharp bounds in terms of $[w]_{A_p}$ constant on weighted L^p spaces for Haar Shift Operators, respectively. As a corollary to their main result they deduced sharp A_p inequalities for T being either the Hilbert transform in dimension $d = 1$, the Beurling transform in dimension $d = 2$, or a Riesz transform in any dimension $d \ge 2$. Let T_* denote the maximal truncations of these operators. They proved weighted weak and strong-type L_w^p inequalities:

$$\|T_*\|_{L_w^{p,\infty}} \le [w]_{A_p}\|f\|_{L_w^p}, \quad 1 < p < 2,$$

and

$$\|T_*\|_{L_w^p} \le [w]_{A_p}^{\max\{1, \frac{1}{p-1}\}}\|f\|_{L_w^p}, \quad 1 < p < \infty.$$

These estimates are sharp in the power of the A_p characteristic of the weight w, and are consistent with the best possible bounds without the truncations.

In the work of Dragicević, Grafakos, Pereyra, and Petermichl [5], sharp L_w^p estimates in terms of $[w]_{A_p}$ in the Rubio de Francia extrapolation theorem [9] have been established. In particular, the main result of [5] shows that if a sublinear operator T is bounded on L^2_w with the linear bound for $\|T\|_{L^2_w}$ in terms of $[w]_{A_2}$, then T is bounded on L^p_w for $1 < p < \infty$, and $\|T\|_{L^p_w}$ is at most a multiple of $[w]_{A_p}^{\alpha_p}$ with $\alpha_p = \max\{1, \frac{1}{p-1}\}$. Therefore, the sharp L^2_w bound for the Hilbert and Riesz transforms along with extrapolation shows that for these operators the best possible exponent α_p can be achieved for all $p > 1$. For more general singular integrals, the question about the best power of $[w]_{A_p}$ in the operator norm on L^p_w is still open.[1]

In [21] and [22], Lerner, Ombrosi, and Pérez derived some results related to the weak Muckenhoupt and Wheeden conjecture for the Calderón–Zygmund operator T, they proved that

$$\|T\|_{L^p(w)\to L^p(w)} \le Cpp'[w]_{A_1} \quad (1 < p < \infty),$$
$$\|T\|_{L^1(w)\to L^{1,\infty}(w)} \le C[w]_{A_1} \quad \big(1 + \log[w]_{A_1}\big).$$

Motivated by these results and recent works on discrete Littlewood–Paley theory and Calderón's identity in multiparameter settings [11] and [12], in the present paper we will describe the explicit dependence of the corresponding $H^p_w \to L^p_w$ $(0 < p \le 1)$ and $H^p_w \to H^p_w$ $(0 < p < \infty)$ operator norms of singular integrals in terms of the A_q characteristic constant of $w \in A_q$ for arbitrary $q > q_w = \inf\{s : w \in A_s\}$.

A singular integral operator is defined as follows.

Definition 1.1 A one-parameter kernel on $\mathbf{R}^n$ is a distribution K on $\mathbf{R}^n$ which coincides with a C^∞ function away from the origin and satisfies

(1) (Differential Inequalities) For all multi-indices α, and $\forall x \neq 0$,

$$|\partial^\alpha K(x)| \le C_\alpha |x|^{-n-|\alpha|}. \tag{1.1}$$

(2) (Cancellation Condition) For any normalized bump function ϕ on $\mathbf{R}^n$ and any $R > 0$,

$$\left|\int_{\mathbf{R}^n} K(x)\phi(Rx)dx\right| \le C, \tag{1.2}$$

where C is a constant independent of ϕ and $R > 0$. An operator with a one-parameter kernel is called a (one-parameter) singular integral operator.

Remark 1.1 There is another way to describe the cancellation condition (1.2), that is

$$\left|\int_{\varepsilon<|x|<N} K(x)dx\right| \le C, \quad \text{for any } 0 < \varepsilon < N < \infty. \tag{1.3}$$

[1] Recent work of T. Hytonen, M. Lacey, M. Reguera, E. Sawyer, I. Uriarte-Tuero, A. Vagharshakyan [13] has obtained such a sharp bound under the assumption that the kernel of the singular integral satisfies some smoothness condition by proving a weighted weak type result and combining with the result of Pérez, Treil, and Volberg [24].

Under the hypothesis of condition (1.1), the L^2 boundedness of T holds if and only if any one of the cancellation conditions (1.2) or (1.3) holds (see [29]).

Fefferman and Stein [6] first obtained the H^p boundedness of these operators for $0 < p \leq 1$. In the weighted case, when $w \in A_1, n/(n+1) < p \leq 1$, Lin and Lee [17] applied the weighted molecular theory and atomic decomposition to obtain the H^p_w boundedness of these operators.

The aim of this paper is to obtain H^p_w boundedness of T by only assuming $w \in A_\infty$ and derive the explicit operator norm bounds of the singular integrals on weighted Hardy spaces. This is accomplished by using discrete Littlewood–Paley theory similar to that developed earlier in [11]. Indeed, boundedness of singular integrals on weighted multiparameter Hardy spaces $H^p_w(\mathbb{R}^n \times \mathbb{R}^m)$ has been established in [4] by only assuming $w \in A_\infty(\mathbb{R}^n \times \mathbb{R}^m)$. Generalization of such results to weighted Hardy spaces of arbitrary number of parameters has been done in [28]. However, no explicit constants for the bounds of singular integrals are given in [4, 28].

To describe our main results, we begin by recalling some properties of weight functions.

Proposition 1.1 [10] *Let $w \in A_p$ for some $1 \leq p < \infty$. Then*

(1) $[\delta^\lambda(w)]_{A_p} = [w]_{A_p}$, *where* $\delta^\lambda(w)(x) = w(\lambda x_1, \ldots, \lambda x_n)$, $\lambda \in \mathbf{R}$.
(2) $[\tau^z(w)]_{A_p} = [w]_{A_p}$, *where* $\tau^z(w)(x) = w(x - z)$, $z \in \mathbf{R}^n$.
(3) $[\lambda w]_{A_p} = [w]_{A_p}$ *for all* $\lambda > 0$.
(4) *When* $1 < p < \infty$, $\sigma = w^{-1/(p-1)} \in A_{p'}$ *with characteristic constant* $[\sigma]_{A_{p'}} = [w]_{A_p}^{1/(p-1)}$.
(5) $[w]_{A_p} \geq 1$ *for all* $w \in A_p$. *Equality holds if and only if w is a constant.*
(6) *For* $1 \leq p < q < \infty$, *we have* $[w]_{A_q} \leq [w]_{A_p}$. *And* $\lim_{q \to 1^+} [w]_{A_q} = [w]_{A_1}$.
(7) *The measure $w(x)dx$ is doubling: precisely, for all $\lambda > 1$ and all cubes Q we have*

$$w(\lambda Q) \leq \lambda^{np}[w]_{A_p} w(Q).$$

Let ψ be a Schwartz function on $\mathbf{R}^n$ which satisfies

$$\int_{\mathbf{R}^n} \psi(x)x^\alpha dx = 0, \quad \text{for all multi-indices}\, \alpha, \tag{1.4}$$

and

$$\sum_{j \in \mathbf{Z}} |\hat{\psi}(2^{-j}\xi)|^2 = 1, \quad \text{for all } \xi \neq 0. \tag{1.5}$$

Strictly speaking, the classical Hardy spaces H^p should be defined by using bounded distributions or distributions modulus polynomials; see [10] and [29]. For our purpose here, we need to introduce some new class which is similar to distributions modulus polynomials.

Definition 1.2 A function $f(x)$ defined on $\mathbf{R}^n$ is said to be in $\mathcal{S}_M(\mathbf{R}^n)$ where M is a positive integer, if $f(x)$ satisfies the following conditions:

(i) For $|\alpha| \le M-1$,

$$|D^\alpha f(x)| \le C\frac{1}{(1+|x|)^{n+M+|\alpha|}};$$

(ii) For $|x-x'| \le \frac{1}{2}(1+|x|)$ and $|\nu| = M$,

$$|D^\nu f(x) - D^\nu f(x')| \le C\frac{|x-x'|}{(1+|x|)^{n+2M}};$$

(iii) For $|\alpha| \le M-1$,

$$\int_{\mathbf{R}^n} f(x)x^\alpha dx = 0.$$

If $f \in \mathcal{S}_M(\mathbf{R}^n)$ the norm of f in $\mathcal{S}_M(\mathbf{R}^n)$ is then defined by

$$\|f\|_{\mathcal{S}_M(\mathbf{R}^n)} = \inf\big\{C : \text{(i) and (ii) hold}\big\}.$$

It is easy to check that $\mathcal{S}_M(\mathbf{R}^n)$ with this norm is a Banach space. Denote by $(\mathcal{S}_M(\mathbf{R}^n))'$ the dual of $\mathcal{S}_M(\mathbf{R}^n)$.

For $f \in (S_M(\mathbf{R}^n))'$, define the Littlewood–Paley square function of f by

$$g(f)(x) = \left\{\sum_{j\in\mathbf{Z}} |\psi_j * f(x)|^2\right\}^{\frac{1}{2}}, \tag{1.6}$$

where $\psi_j(x) = 2^{-jn}\psi(2^{-j}x)$.

Now we give the definition of one-parameter weighted Hardy spaces H^p_w on $\mathbf{R}^n$.

Definition 1.3 Let $0 < p < \infty$, $w \in A_\infty$. Let $M = [(2q_w/p - 1)n] + 1$, where $[\cdot]$ denotes the integer function. The one-parameter weighted Hardy spaces are defined by

$$H^p_w\big(\mathbf{R}^n\big) = \big\{f \in (\mathcal{S}_M)' : \ g(f) \in L^p_w\big(\mathbf{R}^n\big)\big\}$$

and the norm of f in $H^p_w(\mathbf{R}^n)$ is defined by

$$\|f\|_{H^p_w(\mathbf{R}^n)} = \|g(f)\|_{L^p_w(\mathbf{R}^n)}.$$

Our main result is the following theorem.

Theorem 1.1 *Let $w \in A_\infty$. The one-parameter singular integral operator T is bounded on H^p_w for $0 < p < \infty$ and bounded from H^p_w to L^p_w for $0 < p \le 1$. Namely, if r satisfies $\frac{n}{n+M} < r < \min\{\frac{p}{q_w}, 1\}$ and $q > q_w$ (where q_w is the critical index of the* *weight w defined above*), *then*

$$\|T(f)\|_{H^p_w} \leq C(n,p,r)K_1([w]_{A_{p/r}})\|f\|_{H^p_w}, \quad 0<p<\infty,$$

$$\|T(f)\|_{L^p_w(\mathbf{R}^n)} \leq C(n,p,q,r)K_1([w]_{A_q})^2 K_2([w]_{A_{p/r}})[w]_{A_q}^{\frac{1}{p}+\max\{1,\frac{q'}{2}\}}\|f\|_{H^p_w(\mathbf{R}^n)},$$
$$0<p\leq 1,$$

where M is the constant in Definition 1.2 and constants $K_1([w]_{A_q})$ and $K_2([w]_{A_{p/r}})$ are defined as follows:

$$K_1([w]_{A_q}) = \begin{cases} [w]_{A_q}^{\frac{1}{q-1}}, & \text{if } q\leq 2, \\ [w]_{A_q}, & \text{if } q>2, \end{cases} \tag{1.7}$$

and

$$K_2([w]_{A_{p/r}}) = \begin{cases} ([w]_{A_{p/r}})^{\frac{r}{p-r}}, & \text{if } p\leq 2, \\ ([w]_{A_{p/r}})^{\frac{r}{2-r}}, & \text{if } p>2. \end{cases} \tag{1.8}$$

In [4], the authors showed that the weighted Hardy spaces defined by discrete Littlewood–Paley operators are the same as the classical ones defined by a smooth maximal function (see [7] and [31]). Let $\varphi\in\mathcal{S}(\mathbf{R}^n)$ with $\int\varphi(x)dx=1$ and the maximal function defined as follows

$$f^*(x)=\sup_{t>0}|\varphi_t * f(x)|$$

where $\varphi_t(x)=t^{-n}\varphi(x/t)$. Then weighted Hardy space $\mathcal{H}^p_w(\mathbf{R}^n)$ consists of those tempered distributions for which $f^*\in L^p_w$ with $\|f\|_{\mathcal{H}^p_w}=\|f^*\|_{L^p_w}$.

We end the introduction with the following remarks. First of all, a sharp contrast with the weighted L^p boundedness results (where $w\in A_p$ was often required) is that we establish the weighted boundedness of singular integrals on Hardy spaces $H^p_w(\mathbf{R}^n)$ by only assuming $w\in A_\infty$. This also significantly improves the earlier known results on weighted Hardy spaces (see, e.g., [17]). This is accomplished by employing the discrete Littlewood–Paley analysis. We mention in passing that consideration of weighted Hardy spaces $H^p_w(\mathbb{R}^n)$ with $w\in A_\infty(\mathbb{R}^n)$ was given earlier; see [8, 33], and also the more recent work [2, 4, 18]. Second, we are not aware if our results of the operator norm bounds for the singular integrals are sharp or not. In particular, unlike in the case of L^p bounds ($1<p<\infty$) the definition of the weighted Hardy spaces depends on the choice of the Schwartz functions we use. Nonetheless, we are able to determine a nice bound when the definition is given in terms of the discrete Littlewood–Paley square functions. As a consequence, we are also deriving the bounds when an equivalent definition is taken into account using the discrete Littlewood–Paley analysis.

The organization of the paper is as follows. In Sect. 2, we first establish the discrete Calderón identity (Theorem 2.1). Then we prove that the weighted Hardy spaces are well defined by proving a Min-Max comparison principle with an explicit bound. Next, we obtain the bound control of the weighted L^p_w norms of a function in a dense class of H^p_w by their weighted H^p_w norms (Theorem 2.3). To do this, we need to establish an alternative discrete Calderón identity with Schwartz function with compact support. Finally, we prove Theorem 1.1 to conclude Sect. 2.

2 Boundedness on Weighted Hardy Spaces $H^p_w(\mathbf{R}^n)$

In this section, we shall prove the boundedness of singular integrals on weighted Hardy Spaces $H^p_w(\mathbf{R}^n)$. For our purpose, we introduce some new Littlewood–Paley g function. Let ϕ be a C_0^∞ function on $\mathbf{R}^n$ supported in the unit ball and satisfying condition (1.5) and

$$\int_{\mathbf{R}^n} \phi(x)x^\alpha dx = 0, \quad \text{for } |\alpha| \le M_0, \tag{2.1}$$

where $M_0 \ge M$ and M is the same as in the definition of H^p_w.

We introduce discrete Littlewood–Paley g function and its maximal analogue by

$$g_d(f)(x) = \left\{ \sum_j \sum_Q |\phi_j * f(x_Q)|^2 \chi_Q(x) \right\}^{\frac{1}{2}}$$

and

$$g_d^{\sup}(f)(x) = \left\{ \sum_j \sum_Q \sup_{u\in Q} |\phi_j * f(u)|^2 \chi_Q(x) \right\}^{\frac{1}{2}}$$

respectively, where x_Q is any point in Q, $\phi_j(x) = 2^{-jn}\phi(2^j x)$ and the summation of Q is taken over all dyadic cubes Q with side length 2^{-j-N} in $\mathbf{R}^n$ for each $j \in \mathbf{Z}$ and a fixed large integer N.

We need the following weighted Fefferman–Stein vector-valued inequality, for every $1 < p, r < \infty$, $w \in A_p$ (see [10], and also [1] for an earlier version of such an inequality without the explicit bounds):

$$\left\| \left(\sum_j |M(f_j)|^r \right)^{1/r} \right\|_{L^p_w} \le K\big(n, r, p, [w]_{A_p}\big) \left\| \left(\sum_j |f_j|^r \right)^{1/r} \right\|_{L^p_w} \tag{2.2}$$

for all sequence of functions $\{f_j\}$ in L^p_w, where if we set $N(t) = t^{\frac{1}{r-1}}$,

$$K(n, r, p, [w]_{A_p}) = \begin{cases} 2N(K_1(n, p, r)[w]_{A_p}^{\frac{r-1}{p-1}}), & \text{if } p < r, \\ 2^{\frac{pr}{r(p-1)}} N(K_2(n, p, r)[w]_{A_p}), & \text{if } p \ge r \end{cases}$$

and $K_1(n, p, r)$, $K_2(n, p, r)$ are constants that depend only on n, p, r.

Proposition 2.1 *If $M_0 \geq M$ in (2.1), then weighted Hardy spaces can be characterized by these discrete square functions. That is, for any $0 < p < \infty$,*

$$\|f\|_{H^p_w} \approx \|g_d(f)\|_{L^p_w}.$$

It was pointed out in [19] that if $1 < r < \infty$ and $w \in A_r$, we have the following weighted version of the Littlewood–Paley–Stein inequality:

$$\|g\|_{L^r_w \to L^r_w} \approx \|g_d\|_{L^r_w \to L^r_w} \leq C_n [w]_{A_r}^{\max\{1, \frac{r}{2}\}\frac{1}{r-1}}.$$

By the duality argument together with Calderón's identity, we also have

$$\|f\|_{L^r_w} \leq C'_n [w]_{A_r}^{\max\{1, \frac{r'}{2}\}} \|g(f)\|_{L^r_w}.$$

In fact, let $\sigma(x) = w(x)^{-r'/r}$, then $\sigma \in A_{r'}$.

$$\begin{aligned}
\|f\|_{L^r_w} &= \sup_{\|h\|_{L^{r'}_\sigma} \leq 1} \left| \int f(x)h(x)dx \right| = \sup_{\|h\|_{L^{r'}_\sigma} \leq 1} \left| \int \Big(\sum_j \psi_j * \psi_j * f \Big)(x)h(x)dx \right| \\
&\leq \sup_{\|h\|_{L^{r'}_\sigma} \leq 1} \int g(f)(x)g(h)(x)dx \leq \sup_{\|h\|_{L^{r'}_\sigma} \leq 1} \|g(f)\|_{L^r_w} \|g(h)\|_{L^{r'}_\sigma} \\
&\leq C'_n [\sigma]_{A_{r'}}^{\max\{1, \frac{r'}{2}\}\frac{1}{r'-1}} \|g(f)\|_{L^r_w} = C'_n [w]_{A_r}^{\max\{1, \frac{r'}{2}\}} \|g(f)\|_{L^r_w}.
\end{aligned}$$

Lemma 2.1 [11] *If ψ and ϕ are in the class $\mathcal{S}_M(\mathbf{R}^n)$, then for any given positive integers L, K, there exists a constant $C = C(L, K)$ depending only on L, K such that*

$$|\psi_t * \phi_{t'}(x)| \leq C \left(\frac{t}{t'} \wedge \frac{t'}{t} \right)^L \frac{(t \vee t')^K}{(t \vee t' + |x|)^{n+K}}.$$

Lemma 2.2 *Let I, I' be dyadic cubes in $\mathbf{R}^n$ such that $l(I) = 2^{-j-N}$, $l(I') = 2^{-j'-N}$. Then for any $u, u^* \in I$ and any r satisfying $\frac{n}{n+K} < r \leq 1$, we have*

$$\begin{aligned}
&\sum_{I'} \frac{2^{-|j-j'|L} |I'| 2^{-(j \wedge j')K}}{(2^{-j \wedge j'} + |u - x_{I'}|)^{n+K}} |\psi_{j'} * f(x_{I'})| \\
&\quad \leq C 2^{-|j-j'|L} 2^{(\frac{1}{r}-1)Nn} 2^{(\frac{1}{r}-1)n(j'-j)_+} \left(M \Big(\sum_{I'} |\psi_{j'} * f(x_{I'})| \chi_{I'} \Big)^r (u^*) \right)^{1/r},
\end{aligned}$$

where $(j' - j)_+ = \max\{j' - j, 0\}$, $x_{I'} \in I'$ and C is a constant depending on dimension n.

Proof We set

$$A_0 = \left\{ I' : l(I') = 2^{-j'-N},\ \frac{|u - x_{I'}|}{2^{-j\wedge j'}} \le 1 \right\},$$

and for $l \ge 1$,

$$A_l = \left\{ I' : l(I') = 2^{-j'-N},\ 2^{l-1} < \frac{|u - x_{I'}|}{2^{-j\wedge j'}} \le 2^l \right\}.$$

Then

$$\begin{aligned}
&\sum_{I'} \frac{2^{-|j-j'|L}|I'|2^{-(j\wedge j')K}}{(2^{-j\wedge j'} + |u - x_{I'}|)^{n+K}} |\psi_{j'} * f(x_{I'})| \\
&\le \sum_{l\ge 0} 2^{-l(n+K)} 2^{-|j-j'|L} 2^{-n(j'+N)} 2^{(j\wedge j')n} \sum_{I'\in A_l} |\psi_{j'} * f(x_{I'})| \\
&\le \sum_{l\ge 0} 2^{-l(n+K)} 2^{-|j-j'|L} 2^{-n(j'+N)} 2^{(j\wedge j')n} \left(\sum_{I'\in A_l} |\psi_{j'} * f(x_{I'})|^r \right)^{1/r} \\
&= \sum_{l\ge 0} 2^{-l(n+K)} 2^{-|j-j'|L} 2^{-n(j'+N)} 2^{(j\wedge j')n} \\
&\quad \times \left(\int_{I'} |I'|^{-1} \sum_{I'\in A_l} |\psi_{j'} * f(x_{I'})|^r \chi_{I'} \right)^{1/r} \\
&\le \sum_{l\ge 0} 2^{-|j-j'|L} 2^{-l(n+K-\frac{n}{r})} 2^{(\frac{1}{r}-1)Nn} 2^{(\frac{1}{r}-1)n(j'-j)_+} \\
&\quad \times \left(M\left(\sum_{I'\in A_l} |\psi_{j'} * f(x_{I'})|^r \chi_{I'} \right)(u^*) \right)^{1/r} \\
&= C 2^{-|j-j'|L} 2^{(\frac{1}{r}-1)Nn} 2^{(\frac{1}{r}-1)n(j'-j)_+} \left(M\left(\sum_{I'} |\psi_{j'} * (x_{I'})| \chi_{I'} \right)^r (u^*) \right)^{1/r}
\end{aligned}$$

the last inequality follows from the assumption that $r > \frac{n}{n+K}$ which can be done by choosing K big enough. □

With the almost orthogonality estimate (Lemma 2.1) and Lemma 2.2, we now give the following discrete Calderón reproducing formula.

Theorem 2.1 *Suppose that* ψ_j *is the same as in* (1.6). *Then for any* $M \ge 1$, *we can choose a large* N *depending on* M *and* ψ *such that the following discrete Calderón reproducing identity*:

$$f(x) = \sum_j \sum_I |I| \tilde{\psi}_j(x, x_I) \psi_j * f(x_I) \tag{2.3}$$

holds in $\mathcal{S}_M(\mathbf{R}^n)$ *and in the dual space* $(\mathcal{S}_M)'$, *where* $\tilde{\psi}_j(x, x_I) \in \mathcal{S}_M(\mathbf{R}^n)$, I'*s are dyadic cubes with side-length* $l(I) = 2^{-j-N}$, *and* x_I *is a fixed point in* I.

Proof For $f \in \mathcal{S}_M$, we use the discrete Calderón identity $f(x) = \sum_j \psi_j * \psi_j * f(x)$ as follows. We rewrite

$$f(x) = \sum_j \sum_I \int_I \psi_j(x-u)(\psi_j * f)(u)du$$
$$= \sum_j \sum_I \left[\int_I \psi_j(x-u)du\right](\psi_j * f)(x_I) + \mathcal{R}(f)(x).$$

We shall show that $\mathcal{R}$ is bounded on $\mathcal{S}_M$ with a small operator norm when the I's are dyadic cubes with side-length 2^{-j-N} for a large N, and $x_I \in I$.

$$\mathcal{R}(f)(x) = \sum_{j,I} \int_I \psi_j(x-u)[(\psi_j * f)(u) - (\psi_j * f)(x_I)]du$$
$$= \sum_{j,I} \int_I \psi_j(x-u)\bigg(\int \psi_j(u-u')f(u')du'$$
$$- \int \psi_j(x_I - u')f(u')du'\bigg)du$$
$$= \int \bigg(\sum_{j,I} \int_I \psi_j(x-u)[\psi_j(u-u') - \psi_j(x_I-u')]du f(u')\bigg)du'$$
$$= \int \mathcal{R}(x, u', x_I) f(u')du'$$

where $\mathcal{R}(x, u', x_I)$ is the kernel of $\mathcal{R}$. It is not difficult to check that

$$\sum_I \int_I \psi_j(x-u)\big[\psi_j\big(u-u'\big) - \psi_j\big(x_I - u'\big)\big]du$$

satisfies all conditions of $\psi_j(x - x_I)$ but with the constant of $\mathcal{S}_M(\mathbf{R}^n)$ norm replaced by $C2^{-N}$. This follows from the smooth conditions of ψ_j and the fact that u, $x_I \in I$, $l(I) = 2^{-j-N}$. Then $\mathcal{R}(f)(x) \in \mathcal{S}_M(\mathbf{R}^n)$ and

$$\|\mathcal{R}(f)\|_{\mathcal{S}_M(\mathbf{R}^n)} \le C2^{-N}\|f\|_{\mathcal{S}_M(\mathbf{R}^n)}. \tag{2.4}$$

Thus if we set

$$T(f)(x) = \sum_j \sum_I \left[\int_I \psi(x-u)du\right](\psi_j * f)(x_I),$$

then $T^{-1} = (I - \mathcal{R})^{-1}$ exists and

$$f(x) = T^{-1}T(f)(x) = \sum_{i=0}^{\infty} \mathcal{R}^i T(f)(x)$$
$$= \sum_{j,I} \left[\sum_{i=0}^{\infty} \mathcal{R}^i \int_I \psi_j(\cdot - u)du\right](x)(\psi_j * f)(x_I).$$

Set $[\sum_{i=0}^{\infty}\mathcal{R}^i\int_I\psi_j(\cdot-u)du](x)=|I|\tilde{\psi}_j(x,x_I)$. Then it follows from (2.4) that $\tilde{\psi}_j\in\mathcal{S}_M(\mathbf{R}^n)$. Thus, the discrete Calderón identity on $\mathcal{S}_M(\mathbf{R}^n)$ is obtained. The proof of Theorem 2.1 is completed from the duality argument. □

Now we are ready to give the following Plancherel–Pôlya-type inequality, i.e., the Min-Max inequality.

Theorem 2.2 *Let $\psi,\varphi\in\mathcal{S}_M(\mathbf{R}^n)$. Suppose ψ_j and φ_j satisfy the same conditions as in (1.6). Then for $0<p<\infty$, $w\in A_\infty$, $f\in(\mathcal{S}_M)'(\mathbf{R}^n)$, and for any r satisfying $\frac{n}{n+K}<r<\min\{\frac{p}{q_w},1\}$,*

$$\begin{aligned}&\left\|\left\{\sum_{j,I}\inf_{u\in I}|\varphi_j*f(u)|^2\chi_I(x)\right\}^{1/2}\right\|_{L^p_w}\\&\le\left\|\left\{\sum_{j,I}\sup_{u\in I}|\psi_j*f(u)|^2\chi_I(x)\right\}^{1/2}\right\|_{L^p_w}\\&\le C(n,p,r)K_2([w]_{A_{p/r}})\left\|\left\{\sum_{j,I}\inf_{u\in I}|\varphi_j*f(u)|^2\chi_I(x)\right\}^{1/2}\right\|_{L^p_w}\end{aligned}$$

where $I\in\mathbf{R}^n$ are dyadic cubes with side-length $l(I)=2^{-j-N}$ for a fixed large integer N, and $K_2([w]_{A_{p/r}})$ is as given in (1.8).

Proof The discrete Calderón reproducing formula (2.3) on $\mathcal{S}_M(\mathbf{R}^n)$ implies that

$$(\psi_j*f)(u)=\sum_{j',I'}|I'|(\psi_j*\tilde{\varphi}_j)(u,x_{I'})(\varphi_j*f)(x_{I'}).$$

From the almost orthogonality estimates in Lemma 2.1 by choosing $t=2^{-j}, t'=2^{-j'}$ and from Lemma 2.2, we have that for any given positive integers L,K and for any $u,u^*\in I$,

$$\begin{aligned}|\psi_j*f(u)|&\le C\sum_{j',I'}\frac{2^{-|j-j'|L}2^{-(j\wedge j')K}|I'|}{(2^{-j\wedge j'}+|u-x_I|)^{n+K}}|\varphi_{j'}*f(x_{I'})|\\&\le C\sum_{j'}2^{-|j-j'|L}\left(M\left[\left(\sum_{I'}|\varphi_{j'}*f(x_I)|\chi_{I'}\right)^r\right]\right)^{1/r}(u^*).\end{aligned}$$

Summing over j,I yields that

$$\left(\sum_{j,I}\sup_{u\in I}|\psi_j*f(u)|^2\chi_I\right)^{1/2}\le\left(\sum_{j'}\left\{M\left[\sum_{I'}|\varphi_{j'}*f(x_{I'})|\chi_{I'}\right]^r\right\}^{2/r}\right)^{1/2}.$$

Since $\frac{n}{n+K}<r<\min\{\frac{p}{q_w},1\}$, it means that $q_w<p/r$, we have $w\in A_{p/r}$. The Hölder inequality and the $L^{p/r}_w(l^{2/r})$ boundedness of M, i.e., the weighted

Fefferman–Stein vector-valued inequality (2.2), yield

$$\begin{aligned}&\left\|\left(\sum_{j,I}\sup_{u\in I}|\psi_j * f(u)|^2\chi_I\right)^{1/2}\right\|_{L^p_w}\\&\le C\left\|\left(\sum_{j'}\left\{M\left[\sum_{I'}\inf_{u\in I'}|\varphi_{j'} * f(u)|\chi_{I'}\right]^r\right\}^{2/r}\right)^{1/2}\right\|_{L^p_w}\\&\le C(n,p,r)K_2([w]_{A_{p/r}})\left\|\left(\sum_{j',I'}\inf_{u\in I'}|\varphi_{j'} * f(u)|^2\chi_{I'}\right)^{1/2}\right\|_{L^p_w},\end{aligned}$$

where we use the fact that $x_{I'}$ is arbitrary in I'. □

From this theorem, we know that the definition of weighted Hardy spaces is independent of the particular choice of ψ_j. Moreover, it can be characterized by the discrete Littlewood–Paley square function defined by

$$\mathcal{G}^d(f)(x) = \left\{\sum_{j,I}|\psi_j * f(x_I)|^2\chi_I(x)\right\}^{\frac{1}{2}}, \quad x_I \in I.$$

That is, a distribution f belongs to $H^p_w(\mathbf{R}^n)$ if and only if $\mathcal{G}^d(f)\in L^p_w(\mathbf{R}^n)$, and

$$\|f\|_{H^p_w(\mathbf{R}^n)} \approx \|\mathcal{G}^d(f)\|_{L^p_w(\mathbf{R}^n)}.$$

Proposition 3.2 in [4] tell us that when $w\in A_\infty$, $\mathcal{S}_M(\mathbf{R}^n)$ is dense in $H^p_w(\mathbf{R}^n)$ for $0<p<\infty$.

Theorem 2.3 *If $f\in L^2(\mathbf{R}^n)\cap H^p_w(\mathbf{R}^n)$, $0<p\le 1$, then $f\in L^p_w(\mathbf{R}^n)$, and there exists a constant $C(n,p,q)>0$ such that*

$$\|f\|_{L^p_w(\mathbf{R}^n)} \le C(n,p,q)K_1\big([w]_{A_q}\big)^2[w]_{A_q}^{\frac{1}{p}+\max\{1,\frac{q'}{2}\}}\|f\|_{H^p_w(\mathbf{R}^n)},$$

where q is fixed such that $q>q_w$, $K_1([w]_{A_q})$ is given as in (1.7).

To prove this theorem, we need a new version of Calderón-type identity. To be more precise, take $\phi\in C_0^\infty$ with

$$\int_{\mathbf{R}^n}\phi(x)x^\alpha dx = 0, \quad \text{for all } \alpha \text{ satisfying } 0\le|\alpha|\le M_0$$

where M_0 is a large positive integer which will be determined later (indeed $M_0 > (2q_w/p-1)n$ suffices), and

$$\sum_j|\hat{\phi}(2^{-j}\xi)|^2 = 1, \quad \text{for all } \xi\in\mathbf{R}^n\setminus\{0\}.$$

Moreover, we may assume that ϕ is supported in the unit ball of $\mathbf{R}^n$.

We need a discrete Calderón reproducing formula in terms of ϕ.

Lemma 2.3 *There exists an operator T_N^{-1} such that*

$$f(x)=\sum_{j,I}|I|\tilde{\phi}_j(x-x_I)\phi_j*\big(T_N^{-1}(f)\big)(x_I) \tag{2.5}$$

where T_N^{-1} is bounded on $L^2(\mathbf{R}^n)$ and $H_w^p(\mathbf{R}^n)$, $0<p<\infty$, and the series converges in $L^2(\mathbf{R}^n)$.

Proof As in the proof of Theorem 2.1, for $f\in L^2(\mathbf{R}^n)$, the operator $\mathcal{R}$ is defined by the following:

$$\begin{aligned}f(x)&=\sum_j\sum_I\int_I\psi(x-u)(\psi_j*f)(u)du\\&=\sum_j\sum_I\bigg[\int_I\psi_j(x-u)du\bigg](\psi_j*f)(x_I)+\mathcal{R}(f)(x).\end{aligned}$$

We claim that for $0<p<\infty$, there is a constant $C>0$ such that

$$\|\mathcal{R}(f)\|_{L^2(\mathbf{R}^n)}\le C2^{-N}\|f\|_{L^2(\mathbf{R}^n)},$$

and

$$\|\mathcal{R}(f)\|_{H_w^p(\mathbf{R}^n)}\le C2^{-N}K_2\big([w]_{A_{p/r}}\big)\|f\|_{H_w^p(\mathbf{R}^n)}.$$

Assume the claim for the moment. Set

$$T_N(f)(x)=\sum_{j,I}\bigg[\int_I\phi_j(x-u)du\bigg](\phi_j*f)(x_I).$$

The proof in Theorem 2.1 shows that if N is large enough, then both T_N and $(T_N)^{-1}$ are bounded on $H_w^p(\mathbf{R}^n)\cap L^2$. Thus,

$$f(x)=\sum_{j,I}|I|\tilde{\phi}_j(x-x_I)\big(\psi*T_N^{-1}(f)\big)(x_I),$$

where $\tilde{\phi}_j\in\mathcal{S}_M(\mathbf{R}^n)$ and the series converges in $L^2(\mathbf{R}^n)$.

Now we prove the claim. Suppose $f\in L^2(\mathbf{R}^n)$. By Theorem 2.1,

$$\begin{aligned}\|\mathcal{G}(\mathcal{R}(f))\|_{L_w^p}&\le C\bigg\|\bigg\{\sum_{j,I}|\psi_j*\mathcal{R}(f)|^2\chi_I\bigg\}^{1/2}\bigg\|_{L_w^p}\\&=C\bigg\|\bigg\{\sum_{j,I}\sum_{j'I'}|I'|\psi_j*\mathcal{R}(\tilde{\psi}_{j'}(\cdot,x_{I'})\cdot(\psi_{j'}*f)(x_{I'}))|^2\chi_I\bigg\}^{1/2}\bigg\|_{L_w^p}.\end{aligned}$$

By the almost orthogonality estimate

$$\left|\left(\psi_j * \mathcal{R}\left(\tilde{\psi}_{j'}(\cdot, x_{I'})\right)\right)(x)\right| \le C2^{-N}2^{-|j-j'|M} \frac{2^{-(j\wedge j')M}}{(2^{-j\wedge j'}+|x-x_{I'}|)^{n+M}}.$$

Then from Lemma 2.2, Hölder's inequality, and the $L_w^{p/r}(l^{2/r})$ boundedness of the maximal operator ($w \in A_{p/r}$), we have

$$\begin{aligned}
\|\mathcal{R}(f)\|_{H_w^p} &\le 2^{-N}\left\|\left(\sum_{j'}\left[M\left(\sum_{I'}|\psi_j * f(x_{I'})|\chi_{I'}\right)^r\right]^{2/r}\right)^{1/2}\right\|_{L_w^p} \\
&\le C2^{-N}K_2([w]_{A_{p/r}})\left\|\left(\sum_{j'I'}|\psi_{j'} * f(x_{I'})|^2\chi_{I'}\right)^{1/2}\right\|_{L_w^p} \\
&\le C2^{-N}K_2([w]_{A_{p/r}})\|f\|_{H_w^p}.
\end{aligned}$$

Another inequality in the claim follows immediately by taking $w=1$ and $p=2$ in the above inequality. Then the proof of Lemma 2.3 is completed. □

Using a similar argument as in the proof of Theorem 2.2, we can get

Corollary 2.1 *Suppose $w \in A_\infty$. If $f \in L^2 \cap H_w^p(\mathbf{R}^n)$, $0<p<\infty$, then*

$$\|f\|_{H_w^p(\mathbf{R}^n)} \approx \left\|\left\{\sum_{j,I}\left|\left(\phi_j * T_N^{-1}(f)\right)(x_I)\right|^2\chi_I\right\}^{1/2}\right\|_{L_w^p}.$$

Now we are ready to prove Theorem 2.3.

Proof of Theorem 2.3 We may assume $w \in A_q$ for some $2<q<\infty$. Define a square function by

$$\tilde{g}(f)(x) = \left\{\sum_{j,I}\left|\phi_j * \left(T_N^{-1}(f)(x_I)\right)\right|^2\chi_I(x)\right\}^{1/2}.$$

By Corollary 2.1, for $f \in L^2 \cap H_w^p$, we have

$$\|\tilde{g}(f)\|_{L_w^p} \le C\|f\|_{H_w^p}.$$

Let $f \in L^2 \cap H_w^p$, set

$$\Omega_i = \left\{x \in \mathbf{R}^n : \tilde{g}(f)(x) > 2^i\right\}.$$

Denote

$$B_i = \left\{(j,I) : w(I\cap\Omega_i) > \frac{1}{2}w(I),\ w(I\cap\Omega_{i+1}) \le \frac{1}{2}w(I)\right\}$$

where I are dyadic cubes with side-length $l(I)=2^{-j-N}$.

We use ϕ_I to denote ϕ_j when $l(I)=2^{-j-N}$. By the discrete Calderón reproducing formula (2.5), we can write

$$f(x)=\sum_i\sum_{(j,I)\in B_i}|I|\tilde{\phi}_I(x-x_I)\phi_I*\big(T_N^{-1}(f)\big)(x_I)$$

where the series converges in L^2 norm and hence almost everywhere and also w almost everywhere.

We claim

$$\begin{aligned}&\Big\|\sum_{(j,I)\in B_i}|I|\tilde{\phi}_I(x-x_I)\phi_I*\big(T_N^{-1}(f)\big)(x_I)\Big\|_{L_w^p}^p\\&\quad\le C(n,p,q)K_1\big([w]_{A_q}\big)^{2p}[w]_{A_q}^{1+\max\{1,q'/2\}p}2^{pi}w(\Omega_i)\end{aligned}\tag{2.6}$$

which together with the fact that $0<p\le 1$ yields

$$\begin{aligned}\|f\|_{L_w^p}^p&\le\sum_i\Big\|\sum_{(j,I)\in B_i}|I|\tilde{\phi}_I(x-x_I)\phi_I*(T_N^{-1}(f))(x_I)\Big\|_{L_w^p}^p\\&\le CK_1([w]_{A_p})^{2p}[w]_{A_q}^{1+\max\{1,q'/2\}p}\sum_i 2^{pi}w(\Omega_i)\\&\le CK_1([w]_{A_p})^{2p}[w]_{A_q}^{1+\max\{1,q'/2\}p}\|\tilde{g}(f)\|_{L_w^p}^p\\&\le CK_1([w]_{A_p})^{2p}[w]_{A_q}^{1+\max\{1,q'/2\}p}\|f\|_{H_w^p}^p.\end{aligned}$$

To finish the proof, it remains to show the claim (2.6). Note that for $(j,I)\in B_i$, if $x\in I$, then $M(\chi_{\Omega_i})(x)\ge 1/2$. And note that if ϕ is supported in the unit ball, then $\phi_j(x-x_I)$ is supported in $\tilde{\Omega}_i=\{x: M(\chi_{\Omega_i})(x)>\frac{1}{100}\}$. Thus for any fixed $q>q_w$, by Hölder's inequality,

$$\begin{aligned}&\Big\|\sum_{(j,I)\in B_i}|I|\tilde{\phi}_I(x-x_I)\phi_I*(T_N^{-1}(f))(x_I)\Big\|_{L_w^p}^p\\&\quad\le Cw(\tilde{\Omega}_i)^{1-p/q}\Big\|\sum_{(j,I)\in B_i}|I|\tilde{\phi}_I(x-x_I)\phi_I*(T_N^{-1}(f))(x_I)\Big\|_{L_w^q}^p.\end{aligned}$$

By the duality argument, for all $h\in L^{q'}(w^{1-q'})$ with $\|h\|_{L^{q'}(w^{1-q'})}\le 1$.

$$\begin{aligned}&\Big|\Big\langle\sum_{(j,I)\in B_i}|I|\tilde{\phi}_I(x-x_I)\phi_I*(T_N^{-1}(f))(x_I),\,h\Big\rangle\Big|\\&\quad=\Big|\Big\langle\sum_{(j,I)\in B_i}|I|(\tilde{\phi}_I*h(x_I)),(\phi_I*(T_N^{-1}(f))(x_I))\Big\rangle\Big|\end{aligned}$$

$$
\begin{aligned}
&= \left| \sum_{(j,I)\in B_i} \int (\tilde{\phi}_I * h(x_I))(\phi_I * (T_N^{-1}(f))(x_I))\chi_I(x)dx \right| \\
&\le \left(\int \left(\sum_{(j,I)\in B_i} |\phi_I * (T_N^{-1}(f))(x_I)|^2 \chi_I(x) \right)^{q/2} w(x)dx \right)^{1/q} \\
&\quad \times \left(\int \left(\sum_{(j,I)\in B_i} |\tilde{\phi}_I * h(x_I)|^2 \chi_I(x) \right)^{q'/2} w(x)^{1-q'}dx \right)^{1/q'} \\
&= \Lambda_1 \cdot \Lambda_2.
\end{aligned}
$$

We first estimate Λ_2. Since $w \in A_q$ implies $w^{1-q'} \in A_{q'}$, by the weighted Fefferman–Stein inequality, we have the following estimate:

$$
\begin{aligned}
\Lambda_2 &\le \left\| \left\{ \sum_{(j,I)\in B_i} \big(M(\tilde{\phi}_I * h)\big)^2 \chi_I(x) \right\}^{1/2} \right\|_{L^{q'}(w^{1-q'})} \\
&\le C_1 K_1\big([w]_{A_q}\big) \|g(h)\|_{L^{q'}(w^{1-q'})} \le C_1 K_1\big([w]_{A_q}\big) [w]_{A_q}^{\max\{1,q'/2\}}. \qquad (2.7)
\end{aligned}
$$

As for Λ_1, since $\chi_I(x) \le 2M(\chi_{I\cap(\tilde{\Omega}_i \setminus \Omega_{i+1})})(x)$, then using the weighted Fefferman–Stein inequality (2.2) again, we have

$$
\begin{aligned}
\Lambda_1^q &= \left\| \left\{ \sum_{(j,I)\in B_i} |\phi_I * T_N^{-1}(f)(x_I)|^2 \chi_I(x) \right\}^{1/2} \right\|_{L^q(w)}^q \\
&= \int \left\{ \sum_{(j,I)\in B_i} |\phi_I * T_N^{-1}(f)(x_I)|^2 \chi_I(x) \right\}^{q/2} w(x)dx \\
&\le C \int \left\{ \sum_{(j,I)\in B_i} |\phi_I * T_N^{-1}(x_I) M(\chi_{I\cap \tilde{\Omega}_i \setminus \Omega_{i+1}})(x)|^2 \right\}^{q/2} w(x)dx \\
&\le C_2 K_1([w]_{A_q})^q \int_{\tilde{\Omega}_i \setminus \Omega_{i+1}} \left\{ \sum_{(j,I)\in B_i} |\phi_I * T_N^{-1}(x_I)|^2 \chi_I(x) \right\}^{q/2} w(x)dx \\
&\le C_2 K_1([w]_{A_q})^q 2^{iq} w(\tilde{\Omega}_i). \qquad (2.8)
\end{aligned}
$$

Note that $\Omega_i \subset \tilde{\Omega}_i$, and by the weak $L^q(w)$ boundedness of the maximal operator, $w(\tilde{\Omega}_i) \le C[w]_{A_q} w(\Omega_i)$. Combining these estimates for Λ_1 and Λ_2 proves claim (2.6). Thus we complete the proof of Theorem 2.3. □

Proof of Theorem 1.1 Since $L^2 \cap H_w^p$ is dense in H_w^p, by the standard density argument, we assume $f \in L^2 \cap H_w^p(\mathbf{R}^n)$. Using Lemma 2.3 and Corollary 2.1, we have for $0 < p < \infty$

$$\begin{aligned}
\|T(f)\|_{H^p_w} &\le \left\| \left\{ \sum_{j,I} |\phi_j * K * f(x)|^2 \chi_I(x) \right\}^{1/2} \right\|_{L^p_w} \\
&= C \left\| \left\{ \sum_{j,I} \left[\sum_{j'I'} |I'| (\phi_j * K * \tilde{\phi}_{j'}(\cdot - x_{I'}))(x) \right.\right.\right. \\
&\qquad \left.\left.\left. \times (\phi_{j'} * T_N^{-1}(f))(x_{I'}) \right]^2 \chi_I(x) \right\}^{1/2} \right\|_{L^p_w} \\
&\le C \left\| \left(\sum_{j'} \left\{ M \left[\sum_{I'} |\phi_{j'} * (T_N^{-1}(f))(x_{I'})| \chi_{I'} \right]^r \right\}^{2/r} \right)^{1/2} \right\|_{L^p_w} \\
&\le C K_2([w]_{A_{p/r}}) \left\| \left(\sum_{j',I'} |\phi_{j'} * (T_N^{-1}(f))(x_{I'})|^2 \chi_{I'} \right)^{1/2} \right\|_{L^p_w} \\
&\le C K_2([w]_{A_{p/r}}) \|f\|_{H^p_w}
\end{aligned}$$

where in the second-to-the-last inequality, we use the following almost orthogonality estimate:

$$\left| \left(\phi_j * \tilde{\phi}_j(\cdot - x_{I'}) \right)(x) \right| \le C 2^{-|j-j'|M} \frac{2^{-(j\wedge j')M}}{(2^{-(j\wedge j')} + |x - x_{I'}|)^{n+M}}.$$

When $0 < p \le 1$, since T is bounded on $L^2 \cap H^p_w(\mathbf{R}^n)$, we have $T(f) \in L^2 \cap H^p_w(\mathbf{R}^n)$ whenever $f \in L^2 \cap H^p_w(\mathbf{R}^n)$. Thus from Theorem 2.3,

$$\begin{aligned}
\|T(f)\|_{L^p_w(\mathbf{R}^n)} &\le C K_1([w]_{A_q})^2 [w]_{A_q}^{\frac{1}{p} + \max\{1, \frac{q'}{2}\}} \|T(f)\|_{H^p_w(\mathbf{R}^n)} \\
&\le C K_1([w]_{A_q})^2 K_2([w]_{A_{p/r}}) [w]_{A_q}^{\frac{1}{p} + \max\{1, \frac{q'}{2}\}} \|f\|_{H^p_w(\mathbf{R}^n)}.
\end{aligned}$$

By a density argument again, we complete the proof. □

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