Calc. Var. (2011) 42:563–577
DOI 10.1007/s00526-011-0398-7

Calculus of Variations

Symmetry and regularity of extremals of an integral equation related to the Hardy–Sobolev inequality

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Received: 20 July 2010 / Accepted: 18 December 2010 / Published online: 28 January 2011

Abstract Let α be a real number satisfying $0 < \alpha < n$, $0 \leq t < \alpha, \alpha^*(t) = \frac{2(n-t)}{n-\alpha}$. We consider the integral equation

$$u(x) = \int_{\mathbb{R}^n} \frac{u^{\alpha^*(t)-1}(y)}{|y|^t|x-y|^{n-\alpha}}\,dy, \tag{1}$$

which is closely related to the *Hardy–Sobolev* inequality. In this paper, we prove that every positive solution $u(x)$ is radially symmetric and strictly decreasing about the origin by the method of moving plane in integral forms. Moreover, we obtain the regularity of solutions to the following integral equation

$$u(x) = \int_{\mathbb{R}^n} \frac{|u(y)|^p u(y)}{|y|^t|x-y|^{n-\alpha}}\,dy \tag{2}$$

that corresponds to a large class of PDEs by regularity lifting method.

Mathematics Subject Classification (2000) 35B06 · 35B65

1 Introduction

In this paper, we consider a class of integral equations related to the *Hardy–Sobolev* inequality. Let α be a real number satisfying $0 < \alpha < n$, $0 \leq t < \alpha, \alpha^*(t) = \frac{2(n-t)}{n-\alpha}$. Let $\mathbb{R}^n$ be

Communicated by L. Ambrosio.

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n-dimensional Euclidean space. The main purpose of this paper is to study the symmetry and regularity of extremals of the following integral equation:

$$u(x) = \int_{\mathbb{R}^n} \frac{u^{\alpha^*(t)-1}(y)}{|y|^t |x-y|^{n-\alpha}}\, dy. \tag{3}$$

We will show that solutions to the following differential equation of fractional order satisfy the above integral equation (3):

$$\begin{cases} (-\triangle)^{\frac{\alpha}{2}} u = \frac{u^{\alpha^*(t)-1}}{|y|^t} & \text{in } \mathbb{R}^n, \\ u > 0, \\ u \in H^{\frac{\alpha}{2},2}(\mathbb{R}^n), \end{cases} \tag{4}$$

where $\|u\|_{H^{\frac{\alpha}{2},2}} = \|((1+|.|^2)^{\frac{\alpha}{4}} \hat{u})^{\vee}\|_{L^2(\mathbb{R}^n)}$.

When $t = 0$, (3) becomes

$$u(x) = \int_{\mathbb{R}^n} \frac{u^{\frac{n+\alpha}{n-\alpha}}(y)}{|x-y|^{n-\alpha}}\, dy. \tag{5}$$

The integral equation (5) arises as an Euler-Lagrange equation in the context of the *Hardy–Littlewood–Sobolev* inequality which has been extensively explored by Chen, Li and Ou [10] and Li [17] recently, where regularity and extremal functions of (5) are also obtained. (5) is actually equivalent to the following partial differential equation

$$(-\triangle)^{\frac{\alpha}{2}} u = u^{\frac{n+\alpha}{n-\alpha}}. \tag{6}$$

When $\alpha = 2$ and $0 \le t < 2$, the differential equation (4) and the integral equation (3) are closely related to the well-known *Hardy–Sobolev* type inequalities. Such inequalities of second order have been extensively studied by many authors (see [3,6,7,13,20] and the references therein).

In fact, when $\alpha = 2$ and $t = 0$, (3) is closely related to the Euler-Lagrange equation of the extremal functions of the classical *Sobolev* inequality. Namely, (3) turns out to be the corresponding integral equation of the well-known *Sobolev* inequality

$$\left(\int_{\mathbb{R}^n} |u|^{\frac{2n}{n-2}}\, dy \right)^{\frac{n-2}{2n}} \le C \left(\int_{\mathbb{R}^n} |\triangledown u(y)|^2\, dy \right)^{\frac{1}{2}}$$

where the best constant C and the extremal functions were identified by Aubin [2] and Talenti [23].

Moreover, when $\alpha = 2$ and $0 < t < 2$, (3) is closely related to the Euler–Lagrange equation of the extremal functions of the classical *Hardy–Sobolev* inequality which is a special case of *Caffarelli–Kohn–Nirenberg* inequality [5]. The classical *Hardy–Sobolev* inequality is stated as follows: There is a positive constant C such that

$$\left(\int_{\mathbb{R}^n} \frac{|u|^{2^*(t)}}{|y|^t} dy \right)^{\frac{2}{2*(t)}} \le C \int_{\mathbb{R}^n} |\triangledown u(y)|^2\, dy \tag{7}$$

for any $u(x) \in D^{1,2}(\mathbb{R}^n)$. Furthermore, the best constant in (7) is achieved and the extremal function is identified by Lieb [19] up to dilation and translation by

$$u(x) = \frac{1}{(1 + |x|^{2-t})^{\frac{n-2}{2-t}}}.$$

We should note that the inequality (7) is a special case of the following more general *Caffarelli–Kohn–Nirenberg* inequality [5]: If $0 \le a < \frac{n-2}{2}$, $a \le b < a + 1$, $p = \frac{2n}{n-2+2(b-a)}$, then

$$\left(\int\limits_{\mathbb{R}^n} \frac{|u(x)|^p}{|x|^{bp}} dx \right)^{\frac{2}{p}} \le C_{a,b} \int\limits_{\mathbb{R}^n} |x|^{-2a} |\nabla u(x)|^2 dx \tag{8}$$

for any $u \in C_0^\infty(\mathbb{R}^n)$. Positive solutions of the associated Euler equation to (8) on an appropriate weighted Sobolev space turn out to be symmetric and they can be identified explicitly by solving an ODE and they are of the same form as in the case $a = 0$. We refer the reader to [6,7], etc.

In Sect. 1, we prove the following theorem by the moving plane method in integral forms.

Theorem 1 *Assume $u(x) \in L^{\frac{2n}{n-\alpha}}(\mathbb{R}^n)$ is a positive solution of* (3)*, then $u(x)$ is radially symmetric and strictly decreasing about the origin.*

In fact, we will derive the same result under the weaker condition that $u(x) \in L_{loc}^{\frac{2n}{n-\alpha}}(\mathbb{R}^n)$ as indicated in Remark 2.1.

It is well known that the moving plane method was invented by the Soviet mathematician Alexandrov in the 1950s. Then it was further developed by Serrin [21], Gidas et al. [15], Caffarelli et al. [4], Chen and Li [8], Chang and Yang [14] and many others. Recently, Chen et al. [10] applied the moving plane method to integral equations to obtain the symmetry, monotonicity and nonexistence properties of the solutions to the integral equations, see also Li [17] using moving sphere method. Instead of the extensive use of maximum principle of differential equation, moving plane method in integral form explores various specific features of the integral equation itself. By virtue of *Hardy–Littlewood–Sobolev* inequality or *Weighted–Hardy–Littlewood–Sobolev* inequality and comparison of solution to the integral equation (3) and its reflection with the plane, the plane can be started to move from infinity. Furthermore the plane has to be moved to a critical point. Hence symmetry and monotonicity properties of solutions to (3) are consequently derived.

In Sect. 2, we study the regularity of extremal functions of the following integral equation

$$u(x) = \int\limits_{\mathbb{R}^n} \frac{|u(y)|^p u(y)}{|y|^t |x-y|^{n-\alpha}} dy, \tag{9}$$

where $p > 0$. In [11] and [17], the authors consider the nonnegative solutions of the above integral equation in the case $t = 0$ and $p = \frac{2\alpha}{n-\alpha}$. They prove $u \in C^\infty(\mathbb{R}^n)$.

In this paper, we consider the symmetry and regularity of solutions to the integral equation (9) in the case $0 < t < \alpha$. Thus, the second main result of our paper is as follows:

Theorem 2 *If $u(x) \in L^{\frac{pn}{\alpha-t}}(\mathbb{R}^n)$ is the solution of* (9)*, then $u(x) \in L^\infty(\mathbb{R}^n)$. Moreover $u(x) \in C^{[\alpha-t],\beta}(\mathbb{R}^n)$ for any $\beta < \alpha - t - [\alpha - t]$. In particular, $u(x)$ is C^∞ in $\mathbb{R}^n \setminus \{0\}$, where $[\alpha - t]$ is the greatest integer function.*

If $p = \alpha^*(t) - 2$, then $\frac{pn}{\alpha-t} = \frac{2n}{n-\alpha}$, which is exactly the critical *Sobolev* imbedding exponent of $H^{\frac{\alpha}{2},2}(\mathbb{R}^n)$. If $\alpha = 2$ and p is as above, our Theorem 2 provides a new proof of regularity of extremal function of *Hardy–Sobolev* inequality, since we prove it through the corresponding integral equation. See [20] for Hölder's regularity of solutions of (4) in case of $\alpha = 2$. Moreover our Theorem 2 not only proves the regularity of extremal function of (4), but can also be applied to regularity for more general integral equations or corresponding partial differential equations. The method used here is called Regularity Lifting theorem based on contraction map theorem. The related lemma is presented in Sect. 2. It is a simple method to boost regularity of solutions. We refer the reader to [11] for more details.

In Sect. 3, we will study the relations of (3) and (4).

Theorem 3 *If $u(x)$ is a solution to* (4)*, then $u(x)$ satisfies* (3).

In [10], the authors prove the equivalence of (5) and (6), namely the special case of $t = 0$. We will prove that (4) and (3) are actually equivalent in Sect. 3. Moreover, if α is an even number, we then give a new and relatively easy way to derive (3) from (4) by choosing an appropriate cut-off function.

Combining Theorems 1 and 3, we get the following

Corollary 1 *Assume $u(x) \in L^{\frac{2n}{n-\alpha}}(\mathbb{R}^n)$ is a positive solution of* (4)*, then $u(x)$ is radially symmetric and strictly decreasing about the origin.*

Throughout this paper, the positive constant C is frequently used in the paper. It may differ from line to line, even within the same line and it has nothing to do with $u(x)$.

2 The proof of Theorem 1

For the convenience of the reader, we present the following *Weighted Hardy–Littlewood–Sobolev* inequality (see [16]).

Lemma 2.1 *Let* $1 < l, m < \infty$, $0 < \nu < n$, $\tau + \beta \geq 0$, $\frac{1}{l} + \frac{1}{m} + \frac{\nu+\beta+\tau}{n} = 2$ *and* $1 - \frac{1}{m} - \frac{\nu}{n} \leq \frac{\tau}{n} < 1 - \frac{1}{m}$. *Then the weighted HLS inequality states*

$$\left| \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{f(x)g(y)}{|x|^\tau |x-y|^\nu |y|^\beta} dxdy \right| \leq C\|f\|_{L^m}\|g\|_{L^l}.$$

The *Weighted Hardy–Littlewood–Sobolev* inequality can also be written in another form. Let $Tg(x) = \int_{\mathbb{R}^n} \frac{g(y)}{|x|^\tau|x-y|^\nu|y|^\beta}dy$, then

$$\|Tg(x)\|_{L^\gamma} = sup_{\|f\|_{L^m}=1} < Tg(x), \quad f(x) >\leq C\|g\|_{L^l}, \tag{10}$$

where $\frac{1}{l} + \frac{\nu+\beta+\tau}{n} = 1 + \frac{1}{\gamma}$, $\frac{1}{m} + \frac{1}{\gamma} = 1$.

In order to prove our theorems, we first introduce some notations. For any real number λ, define

$$\Sigma_\lambda = \{x = (x_1, x_2, \ldots, x_n), x_1 \leq \lambda\},$$
$$T_\lambda = \{x|x_1 = \lambda\}.$$

Let $x \in \Sigma_\lambda$ and $x_\lambda = (2\lambda - x_1, x_2, \ldots, x_n)$. Moreover, define $u_\lambda(x) = u(x_\lambda)$.

Lemma 2.2 *For any solution $u(x)$ of (3), we have*

$$u(x) - u_\lambda(x) = \int_{\Sigma_\lambda} [\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x_\lambda - y|^{n-\alpha}}][\frac{u^{\alpha^*(t)-1}(y)}{|y|^t} - \frac{u_\lambda^{\alpha^*(t)-1}(y)}{|y_\lambda|^t}]\,dy. \quad (11)$$

Proof Since $|x - y_\lambda| = |x_\lambda - y|$,

$$u(x) = \int_{\Sigma_\lambda} \frac{u^{\alpha^*(t)-1}(y)}{|y|^t|x-y|^{n-\alpha}}dy + \int_{\Sigma_\lambda} \frac{u_\lambda^{\alpha^*(t)-1}(y)}{|y_\lambda|^t|x-y_\lambda|^{n-\alpha}}\,dy,$$

$$u(x_\lambda) = \int_{\Sigma_\lambda} \frac{u^{\alpha^*(t)-1}(y)}{|y|^t|x_\lambda-y|^{n-\alpha}}dy + \int_{\Sigma_\lambda} \frac{u_\lambda^{\alpha^*(t)-1}(y)}{|y_\lambda|^t|x-y|^{n-\alpha}}\,dy.$$

By considering $u(x) - u(x_\lambda)$ and making a simple calculation, the proof of the lemma is completed. □

We now outline the ideas of the moving plane method in our proof. To prove Theorem 1, We compare the value of $u(x)$ with $u_\lambda(x)$ in Σ_λ. The proof consists of two steps. In *step* 1, we show that for sufficiently negative λ,

$$u(x) \le u_\lambda(x). \quad (12)$$

Thus we can start to move the plane T_λ along the x_1 direction continuously from near negative infinity to the right as long as (12) holds. In *step* 2, We show that the plane can move to the limit case $x_1 = 0$, hence $u(x) \le u_0(x)$ for $x \in \Sigma_0$. If we choose to move the plane from positive infinity to the left and carry on the same procedure as done in Steps 1 and 2, we can also prove that $u(x) \ge u_0(x)$ for $x \in \mathbb{R}^n \setminus \Sigma_0$. Therefore $u(x)$ is symmetric about the plane T_0. Since the direction of x_1 can be chosen arbitrarily, we deduce that $u(x)$ is symmetric and decreasing about the origin.

Proof of Theorem : *Step 1:* We show that for sufficiently negative λ,

$$u(x) \le u(x_\lambda), \quad \forall x \in \Sigma_\lambda. \quad (13)$$

Define

$$w_\lambda(x) = u(x) - u(x_\lambda)$$

and

$$\Sigma_\lambda^- = \{x \in \Sigma_\lambda | u(x) > u(x_\lambda)\}.$$

From Lemma (2.2), we have

$$\begin{aligned} u(x) - u(x_\lambda) = &\int_{\Sigma_\lambda \setminus \Sigma_\lambda^-} \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x_\lambda - y|^{n-\alpha}}\right)\left[\frac{u^{\alpha^*(t)-1}}{|y|^t} - \frac{u_\lambda^{\alpha^*(t)-1}}{|y_\lambda|^t}\right] dy \\ &+ \int_{\Sigma_\lambda^-} \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x_\lambda - y|^{n-\alpha}}\right)\left[\frac{u^{\alpha^*(t)-1}}{|y|^t} - \frac{u_\lambda^{\alpha^*(t)-1}}{|y_\lambda|^t}\right] dy. \end{aligned}$$

Since $|x-y|<|x_\lambda-y|$ and $|y|>|y_\lambda|$ in Σ_λ, by the definition of Σ_λ^-,

$$u(x)-u(x_\lambda)\le\int\limits_{\Sigma_\lambda^-}\left(\frac{1}{|x-y|^{n-\alpha}}-\frac{1}{|x_\lambda-y|^{n-\alpha}}\right)\left[\frac{u^{\alpha^*(t)-1}-u_\lambda^{\alpha^*(t)-1}}{|y|^t}\right]dy.$$

Moreover, from the Mean Value Theorem,

$$u(x)-u(x_\lambda)\le\ C\int\limits_{\Sigma_\lambda^-}\frac{1}{|x-y|^{n-\alpha}}\frac{1}{|y|^t}u^{\alpha^*(t)-2}(u-u_\lambda)\,dy. \tag{14}$$

By virtue of the *Weighted–Hardy–Littlewood–Sobolev* inequality, i.e. Lemma 2.1 (see also (10)) in the case of $\tau=0$ in (14), for any $q>\frac{n}{n-\alpha}$ (without loss of generality, let $q=\frac{2n}{n-\alpha}$), we have

$$\|w_\lambda\|_{L^q(\Sigma_\lambda^-)}\le C\|u^{\alpha^*(t)-2}w_\lambda\|_{L^{\frac{nq}{\alpha q+n-qt}}(\Sigma_\lambda^-)}.$$

Then from *Hölder*'s inequality, we get

$$\|w_\lambda\|_{L^q(\Sigma_\lambda^-)}\le C\|u\|^{\frac{2(\alpha-t)}{n-\alpha}}_{L^{\frac{2n}{n-\alpha}}(\Sigma_\lambda^-)}\|w_\lambda\|_{L^q(\Sigma_\lambda^-)}.$$

Since $u\in L^{\frac{2n}{n-\alpha}}(\mathbb{R}^n)$, we can choose sufficiently negative λ such that

$$C\|u\|^{\frac{2(\alpha-t)}{n-\alpha}}_{L^{\frac{2n}{n-\alpha}}(\Sigma_\lambda^-)}\le\frac{1}{2}.$$

Therefore

$$\|w_\lambda\|_{L^q(\Sigma_\lambda^-)}\le\frac{1}{2}\|w_\lambda\|_{L^q(\Sigma_\lambda^-)}.$$

This implies Σ_λ^- must be of measure zero. Hence (13) is verified.

Step 2: Assuming that the plane can move to the critical point $\lambda_0<0$. If there exists some point x_0 in Σ_{λ_0} such that $u(x_0)=u_{\lambda_0}(x_0)$, from Lemma (2.2), we have

$$0=u(x_0)-u_{\lambda_0}(x_0)=\int\limits_{\Sigma_{\lambda_0}}[\frac{1}{|x_0-y|^{n-\alpha}}-\frac{1}{|x_{\lambda_0}-y|^{n-\alpha}}][\frac{u^{\alpha^*(t)-1}}{|y|^t}-\frac{u_{\lambda_0}^{\alpha^*(t)-1}}{|y_{\lambda_0}|^t}]\,dy.$$

Let $x_{\lambda_0}=(x_0)_{\lambda_0}$. Since $|y|>|y_{\lambda_0}|$ in Σ_{λ_0},

$$\frac{u(y)^{\alpha^*(t)-1}}{|y|^t}<\frac{u_{\lambda_0}(y)^{\alpha^*(t)-1}}{|y_{\lambda_0}|^t}\quad \text{in}\,\Sigma_{\lambda_0}.$$

Moreover $|x_0-y|<|x_{\lambda_0}-y|$ in Σ_{λ_0}, we infer that

$$u(x)\equiv u_{\lambda_0}(x)\equiv 0,\quad \forall x\in\Sigma_{\lambda_0}.$$

This also implies that $u(x)\equiv 0$. But it is impossible. Hence

$$u(x)<u_{\lambda_0}(x),\quad \forall x\in\Sigma_{\lambda_0}.$$

We Claim that

$$\lambda_0=sup\{\lambda|u(x)-u_\lambda(x)\le 0,\quad \forall x\in\Sigma_\lambda\}=0. \tag{15}$$

If not, i.e. $\lambda_0 < 0$, we prove that the plane can be moved to the right a little bit further. Since $u \in L^{\frac{2n}{n-\alpha}}(\mathbb{R}^n)$, for any small ϵ, there exists a large enough ball $\mathbb{B}_R(0)$ such that

$$\int\limits_{\mathbb{R}^n \setminus \mathbb{B}_R(0)} u^{\frac{2n}{n-\alpha}}\, dx < \epsilon.$$

By virtue of *Lusin*'s theorem, for any δ, there exists a closed set F_δ such that $w_{\lambda_0}|_{F_\delta}$ is continuous, with $F_\delta \subset \mathbb{B}_R(0) \bigcap \Sigma_{\lambda_0} = E$ and $m(E - F_\delta) < \delta$. As $w_{\lambda_0}(x) < 0$ in the interior of Σ_{λ_0}, $w_{\lambda_0}(x) < 0$ in F_δ.

Choosing ϵ_1 sufficiently small, for any $\lambda \in [\lambda_0\,,\ \lambda_0 + \epsilon_1)$, it holds that

$$w_\lambda < 0, \quad \forall\, x \in F_\delta.$$

It follows that, for such λ,

$$\Sigma_\lambda^- \subset M := (\mathbb{R}^n \setminus \mathbb{B}_R(0)) \cup (E \setminus F_\delta) \cup [(\Sigma_\lambda \setminus \Sigma_{\lambda_0}^-) \cap \mathbb{B}_R(0)].$$

Choosing ϵ, δ and ϵ_1 so small and from the absolute continuity of integration, we have

$$C\|u\|_{L^{\frac{2n}{n-\alpha}}(M)}^{\frac{2(\alpha-t)}{n-\alpha}} \leq \frac{1}{2}.$$

Finally

$$||w_\lambda\|_{L^q(\Sigma_\lambda^-)} \leq C\|u\|_{L^{\frac{2n}{n-\alpha}}(\Sigma_\lambda^-)}^{\frac{2(\alpha-t)}{n-\alpha}} \|w_\lambda\|_{L^q(\Sigma_\lambda^-)} < \frac{1}{2}\|w_\lambda\|_{L^q(\Sigma_\lambda^-)}.$$

This again implies Σ_λ^- must be empty. It contradicts with the assumption that $\lambda_0 < 0$. Therefore, (15) is verified.

On the other hand, we can also move the plane from positive infinity to zero by the similar procedure, hence $u(x)$ is symmetric and monotonic with respect to $x_1 = 0$. Moreover, since the x_1 direction can be chosen arbitrarily, $u(x)$ is radial symmetric and strictly monotonic with respect to the origin. We thus have completed the proof of Theorem 1. □

Remark 2.1 Since (3) is invariant under the Kelvin transform, we can also prove that $u(x)$ is symmetric with respect to the origin under a weaker assumption that $u(x) \in L_{loc}^{\frac{2n}{n-\alpha}}(\mathbb{R}^n)$. Assume

$$v(x) = \frac{1}{|x|^{n-\alpha}} u\left(\frac{x}{|x|^2}\right),$$

then $v(x)$ satisfies (3) and $v(x) \in L_{oc}^{\frac{2n}{n-\alpha}}(\Omega)$, where Ω is any domain with positive distance from the origin. Carrying out the first and second steps for $v(x)$ as we did before, we conclude that $v(x)$ is symmetric and monotonic with respect to the origin. This implies that $u(x)$ is symmetric with respect to the origin.

3 The proof of Theorem 2

In this section, we prove the regularity for functions satisfying (9) by the contraction map. We present the regularity lift lemma below. See also ([9]).

Let Z be a given vector space, $\|.\|_X$ and $\|.\|_Y$ be two norms on Z. Define a new norm $\|.\|_Z$ by

$$\|.\|_Z = \sqrt[p]{\|.\|_X^p + \|.\|_Y^p}.$$

For simplicity, we assume that Z is complete with respect to the norm $\|.\|_Z$. Let X and Y be the completion of Z under $\|.\|_X$ and $\|.\|_Y$, respectively. Here one can choose p, $1 \le p \le \infty$, according to what one needs. It's easy to see that $Z = X \bigcap Y$.

Lemma 3.1 (Regularity Lifting) *Let T be a contraction map from X into itself and from Y into itself. Assume that $f \in X$ and that there exists a function $g \in Z$ such that $f = Tf + g$, then f also belongs to Z.*

Proof of Theorem 2: We define a linear operator

$$T_u w(x) = \int_{\mathbb{R}^n} \frac{|u(y)|^p w}{|y|^t |x-y|^{n-\alpha}}\, dy.$$

For any positive real number a, define

$$\begin{cases} u_a(x) = u(x), & \text{if } |u(x)| > a \text{ or } |x| > a, \\ u_a(x) = 0, & \text{otherwise.} \end{cases} \tag{16}$$

Let $u_b(x) = u(x) - u_a(x)$. Since u satisfies (9),

$$u_a(x) = \int_{\mathbb{R}^n} \frac{|u_a|^p u_a}{|y|^t |x-y|^{n-\alpha}} dy + I(x), \tag{17}$$

where $I(x) = \int_{\mathbb{R}^n} \frac{|u_b|^p u_b}{|y|^t |x-y|^{n-\alpha}}\, dy - u_b(x)$.

As to $I(x)$, for $s > \frac{n}{n-\alpha}$, we Claim that

$$I(x) \in L^\infty(\mathbb{R}^n) \bigcap L^s(\mathbb{R}^n).$$

Obviously, $u_b(x) \in L^\infty(\mathbb{R}^n) \bigcap L^s(\mathbb{R}^n)$.

Thus, we only need to prove $A(x) := \int_{\mathbb{R}^n} \frac{|u_b|^p u_b}{|y|^t |x-y|^{n-\alpha}} dy \in L^\infty(\mathbb{R}^n) \bigcap L^s(\mathbb{R}^n)$.

By the definition of $u_b(x)$, for $\forall x \in \mathbb{R}^n$,

$$|A(x)| \le C \int_{|y| \le a} \frac{1}{|y|^t |x-y|^{n-\alpha}}\, dy.$$

If $x \in \mathbb{R}^n \setminus \mathbb{B}_{2a}(0)$, $|x-y| \ge |y|$, then

$$\int_{|y| \le a} \frac{1}{|y|^t |x-y|^{n-\alpha}}\, dy \le \int_{|y| \le a} \frac{1}{|y|^{n-\alpha+t}}\, dy < \infty.$$

If $x \in \mathbb{B}_{2a}(0)$,

$$\int_{|y| \le a} \frac{1}{|y|^t |x-y|^{n-\alpha}}\, dy \le \int_{|y| \le a} \frac{1}{|y|^{n-\alpha+t}}\, dy + \int_{|x-y| \le 3a} \frac{1}{|x-y|^{n-\alpha+t}}\, dy < \infty,$$

hence $A(x) \in L^\infty(\mathbb{R}^n)$. Using *Weighted Hardy–Littlewood–Sobolev* inequality,

$$\|A(x)\|_{L^s} \le \||u_b|^p u_b\|_{L^{\frac{ns}{\alpha s + n - st}}(\mathbb{B}_a)} < \infty,$$

then $A(x) \in L^s(\mathbb{R}^n)$. Therefore, we have proved the claim.

Next for any $q > \frac{n}{n-\alpha}$, we show that $T_{u_a} w$ is a contraction map. Applying the *Weighted Hardy–Littlewood–Sobolev* inequality and, then Hölder's inequality, we get

$$\|T_{u_a} w\|_{L^q} \le \| |u_a|^p w\|_{L^{\frac{nq}{\alpha q+n-qt}}} \le \| |u_a|^p\|_{L^{\frac{n}{\alpha-t}}} \|w\|_{L^q}. \tag{18}$$

Since $u(x) \in L^{\frac{np}{\alpha-t}}$, for sufficiently large a, we deduce

$$\|T_{u_a} w(x)\|_{L^q} \le \frac{1}{2}\|w\|_{L^q}, \tag{19}$$

which shows that $T_{u_a} w$ is a contraction map. Applying (19) to both the case of $q = \frac{np}{\alpha-t}$ and the case of any $q_0 > \frac{n}{n-\alpha}$, and by the contraction map lemma (i.e. Lemma 3.1), we can conclude that $u_a \in L^q \bigcap L^{q_0}$.

Furthermore, we Claim that $u \in L^\infty(\mathbb{R}^n)$. Since $u(x) = u_a(x) + u_b(x)$ and $u_b(x) \in L^\infty(\mathbb{R}^n)$, we only need to verify $u_a(x) \in L^\infty(\mathbb{R}^n)$. Due to (17) and $I(x) \in L^\infty(\mathbb{R}^n)$, it is equivalent to verify that

$$B(x) := \int_{\mathbb{R}^n} \frac{|u_a|^p u_a}{|y|^t |x-y|^{n-\alpha}} dy \in L^\infty(\mathbb{R}^n).$$

Note

$$B(x) \le \int_{\mathbb{B}_a(0)} \frac{|u_a|^{p+1}}{|y|^t |x-y|^{n-\alpha}} dy + \int_{\mathbb{R}^n \setminus \mathbb{B}_a(0)} \frac{|u_a|^{p+1}}{|y|^t |x-y|^{n-\alpha}} dy. \tag{20}$$

If $x \in \mathbb{R}^n \setminus \mathbb{B}_{2a}(0)$, then $|x-y| > |y|$ and

$$\int_{\mathbb{B}_a(0)} \frac{|u_a|^{p+1}}{|y|^t |x-y|^{n-\alpha}} dy \le \int_{\mathbb{B}_a(0)} \frac{|u_a|^{p+1}}{|y|^{n-\alpha+t}} dy < \infty \tag{21}$$

by Hölder's inequality with the help of $u_a \in L^q \bigcap L^{q_0}$.
If $x \in \mathbb{B}_{2a}(0)$, similarly

$$\int_{\mathbb{B}_a(0)} \frac{|u_a|^{p+1}}{|y|^t |x-y|^{n-\alpha}} dy \le \int_{\mathbb{B}_a(0)} \frac{|u_a|^{p+1}}{|y|^{n-\alpha+t}} dy + \int_{\mathbb{B}_{3a}(x)} \frac{|u_a|^{p+1}}{|x-y|^{n-\alpha+t}} dy < \infty \tag{22}$$

by Hölder's inequality and the property that $u_a \in L^q \bigcap L^{q_0}$.

Combining (21) and (22), for any $x \in \mathbb{R}^n$, we derive that

$$\int_{\mathbb{B}_a(0)} \frac{|u_a|^{p+1}}{|y|^t |x-y|^{n-\alpha}} dy < \infty. \tag{23}$$

Next, for $\forall x \in \mathbb{R}^n$,

$$\begin{aligned} \int_{\mathbb{R}^n \setminus \mathbb{B}_a(0)} \frac{|u_a|^{p+1}}{|y|^t |x-y|^{n-\alpha}} dy &\le \frac{1}{a^t} \int_{\mathbb{B}_a(x)} \frac{|u_a|^{p+1}}{|x-y|^{n-\alpha}} dy \\ &\quad + \int_{(R^n \setminus \mathbb{B}_a(0)) \bigcap (\mathbb{R}^n \setminus B_a(x))} \frac{|u_a|^{p+1}}{|y|^t |x-y|^{n-\alpha}} dy. \end{aligned} \tag{24}$$

By virtue of Hölder's inequality, it is easy to see that

$$\frac{1}{a^t}\int\limits_{\mathbb{B}_a(x)} \frac{|u_a|^{p+1}}{|x-y|^{n-\alpha}}\,dy < \infty. \tag{25}$$

Since

$$\int\limits_{(R^n\setminus\mathbb{B}_a(0))\bigcap(\mathbb{R}^n\setminus B_a(x))} \frac{|u_a|^{p+1}}{|y|^t|x-y|^{n-\alpha}}\,dy \le \int\limits_{\mathbb{R}^n\setminus\mathbb{B}_a(x)} \frac{|u_a|^{p+1}}{|x-y|^{n-\alpha+t}}\,dy + \int\limits_{\mathbb{R}^n\setminus\mathbb{B}_a(0)} \frac{|u_a|^{p+1}}{|y|^{n-\alpha+t}}\,dy,$$

we can also prove that

$$\int\limits_{(R^n\setminus\mathbb{B}_a(0))\bigcap(\mathbb{R}^n\setminus B_a(x))} \frac{|u_a|^{p+1}}{|y|^t|x-y|^{n-\alpha}}\,dy < \infty \tag{26}$$

by Hölder's inequality and the estimate $u_a \in L^q \bigcap L^{q_0}$.

Through (24), (25) and (26), we deduce that

$$\int\limits_{\mathbb{R}^n\setminus\mathbb{B}_a(0)} \frac{|u_a|^{p+1}}{|y|^t|x-y|^{n-\alpha}}\,dy < \infty. \tag{27}$$

Therefore, from (20), (23) and (27), we have $B(x) \in L^\infty(\mathbb{R}^n)$. Hence the claim that $u(x) \in L^\infty(\mathbb{R}^n)$ is verified.
To show the higher regularity, we first show that $u(x) \in C^\infty(\mathbb{R}^n \setminus \{0\})$.

For any $x \in \mathbb{R}^n \setminus \{0\}$, we choose a ball $\mathbb{B}_{3r}(x)$ with radius $3r$ such that $0 \notin \bar{\mathbb{B}}_{3r}(x)$, then

$$u(x) = \int\limits_{\mathbb{R}^n\setminus\mathbb{B}_{3r}(x)} \frac{|u|^p u}{|y|^t|x-y|^{n-\alpha}}\,dy + \int\limits_{\mathbb{B}_{3r}(x)} \frac{|u|^p u}{|y|^t|x-y|^{n-\alpha}}\,dy. \tag{28}$$

We show that $R(x) := \int_{\mathbb{R}^n\setminus\mathbb{B}_{3r}(x)} \frac{|u|^p u}{|y|^t|x-y|^{n-\alpha}}\,dy \in C^\infty(\mathbb{R}^n \setminus \{0\})$.
Let $F(x,y) := \frac{|u(y)|^p u(y)}{|y|^t|x-y|^{n-\alpha}}\chi_{\mathbb{R}^n\setminus\mathbb{B}_{3r}(x)}$. For fixed x, if h is small enough, considering

$$\begin{aligned}
\left|\frac{F(x+he_i,y)-F(x,y)}{h}\right| &= \left|\frac{\frac{|u(y)|^p u(y)}{|y|^t}\left(\frac{\chi_{\mathbb{R}^n\setminus\mathbb{B}_{3r}(x+he_i)}}{|x+he_i-y|^{n-\alpha}} - \frac{\chi_{\mathbb{R}^n\setminus\mathbb{B}_{3r}(x)}}{|x-y|^{n-\alpha}}\right)}{h}\right| \\
&\le C\frac{|u(y)|^{p+1}\chi_{\mathbb{R}^n\setminus\mathbb{B}_{3r}(x+\theta he_i)}}{|y|^t|x+\theta he_i-y|^{n-\alpha+1}} \\
&\le C\frac{|u(y)|^{p+1}}{|y|^t|x+\theta he_i-y|^{n-\alpha+1}}\chi_{\mathbb{R}^n\setminus\mathbb{B}_{2r}(x)} \\
&\le C\left\{\frac{|u(y)|^{p+1}}{|y|^t r^{n-\alpha+1}}\chi_{\mathbb{B}_\epsilon(0)} \right. \\
&\quad \left. + \frac{|u(y)|^{p+1}}{|y|^t|x-y|^{n-\alpha+1}}\chi_{\mathbb{R}^n\setminus(\mathbb{B}_r(x)\cup\mathbb{B}_\epsilon(0))}\right\},
\end{aligned} \tag{29}$$

where $e_i = \{0, \ldots, 1, \ldots, 0\}$ is the ith unit vector, $0 < \theta < 1$ and ϵ is so small that $\mathbb{B}_\epsilon(0) \cap \mathbb{B}_{3r}(x) = \varnothing$. Since $u \in L^\infty(\mathbb{R}^n)$, it is easy to verify

$$\int_{\mathbb{B}_\epsilon(0)} \frac{|u|^{p+1}}{|y|^t r^{n-\alpha+1}} dy < \infty, \tag{30}$$

for fixed r.
For such fixed r and ϵ, let

$$\int_{\mathbb{R}^n \setminus (\mathbb{B}_r(x) \cup \mathbb{B}_\epsilon(0))} \frac{|u|^{p+1}}{|y|^t |x-y|^{n-\alpha+1}} dy < \int_{\mathbb{R}^n \setminus \mathbb{B}_r(x)} \frac{|u|^{p+1}}{|x-y|^{n-\alpha+1+t}} dy + \int_{\mathbb{R}^n \setminus \mathbb{B}_\epsilon(0)} \frac{|u|^{p+1}}{|y|^{n-\alpha+1+t}} dy.$$

Since $u \in L^q$ for any $q > \frac{np}{\alpha - t}$, using Hölder's inequality,

$$\int_{\mathbb{R}^n \setminus \mathbb{B}_r(x)} \frac{|u|^{p+1}}{|x-y|^{n-\alpha+1+t}} dy < \infty,$$

$$\int_{\mathbb{R}^n \setminus \mathbb{B}_\epsilon(0)} \frac{|u|^{p+1}}{|y|^{n-\alpha+1+t}} dy < \infty,$$

then

$$\int_{\mathbb{R}^n \setminus (\mathbb{B}_r(x) \cup \mathbb{B}_\epsilon(0))} \frac{|u|^{p+1}}{|y|^t |x-y|^{n-\alpha+1}} dy < \infty. \tag{31}$$

With (29), (30), (31) and the Lebesgue dominated convergence theorem, $R(x) \in C^1(\mathbb{R}^n \setminus \{0\})$. Continuing this process,

$$R(x) \in C^\infty(\mathbb{R}^n \setminus \{0\}). \tag{32}$$

By standard singular integral estimates (Chap. 10 in [18]),

$$\int_{\mathbb{B}_{2r}(x)} \frac{|u|^p u}{|y|^t |x-y|^{n-\alpha}} dy \in C^{\beta_1}(\mathbb{R}^n \setminus \{0\}) \tag{33}$$

for any $\beta_1 < \alpha$. Combining (28), (32) and (33), $u(x) \in C^{\beta_1}(\mathbb{R}^n \setminus \{0\})$. By the bootstrap technique, we can prove that $u(x) \in C^\infty(\mathbb{R}^n \setminus \{0\})$.

The difficulty of regularity occurs around the origin. We note that

$$u(x) - u(0) = \int_{\mathbb{R}^n \setminus \mathbb{B}_{2r}(0)} \frac{|u|^p u}{|y|^t} [\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|y|^{n-\alpha}}] dy + \int_{B_{2r}(0)} \frac{|u|^p u}{|y|^t} [\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|y|^{n-\alpha}}] dy.$$

If $\alpha - t \leq 1, 0 \in D$ and the domain $D \subset \mathbb{B}_r(0)$, we show that $u \in C^{0,\beta}(D)$ for any $\beta < \alpha - t$. Indeed, from the following property

$$||x-z|^{-c} - |y-z|^{-c}| \leq c|x-y|^b(|x-z|^{-c-b} + |y-z|^{-c-b}),$$

where $c > 0$ and $0 < b < 1$(see also chapter 10 in [18]),

$$sup_{x\in D}\frac{|u(x)-u(0)|}{|x|^\beta} \leq Csup_{x\in D}\int\limits_{\mathbb{R}^n\setminus\mathbb{B}_{2r}(0)} \frac{|u|^{p+1}}{|y|^t|x-y|^{n-\alpha+\beta}}dy$$
$$+ Csup_{x\in D}\int\limits_{\mathbb{B}_{2r}(0)} \frac{|u|^{p+1}}{|y|^t|x-y|^{n-\alpha+\beta}}dy.$$

With the help of Hölder's inequality as in (27), we have

$$sup_{x\in D}\int\limits_{\mathbb{R}^n\setminus\mathbb{B}_{2r}(0)} \frac{|u|^{p+1}}{|y|^t|x-y|^{n-\alpha+\beta}}dy < \infty. \tag{34}$$

From Hölder's inequality again and the property $u \in L^\infty(\mathbb{R}^n)$, we show

$$sup_{x\in D}\int\limits_{\mathbb{B}_{2r}(0)} \frac{|u|^{p+1}}{|y|^t|x-y|^{n-\alpha+\beta}}dy < \infty \tag{35}$$

for any $\beta < \alpha - t$. Therefore, (34) and (35) imply that $u \in C^{0,\beta}(\mathbb{R}^n)$.

If $1 < \alpha - t < 2$, similarly we can prove $u(x) \in C^{1,\beta}(D)$ for $\beta < \alpha - t$, then $u(x) \in C^{1,\beta}(\mathbb{R}^n)$. For more general $0 < t < \alpha$, we conclude that $u \in C^{[\alpha-t],\beta}(\mathbb{R}^n)$ for any $\beta < \alpha - t - [\alpha - t]$. Hence, the proof of Theorem 2 is complete. □

4 The proof of Theorem 3

Proof of Theorem 3:

The proof follows from the properties of the Riesz potential and the Fourier transform of the Riesz potential and is quite similar to that in [10]. For the completeness and the convenience of the reader, we include a proof here.

Let $\phi \in C_0^\infty(\mathbb{R}^n)$ and $\psi(x) = \int_{\mathbb{R}^n}\frac{\phi(y)}{|x-y|^{n-\alpha}}dy$. Then $(-\Delta)^{\frac{\alpha}{2}}\psi = \phi$. Then $\psi \in H^\alpha(\mathbb{R}^n)$. So if u is a solution to the differential equation, then

$$\int\limits_{\mathbb{R}^n}(-\Delta)^{\frac{\alpha}{4}}u(-\Delta)^{\frac{\alpha}{4}}\psi dx = \int\limits_{\mathbb{R}^n}\frac{u^{\alpha^*(t)}(y)}{|y|^t}\psi(y)dy.$$

This implies

$$\int\limits_{\mathbb{R}^n}u(x)(-\Delta)^{\frac{\alpha}{2}}\psi(x)dx = \int\limits_{\mathbb{R}^n}\left(\int\limits_{\mathbb{R}^n}\frac{u^{\alpha^*(t)}(y)}{|y|^t}\frac{1}{|x-y|^{n-\alpha}}dy\right)\phi(x)dx$$

for all $\phi \in C_0^\infty(\mathbb{R}^n)$. Therefore, $u(x) = \int_{\mathbb{R}^n}\frac{u^{\alpha^*(t)}(y)}{|y|^t}\frac{1}{|x-y|^{n-\alpha}}dy$, namely, the integral equation holds.

On the other hand, if the integral equation (3) i.e. $u(x) = \int_{\mathbb{R}^n} \frac{u^{\alpha^*(t)}(y)}{|y|^t} \frac{1}{|x-y|^{n-\alpha}} dy$ holds, then by taking Fourier transform on both sides we get

$$\hat{u}(\xi) = c_n |\xi|^{-\alpha} \left(\widehat{\frac{u^{\alpha^*(t)}(y)}{|y|^t}} \right)(\xi).$$

This implies that

$$|\xi|^{\alpha} \hat{u}(\xi) = c_n \left(\widehat{\frac{u^{\alpha^*(t)}(y)}{|y|^t}} \right)(\xi).$$

Thus for any $\phi \in C_0^{\infty}(\mathbb{R}^n)$ we have

$$\begin{aligned}
\int_{\mathbb{R}^n} (-\Delta)^{\frac{\alpha}{2}} u \phi &= \int_{\mathbb{R}^n} (-\Delta)^{\frac{\alpha}{4}} u (-\Delta)^{\frac{\alpha}{4}} \phi \\
&= c_n \int_{\mathbb{R}^n} |\xi|^{\alpha} \hat{u}(\xi) \hat{\phi}(\xi) d\xi = c_n \int_{\mathbb{R}^n} \left(\widehat{\frac{u^{\alpha^*(t)}(y)}{|y|^t}} \right)(\xi) \hat{\phi}(\xi) d\xi \\
&= c_n \int_{\mathbb{R}^n} \frac{u^{\alpha^*(t)}(y)}{|y|^t} \phi(y) dy.
\end{aligned}$$

Therefore,

$$(-\Delta)^{\frac{\alpha}{2}} u = \frac{u^{\alpha^*(t)}(y)}{|y|^t},$$

namely u satisfies (4). □

We conclude this section by giving another proof of the fact that (4) implies (3) in the case of $\alpha = 2m$, where m is a positive integer.

Proof We argue as follows. By the regularity theorem in [1], $u \in W_{loc}^{2m,s}(\mathbb{R}^n)$ for some $1 < s < \frac{2n}{n+2m}$. For any $x \in \mathbb{R}^n$, we multiply both sides of (4) by $\frac{1}{|x-y|^{n-2m}} \psi(x-y)$. The cut-off function $\psi(x-y) = \eta\left(\frac{|x-y|}{R}\right)$. When $0 < r < 1$, $\eta(r) = 1$, while $\eta(r) = 0$ when $r > 2$. Moreover $0 < \eta^{(i)}(r) < 2$ in $(0,\ 2)$ for $i = 1, \ldots, 2m$. Then

$$\int_{\mathbb{R}^n} (-\triangle)^m u \frac{\psi(x-y)}{|x-y|^{n-2m}} dy = \int_{\mathbb{R}^n} u^{(2m)^*(t)-1} \frac{\psi(x-y)}{|y|^t |x-y|^{n-2m}} dy.$$

Using integration by parts several times, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} u(-\triangle)^m \left(\frac{1}{|x-y|^{n-2m}} \right) \psi + \Sigma_{i=1}^{2m} C_i \int_{\mathbb{R}^n} u|x-y|^{-n+i} \eta^{(i)} \left(\frac{|x-y|}{R} \right) R^{-i} \\
&\quad = \int_{\mathbb{R}^n} u^{(2m)^*(t)-1} \frac{1}{|y|^t} \frac{\psi(x-y)}{|x-y|^{n-2m}},
\end{aligned}$$

where C_i is some constant independent of $u(x)$. Since $u(x) \in L^{\frac{2n}{n-2m}}$, using Hölder's inequality for the second integration,

$$\int_{\mathbb{R}^n} u|x-y|^{-n+i}\eta^{(i)}R^{-i} \le C_i R^{-i}\left(\int_{\mathbb{R}^n} |u|^{\frac{2n}{n-2m}}\right)^{\frac{n-2m}{2n}}\left(\int_{\mathbb{B}_{2R}\setminus\mathbb{B}_R} |x-y|^{\frac{2n(i-n)}{n+2m}}\right)^{\frac{n+2m}{2n}}$$

$$\le C_i R^{-i}\left(\int_R^{2R} r^{\frac{2n(i-n)}{n+2m}} r^{n-1}\right)^{\frac{n+2m}{2n}} \to 0$$

as $R \to \infty$. Therefore,

$$u(x) = \int_{\mathbb{R}^n} u^{(2m)^*(t)-1}\frac{1}{|y|^t}\frac{1}{|x-y|^{n-2m}}\,dy.$$

□

Acknowledgments The authors wish to thank the referee for his careful reading and pointing out numerous typos. Research is partially supported by a US NSF Grant DMS-0901761.

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