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Multilinear Calderón–Zygmund operators with kernels of Dini's type and applications

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ABSTRACT

The main purpose of this paper is to establish a number of results concerning boundedness of multi-linear Calderón–Zygmund operators with kernels of mild regularity. Let T be a multilinear Calderón–Zygmund operator of type $\omega(t)$ with ω being nondecreasing and $\omega \in \mathrm{Dini}(1)$, but without assuming ω to be concave. We obtain the end-point weak-type estimates for multilinear operator T. The multiple-weighted norm inequalities for multilinear operator T and multilinear commutators of T with *BMO* functions are also established.

As applications, multiple-weighted norm estimates for para-products and bilinear pseudo-differential operators with mild regularity and their commutators are obtained.

Moreover, some boundedness properties of the multilinear operators are also established on variable exponent Lebesgue spaces.

Our results improve most of the earlier ones in the literature by removing the assumption of concavity of $\omega(t)$ and weakening the assumption of $\omega \in \mathrm{Dini}(1/2)$ to $\omega \in \mathrm{Dini}(1)$.

1. Introduction and main results

The multilinear Calderón–Zygmund theory was first studied by Coifman and Meyer in [1–3]. This theory was then further investigated by many authors in the last few decades, see for example [4–9], for the theory of multilinear Calderón–Zygmund operators with kernels satisfying the standard estimates. Recently, there are a number of studies concerning multilinear singular integrals which possess rough associated kernels so that they do not belong to the standard Calderón–Zygmund classes. See, for example [10–14] and the references therein. We also mention that the L^p estimates for multi-linear and multi-parameter Coifman–Meyer Fourier multipliers have been established in [15–18].

Recently, Lerner et al. [7] developed a multiple-weight theory that adapts to the multilinear Calderón–Zygmund operators. They established the multiple-weighted norm inequalities for the multilinear Calderón–Zygmund operators and their commutators.

In 2009, Maldonado and Naibo [13] established the weighted norm inequalities, with the Muckenhoupt weights, for the bilinear Calderón–Zygmund operators of type $\omega(t)$, and applied them to the study of para-products and bilinear pseudo-differential operators with mild regularity.

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Motivated by [5,7,13], we will consider the m-linear Calderón–Zygmund operators of type $\omega(t)$ and their commutators, and give some applications to the para-products and the bilinear pseudo-differential operators with mild regularity. In addition, some boundedness properties of the multilinear operators involved on variable exponent Lebesgue spaces are also obtained.

We now give the definition of the multilinear Calderón–Zygmund operators of type $\omega(t)$.

Throughout this paper, we always assume that $\omega(t) : [0, \infty) \to [0, \infty)$ is a nondecreasing function with $0 < \omega(1) < \infty$. For $a > 0$, we say that $\omega \in \mathrm{Dini}(a)$, if

$$|\omega|_{\mathrm{Dini}(a)} := \int_0^1 \frac{\omega^a(t)}{t} dt < \infty.$$

Obviously, $\mathrm{Dini}(a_1) \subset \mathrm{Dini}(a_2)$ provided $0 < a_1 < a_2$, and, if $\omega \in \mathrm{Dini}(1)$ then

$$\sum_{j=0}^{\infty} \omega(2^{-j}) \approx \int_0^1 \frac{\omega(t)}{t} dt < \infty,$$

here and in what follows, $X \approx Y$ means there is a constant $C > 0$ such that $C^{-1}Y \le X \le CY$.

Definition 1.1. A locally integrable function $K(x, y_1, \dots, y_m)$, defined away from the diagonal $x = y_1 = \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$, is called an m-linear Calderón–Zygmund kernel of type $\omega(t)$, if there exists a constant $A > 0$ such that

$$|K(x, y_1, \dots, y_m)| \le \frac{A}{(|x-y_1| + \cdots + |x-y_m|)^{mn}} \tag{1.1}$$

for all $(x, y_1, \dots, y_m) \in (\mathbb{R}^n)^{m+1}$ with $x \ne y_j$ for some $j \in \{1, 2, \dots, m\}$, and

$$|K(x, y_1, \dots, y_m) - K(x', y_1, \dots, y_m)| \le \frac{A}{(|x-y_1| + \cdots + |x-y_m|)^{mn}} \omega\left(\frac{|x-x'|}{|x-y_1| + \cdots + |x-y_m|}\right) \tag{1.2}$$

whenever $|x - x'| \le \frac{1}{2} \max_{1\le j\le m} |x - y_j|$, and

$$\begin{aligned} &|K(x, y_1, \dots, y_j, \dots, y_m) - K(x, y_1, \dots, y_j', \dots, y_m)| \\ &\quad \le \frac{A}{(|x-y_1| + \cdots + |x-y_m|)^{mn}} \omega\left(\frac{|y_j - y_j'|}{|x-y_1| + \cdots + |x-y_m|}\right) \end{aligned} \tag{1.3}$$

whenever $|y_j - y_j'| \le \frac{1}{2} \max_{1\le i\le m} |x - y_i|$.

We say $T : \mathscr{S}(\mathbb{R}^n) \times \cdots \times \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$ is an m-linear operator with an m-linear Calderón–Zygmund kernel of type $\omega(t)$, $K(x, y_1, \dots, y_m)$, if

$$T(f_1, \dots, f_m)(x) = \int_{(\mathbb{R}^n)^m} K(x, y_1, \dots, y_m) f_1(y_1) \cdots f_m(y_m) dy_1 \cdots dy_m$$

whenever $x \notin \bigcap_{j=1}^m \mathrm{supp}\, f_j$ and each $f_j \in C_c^\infty(\mathbb{R}^n)$, $j = 1, \dots, m$.

If T can be extended to a bounded multilinear operator from $L^{q_1}(\mathbb{R}^n) \times \cdots \times L^{q_m}(\mathbb{R}^n)$ to $L^{q,\infty}(\mathbb{R}^n)$ for some $1 < q, q_1, \dots, q_m < \infty$ with $1/q_1 + \cdots + 1/q_m = 1/q$, or, from $L^{q_1}(\mathbb{R}^n) \times \cdots \times L^{q_m}(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ for some $1 < q_1, \dots, q_m < \infty$ with $1/q_1 + \cdots + 1/q_m = 1$, then T is called an m-linear Calderón–Zygmund operator of type $\omega(t)$, abbreviated to m-linear ω-CZO.

Obviously, when $\omega(t) = t^\varepsilon$ for some $\varepsilon > 0$, the m-linear ω-CZO is exactly the multilinear Calderón–Zygmund operator studied by Grafakos and Torres in [5]. The linear Calderón–Zygmund operator of type $\omega(t)$ was studied by Yabuta [19]. The bilinear case in this form was considered by Maldonado and Naibo in [13].

In what follows, the letter C always stands for a constant independent of the main parameter and not necessarily the same at each occurrence. A cube Q in $\mathbb{R}^n$ always means a cube whose sides are parallel to the coordinate axes and denote its side length by $\ell(Q)$. For some $t > 0$, the notation tQ stands for the cube with the same center as Q and with side length $\ell(tQ) = t\ell(Q)$. For $1 \le p \le \infty$, let p' be the conjugate index of p, that is, $1/p + 1/p' = 1$. And we will occasionally use the notations $\vec{f} = (f_1, \dots, f_m)$, $T(\vec{f}) = T(f_1, \dots, f_m)$, $d\vec{y} = dy_1 \cdots dy_m$ and $(x, \vec{y}) = (x, y_1, \dots, y_m)$ for simplicity. For a set E and a positive integer l, we will use the notation $(E)^l = \underbrace{E \times \cdots \times E}_{l}$ sometimes.

1.1. Boundedness of m-linear ω-CZO

Our first result on multilinear operators with multilinear Calderón–Zygmund kernel of type ω is the following end-point weak-type estimates on the product of Lebesgue spaces.

Theorem 1.1. *Let $\omega \in \mathrm{Dini}(1)$ and T be an m-linear operator with an m-linear Calderón–Zygmund kernel of type $\omega(t)$, $K(x, y_1, \ldots, y_m)$. Suppose that for some $1 \le q_1, \ldots, q_m \le \infty$ and some $0 < q < \infty$ with*

$$\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m},$$

T maps $L^{q_1}(\mathbb{R}^n) \times \cdots \times L^{q_m}(\mathbb{R}^n)$ into $L^{q,\infty}(\mathbb{R}^n)$. Then T can be extended to a bounded operator from $L^1(\mathbb{R}^n) \times \cdots \times L^1(\mathbb{R}^n)$ into $L^{1/m,\infty}(\mathbb{R}^n)$. Moreover, there is a constant $C_{m,n,|\omega|_{\mathrm{Dini}(1)}}$ (that depends only on the parameters indicated) such that

$$\|T\|_{L^1 \times \cdots \times L^1 \to L^{1/m,\infty}} \le C_{m,n,|\omega|_{\mathrm{Dini}(1)}} \big(A + \|T\|_{L^{q_1} \times \cdots \times L^{q_m} \to L^{q,\infty}}\big),$$

where A is the constant appearing in (1.1)–(1.3).

Remark 1.1. When $\omega(t) = t^{\varepsilon}$ for some $\varepsilon > 0$, Theorem 1.1 was proved in [5]. For the bilinear case, Theorem 1.1 was proved in [13] when ω is concave and $\omega \in \mathrm{Dini}(1/2)$. Comparing our Theorem 1.1 with Theorem 6.1 of [13], we remove the hypothesis that ω is concave and reduce the condition $\omega \in \mathrm{Dini}(1/2)$ to a weaker condition $\omega \in \mathrm{Dini}(1)$.

Recently, Pérez and Torres [14] introduced the minimal regularity conditions, so-called the bilinear geometric Hörmander conditions (BGHC), on the kernels of bilinear operators. They showed that the BGHC is sufficient for the existence of end-point estimates of the bilinear Calderón–Zygmund operators. Instead of (1.3), they considered the following condition

$$|K(x, y_1, y_2) - K(x, y_1', y_2')| \le \frac{C}{(|x-y_1| + |x-y_2|)^{2n}} \omega\left(\frac{|y_1 - y_1'| + |y_2 - y_2'|}{|x-y_1| + |x-y_2|}\right) \tag{1.4}$$

whenever $|y_1 - y_1'| \le \frac{1}{2}|x - y_1|$ and $|y_2 - y_2'| \le \frac{1}{2}|x - y_2|$. They verified that $\omega \in \mathrm{Dini}(1)$ together with condition (1.4) implies the BGHC. Obviously, if K satisfies (1.4) then it also satisfies (1.3).

Remark 1.2. Though some ideas of the proof of Theorem 1.1 are from [5,13,14], there are some substantial differences and modifications in our arguments. For instance, the concavity of ω, which implies the doubling property of ω, is needed in [13], while in Theorem 1.1 we do not require ω to be concave. In addition, the Marcinkiewicz function is a basic tool in studying the weak-type estimates for the multilinear Calderón–Zygmund operators, see [5,11] for example, and we make no use of it in the proof of Theorem 1.1. Moreover, our (1.3) is a weaker assumption than (1.4), and our estimates appear to be more delicate and complicated.

To state the weighted norm inequalities for the multilinear Calderón–Zygmund operators of type $\omega(t)$, we first recall the definition of multiple-weights introduced by Lerner et al. [7].

Definition 1.2. Let $\vec{P} = (p_1, \ldots, p_m)$ and $1/p = 1/p_1 + \cdots + 1/p_m$ with $1 \le p_1, \ldots, p_m < \infty$. Given $\vec{w} = (w_1, \ldots, w_m)$ with each w_j being nonnegative measurable, set

$$\nu_{\vec{w}} = \prod_{j=1}^{m} w_j^{p/p_j}.$$

We say that $\vec{w}$ satisfies the $A_{\vec{P}}$ condition and write $\vec{w} \in A_{\vec{P}}$, if

$$\sup_Q \left(\frac{1}{|Q|}\int_Q \nu_{\vec{w}}(x)dx\right)^{1/p} \prod_{j=1}^{m} \left(\frac{1}{|Q|}\int_Q w_j(x)^{1-p_j'}dx\right)^{1/p_j'} < \infty,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$, and the term $\left(\frac{1}{|Q|}\int_Q w_j(x)^{1-p_j'}dx\right)^{1/p_j'}$ is understood as $\left(\inf_Q w_j\right)^{-1}$ when $p_j = 1$.

From now on, we will use the following notations. For $0 < p < \infty$ and $w \in A_\infty$, denote by $L^p(w)$ the collection of all functions f satisfying

$$\|f\|_{L^p(w)} := \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)dx\right)^{1/p} < \infty.$$

And, denote by $L^{p,\infty}(w)$ the weak space with norm

$$\|f\|_{L^{p,\infty}(w)} := \sup_{t>0} t\, w\big(\{x \in \mathbb{R}^n : |f(x)| > t\}\big)^{1/p},$$

where $w(E) := \int_E w(x)dx$ for a measurable set $E \subset \mathbb{R}^n$.

Our next theorem concerns the multiple-weighted norm inequalities and weak-type estimates.

Theorem 1.2. *Let T be an m-linear ω-CZO with $\omega \in \mathrm{Dini}(1)$. Let $\vec{P} = (p_1, \dots, p_m)$ with $1/p = 1/p_1 + \cdots + 1/p_m$, and $\vec{w} \in A_{\vec{P}}$.*
(1) *If $1 < p_j < \infty$ for all $j = 1, \dots, m$, then*

$$\left\|T(\vec{f})\right\|_{L^p(\nu_{\vec{w}})} \le C \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(w_j)}.$$

(2) *If $1 \le p_j < \infty$ for all $j = 1, \dots, m$, and at least one of the $p_j = 1$, then*

$$\left\|T(\vec{f})\right\|_{L^{p,\infty}(\nu_{\vec{w}})} \le C \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(w_j)}.$$

Remark 1.3. When $\omega \in \mathrm{Dini}(1/2)$ and ω is concave, the first part of Theorem 1.2 was proved in [13, Theorem 6.8] for $m = 2$ and $\nu_{\vec{w}} = w_1 = w_2 \in A_{\min(p_1,p_2)}$. So, the first part of Theorem 1.2 improves the corresponding result in [13] in two aspects. Firstly, we reduce the condition $\omega \in \mathrm{Dini}(1/2)$ to a weaker condition $\omega \in \mathrm{Dini}(1)$ and remove the hypothesis that ω is concave. Secondly, we extend the weights from the classical A_p class to the $A_{\vec{P}}$ class. Moreover, when $\omega(t) = t^{\varepsilon}$ for some $\varepsilon > 0$, Theorem 1.2 was proved in [7].

1.2. Commutators of m-linear ω-CZO

Let T be an m-linear operator, given a collection of locally integrable functions $\vec{b} = (b_1, \dots, b_m)$, the m-linear commutator of T with $\vec{b}$ is defined by

$$T_{\vec{b}}(f_1, \dots, f_m) = \sum_{j=1}^{m} T_{\vec{b}}^{j}(\vec{f})$$

where

$$T_{\vec{b}}^{j}(\vec{f}) = b_j T(f_1, \dots, f_j, \dots, f_m) - T(f_1, \dots, b_j f_j, \dots, f_m).$$

We will use the notation $\vec{b} \in BMO^m$ stands for $b_j \in BMO(\mathbb{R}^n)$ for $j = 1, \dots, m$, and denote by $\|\vec{b}\|_{BMO^m} = \max_{1\le j\le m} \|b_j\|_{BMO(\mathbb{R}^n)}$.

Theorem 1.3. *Let T be an m-linear ω-CZO and $T_{\vec{b}}$ be the m-linear commutator of T with $\vec{b} \in BMO^m$. Let $\vec{w} \in A_{\vec{P}}$ with $1/p = 1/p_1 + \cdots + 1/p_m$ and $1 < p_j < \infty$, $j = 1, \dots, m$. If ω satisfies*

$$\int_0^1 \frac{\omega(t)}{t}\left(1 + \log\frac{1}{t}\right) dt < \infty, \tag{1.5}$$

then there exists a constant $C > 0$ such that

$$\left\|T_{\vec{b}}(\vec{f})\right\|_{L^p(\nu_{\vec{w}})} \le C\|\vec{b}\|_{BMO^m} \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(w_j)}.$$

It is easy to check that if ω satisfies (1.5), then $\omega \in \mathrm{Dini}(1)$ and

$$\sum_{k=1}^{\infty} k \cdot \omega(2^{-k}) \approx \int_0^1 \frac{\omega(t)}{t}\left(1 + \log\frac{1}{t}\right) dt < \infty.$$

We remark that, since the commutator has more singularity, the more regular conditions imposed on the kernel is reasonable. In addition, although condition (1.5) is stronger than the Dini(1) condition, a standard Calderón–Zygmund kernel is also a Calderón–Zygmund kernel of type $\omega(t)$ with ω satisfying (1.5) in the linear case, so does in the multilinear case.

For the multiple-weighted weak-type estimate, we have the following result.

Theorem 1.4. *Let T be an m-linear ω-CZO with ω satisfying (1.5) and $T_{\vec{b}}$ be the m-linear commutator of T with $\vec{b} \in BMO^m$. If $\vec{w} \in A_{(1,\dots,1)}$, then there exists a constant $C > 0$ depending on $\|\vec{b}\|_{BMO^m}$ such that*

$$\nu_{\vec{w}}\left(\left\{x \in \mathbb{R}^n : \left|T_{\vec{b}}(\vec{f})(x)\right| > \lambda^m\right\}\right) \le C \prod_{j=1}^{m} \left(\int_{\mathbb{R}^n} \Phi\left(\frac{|f_j(x)|}{\lambda}\right) w_j(x) dx\right)^{1/m}$$

here and in the sequel, $\Phi(t) = t(1 + \log^+ t)$.

Remark 1.4. For the linear commutator of the Calderón–Zygmund operator of type $\omega(t)$ with $b \in BMO(\mathbb{R}^n)$, see [20,21]. Lerner et al. [7] proved that the conclusions of Theorems 1.3 and 1.4 hold when $\omega(t) = t^{\varepsilon}$ for some $\varepsilon > 0$.

The remainder of this paper will be organized as follows. In Section 2, we will apply the main results to the para-products associated with ω-molecules and the bilinear pseudo-differential operators with mild regularity. In Section 3, some boundedness properties of the multilinear operators involved on variable exponent Lebesgue spaces are given. In Section 4, we recall some basic definitions and known results needed. The remaining sections are devoted to proving the theorems of this paper.

2. Applications

In this section, we will apply the results stated above to the para-products associated with ω-molecules and the bilinear pseudo-differential operators with mild regularity.

2.1. Para-products with mild regularity

For $v \in \mathbb{Z}$ and $\kappa = (k_1, \dots, k_n) \in \mathbb{Z}^n$, let $P_{v\kappa}$ be the dyadic cube

$$P_{v\kappa} := \left\{(x_1, \dots, x_n) \in \mathbb{R}^n : k_i \le 2^v x_i < k_i + 1,\ i = 1, \dots, n\right\}.$$

The lower left-corner of $P := P_{v\kappa}$ is $x_P = x_{v\kappa} := 2^{-v\kappa}$ and the Lebesgue measure of P is $|P| = 2^{-vn}$. We set

$$\mathcal{D} = \left\{P_{v\kappa} : v \in \mathbb{Z},\ \kappa \in \mathbb{Z}^n\right\}$$

as the collection of all dyadic cubes.

Definition 2.1 (*[13]*). Let $\omega : [0, \infty) \to [0, \infty)$ be a nondecreasing and concave function. An ω-molecule associated to a dyadic cube $P = P_{v\kappa}$ is a function $\phi_P = \phi_{v\kappa} : \mathbb{R}^n \to \mathbb{C}$ such that, for some $A_0 > 0$ and $N > n$, it satisfies the decay condition

$$|\phi_P(x)| \le \frac{A_0 2^{vn/2}}{(1 + 2^v|x - x_P|)^N}, \quad x \in \mathbb{R}^n, \tag{2.1}$$

and the mild regularity condition

$$|\phi_P(x) - \phi_P(y)| \le A_0 2^{vn/2}\omega(2^v|x - y|)\left[\frac{1}{(1 + 2^v|x - x_P|)^N} + \frac{1}{(1 + 2^v|y - x_P|)^N}\right] \tag{2.2}$$

for all $x, y \in \mathbb{R}^n$.

Definition 2.2 (*[13]*). Given three families of ω-molecules $\{\phi_Q^j\}_{Q\in\mathcal{D}}$, $j = 1, 2, 3$, the para-product $\Pi(f, g)$ associated to these families is defined by

$$\Pi(f, g) = \sum_{Q\in\mathcal{D}} |Q|^{-1/2}\langle f, \phi_Q^1\rangle\langle g, \phi_Q^2\rangle\phi_Q^3, \quad f, g \in \mathscr{S}(\mathbb{R}^n). \tag{2.3}$$

The term para-product was coined by Bony in [22], which has been studied extensively and has experienced remarkable development in recent years. Operators of the form (2.3) have been studied by many authors. Some developments on para-products and their applications can be found in [23,18,24,25,13,15,16], and references therein.

In [23], it is showed that para-products associated to smooth molecules can be realized as bilinear Calderón–Zygmund operators. In [13], some sufficient conditions on ω were given so that the para-products built from ω-molecules can be realized as bilinear ω-CZOs.

As applications of the results stated in Section 1, we consider the multiple-weighted norm inequalities of para-products associated to ω-molecules and the multilinear commutator with *BMO* functions. Our results can be stated as follows.

Theorem 2.1. *Let ω be concave with $\omega \in \mathrm{Dini}(1)$, and $\{\phi_Q^j\}_{Q\in\mathcal{D}}, j = 1, 2, 3$, be three families of ω-molecules with decay $N > 10n$ and such that at least two of them, say $j = 1, 2$, enjoy the following cancellation property*

$$\int_{\mathbb{R}^n} \phi_Q^j(x)dx = 0, \quad Q \in \mathcal{D},\ j = 1, 2.$$

Let $\vec{P} = (p_1, p_2)$ with $1/p = 1/p_1 + 1/p_2$, and $\vec{w} \in A_{\vec{P}}$.

(1) *If $1 < p_1, p_2 < \infty$, then*

$$\left\| \Pi(f_1, f_2)\right\|_{L^p(v_{\vec{w}})} \le C\|f_1\|_{L^{p_1}(w_1)}\|f_2\|_{L^{p_2}(w_2)}.$$

(2) *If* $1 \le p_1, p_2 < \infty$ *and at least one of the* $p_j = 1$, *then*

$$\left\| \Pi(f_1, f_2) \right\|_{L^{p,\infty}(\nu_{\vec{w}})} \le C \|f_1\|_{L^{p_1}(w_1)} \|f_2\|_{L^{p_2}(w_2)}.$$

Remark 2.1. Theorem 2.1 improves the corresponding result of [13] in two aspects. Firstly, we reduce the condition $\omega \in \mathrm{Dini}(1/2)$ to $\omega \in \mathrm{Dini}(1)$, which is weaker than the former one. Secondly, we extend the weights from the Muckenhoupt A_p class to the multiple-weight class $A_{\vec{P}}$.

Let $\vec{b} = (b_1, b_2) \in BMO^2$, we define the commutators of the para-product Π with $\vec{b}$ by

$$\Pi_{\vec{b}}(f_1, f_2)(x) = \left[b_1(x)\Pi(f_1, f_2)(x) - \Pi(b_1 f_1, f_2)(x) \right] + \left[b_2(x)\Pi(f_1, f_2)(x) - \Pi(f_1, b_2 f_2)(x) \right].$$

Theorem 2.2. *Under the assumption of Theorem 2.1, if, in addition, we assume that* ω *satisfies (1.5) and* $1 < p_1, p_2 < \infty$, *then there exists a constant* $C > 0$ *such that*

$$\left\| \Pi_{\vec{b}}(f_1, f_2) \right\|_{L^p(\nu_{\vec{w}})} \le C \|\vec{b}\|_{BMO^2} \|f_1\|_{L^{p_1}(w_1)} \|f_2\|_{L^{p_2}(w_2)}.$$

Theorem 2.3. *Let* ω *and* ϕ_Q^j *be the same as in Theorem 2.1 and* ω *satisfy (1.5). If* $\vec{w} \in A_{(1,1)}$, *then there exists a constant* $C > 0$ *depending on* $\|\vec{b}\|_{BMO^2}$ *such that*

$$\nu_{\vec{w}} \left(\left\{ x \in \mathbb{R}^n : \left| \Pi_{\vec{b}}(f_1, f_2)(x) \right| > \lambda^2 \right\} \right) \le C \prod_{j=1}^{2} \left(\int_{\mathbb{R}^n} \Phi\left(\frac{|f_j(x)|}{\lambda} \right) w_j(x) dx \right)^{1/2}.$$

2.2. Bilinear pseudo-differential operators with mild regularity

Let $m \in \mathbb{R}$, $0 \le \delta, \rho \le 1$ and $\alpha, \beta, \gamma \in \mathbb{Z}_+^n$. A bilinear pseudo-differential operator T_σ with a bilinear symbol $\sigma(x, \xi, \eta)$, a priori defined from $\mathscr{S}(\mathbb{R}^n) \times \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$, is given by

$$T_\sigma(f_1, f_2)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{ix\cdot(\xi+\eta)} \sigma(x, \xi, \eta) \hat{f}_1(\xi) \hat{f}_2(\eta) d\xi d\eta.$$

We say that a symbol $\sigma(x, \xi, \eta)$ belongs to the bilinear Hörmander class $BS^m_{\rho,\delta}$ if

$$\left| \partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma(x, \xi, \eta) \right| \le C_{\alpha,\beta} (1 + |\xi| + |\eta|)^{m+\delta|\alpha|-\rho(|\beta|+|\gamma|)}, \quad x, \xi, \eta \in \mathbb{R}^n$$

for all multi-indices α, β and γ and some constant $C_{\alpha,\beta}$.

For $\Omega : [0, \infty) \to [0, \infty)$, $m \in \mathbb{R}$ and $0 \le \rho \le 1$, we say that a symbol $\sigma \in BS^m_{\rho,\omega,\Omega}$ if

$$\left| \partial_\xi^\alpha \partial_\eta^\beta \sigma(x, \xi, \eta) \right| \le C_{\alpha,\beta} (1 + |\xi| + |\eta|)^{m-\rho(|\alpha|+|\beta|)} \tag{2.4}$$

and

$$\left| \partial_\xi^\alpha \partial_\eta^\beta \left(\sigma(x+h, \xi, \eta) - \sigma(x, \xi, \eta) \right) \right| \le C_{\alpha,\beta} \omega(|h|) \Omega(|\xi| + |\eta|) (1 + |\xi| + |\eta|)^{m-\rho(|\alpha|+|\beta|)} \tag{2.5}$$

for all $x, \xi, \eta \in \mathbb{R}^n$. Obviously, $BS^m_{\rho,0} \subset BS^m_{\rho,\omega,\Omega}$.

The study of bilinear pseudo-differential operators grew from the early works of Coifman and Meyer [1–3]. In 1978, Coifman and Meyer [3] considered a bilinear pseudo-differential operator with symbol $\sigma \in BS^0_{1,\omega,1}$ and $\omega \in \mathrm{Dini}(2)$ in unweighted Lebesgue spaces. In 2009, Maldonado and Naibo [13] studied the weighted norm inequalities for pseudo-differential operator with associated symbol $\sigma(x, \xi, \eta) \in BS^0_{1,\omega,\Omega}$ (see Theorem 1.1 in [13]).

The purpose of this subsection is to apply the results stated in Section 1 to the bilinear pseudo-differential operators with associated symbol $\sigma \in BS^0_{1,\omega,\Omega}$. We obtain the multiple-weighted norm inequalities for the bilinear pseudo-differential operators and their commutators. Our results can be stated as follows.

Theorem 2.4. *Let* $a \in (0, 1)$, ω *be concave with* $\omega \in \mathrm{Dini}(a/2)$, *and* $\Omega : [0, \infty) \to [0, \infty)$ *be nondecreasing such that*

$$\sup_{0<t<1} \omega^{1-a}(t) \Omega(1/t) < \infty. \tag{2.6}$$

Suppose that $\vec{P} = (p_1, p_2)$ *with* $1/p = 1/p_1 + 1/p_2$ *and* $\vec{v}_w \in A_{\vec{P}}$. *If* $\sigma \in BS^0_{1,\omega,\Omega}$ *with* $|\alpha| + |\beta| \le 4n + 4$, *then the following boundedness properties hold:*

(1) *If* $1 < p_1, p_2 < \infty$, *then*

$$\|T_\sigma(f_1, f_2)\|_{L^p(\nu_{\vec{w}})} \le C \|f_1\|_{L^{p_1}(w_1)} \|f_2\|_{L^{p_2}(w_2)}.$$

(2) *If* $1 \le p_1, p_2 < \infty$ *and at least one of the* $p_j = 1$, *then*

$$\|T_\sigma(f_1, f_2)\|_{L^{p,\infty}(\nu_{\vec{w}})} \le C \|f_1\|_{L^{p_1}(w_1)} \|f_2\|_{L^{p_2}(w_2)}.$$

Remark 2.2. The first part of Theorem 2.4 when $\vec{v}_w = w_1 = w_2 \in A_{\min(p_1,p_2)}$ and the second part when $\vec{v}_w = w_1 = w_2 \in A_1$ and $p_1 = p_2 = 1$ were obtained in Theorem 1.1 of [13].

Let $\vec{b} = (b_1, b_2) \in BMO^2$, the commutator of the bilinear pseudo-differential operator T_σ with $\vec{b} = (b_1, b_2)$ is defined by

$$T_\sigma^{\vec{b}}(f_1, f_2)(x) = \left[b_1(x)T_\sigma(f_1, f_2)(x) - T_\sigma(b_1 f_1, f_2)(x)\right] + \left[b_2(x)T_\sigma(f_1, f_2)(x) - T_\sigma(f_1, b_2 f_2)(x)\right].$$

Theorem 2.5. *Under the assumption of Theorem 2.4, if, in addition, we assume that $\omega^a(t)$ satisfies (1.5) and $1 < p_1, p_2 < \infty$, then there exists a constant $C > 0$ such that*

$$\left\| T_\sigma^{\vec{b}}(f_1, f_2) \right\|_{L^p(\nu_{\vec{w}})} \le C \|\vec{b}\|_{BMO^2} \|f_1\|_{L^{p_1}(w_1)} \|f_2\|_{L^{p_2}(w_2)}.$$

Theorem 2.6. *Let a, ω and Ω be the same as in Theorem 2.4, and, in addition, we assume that $\omega^a(t)$ satisfies (1.5). If $\sigma \in BS^0_{1,\omega,\Omega}$ with $|\alpha| + |\beta| \le 4n + 4$ and $\vec{w} \in A_{(1,1)}$, then, there exists a constant $C > 0$, depending on $\|\vec{b}\|_{BMO^2}$, such that*

$$\nu_{\vec{w}}\left(\left\{x \in \mathbb{R}^n : \left|T_\sigma^{\vec{b}}(f_1, f_2)(x)\right| > \lambda^2\right\}\right) \le C \prod_{j=1}^{2} \left(\int_{\mathbb{R}^n} \Phi\left(\frac{|f_j(x)|}{\lambda}\right) w_j(x)dx\right)^{1/2}.$$

3. On variable exponent Lebesgue spaces

In this section, we will study the boundedness properties of m-linear ω-CZOs, the para-products and the bilinear pseudo-differential operators with mild regularity and their commutators on variable exponent Lebesgue spaces. We first recall some definitions and notations.

Definition 3.1. Let $p(\cdot) : \mathbb{R}^n \to [1, \infty)$ be a measurable function. The variable exponent Lebesgue space, $L^{p(\cdot)}(\mathbb{R}^n)$, is defined by

$$L^{p(\cdot)}(\mathbb{R}^n) = \left\{f \text{ measurable} : \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx < \infty \text{ for some constant } \lambda > 0\right\}.$$

It is well known that the set $L^{p(\cdot)}(\mathbb{R}^n)$ becomes a Banach space with respect to the norm

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx \le 1\right\}.$$

Denote by $\mathscr{P}(\mathbb{R}^n)$ the set of all measurable functions $p(\cdot) : \mathbb{R}^n \to [1, \infty)$ such that

$$1 < p_- := \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} p(x) \quad \text{and} \quad p_+ := \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p(x) < \infty,$$

and by $\mathscr{B}(\mathbb{R}^n)$ the set of all $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ such that the Hardy–Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

The theory of function spaces with variable exponent has been intensely investigated in the past twenty years since some elementary properties were established by Kováčik and Rákosník in [26]. In 2003, Diening and Růžička [27] studied the Calderón–Zygmund operators on variable exponent Lebesgue spaces and gave some applications to problems related to fluid dynamics. In 2006, by applying the theory of weighted norm inequalities and extrapolation, Cruz-Uribe et al. [28] showed that many classical operators in harmonic analysis are bounded on the variable exponent Lebesgue space. For more information on function spaces with variable exponent, we refer to [29,30].

For the m-linear ω-CZO and its commutator, we have the following results.

Theorem 3.1. *Let T be an m-linear ω-CZO with $\omega \in \mathrm{Dini}(1)$. If $p(\cdot), p_1(\cdot), \ldots, p_m(\cdot) \in \mathscr{B}(\mathbb{R}^n)$ so that*

$$\frac{1}{p(\cdot)} = \frac{1}{p_1(\cdot)} + \cdots + \frac{1}{p_m(\cdot)}. \tag{3.1}$$

Then there exists a positive constant C such that

$$\left\|T(\vec{f})\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C \prod_{j=1}^{m} \|f_j\|_{L^{p_j(\cdot)}(\mathbb{R}^n)}.$$

Theorem 3.2. *Let T be an m-linear ω-CZO with ω satisfying (1.5) and $T_{\vec{b}}$ be the m-linear commutator of T with $\vec{b} \in BMO^m$. If $p(\cdot), p_1(\cdot), \ldots, p_m(\cdot) \in \mathscr{B}(\mathbb{R}^n)$ so that (3.1) holds. Then there exists a constant $C > 0$ such that*

$$\left\|T_{\vec{b}}(\vec{f})\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C \prod_{j=1}^{m} \|f_j\|_{L^{p_j(\cdot)}(\mathbb{R}^n)}.$$

For the para-product defined by (2.3) and its commutator, the similar boundedness properties also hold.

Theorem 3.3. *Let ω, $\{\phi_Q^j\}_{Q\in\mathcal{D}}$, $j=1,2,3$, and N be the same as in Theorem 2.1. If $p(\cdot), p_1(\cdot), p_2(\cdot) \in \mathscr{B}(\mathbb{R}^n)$ so that*

$$\frac{1}{p(\cdot)} = \frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)}, \tag{3.2}$$

then there exists a constant $C > 0$ such that

$$\left\|\Pi(f_1,f_2)\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|f_1\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|f_2\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}.$$

Theorem 3.4. *Let ω, $\{\phi_Q^j\}_{Q\in\mathcal{D}}$, $j=1,2,3$, and N be the same as in Theorem 2.1 and, in addition, we assume that ω satisfies (1.5). If $p(\cdot), p_1(\cdot), p_2(\cdot) \in \mathscr{B}(\mathbb{R}^n)$ so that (3.2) holds, then there exists a constant $C > 0$ such that*

$$\left\|\Pi_{\vec{b}}(f_1,f_2)\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|f_1\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|f_2\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}.$$

For the bilinear pseudo-differential operators with mild regularity stated above and their commutators, there hold the following results.

Theorem 3.5. *Let a, ω and Ω be the same as in Theorem 2.4. Suppose that $\sigma \in BS^0_{1,\omega,\Omega}$ with $|\alpha|+|\beta| \le 4n+4$. If $p(\cdot), p_1(\cdot), p_2(\cdot) \in \mathscr{B}(\mathbb{R}^n)$ so that (3.2) holds, then there exists a constant $C > 0$ such that*

$$\left\|T_\sigma(f_1,f_2)\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|f_1\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|f_2\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}.$$

Theorem 3.6. *Let a, ω and Ω be the same as in Theorem 2.4 and, in addition, we assume that $\omega^a(t)$ satisfies (1.5). Suppose that $\sigma \in BS^0_{1,\omega,\Omega}$ with $|\alpha|+|\beta| \le 4n+4$. If $p(\cdot), p_1(\cdot), p_2(\cdot) \in \mathscr{B}(\mathbb{R}^n)$ so that (3.2) holds, then there exists a constant $C > 0$ such that*

$$\left\|T_\sigma^{\vec{b}}(f_1,f_2)\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|f_1\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|f_2\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}.$$

We will prove Theorems 3.1–3.6 in the last section.

4. Notations and preliminaries

4.1. Sharp maximal function and A_p weights

Let f be a locally integral function. For a cube Q, denote by $f_Q = \frac{1}{|Q|}\int_Q f(y)dy$. The sharp maximal function of Fefferman and Stein [31] is defined by

$$M^\sharp(f)(x) = \sup_{Q\ni x}\inf_c \frac{1}{|Q|}\int_Q |f(y)-c|dy \approx \sup_{Q\ni x}\frac{1}{|Q|}\int_Q |f(y)-f_Q|dy.$$

Let M be the usual Hardy–Littlewood maximal operator, for $0<\delta<\infty$, we define the maximal functions M_δ and $M_\delta^\sharp$ by

$$M_\delta(f) = \left[M(|f|^\delta)\right]^{1/\delta} \quad \text{and} \quad M_\delta^\sharp(f) = \left[M^\sharp\left(|f|^\delta\right)\right]^{1/\delta}.$$

Let w be a nonnegative locally integrable function defined in $\mathbb{R}^n$. We say that $w \in A_p$, $1<p<\infty$, if there is a constant $C>0$ such that for any cube Q, there has

$$\left(\frac{1}{|Q|}\int_Q w(x)dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{1-p'}dx\right)^{p-1} \le C.$$

We say that $w \in A_1$ if there is a constant $C>0$ such that $Mw(x) \le Cw(x)$ almost everywhere. And we defined $A_\infty = \bigcup_{p\ge 1} A_p$. See [32] or [33] for more information about the Muckenhoupt weight class A_p.

The following relationships between $M_\delta^\sharp$ and M_δ to be used is a version of the classical ones due to Fefferman and Stein [31], see also [7] page 1228.

Lemma 4.1. (1) *Let $0<p,\delta<\infty$ and $w \in A_\infty$. Then there exists a constant $C>0$ (depending on the A_∞ constant of w) such that*

$$\int_{\mathbb{R}^n}\left[M_\delta(f)(x)\right]^p w(x)dx \le C\int_{\mathbb{R}^n}\left[M_\delta^\sharp(f)(x)\right]^p w(x)dx,$$

for every function f such that the left-hand side is finite.

(2) *Let* $0 < \delta < \infty$ *and* $w \in A_\infty$. *If* $\varphi : (0, \infty) \to (0, \infty)$ *is doubling, then there exists a constant* $C > 0$ *(depending on the* A_∞ *constant of* w *and the doubling condition of* φ*) such that*

$$\sup_{\lambda>0} \varphi(\lambda) w\left(\left\{y \in \mathbb{R}^n : M_\delta(f)(y) > \lambda\right\}\right) \le C \sup_{\lambda>0} \varphi(\lambda) w\left(\left\{y \in \mathbb{R}^n : M_\delta^\sharp(f)(y) > \lambda\right\}\right),$$

for every function f *such that the left-hand side is finite.*

4.2. Some facts of the Orlicz spaces

We need some basic facts from the theory of Orlicz spaces. For more information about Orlicz spaces, we refer to [34].

A function Ψ defined on $[0, \infty)$ is called a Young function, if Ψ is a continuous, increasing and convex function with $\Psi(0) = 0$ and $\Psi(t) \to \infty$ as $t \to \infty$. The Orlicz space $L_\Psi(\mathbb{R}^n)$ is defined to be the set of all measurable functions f such that for some $\lambda > 0$,

$$\int_{\mathbb{R}^n} \Psi\left(\frac{|f(x)|}{\lambda}\right) dx < \infty.$$

The space $L_\Psi(\mathbb{R}^n)$ is a Banach space when endowed with the Luxemburg norm

$$\|f\|_\Psi = \|f\|_{L_\Psi} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \Psi\left(\frac{|f(x)|}{\lambda}\right) dx \le 1\right\}.$$

The Ψ-average of a function f on a cube Q is defined by

$$\|f\|_{\Psi,Q} = \inf\left\{\lambda > 0 : \frac{1}{|Q|}\int_Q \Psi\left(\frac{|f(x)|}{\lambda}\right) dx \le 1\right\}.$$

The following generalized Jensen's inequality holds (see (2.10) in [7]).

Lemma 4.2. *If* Ψ_1 *and* Ψ_2 *are two Young functions with* $\Psi_1(t) \le \Psi_2(t)$, *for* $t \ge t_0 > 0$, *then there is a constant* $C > 0$ *such that* $\|f\|_{\Psi_1,Q} \le C\|f\|_{\Psi_2,Q}$.

For Young functions $\Phi(t) = t(1 + \log^+ t)$ and $\Psi(t) = e^t - 1$, the corresponding averages will be denoted by

$$\|\cdot\|_{\Phi,Q} = \|\cdot\|_{L(\log L),Q} \quad \text{and} \quad \|\cdot\|_{\Psi,Q} = \|\cdot\|_{\exp L,Q}.$$

The following inequality holds (see (2.13) in [7])

$$\frac{1}{|Q|}\int_Q |f(x)g(x)|dx \le C\|f\|_{\exp L,Q}\|g\|_{L(\log L),Q}. \tag{4.1}$$

Let $Q \subset \mathbb{R}^n$ be a cube and $b \in BMO(\mathbb{R}^n)$, the generalized Hölder inequality (4.1) together with John–Nirenberg's inequality implies that (see (2.14) in [7])

$$\frac{1}{|Q|}\int_Q |b(x) - b_Q|\,|f(x)|dx \le C\|b\|_{BMO}\|f\|_{L(\log L),Q}. \tag{4.2}$$

The maximal function related to Young function $\Phi(t) = t(1 + \log^+ t)$ is defined by

$$M_{L(\log L)}(f)(x) = \sup_{Q\ni x} \|f\|_{L(\log L),Q},$$

where the supremum is taken over all the cubes containing x.

4.3. Multilinear maximal functions and multiple weights

The following multilinear maximal functions that adapts to the multilinear Calderón–Zygmund theory are introduced by Lerner et al. in [7].

Definition 4.1. For all locally integrable functions $\vec{f} = (f_1, \ldots, f_m)$ and $x \in \mathbb{R}^n$, the multilinear maximal functions $\mathcal{M}$ and $\mathcal{M}_r$ are defined by

$$\mathcal{M}(\vec{f})(x) = \sup_{Q\ni x} \prod_{j=1}^m \frac{1}{|Q|}\int_Q |f_j(y_j)|dy_j,$$

and

$$\mathcal{M}_r(\vec{f})(x) = \sup_{Q\ni x} \prod_{j=1}^m \left(\frac{1}{|Q|}\int_Q |f_j(y_j)|^r dy_j\right)^{1/r}, \quad \text{for } r > 1,$$

the maximal functions related to Young function $\Phi(t) = t(1 + \log^+ t)$ are defined by

$$\mathcal{M}^i_{L(\log L)}(\vec{f})(x) = \sup_{Q \ni x} \|f_i\|_{L(\log L),Q} \prod_{j \neq i}^{m} \frac{1}{|Q|} \int_Q |f_j(y_j)| dy_j$$

and

$$\mathcal{M}_{L(\log L)}(\vec{f})(x) = \sup_{Q \ni x} \prod_{j=1}^{m} \|f_j\|_{L(\log L),Q},$$

where the supremum is taken over all the cubes Q containing x.

Obviously, if $r > 1$, then the following pointwise estimates hold

$$\mathcal{M}(\vec{f})(x) \leq C\mathcal{M}^i_{L(\log L)}(\vec{f})(x) \leq C'\mathcal{M}_{L(\log L)}(\vec{f})(x) \leq C''\mathcal{M}_r(\vec{f})(x). \tag{4.3}$$

The first two inequalities in (4.3) follows from

$$\frac{1}{|Q|} \int_Q |f_j(y_j)| dy_j \leq \|f_j\|_{L(\log L),Q},$$

and the last one follows from the generalized Jensen's inequality (Lemma 4.2).

In [7], the characterizations of the multiple-weight class $A_{\vec{P}}$ in terms of the multilinear maximal function $\mathcal{M}$ are proved in Theorems 3.3 and 3.7. We restate it as follows.

Lemma 4.3 ([7]). *Let $\vec{P} = (p_1, \ldots, p_m),\ 1 \leq p_1, \ldots, p_m < \infty$ and $1/p = 1/p_1 + \cdots + 1/p_m$.*

(1) *If $1 < p_1, \ldots, p_m < \infty$, then $\mathcal{M}$ is bounded from $L^{p_1}(w_1) \times \cdots \times L^{p_m}(w_m)$ to $L^p(\nu_{\vec{w}})$ if and only if $\vec{w} = (w_1, \ldots, w_m) \in A_{\vec{P}}$.*
(2) *If $1 \leq p_1, \ldots, p_m < \infty$, then $\mathcal{M}$ is bounded from $L^{p_1}(w_1) \times \cdots \times L^{p_m}(w_m)$ to $L^{p,\infty}(\nu_{\vec{w}})$ if and only if $\vec{w} = (w_1, \ldots, w_m) \in A_{\vec{P}}$.*

The characterization of the multiple-weight class $A_{\vec{P}}$ in terms of the Muckenhoupt weights, which will be used later, is also established in Theorem 3.6 of [7].

Lemma 4.4 ([7]). *Let $\vec{w} = (w_1, \ldots, w_m),\ \vec{P} = (p_1, \ldots, p_m)$ and $1/p = 1/p_1 + \cdots + 1/p_m$ with $1 \leq p_1, \ldots, p_m < \infty$. Then $\vec{w} \in A_{\vec{P}}$ if and only if*

$$\begin{cases} w_j^{1-p'_j} \in A_{mp'_j}, \quad j = 1, \ldots, m, \\ \nu_{\vec{w}} \in A_{mp}, \end{cases}$$

where the condition $w_j^{1-p'_j} \in A_{mp'_j}$ in the case $p_j = 1$ is understood as $w_j^{1/m} \in A_1$.

The following boundedness property of $\mathcal{M}_r$ is contained in the proof of Theorem 3.18 of [7] page 1258.

Lemma 4.5. *Let $\vec{P} = (p_1, \ldots, p_m),\ 1 < p_1, \ldots, p_m < \infty$ and $1/p = 1/p_1 + \cdots + 1/p_m$. If $\vec{w} \in A_{\vec{P}}$, then there exists a constant $r > 1$ such that, $\mathcal{M}_r$ is bounded from $L^{p_1}(w_1) \times \cdots \times L^{p_m}(w_m)$ to $L^p(\nu_{\vec{w}})$.*

4.4. Kolmogorov's inequality

Finally, we will also need the following Kolmogorov's inequality (see page 485 in [32] or (2.16) in [7]).

Lemma 4.6. *Let $0 < p < q < \infty$, then there is a positive constant $C = C_{p,q}$ such that for any measurable function f there has*

$$|Q|^{-1/p} \|f\|_{L^p(Q)} \leq C|Q|^{-1/q} \|f\|_{L^{q,\infty}(Q)}.$$

5. Proof of Theorem 1.1

Proof. Set $B = \|T\|_{L^{q_1} \times \cdots \times L^{q_m} \to L^{q,\infty}}$. Fix $\lambda > 0$ and consider functions $f_j \in L^1(\mathbb{R}^n), j = 1, \ldots, m$. Without loss of generality, we may assume that $\|f_j\|_{L^1(\mathbb{R}^n)} = 1$ for $1 \leq j \leq m$. We need to show that there is a constant $C = C_{m,n,|\omega|_{\mathrm{Dini}(1)}} > 0$ such that

$$|\{x \in \mathbb{R}^n : |T(f_1, \ldots, f_m)(x)| > \lambda\}| \leq C(A + B)^{1/m} \lambda^{-1/m}. \tag{5.1}$$

Let γ be a positive number to be determined later. Applying the Calderón–Zygmund decomposition to each function f_j at height $(\gamma\lambda)^{1/m}$ to obtain a sequence of pairwise disjoint cubes $\{Q_{j,k_j}\}_{k_j=1}^{\infty}$ and a decomposition

$$f_j = g_j + b_j = g_j + \sum_{k_j} b_{j,k_j}$$

such that for all $j = 1, \dots, m$,

(P1) $\mathrm{supp}(b_{j,k_j}) \subset Q_{j,k_j}$,

(P2) $\int_{\mathbb{R}^n} b_{j,k_j}(x)dx = 0$,

(P3) $\int_{\mathbb{R}^n} |b_{j,k_j}(x)|dx \le C(\gamma\lambda)^{1/m}|Q_{j,k_j}|$,

(P4) $\left|\bigcup_{k_j} Q_{j,k_j}\right| = \sum_{k_j} |Q_{j,k_j}| \le C(\gamma\lambda)^{-1/m}$,

(P5) $\|b_j\|_{L^1(\mathbb{R}^n)} \le C$,

(P6) $\|g_j\|_{L^s(\mathbb{R}^n)} \le C(\gamma\lambda)^{1/(ms')}$ for $1 \le s \le \infty$.

Let c_{j,k_j} be the center of cube Q_{j,k_j} and $\ell(Q_{j,k_j})$ be its side length. Set $Q^*_{j,k_j} = 8\sqrt{n}Q_{j,k_j}$ and $\Omega^*_j = \bigcup_{k_j} Q^*_{j,k_j}$ for $j = 1, \dots, m$, and $\Omega^* = \bigcup_{j=1}^m \Omega^*_j$. And let

$$\begin{aligned}
&E_1 = \{x \in \mathbb{R}^n : |T(g_1, g_2, \dots, g_m)(x)| > \lambda/2^m\}\\
&E_2 = \{x \in \mathbb{R}^n \setminus \Omega^* : |T(b_1, g_2, \dots, g_m)(x)| > \lambda/2^m\}\\
&E_3 = \{x \in \mathbb{R}^n \setminus \Omega^* : |T(g_1, b_2, \dots, g_m)(x)| > \lambda/2^m\}\\
&\dots\\
&E_{2^m} = \{x \in \mathbb{R}^n \setminus \Omega^* : |T(b_1, b_2, \dots, b_m)(x)| > \lambda/2^m\}.
\end{aligned}$$

It follows from property (P4) that

$$|\Omega^*| \le \sum_{j=1}^m |\Omega^*_j| \le C\sum_{j=1}^m \sum_{k_j} |Q_{j,k_j}| \le C(\gamma\lambda)^{-1/m}.$$

By the $L^{q_1} \times \cdots \times L^{q_m} \to L^{q,\infty}$ boundedness of T and property (P6), we have

$$\begin{aligned}
|E_1| &\le (2^m B)^q \lambda^{-q} \|g_1\|^q_{L^{q_1}(\mathbb{R}^n)} \cdots \|g_m\|^q_{L^{q_m}(\mathbb{R}^n)}\\
&\le CB^q \gamma^{q-1/m} \lambda^{-1/m}.
\end{aligned}$$

Thus,

$$\begin{aligned}
|\{x \in \mathbb{R}^n : |T(\vec{f})(x)| > \lambda\}| &\le \sum_{s=1}^{2^m} |E_s| + C|\Omega^*|\\
&\le \sum_{s=2}^{2^m} |E_s| + CB^q\gamma^{q-1/m}\lambda^{-1/m} + C(\gamma\lambda)^{-1/m}.
\end{aligned} \tag{5.2}$$

So, to complete the proof of Theorem 1.1, we need to give the appropriate estimates for each $|E_s|$, $2 \le s \le 2^m$, to guarantee the validity of (5.1).

For the sake of clarity and showing the difference of the proof from the ones in the literature mentioned above, we split the proof into two cases.

Case 1: *the case* $m = 2$. In this case, $\Omega^*_1 = \bigcup_{k_1} Q^*_{1,k_1}$, $\Omega^*_2 = \bigcup_{k_2} Q^*_{2,k_2}$ and $\Omega^* = \Omega^*_1 \bigcup \Omega^*_2$, and $d\vec{y} = dy_1dy_2$. There leaves only the following three terms to be considered

$$\begin{aligned}
&|E_2| = |\{x \in \mathbb{R}^n \setminus \Omega^* : |T(b_1, g_2)(x)| > \lambda/4\}|,\\
&|E_3| = |\{x \in \mathbb{R}^n \setminus \Omega^* : |T(g_1, b_2)(x)| > \lambda/4\}|,\\
&|E_4| = |\{x \in \mathbb{R}^n \setminus \Omega^* : |T(b_1, b_2)(x)| > \lambda/4\}|.
\end{aligned}$$

We will show that

$$|E_s| \le CA\gamma^{1/2}\lambda^{-1/2} \quad \text{for } s = 2, 3, 4. \tag{5.3}$$

For the term $|E_2|$, by Chebychev's inequality and property (P2), we have

$$\begin{aligned}
|E_2| &\le \frac{4}{\lambda}\sum_{k_1}\int_{\mathbb{R}^n\setminus\Omega^*}|T(b_{1,k_1},g_2)(x)|dx\\
&\le \frac{4}{\lambda}\sum_{k_1}\int_{\mathbb{R}^n\setminus\Omega^*}\left|\int_{(\mathbb{R}^n)^2}\big[K(x,y_1,y_2)-K(x,c_{1,k_1},y_2)\big]b_{1,k_1}(y_1)g_2(y_2)d\vec{y}\right|dx\\
&\le \frac{4}{\lambda}\|g_2\|_{L^\infty}\sum_{k_1}\int_{Q_{1,k_1}}|b_{1,k_1}(y_1)|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n\setminus\Omega^*}\big|K(x,y_1,y_2)-K(x,c_{1,k_1},y_2)\big|dxd\vec{y}. \qquad (5.4)
\end{aligned}$$

For fixed k_1, denote by $\mathscr{Q}^i_{1,k_1}=(2^{i+2}\sqrt{n}Q_{1,k_1})\setminus(2^{i+1}\sqrt{n}Q_{1,k_1}),\ i=1,2,\dots,$ to shorten the notations. Clearly we have $\mathbb{R}^n\setminus\Omega^*\subset\mathbb{R}^n\setminus Q^*_{1,k_1}\subset\bigcup_{i=1}^\infty\mathscr{Q}^i_{1,k_1}$. For any $y_1\in Q_{1,k_1}$ and $y_2\in\mathbb{R}^n$, since ω is nondecreasing then it follows from (1.3) that

$$\begin{aligned}
\int_{\mathbb{R}^n\setminus\Omega^*}\big|K(x,y_1,y_2)-K(x,c_{1,k_1},y_2)\big|dx &\le A\int_{\mathbb{R}^n\setminus\Omega^*}\frac{1}{(|x-y_1|+|x-y_2|)^{2n}}\omega\left(\frac{|y_1-c_{1,k_1}|}{|x-y_1|+|x-y_2|}\right)dx\\
&\le A\sum_{i=1}^\infty\int_{\mathscr{Q}^i_{1,k_1}}\frac{1}{(|x-y_1|+|x-y_2|)^{2n}}\omega\left(\frac{|y_1-c_{1,k_1}|}{|x-y_1|}\right)dx\\
&\le A\sum_{i=1}^\infty\omega(2^{-i})\int_{\mathscr{Q}^i_{1,k_1}}\frac{1}{(|x-y_1|+|x-y_2|)^{2n}}dx, \qquad (5.5)
\end{aligned}$$

where in the last step we use the facts that, for $x\in\mathscr{Q}^i_{1,k_1}$ and $y_1\in Q_{1,k_1}$,

$$|y_1-c_{1,k_1}|\le\frac{1}{2}\sqrt{n}\ell(Q_{1,k_1})\quad\text{and}\quad|x-y_1|\ge 2^{i-1}\sqrt{n}\ell(Q_{1,k_1}).$$

Putting (5.5) into (5.4), and applying properties (P6), (P3) and (P4), we have

$$\begin{aligned}
|E_2| &\le \frac{CA\gamma^{1/2}}{\lambda^{1/2}}\sum_{k_1}\int_{Q_{1,k_1}}|b_{1,k_1}(y_1)|\int_{\mathbb{R}^n}\sum_{i=1}^\infty\omega(2^{-i})\int_{\mathscr{Q}^i_{1,k_1}}\frac{dx}{(|x-y_1|+|x-y_2|)^{2n}}d\vec{y}\\
&\le \frac{CA\gamma^{1/2}}{\lambda^{1/2}}\sum_{k_1}\sum_{i=1}^\infty\omega(2^{-i})\int_{Q_{1,k_1}}|b_{1,k_1}(y_1)|\int_{\mathscr{Q}^i_{1,k_1}}\int_{\mathbb{R}^n}\frac{dy_2}{(|x-y_1|+|x-y_2|)^{2n}}dxdy_1\\
&\le \frac{CA\gamma^{1/2}}{\lambda^{1/2}}\sum_{k_1}\sum_{i=1}^\infty\omega(2^{-i})\int_{Q_{1,k_1}}|b_{1,k_1}(y_1)|\int_{\mathscr{Q}^i_{1,k_1}}\frac{1}{|x-y_1|^n}dxdy_1\\
&\le \frac{CA\gamma^{1/2}}{\lambda^{1/2}}\sum_{k_1}\sum_{i=1}^\infty\omega(2^{-i})\int_{Q_{1,k_1}}|b_{1,k_1}(y_1)|\int_{2^{i+2}\sqrt{n}Q_{1,k_1}}\frac{1}{|2^{i-1}\sqrt{n}Q_{1,k_1}|}dxdy_1\\
&\le \frac{CA\gamma^{1/2}}{\lambda^{1/2}}\sum_{k_1}\sum_{i=1}^\infty\omega(2^{-i})\int_{Q_{1,k_1}}|b_{1,k_1}(y_1)|dy_1\\
&\le CA\gamma^{1/2}\lambda^{-1/2}.
\end{aligned}$$

Similarly, we can obtain that $|E_3|\le CA\gamma^{1/2}\lambda^{-1/2}$.

Now, let us consider $|E_4|$. For this term, our approach is much more different from the ones used by Grafakos and Torres [5] and Maldonado and Naibo [13] (pages 241–242).

By Chebychev's inequality and properties (P1) and (P2), we have

$$\begin{aligned}
|E_4| &\le \frac{4}{\lambda}\int_{\mathbb{R}^n\setminus\Omega^*}\big|T(b_1,b_2)(x)\big|dx\\
&\le \frac{4}{\lambda}\sum_{k_1,k_2}\int_{\mathbb{R}^n\setminus\Omega^*}\int_{Q_{2,k_2}}\int_{Q_{1,k_1}}\big|K(x,y_1,y_2)-K(x,c_{1,k_1},y_2)\big||b_{1,k_1}(y_1)|\,|b_{2,k_2}(y_2)|d\vec{y}dx\\
&\le \frac{4}{\lambda}\sum_{k_1,k_2}\int_{Q_{2,k_2}}\int_{Q_{1,k_1}}\left(\int_{\mathbb{R}^n\setminus\Omega^*}\big|K(x,y_1,y_2)-K(x,c_{1,k_1},y_2)\big|dx\right)|b_{1,k_1}(y_1)|\,|b_{2,k_2}(y_2)|d\vec{y}.
\end{aligned}$$

Let $\mathcal{Q}^i_{1,k_1}$ be as above. For any fixed k_2 denote by $\mathcal{Q}^h_{2,k_2} = (2^{h+2}\sqrt{n}Q_{2,k_2}) \setminus (2^{h+1}\sqrt{n}Q_{2,k_2}),\ h = 1, 2, \ldots$. Then

$$\mathbb{R}^n \setminus \Omega^* \subset \mathbb{R}^n \setminus \big(Q^*_{1,k_1} \bigcup Q^*_{2,k_2}\big) \subset \bigcup_{h=1}^{\infty}\bigcup_{i=1}^{\infty}\Big(\mathcal{Q}^i_{1,k_1} \bigcap \mathcal{Q}^h_{2,k_2}\Big).$$

For any $(y_1, y_2) \in Q_{1,k_1} \times Q_{2,k_2}$, similar to (5.5), we have

$$\begin{aligned}\int_{\mathbb{R}^n\setminus\Omega^*} \big|K(x, y_1, y_2) - K(x, c_{1,k_1}, y_2)\big|dx &\le A \int_{\mathbb{R}^n\setminus\Omega^*} \frac{1}{(|x-y_1|+|x-y_2|)^{2n}}\,\omega\left(\frac{|y_1-c_{1,k_1}|}{|x-y_1|+|x-y_2|}\right) dx \\ &\le A \sum_{h=1}^{\infty}\sum_{i=1}^{\infty} \omega(2^{-i}) \int_{\mathcal{Q}^i_{1,k_1}\bigcap \mathcal{Q}^h_{2,k_2}} \frac{1}{(|x-y_1|+|x-y_2|)^{2n}} dx. \qquad (5.6)\end{aligned}$$

Note that, for any $x \in \mathcal{Q}^i_{1,k_1} \bigcap \mathcal{Q}^h_{2,k_2}$ and $(y_1, y_2) \in Q_{1,k_1} \times Q_{2,k_2}$, there has

$$|x - y_1| \approx 2^{i+1}\sqrt{n}\ell(Q_{1,k_1}) \quad \text{and} \quad |x - y_2| \approx 2^{h+1}\sqrt{n}\ell(Q_{2,k_2}),$$

then, for any $(y_1, y_2) \in Q_{1,k_1} \times Q_{2,k_2}$, the following holds

$$\begin{aligned}\int_{\mathcal{Q}^i_{1,k_1}\bigcap \mathcal{Q}^h_{2,k_2}} \frac{1}{(|x-y_1|+|x-y_2|)^{2n}} dx &\approx \frac{|\mathcal{Q}^i_{1,k_1} \bigcap \mathcal{Q}^h_{2,k_2}|}{\big(2^{i+1}\sqrt{n}\ell(Q_{1,k_1}) + 2^{h+1}\sqrt{n}\ell(Q_{2,k_2})\big)^{2n}} \\ &:= \mathcal{H}(i, k_1; h, k_2). \qquad (5.7)\end{aligned}$$

For any $(y_1, y_2) \in Q_{1,k_1} \times Q_{2,k_2}$, it follows from (5.6) and (5.7) that

$$\int_{\mathbb{R}^n\setminus\Omega^*} \big|K(x, y_1, y_2) - K(x, c_{1,k_1}, y_2)\big|dx \le CA \sum_{h,i=1}^{\infty} \omega(2^{-i})\mathcal{H}(i, k_1; h, k_2). \qquad (5.8)$$

Then, by (5.8) and property (P3) one has

$$\begin{aligned}|E_4| &\le \frac{CA}{\lambda} \sum_{k_1,k_2} \int_{Q_{2,k_2}}\int_{Q_{1,k_1}} \left(\sum_{h,i=1}^{\infty} \omega(2^{-i})\mathcal{H}(i, k_1; h, k_2)\right)|b_{1,k_1}(y_1)|\,|b_{2,k_2}(y_2)|d\vec{y} \\ &\le CA\gamma \sum_{i=1}^{\infty}\omega(2^{-i}) \sum_{k_1,k_2} |Q_{1,k_1}|\,|Q_{2,k_2}| \left(\sum_{h=1}^{\infty} \mathcal{H}(i, k_1; h, k_2)\right) \\ &= CA\gamma \sum_{i=1}^{\infty}\omega(2^{-i}) \sum_{k_1,k_2} \int_{Q_{1,k_1}}\int_{Q_{2,k_2}} \left(\sum_{h=1}^{\infty} \mathcal{H}(i, k_1; h, k_2)\right)d\vec{y}.\end{aligned}$$

Applying (5.7) again and noting that for any fixed k_2 the sequence $\{\mathcal{Q}^h_{2,k_2}\}_{h=1}^{\infty}$ is pairwise disjoint, it follows from property (P4) that

$$\begin{aligned}|E_4| &\le CA\gamma \sum_{i=1}^{\infty}\omega(2^{-i}) \sum_{k_1,k_2} \int_{Q_{1,k_1}}\int_{Q_{2,k_2}} \left(\sum_{h=1}^{\infty}\int_{\mathcal{Q}^i_{1,k_1}\bigcap \mathcal{Q}^h_{2,k_2}} \frac{1}{(|x-y_1|+|x-y_2|)^{2n}} dx\right)d\vec{y} \\ &\le CA\gamma \sum_{i=1}^{\infty}\omega(2^{-i}) \sum_{k_1,k_2} \int_{Q_{1,k_1}}\int_{Q_{2,k_2}} \left(\int_{\mathcal{Q}^i_{1,k_1}} \frac{1}{(|x-y_1|+|x-y_2|)^{2n}} dx\right)d\vec{y} \\ &\le CA\gamma \sum_{i=1}^{\infty}\omega(2^{-i}) \sum_{k_1} \int_{Q_{1,k_1}}\int_{\mathcal{Q}^i_{1,k_1}} \left(\sum_{k_2}\int_{Q_{2,k_2}} \frac{1}{(|x-y_1|+|x-y_2|)^{2n}} dy_2\right)dxdy_1 \\ &\le CA\gamma \sum_{i=1}^{\infty}\omega(2^{-i}) \sum_{k_1} \int_{Q_{1,k_1}}\int_{\mathcal{Q}^i_{1,k_1}} \left(\int_{\mathbb{R}^n} \frac{1}{(|x-y_1|+|x-y_2|)^{2n}} dy_2\right)dxdy_1 \\ &\le CA\gamma \sum_{i=1}^{\infty}\omega(2^{-i}) \sum_{k_1} \int_{Q_{1,k_1}}\int_{\mathcal{Q}^i_{1,k_1}} \frac{1}{|x-y_1|^n} dxdy_1 \\ &\le CA\gamma \sum_{i=1}^{\infty}\omega(2^{-i}) \sum_{k_1} \int_{Q_{1,k_1}}\int_{2^{i+2}\sqrt{n}Q_{1,k_1}} \frac{1}{|2^i\sqrt{n}Q_{1,k_1}|} dxdy_1\end{aligned}$$

$$\le CA\gamma\sum_{i=1}^{\infty}\omega(2^{-i})\sum_{k_1}|Q_{1,k_1}|$$
$$\le CA\gamma^{1/2}\lambda^{-1/2}.$$

It is easy to see that the constants C's involved depend only on m, n and $|\omega|_{\mathrm{Dini}(1)}$. So, (5.3) is proven. Set $\gamma=(A+B)^{-1}$, it follows from (5.2) and (5.3) that

$$\begin{aligned}|\{x\in\mathbb{R}^n:|T(f_1,f_2)(x)|>\lambda\}| &\le \sum_{s=2}^{4}|E_s|+CB^q\gamma^{q-1/2}\lambda^{-1/2}+C(\gamma\lambda)^{-1/2}\\ &\le C(A+B)^{1/2}\lambda^{-1/2},\end{aligned}$$

which is the desired result. The proof of the case $m=2$ is completed.

Case 2: *the case* $m\ge 3$. We need to estimate $|E_s|$ for $2\le s\le 2^m$. Suppose that for some $1\le l\le m$ we have l bad functions and $m-l$ good functions appearing in $T(h_1,\dots,h_m)$, where $h_j\in\{g_j,b_j\}$. For matters of simplicity, we assume that the bad functions appear at the entries $1,\dots,l$, and denote the corresponding term by $|E_s^{(l)}|$ to distinguish it from the other terms. That is, we will consider

$$|E_s^{(l)}|=\left|\left\{x\in\mathbb{R}^n\setminus\Omega^*:|T(b_1,\dots,b_l,g_{l+1},\dots,g_m)(x)|>\lambda/2^m\right\}\right|,$$

and the other terms can be estimated similarly. We will show

$$|E_s^{(l)}|\le CA\gamma(\gamma\lambda)^{-1/m}. \tag{5.9}$$

Recall that $\operatorname{supp}(b_{1,k_1})\subset Q_{1,k_1}$ and c_{1,k_1} is the center of Q_{1,k_1}. Denote by $\prod_{r=1}^{l}Q_{r,k_r}=Q_{1,k_1}\times\cdots\times Q_{l,k_l}$ and $\vec{y}_*=(c_{1,k_1},y_2,\dots,y_m)$ for simplicity. Then it follows from properties (P2) and (P6) that, for any $x\in\mathbb{R}^n\setminus\Omega^*$,

$$\begin{aligned}|T(b_1,\dots,b_l,g_{l+1},\dots,g_m)(x)| &\le \sum_{k_1,\dots,k_l}\left|\int_{(\mathbb{R}^n)^m}K(x,\vec{y})\prod_{r=1}^{l}b_{r,k_r}(y_r)\prod_{r=l+1}^{m}g_r(y_r)d\vec{y}\right|\\ &\le \sum_{k_1,\dots,k_l}\int_{(\mathbb{R}^n)^m}\left|K(x,\vec{y})-K(x,\vec{y}_*)\right|\prod_{r=1}^{l}|b_{r,k_r}(y_r)|\prod_{r=l+1}^{m}|g_r(y_r)|d\vec{y}\\ &\le C(\gamma\lambda)^{\frac{m-l}{m}}\sum_{k_1,\dots,k_l}\int_{(\mathbb{R}^n)^m}\left|K(x,\vec{y})-K(x,\vec{y}_*)\right|\prod_{r=1}^{l}|b_{r,k_r}(y_r)|d\vec{y}.\end{aligned}$$

This together with Chebychev's inequality gives

$$\begin{aligned}|E_s^{(l)}| &\le \frac{2^m}{\lambda}\int_{\mathbb{R}^n\setminus\Omega^*}|T(b_1,\dots,b_l,g_{l+1},\dots,g_m)(x)|dx\\ &\le \frac{C}{\lambda}(\gamma\lambda)^{\frac{m-l}{m}}\int_{\mathbb{R}^n\setminus\Omega^*}\left(\sum_{k_1,\dots,k_l}\int_{(\mathbb{R}^n)^m}\left|K(x,\vec{y})-K(x,\vec{y}_*)\right|\prod_{r=1}^{l}|b_{r,k_r}(y_r)|d\vec{y}\right)dx\\ &\le \frac{C}{\lambda}(\gamma\lambda)^{\frac{m-l}{m}}\sum_{k_1,\dots,k_l}\int_{(\mathbb{R}^n)^m}\left(\int_{\mathbb{R}^n\setminus\Omega^*}\left|K(x,\vec{y})-K(x,\vec{y}_*)\right|dx\right)\prod_{r=1}^{l}|b_{r,k_r}(y_r)|d\vec{y}\\ &\le \frac{C}{\lambda}(\gamma\lambda)^{\frac{m-l}{m}}\sum_{k_1,\dots,k_l}\int_{(\mathbb{R}^n)^{m-l}}\int_{\prod_{r=1}^{l}Q_{r,k_r}}\prod_{r=1}^{l}|b_{r,k_r}(y_r)|\left(\int_{\mathbb{R}^n\setminus\Omega^*}\left|K(x,\vec{y})-K(x,\vec{y}_*)\right|dx\right)d\vec{y}.\end{aligned}$$

Let $\mathscr{Q}_{r,k_r}^{i_r}=(2^{i_r+2}\sqrt{n}Q_{r,k_r})\setminus(2^{i_r+1}\sqrt{n}Q_{r,k_r})$ for $r=1,\dots,l$ and $i_r=1,2,\dots$. Then

$$\mathbb{R}^n\setminus\Omega^*\subset\bigcup_{i_1=1}^{\infty}\cdots\bigcup_{i_l=1}^{\infty}\left(\mathscr{Q}_{1,k_1}^{i_1}\bigcap\cdots\bigcap\mathscr{Q}_{l,k_l}^{i_l}\right)=\bigcup_{i_1,\dots,i_l=1}^{\infty}\left(\bigcap_{r=1}^{l}\mathscr{Q}_{r,k_r}^{i_r}\right).$$

For any $(y_1,\dots,y_l)\in\prod_{r=1}^{l}Q_{r,k_r}$ and any $(y_{l+1},\dots,y_m)\in(\mathbb{R}^n)^{m-l}$, applying (1.3) and the fact that ω is nondecreasing, similar to (5.5) and (5.6), we have

$$\int_{\mathbb{R}^n\setminus\Omega^*}\left|K(x,\vec{y})-K(x,\vec{y}_*)\right|dx\le A\int_{\mathbb{R}^n\setminus\Omega^*}\frac{1}{\left(\sum_{j=1}^{m}|x-y_j|\right)^{mn}}\omega\left(\frac{|y_1-c_{1,k_1}|}{\sum_{j=1}^{m}|x-y_j|}\right)dx$$

$$\leq A \sum_{i_1,\ldots,i_l=1}^{\infty} \int_{\bigcap_{r=1}^{l} \mathcal{Q}_{r,k_r}^{i_r}} \frac{1}{\left(\sum_{j=1}^{m} |x-y_j|\right)^{mn}} \omega\left(\frac{|y_1 - c_{1,k_1}|}{|x-y_1|}\right) dx$$

$$\leq A \sum_{i_1,\ldots,i_l=1}^{\infty} \omega(2^{-i_1}) \int_{\bigcap_{r=1}^{l} \mathcal{Q}_{r,k_r}^{i_r}} \frac{1}{\left(\sum_{j=1}^{m} |x-y_j|\right)^{mn}} dx.$$

Then,

$$|E_s^{(l)}| \leq \frac{CA}{\lambda}(\gamma\lambda)^{\frac{m-l}{m}} \sum_{k_1,\ldots,k_l} \sum_{i_1,\ldots,i_l=1}^{\infty} \omega(2^{-i_1}) \int_{(\mathbb{R}^n)^{m-l}} \int_{\prod_{r=1}^{l} Q_{r,k_r}} \prod_{r=1}^{l} |b_{r,k_r}(y_r)| \left(\int_{\bigcap_{r=1}^{l} \mathcal{Q}_{r,k_r}^{i_r}} \frac{1}{\left(\sum_{j=1}^{m} |x-y_j|\right)^{mn}} dx \right) d\vec{y}$$

$$\leq \frac{CA}{\lambda}(\gamma\lambda)^{\frac{m-l}{m}} \sum_{k_1,\ldots,k_l} \sum_{i_1,\ldots,i_l=1}^{\infty} \omega(2^{-i_1}) \int_{\prod_{r=1}^{l} Q_{r,k_r}} \prod_{r=1}^{l} |b_{r,k_r}(y_r)|$$

$$\times \left\{ \int_{\bigcap_{r=1}^{l} \mathcal{Q}_{r,k_r}^{i_r}} \left(\int_{(\mathbb{R}^n)^{m-l}} \frac{1}{\left(\sum_{j=1}^{m} |x-y_j|\right)^{mn}} dy_{l+1} \cdots dy_m \right) dx \right\} dy_1 \cdots dy_l$$

$$\leq \frac{CA}{\lambda}(\gamma\lambda)^{\frac{m-l}{m}} \sum_{k_1,\ldots,k_l} \sum_{i_1,\ldots,i_l=1}^{\infty} \omega(2^{-i_1}) \int_{\prod_{r=1}^{l} Q_{r,k_r}} \prod_{r=1}^{l} |b_{r,k_r}(y_r)| \left\{ \int_{\bigcap_{r=1}^{l} \mathcal{Q}_{r,k_r}^{i_r}} \frac{1}{\left(\sum_{j=1}^{l} |x-y_j|\right)^{nl}} dx \right\} dy_1 \cdots dy_l.$$

On the other hand, similar to (5.7), for any $(y_1, \ldots, y_l) \in \prod_{r=1}^{l} Q_{r,k_r}$, there has

$$\int_{\bigcap_{r=1}^{l} \mathcal{Q}_{r,k_r}^{i_r}} \frac{1}{\left(\sum_{r=1}^{l} |x-y_r|\right)^{nl}} dx \approx \frac{\left|\bigcap_{r=1}^{l} \mathcal{Q}_{r,k_r}^{i_r}\right|}{\left(\sum_{r=1}^{l} 2^{i_r+1}\sqrt{n}\ell(Q_{r,k_r})\right)^{nl}}. \tag{5.10}$$

Then by (5.10) and the property (P3), we have

$$|E_s^{(l)}| \leq \frac{CA}{\lambda}(\gamma\lambda)^{\frac{m-l}{m}} \sum_{k_1,\ldots,k_l} \sum_{i_1,\ldots,i_l=1}^{\infty} \omega(2^{-i_1}) \int_{\prod_{r=1}^{l} Q_{r,k_r}} \prod_{r=1}^{l} |b_{r,k_r}(y_r)| \left\{ \frac{\left|\bigcap_{r=1}^{l} \mathcal{Q}_{r,k_r}^{i_r}\right|}{\left(\sum_{r=1}^{l} 2^{i_r+1}\sqrt{n}\ell(Q_{r,k_r})\right)^{nl}} \right\} dy_1 \cdots dy_l$$

$$\leq CA\gamma \sum_{k_1,\ldots,k_l} \sum_{i_1,\ldots,i_l=1}^{\infty} \omega(2^{-i_1}) \prod_{r=1}^{l} |Q_{r,k_r}| \left\{ \frac{\left|\bigcap_{r=1}^{l} \mathcal{Q}_{r,k_r}^{i_r}\right|}{\left(\sum_{r=1}^{l} 2^{i_r+1}\sqrt{n}\ell(Q_{r,k_r})\right)^{nl}} \right\}$$

$$= CA\gamma \sum_{k_1,\ldots,k_l} \sum_{i_1,\ldots,i_l=1}^{\infty} \omega(2^{-i_1}) \int_{\prod_{r=1}^{l} Q_{r,k_r}} \left\{ \frac{\left|\bigcap_{r=1}^{l} \mathcal{Q}_{r,k_r}^{i_r}\right|}{\left(\sum_{r=1}^{l} 2^{i_r+1}\sqrt{n}\ell(Q_{r,k_r})\right)^{nl}} \right\} dy_1 \cdots dy_l.$$

Applying (5.10) again, we can see that $|E_s^{(l)}|$ is dominated by

$$A\gamma \sum_{k_1,\ldots,k_l} \sum_{i_1,\ldots,i_l=1}^{\infty} \omega(2^{-i_1}) \int_{\prod_{r=1}^{l} Q_{r,k_r}} \left\{ \int_{\bigcap_{r=1}^{l} \mathcal{Q}_{r,k_r}^{i_r}} \frac{1}{\left(\sum_{r=1}^{l} |x-y_r|\right)^{nl}} dx \right\} dy_1 \cdots dy_l$$

$$= A\gamma \sum_{i_1,\ldots,i_l=1}^{\infty} \omega(2^{-i_1}) \sum_{k_1} \int_{Q_{1,k_1}} \left\{ \int_{\bigcap_{r=1}^{l} \mathcal{Q}_{r,k_r}^{i_r}} \left(\sum_{k_2,\ldots,k_l} \int_{\prod_{r=2}^{l} Q_{r,k_r}} \frac{dy_2 \cdots dy_l}{\left(\sum_{r=1}^{l} |x-y_r|\right)^{nl}} \right) dx \right\} dy_1.$$

Since for any fixed r, the family $\{Q_{r,k_r}\}_{k_r=1}^{\infty}$ is a sequence of pairwise disjoint cubes, then for any $x \in \bigcap_{r=1}^{l} \mathscr{Q}_{r,k_r}^{i_r}$ and $y_1 \in Q_{1,k_1}$, there has

$$\begin{aligned}
\sum_{k_2,\ldots,k_l} \int_{\prod_{r=2}^{l} Q_{r,k_r}} \frac{1}{\left(\sum_{r=1}^{l} |x-y_r|\right)^{nl}} dy_2 \cdots dy_l &= \sum_{k_2,\ldots,k_{l-1}} \int_{\prod_{r=2}^{l-1} Q_{r,k_r}} \left(\sum_{k_l} \int_{Q_{l,k_l}} \frac{1}{\left(\sum_{r=1}^{l} |x-y_r|\right)^{nl}} dy_l \right) dy_2 \cdots dy_{l-1} \\
&\le \sum_{k_2,\ldots,k_{l-1}} \int_{\prod_{r=2}^{l-1} Q_{r,k_r}} \left(\int_{\mathbb{R}^n} \frac{1}{\left(\sum_{r=1}^{l} |x-y_r|\right)^{nl}} dy_l \right) dy_2 \cdots dy_{l-1} \\
&\le C \sum_{k_2,\ldots,k_{l-1}} \int_{\prod_{r=2}^{l-1} Q_{r,k_r}} \frac{1}{\left(\sum_{r=1}^{l-1} |x-y_r|\right)^{n(l-1)}} dy_2 \cdots dy_{l-1} \\
&\le \cdots \le C \sum_{k_2} \int_{Q_{2,k_2}} \frac{1}{(|x-y_1|+|x-y_2|)^{2n}} dy_2 \\
&\le C \int_{\mathbb{R}^n} \frac{1}{(|x-y_1|+|x-y_2|)^{2n}} dy_2 \\
&\le \frac{C}{|x-y_1|^n}.
\end{aligned}$$

Thus

$$\begin{aligned}
|E_s^{(l)}| &\le CA\gamma \sum_{i_1,\ldots,i_l=1}^{\infty} \omega(2^{-i_1}) \sum_{k_1} \int_{Q_{1,k_1}} \left\{ \int_{\bigcap_{r=1}^{l} \mathscr{Q}_{r,k_r}^{i_r}} \frac{1}{|x-y_1|^n} dx \right\} dy_1 \\
&\le CA\gamma \sum_{i_1}^{\infty} \omega(2^{-i_1}) \sum_{k_1} \int_{Q_{1,k_1}} \left\{ \sum_{i_2,\ldots,i_l=1}^{\infty} \int_{\bigcap_{r=1}^{l} \mathscr{Q}_{r,k_r}^{i_r}} \frac{1}{|x-y_1|^n} dx \right\} dy_1. \qquad (5.11)
\end{aligned}$$

On the other hand, noting that for any fixed r the sequence $\{\mathscr{Q}_{r,k_r}^{i_r}\}_{i_r=1}^{\infty}$ is also pairwise disjoint, then for any $y_1 \in Q_{1,k_1}$, there has

$$\begin{aligned}
\sum_{i_2,\ldots,i_l=1}^{\infty} \int_{\bigcap_{r=1}^{l} \mathscr{Q}_{r,k_r}^{i_r}} \frac{1}{|x-y_1|^n} dx &= \sum_{i_2,\ldots,i_{l-1}=1}^{\infty} \left(\sum_{i_l=1}^{\infty} \int_{\bigcap_{r=1}^{l} \mathscr{Q}_{r,k_r}^{i_r}} \frac{1}{|x-y_1|^n} dx \right) \\
&= \sum_{i_2,\ldots,i_{l-1}=1}^{\infty} \left(\int_{\left(\bigcap_{r=1}^{l-1} \mathscr{Q}_{r,k_r}^{i_r}\right) \bigcap \left(\bigcup_{i_l=1}^{\infty} \mathscr{Q}_{l,k_l}^{i_l}\right)} \frac{1}{|x-y_1|^n} dx \right) \\
&\le \sum_{i_2,\ldots,i_{l-1}=1}^{\infty} \int_{\bigcap_{r=1}^{l-1} \mathscr{Q}_{r,k_r}^{i_r}} \frac{1}{|x-y_1|^n} dx \\
&\le \cdots \le \sum_{i_2=1}^{\infty} \int_{\mathscr{Q}_{1,k_1}^{i_1} \bigcap \mathscr{Q}_{2,k_2}^{i_2}} \frac{1}{|x-y_1|^n} dx \\
&= \int_{\mathscr{Q}_{1,k_1}^{i_1} \bigcap \left(\bigcup_{i_2=1}^{\infty} \mathscr{Q}_{2,k_2}^{i_2}\right)} \frac{1}{|x-y_1|^n} dx \\
&\le \int_{\mathscr{Q}_{1,k_1}^{i_1}} \frac{1}{|x-y_1|^n} dx.
\end{aligned}$$

Putting the above estimate into (5.11) and applying property (P4), we have

$$\begin{aligned}
|E_s^{(l)}| &\le CA\gamma \sum_{i_1=1}^{\infty} \omega(2^{-i_1}) \sum_{k_1} \int_{Q_{1,k_1}} \left(\int_{\mathscr{Q}_{1,k_1}^{i_1}} \frac{1}{|x-y_1|^n} dx \right) dy_1 \\
&\le CA\gamma \sum_{i_1=1}^{\infty} \omega(2^{-i_1}) \sum_{k_1} \int_{Q_{1,k_1}} \left(\int_{2^{i_1+2}\sqrt{n}Q_{1,k_1}} \frac{1}{|2^{i_1}\sqrt{n}Q_{1,k_1}|} dx \right) dy_1
\end{aligned}$$

$$\le CA\gamma \sum_{i_1=1}^{\infty} \omega(2^{-i_1}) \sum_{k_1} |Q_{1,k_1}|$$
$$\le CA\gamma(\gamma\lambda)^{-1/m}.$$

This shows that (5.9) holds.

Now, we have proved that each $|E_s|$ satisfies $|E_s| \le CA\gamma(\gamma\lambda)^{-1/m}$. So, by (5.2) we have

$$|\{x \in \mathbb{R}^n : |T(f_1, \ldots, f_m)(x)| > \lambda\}| \le CA\gamma(\gamma\lambda)^{-1/m} + CB^q\gamma^{q-1/m}\lambda^{-1/m} + C(\gamma\lambda)^{-1/m}.$$

Set $\gamma = (A+B)^{-1}$, then (5.1) follows. The proof of Theorem 1.1 is finished. □

Remark 5.1. We would like to note that although there makes no use of the Marcinkiewicz function in the proof of Theorem 6.1 in [13], the authors introduce a new operator $\mathcal{M}^{\omega}_{j,l}$ instead (see [13] page 240), which is similar to the Marcinkiewicz function and plays a similar role in the proof. In the proof of Theorem 1.1, we do not use the Marcinkiewicz function and $\mathcal{M}^{\omega}_{j,l}$. Our approach is more direct and the estimates are more delicate and complicated.

6. Proof of Theorem 1.2

To prove Theorem 1.2, we first establish the following pointwise estimates on the sharp maximal function $M^{\sharp}_{\delta}$ acting on $T(\vec{f})$ in terms of the multilinear maximal function $\mathcal{M}$.

Theorem 6.1. *Let T be an m-linear ω-CZO with $\omega \in \mathrm{Dini}(1)$ and $0 < \delta < 1/m$. Then for all $\vec{f}$ in any product space $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n)$ with $1 \le p_j < \infty$ for $j = 1, \ldots, m$,*

$$M^{\sharp}_{\delta}\big(T(\vec{f})\big)(x) \le C\mathcal{M}(\vec{f})(x).$$

Proof. For a fixed point x and a cube $Q \ni x$. Due to the fact $\big||a|^r - |b|^r\big| \le |a-b|^r$ for $0 < r < 1$, it suffices to prove that, for $0 < \delta < 1/m$,

$$\left(\frac{1}{|Q|}\int_Q \big|T(\vec{f})(z) - c\big|^{\delta} dz\right)^{1/\delta} \le C\mathcal{M}(\vec{f})(x), \tag{6.1}$$

where the constant c is to be determined later.

For each j, we decompose $f_j = f_j^0 + f_j^{\infty}$, where $f_j^0 = f_j\chi_{Q^*}$ and $Q^* = 8\sqrt{n}Q$. Then

$$\begin{aligned}\prod_{j=1}^m f_j(y_j) &= \prod_{j=1}^m \big(f_j^0(y_j) + f_j^{\infty}(y_j)\big)\\ &= \sum_{\alpha_1,\ldots,\alpha_m \in \{0,\infty\}} f_1^{\alpha_1}(y_1)\cdots f_m^{\alpha_m}(y_m)\\ &= \prod_{j=1}^m f_j^0(y_j) + \sum_{(\alpha_1,\ldots,\alpha_m)\in\mathcal{I}} f_1^{\alpha_1}(y_1)\cdots f_m^{\alpha_m}(y_m),\end{aligned}$$

where $\mathcal{I} = \{(\alpha_1, \ldots, \alpha_m) :$ there is at least one $\alpha_j \neq 0\}$. Then we can write

$$T(\vec{f})(z) = T(f_1^0, \ldots, f_m^0)(z) + \sum_{(\alpha_1,\ldots,\alpha_m)\in\mathcal{I}} T(f_1^{\alpha_1}, \ldots, f_m^{\alpha_m})(z). \tag{6.2}$$

Since T is an m-linear ω-CZO with $\omega \in \mathrm{Dini}(1)$, then it follows from Theorem 1.1 that T maps $L^1(\mathbb{R}^n) \times \cdots \times L^1(\mathbb{R}^n)$ into $L^{1/m,\infty}(\mathbb{R}^n)$. Applying Kolmogorov's inequality (Lemma 4.6) with $p = \delta$ and $q = 1/m$, we have

$$\begin{aligned}\left(\frac{1}{|Q|}\int_Q \big|T(f_1^0, \ldots, f_m^0)(z)\big|^{\delta} dz\right)^{1/\delta} &\le C|Q|^{-1/m}\big\|T(f_1^0, \ldots, f_m^0)\big\|_{L^{1/m,\infty}(\mathbb{R}^n)}\\ &\le C|Q|^{-1/m}\prod_{j=1}^m \|f_j^0\|_{L^1(\mathbb{R}^n)}\\ &\le C\prod_{j=1}^m \frac{1}{|Q^*|}\int_{Q^*} |f_j(z)|dz\\ &\le C\mathcal{M}(\vec{f})(x).\end{aligned} \tag{6.3}$$

To estimate the remaining terms in (6.2), we choose

$$c = \sum_{(\alpha_1,\ldots,\alpha_m)\in\mathcal{I}} T(f_1^{\alpha_1},\ldots,f_m^{\alpha_m})(x),$$

and it suffices to show that, for any $z \in Q$, the following estimates hold

$$\sum_{(\alpha_1,\ldots,\alpha_m)\in\mathcal{I}} \left|T(f_1^{\alpha_1},\ldots,f_m^{\alpha_m})(z) - T(f_1^{\alpha_1},\ldots,f_m^{\alpha_m})(x)\right| \le C\mathcal{M}(\vec{f})(x). \tag{6.4}$$

We consider first the case when $\alpha_1 = \cdots = \alpha_m = \infty$. For any $z \in Q$, there has

$$\begin{aligned}\left|T(f_1^{\infty},\ldots,f_m^{\infty})(z) - T(f_1^{\infty},\ldots,f_m^{\infty})(x)\right| &\le \int_{(\mathbb{R}^n\setminus Q^*)^m} |K(z,\vec{y}) - K(x,\vec{y})| \prod_{j=1}^m |f_j^{\infty}(y_j)|d\vec{y} \\ &\le \sum_{k=1}^{\infty}\int_{(\mathcal{Q}_k)^m} |K(z,\vec{y}) - K(x,\vec{y})| \prod_{j=1}^m |f_j^{\infty}(y_j)|d\vec{y}\end{aligned}$$

where $\mathcal{Q}_k = (2^{k+3}\sqrt{n}Q) \setminus (2^{k+2}\sqrt{n}Q)$ for $k = 1, 2, \ldots$.

Noting that, for $x, z \in Q$ and any $(y_1,\ldots,y_m) \in (\mathcal{Q}_k)^m$, there has

$$|z - y_j| \ge 2^k\sqrt{n}\ell(Q) \quad \text{and} \quad |z - x| \le \sqrt{n}\ell(Q),$$

and recalling that ω is nondecreasing, and applying (1.2), we have

$$|K(z,\vec{y}) - K(x,\vec{y})| \le \frac{A}{\left(\sum_{j=1}^m |z-y_j|\right)^{mn}}\,\omega\left(\frac{|z-x|}{\sum_{j=1}^m |z-y_j|}\right) \le \frac{C\omega(2^{-k})}{|2^k\sqrt{n}Q|^m}. \tag{6.5}$$

Then

$$\begin{aligned}\left|T(f_1^{\infty},\ldots,f_m^{\infty})(z) - T(f_1^{\infty},\ldots,f_m^{\infty})(x)\right| &\le C\sum_{k=1}^{\infty}\omega(2^{-k})\int_{(\mathcal{Q}_k)^m}\frac{1}{|2^k\sqrt{n}Q|^m}\prod_{j=1}^m |f_j^{\infty}(y_j)|d\vec{y} \\ &\le C\sum_{k=1}^{\infty}\omega(2^{-k})\prod_{j=1}^m\frac{1}{|2^{k+3}\sqrt{n}Q|}\int_{2^{k+3}\sqrt{n}Q}|f_j(y_j)|dy_j \\ &\le C|\omega|_{\mathrm{Dini}(1)}\mathcal{M}(\vec{f})(x).\end{aligned}$$

What remains to be considered are the terms in (6.4) such that $\alpha_{j_1} = \cdots = \alpha_{j_l} = 0$ for $1 \le l < m$. Set $\mathcal{J} := \{j_1,\ldots,j_l\}$ then $\alpha_j = \infty$ for $j \notin \mathcal{J}$. Thus

$$\begin{aligned}&\left|T(f_1^{\alpha_1},\ldots,f_m^{\alpha_m})(z) - T(f_1^{\alpha_1},\ldots,f_m^{\alpha_m})(x)\right| \\ &\le \int_{(\mathbb{R}^n)^m}|K(z,\vec{y}) - K(x,\vec{y})|\prod_{j=1}^m |f_j^{\alpha_j}(y_j)|d\vec{y} \\ &\le \int_{(Q^*)^l}\prod_{j\in\mathcal{J}}|f_j^0(y_j)|\int_{(\mathbb{R}^n\setminus Q^*)^{m-l}}|K(z,\vec{y}) - K(x,\vec{y})|\prod_{j\notin\mathcal{J}}|f_j^{\infty}(y_j)|d\vec{y} \\ &\le \int_{(Q^*)^l}\prod_{j\in\mathcal{J}}|f_j^0(y_j)|\sum_{k=1}^{\infty}\int_{(\mathcal{Q}_k)^{m-l}}|K(z,\vec{y}) - K(x,\vec{y})|\prod_{j\notin\mathcal{J}}|f_j^{\infty}(y_j)|d\vec{y}.\end{aligned} \tag{6.6}$$

Since for $x, z \in Q$ and any $y_j \in \mathcal{Q}_k$ with $j \notin \mathcal{J}$, there has $|z - y_j| \ge 2^k\sqrt{n}\ell(Q)$, then, similar to (6.5), we have

$$\begin{aligned}|K(z,\vec{y}) - K(x,\vec{y})| &\le \frac{A}{\left(\sum_{j=1}^m |z-y_j|\right)^{mn}}\,\omega\left(\frac{|z-x|}{\sum_{j=1}^m |z-y_j|}\right) \\ &\le \frac{A}{\left(\sum_{j\notin\mathcal{J}} |z-y_j|\right)^{mn}}\,\omega\left(\frac{|z-x|}{\sum_{j\notin\mathcal{J}} |z-y_j|}\right) \\ &\le \frac{C\omega(2^{-k})}{|2^k\sqrt{n}Q|^m}.\end{aligned} \tag{6.7}$$

This together with (6.6) gives

$$
\begin{aligned}
&\left|T(f_1^{\alpha_1},\ldots,f_m^{\alpha_m})(z)-T(f_1^{\alpha_1},\ldots,f_m^{\alpha_m})(x)\right|\\
&\le C\int_{(Q^*)^l}\prod_{j\in\mathcal{J}}|f_j^0(y_j)|\sum_{k=1}^{\infty}\omega(2^{-k})\int_{(\mathcal{Q}_k)^{m-l}}\frac{1}{|2^k\sqrt{n}Q|^m}\prod_{j\notin\mathcal{J}}|f_j^\infty(y_j)|d\vec{y}\\
&\le C\sum_{k=1}^{\infty}\omega(2^{-k})\frac{1}{|2^k\sqrt{n}Q|^m}\left(\prod_{j\in\mathcal{J}}\int_{Q^*}|f_j^0(y_j)|dy_j\right)\left(\prod_{j\notin\mathcal{J}}\int_{2^{k+3}\sqrt{n}Q}|f_j^\infty(y_j)|dy_j\right)\\
&\le C\sum_{k=1}^{\infty}\omega(2^{-k})\left(\prod_{j=1}^{m}\frac{1}{|2^{k+3}\sqrt{n}Q|}\int_{2^{k+3}\sqrt{n}Q}|f_j(y_j)|dy_j\right)\\
&\le C|\omega|_{\mathrm{Dini}(1)}\mathcal{M}(\vec{f})(x).
\end{aligned}
$$

So, (6.4) is proven and then (6.1) follows from (6.2) to (6.4). This concludes the proof. □

Recently, Grafakos et al. [24] proved the following result in the context of *RD*-spaces, which serves as an analog of the classical Fefferman–Stein inequalities (see Lemma 4.11 in [24]). Here, we rewrite their result as follows.

Lemma 6.1 ([24]). *Let $0 < p_0 < \infty$ and $w \in A_\infty$. Then for any p with $p_0 \le p < \infty$ there exists a constant C (depending on n, p and the A_∞ constant of w) such that for all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ with $Mf \in L^{p_0,\infty}(w)$, we have*

$$\|M(f)\|_{L^p(w)} \le C\|M^\sharp(f)\|_{L^p(w)}, \quad \text{if } p_0 < p$$

and

$$\|M(f)\|_{L^{p,\infty}(w)} \le C\|M^\sharp(f)\|_{L^{p,\infty}(w)}, \quad \text{if } p_0 \le p.$$

Now, by Theorem 6.1, Lemma 6.1 and Theorem 1.1, we can get the following result. Since the argument is almost the same as the proof of Proposition 4.13 in [24], we omit the proof.

Theorem 6.2. *Let T be an m-linear ω-CZO with $\omega \in \mathrm{Dini}(1)$, $p \in [1/m,\infty)$ and $w \in A_\infty$. Then there exists a constant $C > 0$ such that*

$$\left\|T(\vec{f})\right\|_{L^p(w)} \le C\left\|\mathcal{M}(\vec{f})\right\|_{L^p(w)}, \quad \text{if } p > 1/m$$

and

$$\left\|T(\vec{f})\right\|_{L^{p,\infty}(w)} \le C\left\|\mathcal{M}(\vec{f})\right\|_{L^{p,\infty}(w)}, \quad \text{if } p \ge 1/m$$

hold for all bounded functions $\vec{f}$ with compact support.

Proof of Theorem 1.2. For the same reason as in the proof of Corollary 3.9 in [7], it is enough to prove Theorem 1.2 is valid for $f_1,\ldots,f_m$ being bounded functions with compact supports. By Lemma 4.4, for $\vec{w} \in A_{\vec{P}}$ there has $\nu_{\vec{w}} \in A_\infty$. Then Theorem 1.2 follows from Theorem 6.2 and the weighted boundedness of $\mathcal{M}$ with multiple-weights (Lemma 4.3). □

7. Proofs of Theorems 1.3 and 1.4

To prove Theorems 1.3 and 1.4, we first establish the pointwise estimates on sharp maximal function acting on the multilinear commutator $T_{\vec{b}}$.

Theorem 7.1. *Let T be an m-linear ω-CZO with ω satisfying (1.5) and $T_{\vec{b}}$ be the m-linear commutator of T with $\vec{b} \in BMO^m$. Assume that $0 < \delta < \varepsilon$ and $0 < \delta < 1/m$. Then, there exists a constant $C > 0$, depending on δ and ε, such that*

$$M_\delta^\sharp\big(T_{\vec{b}}(\vec{f})\big)(x) \le C\|\vec{b}\|_{BMO^m}\left(M_\varepsilon\big(T(\vec{f})\big)(x) + \mathcal{M}_{L(\log L)}(\vec{f})(x)\right)$$

for all m-tuples $\vec{f} = (f_1,\ldots,f_m)$ of bounded measurable functions with compact supports.

Proof. By linearity it is sufficient to consider the commutator with only one symbol, that is, for $\vec{b} = b \in BMO(\mathbb{R}^n)$, we will consider the operator

$$T_b(f_1,\ldots,f_m)(x) = b(x)T(f_1,\ldots,f_m)(x) - T(bf_1,\ldots,f_m)(x).$$

Fix $x \in \mathbb{R}^n$, for any cube Q centered at x, set $Q^* = 8\sqrt{n}Q$. Then for any $z \in Q$

$$T_b(\vec{f})(z) = \big(b(z)-b_{Q^*}\big)T(\vec{f})(z) - T\big((b-b_{Q^*})f_1,\ldots,f_m\big)(z).$$

Since $0 < \delta < 1$, then for any number c, there has

$$\begin{aligned}\left(\frac{1}{|Q|}\int_Q \big|\,|T_b(\vec{f})(z)| - |c|\big|^{\delta} dz\right)^{1/\delta} &\le \left(\frac{C}{|Q|}\int_Q \big|(b(z)-b_{Q^*})T(\vec{f})(z)\big|^{\delta} dz\right)^{1/\delta}\\ &\quad + \left(\frac{C}{|Q|}\int_Q \big|T\big((b-b_{Q^*})f_1,\ldots,f_m\big)(z) - c\big|^{\delta} dz\right)^{1/\delta}\\ &:= \mathrm{I} + \mathrm{II}.\end{aligned}$$

For any $1 < q < \varepsilon/\delta$, by Hölder's and John–Nirenberg's inequalities, we obtain

$$\begin{aligned}\mathrm{I} &\le C\left(\frac{1}{|Q|}\int_Q |b(z)-b_{Q^*}|^{\delta q'} dz\right)^{1/(\delta q')}\left(\frac{1}{|Q|}\int_Q |T(\vec{f})(z)|^{\delta q} dz\right)^{1/(\delta q)}\\ &\le C\|b\|_{BMO} M_{\delta q}\big(T(\vec{f})\big)(x)\\ &\le C\|b\|_{BMO} M_{\varepsilon}\big(T(\vec{f})\big)(x).\end{aligned}$$

To estimate II, we use the similar decomposition to the ones in the proof of Theorem 6.1. For each j, we decompose f_j as $f_j = f_j^0 + f_j^{\infty}$ with $f_j^0 = f_j\chi_{Q^*}$, $j = 1, \ldots, m$. As in the proof of Theorem 6.1, we write

$$\prod_{j=1}^m f_j(y_j) = \prod_{j=1}^m f_j^0(y_j) + \sum_{(\alpha_1,\ldots,\alpha_m)\in\mathcal{I}} f_1^{\alpha_1}(y_1)\cdots f_m^{\alpha_m}(y_m),$$

where $\mathcal{I} = \{(\alpha_1, \ldots, \alpha_m) :$ there is at least one $\alpha_j \neq 0\}$.

Set

$$c = \sum_{(\alpha_1,\ldots,\alpha_m)\in\mathcal{I}} T\big((b-b_{Q^*})f_1^{\alpha_1},\ldots,f_m^{\alpha_m}\big)(x).$$

Then we have

$$\begin{aligned}\mathrm{II} &\le C\left(\frac{1}{|Q|}\int_Q \big|T\big((b-b_{Q^*})f_1^0,\ldots,f_m^0\big)(z)\big|^{\delta} dz\right)^{1/\delta}\\ &\quad + C\sum_{(\alpha_1,\ldots,\alpha_m)\in\mathcal{I}}\left(\frac{1}{|Q|}\int_Q \Big|T\big((b-b_{Q^*})f_1^{\alpha_1},\ldots,f_m^{\alpha_m}\big)(z) - T\big((b-b_{Q^*})f_1^{\alpha_1},\ldots,f_m^{\alpha_m}\big)(x)\Big|^{\delta} dz\right)^{1/\delta}\\ &:= \mathrm{II}_0 + \sum_{(\alpha_1,\ldots,\alpha_m)\in\mathcal{I}} \mathrm{II}_{\alpha_1,\ldots,\alpha_m}.\end{aligned}$$

Noting that $0 < \delta < 1/m$, by Kolmogorov's inequality (Lemma 4.6), and applying the $L^1 \times \cdots \times L^1$ to $L^{1/m,\infty}$ boundedness of T, (4.2) and (4.3), we get

$$\begin{aligned}\mathrm{II}_0 &\le \frac{C}{|Q|^m}\big\|T\big((b-b_{Q^*})f_1^0,\ldots,f_m^0\big)\big\|_{L^{1/m,\infty}(Q)}\\ &\le \frac{C}{|Q|^m}\big\|(b-b_{Q^*})f_1^0\big\|_{L^1(\mathbb{R}^n)}\prod_{j=2}^m \|f_j^0\|_{L^1(\mathbb{R}^n)}\\ &\le \frac{C}{|Q|}\int_Q |b(z)-b_{Q^*}|\,|f_1(z)|dz \prod_{j=2}^m \frac{1}{|Q|}\int_Q |f_j(z)|dz\\ &\le C\|b\|_{BMO}\|f_1\|_{L(\log L),Q^*}\prod_{j=2}^m \frac{1}{|Q^*|}\int_{Q^*}|f_j(z)|dz\\ &\le C\|b\|_{BMO}\mathcal{M}^1_{L(\log L)}(\vec{f})(x)\\ &\le C\|b\|_{BMO}\mathcal{M}_{L(\log L)}(\vec{f})(x).\end{aligned}$$

For $\mathrm{II}_{\alpha_1,\ldots,\alpha_m}$, we consider the term $\mathrm{II}_{\infty,\ldots,\infty}$ first. Set $\mathcal{Q}_k = (2^{k+3}\sqrt{n}Q) \setminus (2^{k+2}\sqrt{n}Q)$ as above. Since $0 < \delta < 1$, it follows from Hölder's inequality that

$$\mathrm{II}_{\infty,\ldots,\infty} \le \frac{C}{|Q|}\int_Q \big|T\big((b-b_{Q^*})f_1^{\infty},\ldots,f_m^{\infty}\big)(z) - T\big((b-b_{Q^*})f_1^{\infty},\ldots,f_m^{\infty}\big)(x)\big|dz$$

$$\le \frac{C}{|Q|}\int_Q\left(\int_{(\mathbb{R}^n\setminus Q^*)^m}\left|K(z,\vec{y})-K(x,\vec{y})\right||b(y_1)-b_{Q^*}|\prod_{j=1}^m|f_j^\infty(y_j)|d\vec{y}\right)dz$$
$$\le \frac{C}{|Q|}\int_Q\left(\sum_{k=1}^\infty\int_{(\mathcal{Q}_k)^m}\left|K(z,\vec{y})-K(x,\vec{y})\right||b(y_1)-b_{Q^*}|\prod_{j=1}^m|f_j(y_j)|d\vec{y}\right)dz.$$

Then, by (6.5) one has

$$\begin{aligned}
\mathrm{II}_{\infty,\ldots,\infty} &\le \frac{C}{|Q|}\int_Q\left(\sum_{k=1}^\infty\omega(2^{-k})\int_{(\mathcal{Q}_k)^m}\frac{|b(y_1)-b_{Q^*}|}{|2^k\sqrt{n}Q|^m}\prod_{j=1}^m|f_j(y_j)|d\vec{y}\right)dz\\
&\le C\sum_{k=1}^\infty\omega(2^{-k})\int_{(2^{k+3}\sqrt{n}Q)^m}\frac{|b(y_1)-b_{Q^*}|\,|f_1(y_1)|}{|2^{k+3}\sqrt{n}Q|^m}\prod_{j=2}^m|f_j(y_j)|d\vec{y}\\
&\le C\sum_{k=1}^\infty\omega(2^{-k})\left(\frac{1}{|2^{k+3}\sqrt{n}Q|}\int_{2^{k+3}\sqrt{n}Q}|b(y_1)-b_{Q^*}|\,|f_1(y_1)|dy_1\right)\\
&\quad\times\left(\prod_{j=2}^m\frac{1}{|2^{k+3}\sqrt{n}Q|}\int_{2^{k+3}\sqrt{n}Q}|f_j(y_j)|dy_j\right).
\end{aligned}\tag{7.1}$$

Recalling that $|b_{2^{k+3}\sqrt{n}Q}-b_{Q^*}|\le Ck\|b\|_{BMO}$, then by (4.2) we have

$$\begin{aligned}
&\frac{1}{|2^{k+3}\sqrt{n}Q|}\int_{2^{k+3}\sqrt{n}Q}|b(y_1)-b_{Q^*}|\,|f_1(y_1)|dy_1\\
&\le \frac{1}{|2^{k+3}\sqrt{n}Q|}\int_{2^{k+3}\sqrt{n}Q}|b(y_1)-b_{2^{k+3}\sqrt{n}Q}|\,|f_1(y_1)|dy_1+\frac{|b_{2^{k+3}\sqrt{n}Q}-b_{Q^*}|}{|2^{k+3}\sqrt{n}Q|}\int_{2^{k+3}\sqrt{n}Q}|f_1(y_1)|dy_1\\
&\le C\|b\|_{BMO}\|f_1\|_{L(\log L),2^{k+3}\sqrt{n}Q}+\frac{Ck\|b\|_{BMO}}{|2^{k+3}\sqrt{n}Q|}\int_{2^{k+3}\sqrt{n}Q}|f_1(y_1)|dy_1\\
&\le C(k+1)\|b\|_{BMO}\|f_1\|_{L(\log L),2^{k+3}\sqrt{n}Q}.
\end{aligned}\tag{7.2}$$

Since ω satisfying (1.5), then it follows from (7.1), (7.2) and (4.3) that

$$\begin{aligned}
\mathrm{II}_{\infty,\ldots,\infty} &\le C\|b\|_{BMO}\mathcal{M}^1_{L(\log L)}(\vec{f})(x)\sum_{k=1}^\infty(k+1)\omega(2^{-k})\\
&\le C\|b\|_{BMO}\mathcal{M}_{L(\log L)}(\vec{f})(x).
\end{aligned}\tag{7.3}$$

From (7.1) to (7.3), there holds the following inequality, which will be used later,

$$\sum_{k=1}^\infty\omega(2^{-k})\int_{(2^{k+3}\sqrt{n}Q)^m}\frac{|b(y_1)-b_{Q^*}|\,|f_1(y_1)|}{|2^{k+3}\sqrt{n}Q|^m}\prod_{j=2}^m|f_j(y_j)|d\vec{y}\le C\|b\|_{BMO}\mathcal{M}_{L(\log L)}(\vec{f})(x).\tag{7.4}$$

Now, let us consider the terms $\mathrm{II}_{\alpha_1,\ldots,\alpha_m}$ such that at least one $\alpha_j=0$ and one $\alpha_i=\infty$. Without loss of generality, we assume that $\alpha_{j_1}=\cdots=\alpha_{j_l}=0$ for some $1\le l<m$ and set $\mathcal{J}:=\{j_1,\ldots,j_l\}$ as before, then $\alpha_j=\infty$ for $j\notin\mathcal{J}$. For any $z\in Q$, similar to (6.6) and applying (6.7) and (7.4), we have

$$\begin{aligned}
&\left|T\big((b-b_{Q^*})f_1^{\alpha_1},\ldots,f_m^{\alpha_m}\big)(z)-T\big((b-b_{Q^*})f_1^{\alpha_1},\ldots,f_m^{\alpha_m}\big)(x)\right|\\
&\le\int_{(Q^*)^l}\prod_{j\in\mathcal{J}}|f_j^0(y_j)|\sum_{k=1}^\infty\int_{(\mathcal{Q}_k)^{m-l}}|K(z,\vec{y})-K(x,\vec{y})|\,|b(y_1)-b_{Q^*}|\prod_{j\notin\mathcal{J}}|f_j^\infty(y_j)|d\vec{y}\\
&\le C\int_{(Q^*)^l}\prod_{j\in\mathcal{J}}|f_j^0(y_j)|\sum_{k=1}^\infty\omega(2^{-k})\int_{(\mathcal{Q}_k)^{m-l}}\frac{|b(y_1)-b_{Q^*}|}{|2^k\sqrt{n}Q|^m}\prod_{j\notin\mathcal{J}}|f_j^\infty(y_j)|d\vec{y}\\
&\le C\sum_{k=1}^\infty\omega(2^{-k})\int_{(2^{k+3}\sqrt{n}Q)^m}\frac{|b(y_1)-b_{Q^*}|\,|f_1(y_1)|}{|2^{k+3}\sqrt{n}Q|^m}\prod_{j=2}^m|f_j(y_j)|d\vec{y}\\
&\le C\|b\|_{BMO}\mathcal{M}_{L(\log L)}(\vec{f})(x).
\end{aligned}$$

Then, it follows from Hölder's inequality that

$$\begin{aligned}\mathrm{II}_{\alpha_1,\ldots,\alpha_m} &\le \frac{C}{|Q|}\int_Q \Big|T\big((b-b_{Q^*})f_1^{\alpha_1},\ldots,f_m^{\alpha_m}\big)(z)-T\big((b-b_{Q^*})f_1^{\alpha_1},\ldots,f_m^{\alpha_m}\big)(x)\Big|dz\\ &\le C\|b\|_{BMO}\mathcal{M}_{L(\log L)}(\vec f)(x).\end{aligned}$$

Combining the above estimates we get the desired result. The proof is completed. □

Remark 7.1. For the linear case, the condition ω satisfying (1.5) is also needed, see [20,21], for details.

Theorem 7.2. *Let T be an m-linear ω-CZO with ω satisfying (1.5) and $T_{\vec b}$ be the m-linear commutator of T with $\vec b \in BMO^m$. If $p > 0$ and $w \in A_\infty$ then there exists a constant $C > 0$, depending on the A_∞ constant of w, such that*

$$\int_{\mathbb{R}^n}\Big|T_{\vec b}(\vec f)(x)\Big|^p w(x)dx \le C\|\vec b\|^p_{BMO^m}\int_{\mathbb{R}^n}\Big[\mathcal{M}_{L(\log L)}(\vec f)(x)\Big]^p w(x)dx$$

and

$$\sup_{t>0}\frac{1}{\Phi(1/t)}w\Big(\big\{y\in\mathbb{R}^n : |T_{\vec b}(\vec f)(y)| > t^m\big\}\Big) \le \sup_{t>0}\frac{1}{\Phi(1/t)}w\Big(\big\{y\in\mathbb{R}^n : \mathcal{M}_{L(\log L)}(\vec f)(y) > t^m\big\}\Big)$$

for all $\vec f = (f_1,\ldots,f_m)$ bounded with compact supports.

Proof. By Theorems 7.1 and 6.1, we can get Theorem 7.2. Since the ideas are almost the same as the ones of the proof of Theorem 3.19 in [7], we omit the details. □

Now, we are in a position to prove Theorems 1.3 and 1.4.

Proof of Theorem 1.3. It is enough to prove that Theorem 1.3 is valid for $f_1,\ldots,f_m$ being bounded functions with compact supports. Since $\vec w \in A_{\vec P}$ then by Lemma 4.4 one has $\nu_{\vec w} \in A_\infty$. It follows from the first part of Theorem 7.2 that

$$\big\|T_{\vec b}(\vec f)\big\|_{L^p(\nu_{\vec w})} \le C\|\vec b\|_{BMO^m}\big\|\mathcal{M}_{L(\log L)}(\vec f)\big\|_{L^p(\nu_{\vec w})}.$$

By (4.3) and Lemma 4.5, for some $r > 1$,

$$\big\|T_{\vec b}(\vec f)\big\|_{L^p(\nu_{\vec w})} \le C\|\vec b\|_{BMO^m}\big\|\mathcal{M}_r(\vec f)\big\|_{L^p(\nu_{\vec w})} \le C\|\vec b\|_{BMO^m}\prod_{j=1}^m\|f_j\|_{L^{p_j}(w_j)}.$$

So complete the proof of Theorem 1.3. □

To prove Theorem 1.4, we need the following result due to Lerner et al. [7].

Lemma 7.1. *Let $\vec w \in A_{(1,\ldots,1)}$. Then there exists a constant C such that*

$$\nu_{\vec w}\Big(\big\{x\in\mathbb{R}^n : \mathcal{M}^i_{L(\log L)}(\vec f)(x) > t^m\big\}\Big) \le C\prod_{j=1}^m\left(\int_{\mathbb{R}^n}\Phi\left(\frac{|f_j(x)|}{t}\right)w_j(x)dx\right)^{1/m}.$$

Proof of Theorem 1.4. Applying Theorem 7.2 and Lemma 7.1 and making use of the same arguments as the ones in the proof of Theorem 3.16 in [7], we can obtain Theorem 1.4. Here we omit the proof. □

8. Proofs of Theorems 2.1–2.6

8.1. Proofs of Theorems 2.1–2.3

To prove Theorems 2.1–2.3, we first recall the following result obtained in [13].

Lemma 8.1 ([13, Theorem 5.3])**.** *Assume that ω is concave and $\omega \in \mathrm{Dini}(1/2)$. Let $\{\phi_Q^j\}_{Q\in\mathcal{D}}$, $j = 1, 2, 3$, be three families of ω-molecules with decay $N > 10n$ and such that at least two of them have cancellation property. Then, the para-product Π has a bilinear Calderón–Zygmund kernel of type $\theta(t)$ with*

$$\theta(t) = A_0^3A_N\omega(C_Nt),\quad t > 0,$$

for some constants A_N and C_N (hence, $\theta \in \mathrm{Dini}(1/2)$). Moreover, Π has the mapping property

$$L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)\to L^1(\mathbb{R}^n).$$

In particular, Π is a bilinear Calderón–Zygmund operator of type $\theta(t)$.

We checked the proof of Lemma 8.1 (see the proof of Theorem 5.3 in [13] pp. 235–237) carefully and found that the condition $\omega \in \mathrm{Dini}(1/2)$ has no use for proving that Π having a bilinear Calderón–Zygmund kernel of type $\theta(t)$, and in the proof of Π mapping $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$, only the condition $\omega \in \mathrm{Dini}(1)$ is needed. For the sake of clarity, we restate Theorem 5.3 of [13] into the following two lemmas.

Lemma 8.2. *Assume that ω is concave, and let $\{\phi_Q^j\}_{Q\in\mathcal{D}}$, $j = 1, 2, 3$, be three families of ω-molecules with decay $N > 10n$ and such that at least two of them have cancellation property. Then, the para-product Π has a bilinear Calderón–Zygmund kernel of type $\theta(t)$ with*

$$\theta(t) = A_0^3 A_{n,N}\,\omega(C_N t), \quad t > 0.$$

Lemma 8.3. *Assume that ω is concave and $\omega \in \mathrm{Dini}(1)$. Let $\{\phi_Q^j\}_{Q\in\mathcal{D}}$, $j = 1, 2, 3$, be three families of ω-molecules such that at least two of them enjoy cancellation property. Then Π is bounded from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$.*

Proof of Theorems 2.1–2.3. By Lemmas 8.2 and 8.3 we see that under the assumption of Theorem 2.1, the para-product Π is a bilinear Calderón–Zygmund operator of type $\theta(t)$ with

$$\theta(t) = A_0^3 A_{n,N}\,\omega(C_N t), \quad t > 0.$$

If $\omega \in \mathrm{Dini}(1)$ (or, ω satisfies (1.5)), then so does $\theta(t)$. Thus, Theorems 2.1–2.3 follow from Theorems 1.2–1.4, respectively. □

8.2. Proofs of Theorems 2.4–2.6

To prove Theorems 2.4–2.6, it suffices to show that the bilinear pseudo-differential operator T_σ stated above is a bilinear Calderón–Zygmund operator of type $\theta(t)$ with $\theta(t) = \omega^a(t)$. To do this, we first give the following remark.

Remark 8.1. In Definition 1.1, we assume that an m-linear Calderón–Zygmund kernel of type ω satisfies (1.2) and (1.3) whenever $|x - x'| \le \frac{1}{2}\max_{1\le j\le m}|x - y_j|$ and $|y_i - y_i'| \le \frac{1}{2}\max_{1\le j\le m}|x - y_j|$, respectively. We note that the constant $\frac{1}{2}$ is not the essential attribute to ensure the validity of Theorems 1.1–1.4. More precisely, if we replace the constant $\frac{1}{2}$ by a constant $\tau \in (0, 1)$ then Theorems 1.1–1.4 are also true.

It is known that the bilinear pseudo-differential operator T_σ stated above has the following kernel representation

$$T_\sigma(f, g)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathcal{K}(x, y_1, y_2) f(y_1) g(y_2)\, dy_1 dy_2, \quad f, g \in \mathscr{S}(\mathbb{R}^n),$$

where

$$\mathcal{K}(x, y_1, y_2) = \hat{\sigma}(x, y_1 - x, y_2 - x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \sigma(x, \xi, \eta) e^{-\xi(y_1 - x)} e^{-\eta(y_2 - x)}\, d\xi\, d\eta.$$

From Theorem 4.1 of [13], the following estimates hold for the associated kernel.

Lemma 8.4. *Let $a \in (0, 1)$, ω and Ω be the same as in Theorem 2.4, and, set $\theta(t) = \omega^a(t)$. If $\sigma \in BS^0_{1,\omega,\Omega}$ with $|\alpha| + |\beta| \le 2n + 2$, then the associated kernel $\mathcal{K}$ of T_σ satisfies (1.1) and (1.3) with ω being replaced by θ, and*

$$|\mathcal{K}(x, y_1, y_2) - \mathcal{K}(x', y_1, y_2)| \le \frac{A}{(|x - y_1| + |x - y_2|)^{2n}}\,\theta\left(\frac{|x - x'|}{|x - y_1| + |x - y_2|}\right)$$

whenever $|x - x'| \le \frac{1}{3}\max\{|x - y_1|, |x - y_2|\}$.

This shows that the associated kernel of T_σ is an m-linear Calderón–Zygmund kernel of type θ with $\theta(t) = \omega^a(t)$ and $\tau = \frac{1}{3}$. In addition, the following boundedness property of T_σ is valid, see Theorem 4.3 in [13].

Lemma 8.5. *Let $a \in (0, 1)$, ω and Ω be the same as in Theorem 2.4. If $\sigma \in BS^0_{1,\omega,\Omega}$ with $|\alpha| + |\beta| \le 4n + 4$, then T_σ is bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ for some $1 < p, q < \infty$ and $1 \le p < \infty$ with $1/p = 1/p_1 + 1/p_2$. Moreover, T_σ is a bilinear Calderón–Zygmund operator of type $\theta(t) = \omega^a(t)$.*

Proof of Theorems 2.4–2.6. Note that if $\omega^a(t) \in \mathrm{Dini}(a/2)$ (or, $\omega^a(t)$ satisfies (1.5)), then $\theta(t) \in \mathrm{Dini}(1/2) \subset \mathrm{Dini}(1)$ (or, $\theta(t)$ satisfies (1.5)). By Lemma 8.5, Theorems 2.4–2.6 follow from Theorems 1.2–1.4, respectively. □

9. Proofs of Theorems 3.1–3.6

We do some preparations for the proof of the theorems. Now, we recall some facts on variable exponent Lebesgue spaces. The first one is the generalized Hölder's inequality, see Lemma 3.2.20 in [30] or Corollary 2.30 in [29].

Lemma 9.1. *Let $r(\cdot), p(\cdot), q(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ so that $1/r(x) = 1/p(x) + 1/q(x)$. Then, for any $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $g \in L^{q(\cdot)}(\mathbb{R}^n)$, there has*

$$\|fg\|_{L^{r(\cdot)}(\mathbb{R}^n)} \le 2\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|g\|_{L^{q(\cdot)}(\mathbb{R}^n)}.$$

When $p(\cdot) \equiv 1$, the constant 2 in the previous inequality can be replaced by $1 + 1/p_- - 1/p_+$, see Kováčik and Rákosník [26] for details. By the induction argument, we can generalize Lemma 9.1 to three or more exponents.

Lemma 9.2. *Let $q(\cdot), q_1(\cdot), \dots, q_m(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ so that*

$$\frac{1}{q(x)} = \frac{1}{q_1(x)} + \cdots + \frac{1}{q_m(x)}.$$

Then any $f_j \in L^{q_j(\cdot)}(\mathbb{R}^n)$, $j = 1, \dots, m$, there has

$$\|f_1 \cdots f_m\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le 2^{m-1}\|f_1\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \cdots \|f_m\|_{L^{q_m(\cdot)}(\mathbb{R}^n)}.$$

The next one is an extrapolation theorem originally due to Cruz-Uribe et al. [28]. Here, we use the following form, see Theorem 7.2.1 in [30].

Lemma 9.3. *Given a family $\mathcal{F}$ of ordered pairs of measurable functions, suppose for some fixed $0 < p_0 < \infty$, every $(f, g) \in \mathcal{F}$ and every $\omega \in A_1$,*

$$\int_{\mathbb{R}^n} |f(x)|^{p_0}\omega(x)dx \le C_0 \int_{\mathbb{R}^n} |g(x)|^{p_0}\omega(x)dx.$$

Let $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ with $p_0 \le p_-$. If $\big(p(\cdot)/p_0\big)' \in \mathscr{B}(\mathbb{R}^n)$, then there exists a constant $C > 0$ such that for all $(f, g) \in \mathcal{F}$,

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|g\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

The following result was proved by Diening in [35].

Lemma 9.4. *Let $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$. Then the following conditions are equivalent:*

(i) $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$,
(ii) $p'(\cdot) \in \mathscr{B}(\mathbb{R}^n)$,
(iii) $p(\cdot)/p_0 \in \mathscr{B}(\mathbb{R}^n)$ *for some* $1 < p_0 < p_-$,
(iv) $\big(p(\cdot)/p_0\big)' \in \mathscr{B}(\mathbb{R}^n)$ *for some* $1 < p_0 < p_-$.

We also need the following density property, see Theorem 3.4.12 in [30].

Lemma 9.5. *If $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$, then $C_0^\infty(\mathbb{R}^n)$ is dense in $L^{p(\cdot)}(\mathbb{R}^n)$.*

Now, we have all the ingredients to prove Theorems 3.1–3.6.

Proof of Theorem 3.1. Since $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$ then, by Lemma 9.4, there exists a p_0 with $1 < p_0 < p_-$ such that $\big(p(\cdot)/p_0\big)' \in \mathscr{B}(\mathbb{R}^n)$. On the other hand, by Theorem 6.2 we see that, for this p_0 and any $w \in A_1$,

$$\int_{\mathbb{R}^n} \big|T(\vec{f})(x)\big|^{p_0} w(x)dx \le C \int_{\mathbb{R}^n} \big[\mathcal{M}(\vec{f})(x)\big]^{p_0} w(x)dx$$

holds for all m-tuples $\vec{f} = (f_1, \dots, f_m)$ of bounded functions with compact support.

Apply Lemma 9.3 to the pair $\big(T(\vec{f}),\ \mathcal{M}(\vec{f})\big)$ and obtain

$$\big\|T(\vec{f})\big\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\big\|\mathcal{M}(\vec{f})\big\|_{L^{p(\cdot)}(\mathbb{R}^n)}. \tag{9.1}$$

By Definition 4.1, it is easy to see that

$$\mathcal{M}(\vec{f})(x) \le \prod_{j=1}^m M(f_j)(x) \quad \text{for } x \in \mathbb{R}^n.$$

This, together with (9.1) and the generalized Hölder's inequality (Lemma 9.2), yields

$$\big\|T(\vec{f})\big\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\prod_{j=1}^m \|M(f_j)\|_{L^{p_j(\cdot)}(\mathbb{R}^n)} \le C\prod_{j=1}^m \|f_j\|_{L^{p_j(\cdot)}(\mathbb{R}^n)},$$

where in the last inequality, we make use of the $L^{p_j(\cdot)}(\mathbb{R}^n)$ boundedness of the Hardy–Littlewood maximal operator M since $p_j(\cdot) \in \mathscr{B}(\mathbb{R}^n)$.

Now, we have showed that Theorem 3.1 is valid for all bounded functions $f_1, \ldots, f_m$ with compact support. Lemma 9.5 concludes the proof of Theorem 3.1. □

Proof of Theorem 3.2. By Lemma 9.5 it suffices to prove Theorem 3.2 for all bounded functions $f_1, \ldots, f_m$ with compact support.

By Lemma 9.4, there exists a p_0 with $1 < p_0 < p_-$ such that $\big(p(\cdot)/p_0\big)' \in \mathscr{B}(\mathbb{R}^n)$ since $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$. It follows from Theorem 7.2 that, for this p_0 and any $w \in A_1$,

$$\int_{\mathbb{R}^n} \left|T_{\vec{b}}(\vec{f})(x)\right|^{p_0} w(x)dx \leq C \int_{\mathbb{R}^n} \left[\mathcal{M}_{L(\log L)}(\vec{f})(x)\right]^{p_0} w(x)dx$$

holds for all bounded functions $f_1, \ldots, f_m$ with compact support.

Applying Lemma 9.3 to the pair $\big(T_{\vec{b}}(\vec{f}),\ \mathcal{M}_{L(\log L)}(\vec{f})\big)$, we get

$$\left\|T_{\vec{b}}(\vec{f})\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C \left\|\mathcal{M}_{L(\log L)}(\vec{f})\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}. \tag{9.2}$$

Recall the pointwise equivalence $M_{L(\log L)}(g)(x) \approx M^2(g)(x)$ for any locally integrable function g (see (21) in [36]) and

$$\mathcal{M}_{L(\log L)}(\vec{f})(x) \leq \prod_{j=1}^{m} \Big(\sup_{Q \ni x} \|f_j\|_{L(\log L),Q}\Big) = \prod_{j=1}^{m} M_{L(\log L)}(f_j)(x),$$

then, by (9.2) and Lemma 9.2 there has

$$\begin{aligned}\left\|T_{\vec{b}}(\vec{f})\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} &\leq C \left\|\prod_{j=1}^{m} M_{L(\log L)}(f_j)\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C \left\|\prod_{j=1}^{m} M^2(f_j)\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\leq C \prod_{j=1}^{m} \|f_j\|_{L^{p_j(\cdot)}(\mathbb{R}^n)}\end{aligned}$$

where in the last inequality, we make use of the $L^{p_j(\cdot)}(\mathbb{R}^n)$ boundedness of M twice. □

Now, Theorem 3.3 (Theorem 3.4) is a direct consequence of Theorem 3.1 (Theorem 3.2) together with Lemmas 8.2 and 8.3. And, Theorem 3.5 (Theorem 3.6) is a direct consequence of Theorem 3.1 (Theorem 3.2) together with Lemma 8.5.

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