L^p-Estimates for a Trilinear Pseudo-Differential Operator with Flag Symbols

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ABSTRACT. In this paper, we establish the L^p-estimate for a trilinear pseudo-differential operator, where the symbol involved is given by the product of two standard symbols from the Hömander class $BS^0_{1,0}$. The study of this operator is motivated by C. Muscalu's analysis on the flag paraproducts that is used to investigate the trilinear Fourier multiplier operator with flag singularities in [11].

1. INTRODUCTION

For $n \geq 1$, we denote by $\mathcal{M}(\mathbb{R}^n)$ the set of all bounded symbols $m \in L^\infty(\mathbb{R}^n)$, smooth away from the origin and satisfying the classical Marcinkiewicz-Mikhlin-Hörmander condition

$$|\partial^\alpha m(\xi)| \lesssim \frac{1}{|\xi|^\alpha}$$

for every $\xi \in \mathbb{R}^n \setminus \{0\}$ and sufficiently many multi-indices α. Denote by T_m the n-linear operator

$$T_m(f_1, \dots, f_n)(x) := \int_{\mathbb{R}^n} m(\xi)\hat{f}_1(\xi_1) \cdots \hat{f}_n(\xi_n) e^{2\pi i(\xi_1+\cdots+\xi_n)\cdot x}\,\mathrm{d}\xi,$$

where $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ and $f_1, \dots, f_n$ are Schwartz functions on $\mathbb{R}$, denoted by $S(\mathbb{R})$. From the classical Coifman-Meyer theorem, we know T_m extends to a bounded n-linear operator from $L^{p_1}(\mathbb{R}) \times \cdots \times L^{p_n}(\mathbb{R})$ to $L^r(\mathbb{R})$ for $1 < p_1, \dots, p_n \leq \infty$ and $1/p_1 + \cdots + 1/p_n = 1/r > 0$. In fact, this property holds for the high dimensions when $f_i \in L^{p_i}(\mathbb{R}^d)$, $i = 1, \dots, n$ and $m \in \mathcal{M}(\mathbb{R}^{nd})$ (see [4,7,9]). The case $p \geq 1$ was proved by Coifman and Meyer [4], and was extended to $p < 1$ by Grafakos and Torres [7] and Kenig and Stein [9]. Moreover, in the multiparameter setting, the same boundedness property is true (see [12–14]; and also [2] for a weaker restriction on the smoothness for the multiplier).

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For the corresponding pseudo-differential variant of the classical Coifman-Meyer theorem, let the symbol $\sigma(x,\xi)$ belong to the bilinear Hörmander symbol class $BS^0_{1,0}$; that is, σ satisfies the condition

$$|\partial_x^{\ell}\partial_{\xi}^{\alpha}\sigma(x,\xi)| \lesssim \frac{1}{(1+|\xi|)^{|\alpha|}} \tag{1.1}$$

for any $x \in \mathbb{R}$, $\xi = (\xi_1,\dots,\xi_n) \in \mathbb{R}^n$, and sufficiently many indices ℓ, α. We have the following result.

Theorem 1.1. *The operator*

$$T_{\sigma}(f_1,\dots,f_n)(x) := \int_{\mathbb{R}^n} \sigma(x,\xi)\hat{f}_1(\xi_1)\cdots f_n(\xi_n)e^{2\pi i(\xi_1+\cdots+\xi_n)\cdot x}\,\mathrm{d}\xi$$

is bounded from $L^{p_1}(\mathbb{R})\times\cdots\times L^{p_n}(\mathbb{R})$ to $L^r(\mathbb{R})$ for $1 < p_1,\dots,p_n \le \infty$ and $1/p_1+\cdots+1/p_n = 1/r > 0$, where $f_1,\dots,f_n \in S(\mathbb{R})$ and σ satisfies (1.1).

For the proof of the above theorem, see [1] for the bilinear, high dimensional case and [12] for the one-dimensional, n-linear case. This boundedness property also holds in the multi-parameter setting (see [6]). Properties of multi-parameter and multilinear pseudo-differential operators of Coifman-Meyer type have also been studied in [8].

For the trilinear Coifman-Meyer type theorem, Muscalu [11] proved the following theorem where the multiplier involved is a product of two symbols and has *flag singularities*. That is, for $m_1, m_2 \in \mathcal{M}(\mathbb{R}^2)$ satisfying

$$\begin{aligned} |\partial_{\xi}^{\alpha}\partial_{\eta}^{\beta}m_1(\xi,\eta)| &\lesssim \frac{1}{(|\xi|+|\eta|)^{\alpha+\beta}}, \\ |\partial_{\eta}^{\beta}\partial_{\zeta}^{\gamma}m_2(\eta,\zeta)| &\lesssim \frac{1}{(|\eta|+|\zeta|)^{\beta+\gamma}}, \end{aligned} \tag{1.2}$$

for every $\xi,\eta,\zeta \in \mathbb{R}$ and sufficiently many indices α, β and γ, we define

$$\begin{aligned} &T_{m_1,m_2}(f_1,f_2,f_3)(x) \\ &\quad := \int_{\mathbb{R}^3} m_1(\xi,\eta)m_2(\eta,\zeta)\hat{f}_1(\xi)\hat{f}_2(\eta)\hat{f}_3(\zeta)e^{2\pi i(\xi+\eta+\zeta)\cdot x}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta, \end{aligned} \tag{1.3}$$

where $f_1,f_2,f_3 \in S(\mathbb{R})$. Then, we have the following result.

Theorem 1.2 ([11]). *The operator defined in* (1.3) *maps $L^{p_1}\times L^{p_2}\times L^{p_3} \to L^r$ for $1 < p_1,p_2,p_3 < \infty$ with $1/p_1+1/p_2+1/p_3 = 1/r$ and $0 < r < \infty$. In addition, T_{m_1,m_2} also maps $L^\infty\times L^p\times L^q \to L^s$, $L^p\times L^\infty\times L^q \to L^s$, $L^\infty\times L^t\times L^\infty \to L^t$ for every $1 < p,q,t < \infty$ and $1/p+1/q = 1/s$.*

Moreover, for the above theorem, the estimates like $L^\infty\times L^\infty\times L^t \to L^t$ or $L^\infty\times L^\infty\times L^\infty \to L^\infty$ are false; these can be checked if we set f_2 to be identically 1.

Our main purpose is to consider a pseudo-differential operator corresponding to the above theorem; that is, let $a(x,\xi,\eta), b(x,\eta,\zeta) \in BS^0_{1,0}$ be symbols satisfying the conditions

$$(1.4)\qquad \begin{aligned} |\partial_x^\ell \partial_\xi^\alpha \partial_\eta^\beta a(x,\xi,\eta)| &\lesssim \frac{1}{(1+|\xi|+|\eta|)^{\alpha+\beta}}, \\ |\partial_x^\ell \partial_\eta^\beta \partial_\zeta^\gamma b(x,\eta,\zeta)| &\lesssim \frac{1}{(1+|\eta|+|\zeta|)^{\beta+\gamma}}, \end{aligned}$$

and for every $x,\xi,\eta,\zeta \in \mathbb{R}$ and sufficiently many indices α, β and γ, define the operator

$$(1.5)\quad \begin{aligned} &T_{ab}(f,g,h)(x) \\ &\quad := \int_{\mathbb{R}^3} a(x,\xi,\eta)b(x,\eta,\zeta)\hat{f}(\xi)\hat{g}(\eta)\hat{h}(\zeta)e^{2\pi i x(\xi+\eta+\zeta)}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta. \end{aligned}$$

It is easy to see that the symbol $a(x,\xi,\eta)\cdot b(x,\eta,\zeta)$ satisfies a less restrictive condition than the condition (1.1) for the symbol σ in Theorem 1.1. The main result of this paper is the following theorem.

Theorem 1.3. *The operator T_{ab} defined as (1.5) is bounded from $L^{p_1}\times L^{p_2}\times L^{p_3}$ to L^r for $1<p_1,p_2,p_3<\infty$ with $1/p_1+1/p_2+1/p_3=1/r$ and $0<r<\infty$. In addition, T_{ab} also maps $L^\infty\times L^p\times L^q\to L^s$, $L^p\times L^\infty\times L^q\to L^s$, $L^\infty\times L^t\times L^\infty\to L^t$ for every $1<p,q,t<\infty$ and $1/p+1/q=1/s$.*

The proof of Theorem 1.3 involves reducing the trilinear pseudo-differential operator with a flag symbol to a localized version, and taking advantage of the *flag paraproducts* from Muscalu's work [11] on the L^p-estimates for the Fourier multipliers with symbols of flag singularity. Specifically, we need to prove Theorem 1.3, which is an equivalent localized version of Theorem 1.3 (see Muscalu and Schlag [12] for the one-parameter case, and [6] for the multi-parameter setting). Moreover, the key to proving the localized result is that conditions (1.4) allow us only to consider the dyadic intervals with lengths at most 1 in the *flag paraproducts*.

More precisely, in Section 3 we will show that our main theorem can be reduced to an estimate for a localized operator

$$\begin{aligned} &T^{0,0}_{ab}(f,g,h)(x) \\ &\quad = \left(\int_{\mathbb{R}^3} a_0(\xi,\eta)b_0(\eta,\zeta)\hat{f}(\xi)\hat{g}(\eta)\hat{h}(\zeta)e^{2\pi i x(\xi+\eta+\zeta)}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta\right)\varphi_0(x), \end{aligned}$$

where $\varphi_0(x)$ is a Schwartz function supported near the origin, and a_0, b_0 satisfy a stronger decay condition than the classical Hörmander-Mikhlin condition.

In Section 4, we will decompose the operator $T^{0,0}_{ab}$ to some operators of different forms. Among these operators, some of them could be reduced to the classical pseudo-differential operators in Theorem 1.1, and the others could be written as *flag paraproducts*, which are used in the proof of Theorem 1.2, in the form of

$$(T_1(f,g,h)\cdot\varphi_0)(x) = \sum_{I\in\mathcal{I}} \frac{1}{|I|^{1/2}}\langle f,\phi_I^1\rangle\langle B_I^1(g,h),\phi_I^2\rangle\phi_I^3\varphi_0,$$

where

$$B_I^1(g,h) = \sum_{J\in\mathcal{J},\,|\omega_J^3|\le|w_I^2|} \frac{1}{|J|^{1/2}}\langle g,\phi_J^1\rangle\langle h,\phi_J^2\rangle\phi_J^3,$$

but with dyadic intervals having lengths at most 1. Then, by taking advantage of the *flag paraproducts* mentioned above, we will be able to prove the desired estimate for the localized version of our theorem in Section 5.

We end this introduction by briefly describing some recent works related to the results in this paper. In our recent paper [10], we study the bi-parameter pseudo-differential variant of Theorem 1.3. In order to study such a bi-parameter pseudo-differential operator, usually a bi-parameter version of Theorem 1.2 has to be established first. However, such a result is hard to prove when the multipliers there have bi-parameter flag singularities involved. Fortunately, it turns out that we can strengthen the conditions on the bi-parameter trilinear multipliers and get a Hölder-type estimate for such strengthened bi-parameter trilinear multipliers. The L^p-estimates for these bi-parameter trilinear multipliers will be sufficient in the study of bi-parameter trilinear pseudo-differential operators of flag symbols. That is, let $m_3, m_4 \in B\mathcal{M}_0(\mathbb{R}^{2n}\times\mathbb{R}^{2n})$ be smooth symbols that satisfy

$$\begin{aligned}
(1.6)\qquad & |\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}\partial_{\eta_1}^{\beta_1}\partial_{\eta_2}^{\beta_2} m_3(\xi,\eta)| \\
&\qquad \lesssim \frac{1}{(1+|\xi_1|+|\eta_1|)^{|\alpha_1|+|\beta_1|}}\frac{1}{(1+|\xi_2|+|\eta_2|)^{|\alpha_2|+|\beta_2|}},\\
(1.7)\qquad & |\partial_{\eta_1}^{\beta_1}\partial_{\eta_2}^{\beta_2}\partial_{\zeta_1}^{\gamma_1}\partial_{\zeta_2}^{\gamma_2} m_4(\eta,\zeta)| \\
&\qquad \lesssim \frac{1}{(1+|\eta_1|+|\zeta_1|)^{|\beta_1|+|\gamma_1|}}\frac{1}{(1+|\eta_2|+|\zeta_2|)^{|\beta_2|+|\gamma_2|}},
\end{aligned}$$

for every $\xi = (\xi_1,\xi_2)$, $\eta = (\eta_1,\eta_2)$, $\zeta = (\zeta_1,\zeta_2) \in \mathbb{R}^n\times\mathbb{R}^n$ and all multi-indices $\alpha = (\alpha_1,\alpha_2)$, $\beta = (\beta_1,\beta_2)$ and $\gamma = (\gamma_1,\gamma_2)$. Then, we can establish the following L^p-estimates (see [10]).

Theorem 1.4. *For $f,g,h \in S(\mathbb{R}^{2n})$, the bi-parameter operators*

$$\begin{aligned}
&T_{m_3,m_4}(f,g,h)(x)\\
&\quad := \int_{\mathbb{R}^{6n}} m_3(\xi,\eta)m_4(\eta,\zeta)\hat{f}(\xi)\hat{g}(\eta)\hat{h}(\zeta)e^{2\pi i(\xi+\eta+\zeta)\cdot x}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta
\end{aligned}$$

map $L^{p_1}\times L^{p_2}\times L^{p_3}\to L^r$ for $1 < p_1,p_2,p_3 < \infty$ with $1/p_1+1/p_2+1/p_3 = 1/r$ and $0 < r < \infty$.

Actually, from the proof of the above theorem, we can get a more general result without much difficulty. For $\ell, n \geq 1$, let $m(\xi) \in C^\infty(\mathbb{R}^{\ell \cdot 2n})$, where $\xi = (\xi_i)_{i=1}^{\ell}$ and $\xi_i = (\xi_i^1, \xi_i^2) \in \mathbb{R}^n \times \mathbb{R}^n$. We say $m \in B\mathcal{M}_0(\mathbb{R}^{\ell \cdot 2n})$ if

$$|\partial_{\xi_1^1,\xi_1^2}^{\alpha_1,\alpha_1'} \cdots \partial_{\xi_l^1,\xi_l^2}^{\alpha_\ell,\alpha_\ell'} m(\xi)| \lesssim \frac{1}{(1+|\xi_1^1|+\cdots+|\xi_\ell^1|)^{|\alpha_1|+\cdots+|\alpha_\ell|}} \times \frac{1}{(1+|\xi_1^2|+\cdots+|\xi_\ell^2|)^{|\alpha_1'|+\cdot+|\alpha_\ell'|}}$$

for every $\xi \in \mathbb{R}^{\ell \cdot 2n}$ and all multi-indices $\alpha_1, \alpha_1', \dots, \alpha_\ell, \alpha_\ell'$. Then, the following result has been proved in [10].

Theorem 1.5. *For integers $n, \ell \geq 1$, let*

$$m(\xi) := \prod_{S \subseteq \{1,\dots,\ell\}} m_S(\xi_S),$$

where $m_S \in B\mathcal{M}_0(\mathbb{R}^{\mathrm{card}(S)\cdot 2n})$, the vector $\xi_S \in \mathbb{R}^{\mathrm{card}(S)\cdot 2n}$ is defined by $\xi_S := (\xi_i)_{i\in S}$, where $\xi_i \in \mathbb{R}^{2n}$, and ξ is the vector $\xi := (\xi_i)_{i=1}^{\ell}$. Every such symbol m can define a ℓ-linear operator

$$T_m^\ell(f_1, \dots, f_\ell)(x) := \int_{\mathbb{R}^{2\ell n}} m(\xi)\widehat{f_1}(\xi_1)\cdots\widehat{f_\ell}(\xi_\ell)e^{2\pi i x(\xi_1+\cdots+\xi_\ell)}\,\mathrm{d}\xi,$$

where $f_1, \dots, f_\ell$ are Schwartz functions on $\mathbb{R}^{2n}$. Then, we also have that T_m^ℓ maps $L^{p_1} \times \cdots \times L^{p_\ell} \to L^p$ if $1 < p_1, \dots, p_\ell < \infty$ and $1/p_1 + \dots 1/p_\ell = 1/p$.

Now, we state the result for L^p-estimates for the corresponding bi-parameter trilinear pseudo-differential operators proved in [10]. Let

$$\begin{aligned} &T_{ab}(f,g,h)(x) \\ &\quad := \int_{\mathbb{R}^6} a(x,\xi,\eta)b(x,\eta,\zeta)\hat{f}(\xi)\hat{g}(\eta)\hat{h}(\zeta)e^{2\pi i x(\xi+\eta+\zeta)}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta, \end{aligned} \tag{1.8}$$

where $f, g, h \in S(\mathbb{R}^2)$, and the smooth symbols $a, b \in BBS_{1,0}^0$ satisfy the following conditions:

$$\begin{aligned} &|\partial_{x_1}^{\ell_1}\partial_{x_2}^{\ell_2}\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}\partial_{\eta_1}^{\beta_1}\partial_{\eta_2}^{\beta_2} a(x,\xi,\eta)| \\ &\qquad \lesssim \frac{1}{(1+|\xi_1|+|\eta_1|)^{|\alpha_1|+|\beta_1|}} \frac{1}{(1+|\xi_2|+|\eta_2|)^{|\alpha_2|+|\beta_2|}}, \\ &|\partial_{x_1}^{\ell_1}\partial_{x_2}^{\ell_2}\partial_{\eta_1}^{\beta_1}\partial_{\eta_2}^{\beta_2}\partial_{\zeta_1}^{\gamma_1}\partial_{\zeta_2}^{\gamma_2} b(x,\eta,\zeta)| \\ &\qquad \lesssim \frac{1}{(1+|\eta_1|+|\zeta_1|)^{|\beta_1|+|\gamma_1|}} \frac{1}{(1+|\eta_2|+|\zeta_2|)^{|\beta_2|+|\gamma_2|}}, \end{aligned}$$

for every $x = (x_1, x_2)$, $\xi = (\xi_1, \xi_2)$, $\eta = (\eta_1, \eta_2)$, $\zeta = (\zeta_1, \zeta_2) \in \mathbb{R} \times \mathbb{R}$, and all multi-indices $\ell = (\ell_1, \ell_2)$, $\alpha = (\alpha_1, \alpha_2)$, $\beta = (\beta_1, \beta_2)$, and $\gamma = (\gamma_1, \gamma_2)$. Our result established in [10] is the following theorem.

Theorem 1.6. *The operators T_{ab} defined as* (1.8) *map $L^{p_1} \times L^{p_2} \times L^{p_3} \to L^r$ for $1 < p_1, p_2, p_3 < \infty$ with $1/p_1 + 1/p_2 + 1/p_3 = 1/r$ and $0 < r < \infty$.*

The main idea in proving Theorem 1.6 is to reduce the bi-parameter trilinear pseudo-differential operator to a localized version. Then, by taking advantage of the L^p-estimates of the bi-parameter trilinear multipliers satisfying (1.6)–(1.7), we can establish Theorem 1.6. We refer the reader to [10] for more details.

2. NOTATION AND PRELIMINARIES

Let $S(\mathbb{R})$ denote the Schwartz space of rapidly decreasing, C^∞-functions in $\mathbb{R}$. Define the Fourier transform of a function f in $S(\mathbb{R})$ as

$$F(f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \cdot \xi} \, \mathrm{d}x$$

extended in the usual way to the space of tempered distribution $S'(\mathbb{R})$, which is the dual space of $S(\mathbb{R})$.

Throughout the paper, we use $A \lesssim B$ to represent that there exists a universal constant $C > 1$ so that $A \le CB$, and use the notation $A \sim B$ to denote that $A \lesssim B$ and $B \lesssim A$.

We call the intervals of the form $[2^k n, 2^k(n+1)]$ in $\mathbb{R}$ *dyadic intervals*, where $k, n \in \mathbb{Z}$. We denote by $\mathbb{D}$ the set of all such dyadic intervals.

Definition 2.1. For $I \in \mathbb{D}$, we define the approximate cutoff function as

$$\tilde{\chi}_I(x) := \left(1 + \frac{\operatorname{dist}(x, I)}{|I|}\right)^{-100}. \tag{2.1}$$

Definition 2.2. Let $I \subseteq \mathbb{R}$ be an arbitrary interval. A smooth function φ is said to be a bump adapted to I if and only if one has

$$|\varphi^{(\ell)}| \le C_\ell C_M \frac{1}{|I|^\ell} \frac{1}{(1 + |x - x_I|/|I|)^M}$$

for every integer $M \in \mathbb{N}$ and sufficiently many derivatives $\ell \in \mathbb{N}$, where x_I denotes the center of I and $|I|$ is the length of I.

If φ_I is a bump adapted to I, we say that $|I|^{1/p}\varphi_I$ is an L^p-normalized bump adapted to I, for $1 \le p \le \infty$.

Definition 2.3. A sequence of L^2-normalized bumps $(\Phi_I)_{I \in \mathbb{D}}$ adapted to dyadic intervals $I \in \mathbb{D}$ is called a non-lacunary sequence if and only if, for each $I \in \mathbb{D}$, there exists an interval $\omega_I = \omega_{|I|}$ symmetric with respect to the origin so that $\operatorname{supp} \widehat{\Phi_I} \subseteq \omega_I$ and $|\omega_I| \sim |I|^{-1}$.

Definition 2.4. A sequence of L^2-normalized bumps $(\Phi_I)_{I\in\mathbb{D}}$ adapted to dyadic intervals $I \in \mathbb{D}$ is called a lacunary sequence if and only if, for each $I \in \mathbb{D}$, there exists an interval $\omega_I = \omega_{|I|}$ so that $\operatorname{supp}\widehat{\Phi_I} \subseteq \omega_I$, $|\omega_I| \sim |I|^{-1} \sim \operatorname{dist}(0, \omega_I)$, and $0 \notin 5\omega_I$.

Definition 2.5. Let $\mathcal{I}, \mathcal{J} \subseteq \mathbb{D}$ be two families of dyadic intervals with lengths at most 1. Suppose that $(\phi_I^j)_{I\in\mathcal{I}}$ for $j = 1, 2, 3$ are three families of L^2-normalized bump functions such that the family $(\phi_I^2)_{I\in\mathcal{I}}$ is non-lacunary while the families $(\phi_I^j)_{I\in\mathcal{I}}$ for $j \neq 2$ are both lacunary, and $(\phi_J^j)_{J\in\mathcal{J}}$ for $j = 1, 2, 3$ are three families of L^2-normalized bump functions, where at least two of the three are lacunary.

We define as in [11] the discrete model operators T_1 and T_{1,k_0} for a positive integer k_0 by

$$T_1(f, g, h) = \sum_{I\in\mathcal{I}} \frac{1}{|I|^{1/2}} \langle f, \phi_I^1\rangle \langle B_I^1(g, h), \phi_I^2\rangle \phi_I^3, \tag{2.2}$$

where

$$B_I^1(g, h) = \sum_{\substack{J\in\mathcal{J}\\ |\omega_J^3|\le|w_I^2|}} \frac{1}{|J|^{1/2}} \langle g, \phi_J^1\rangle \langle h, \phi_J^2\rangle \phi_J^3; \tag{2.3}$$

$$T_{1,k_0}(f, g, h) = \sum_{I\in\mathcal{I}} \frac{1}{|I|^{1/2}} \langle f, \phi_I^1\rangle \langle B_{I,k_0}^1(g, h), \phi_I^2\rangle \phi_I^3, \tag{2.4}$$

where

$$B_{I,k_0}^1(g, h) = \sum_{\substack{J\in\mathcal{J}\\ 2^{k_0}|\omega_J^3|\sim|w_I^2|}} \frac{1}{|J|^{1/2}} \langle g, \phi_J^1\rangle \langle h, \phi_J^2\rangle \phi_J^3. \tag{2.5}$$

3. Reduction to a Localized Version

To prove the theorem, we proceed as follows. First, pick a sequence of smooth functions $(\varphi_n)_n \in \mathbb{Z}$ such that $\operatorname{supp}\varphi_n \subseteq [n-1, n+1]$ and $\sum_{n\in\mathbb{Z}} \varphi_n = 1$. Then, we can decompose the operator T_{ab} in (1.5) as

$$T_{ab} = \sum_{n\in\mathbb{Z}} T_{ab}^n,$$

where

$$T_{ab}^n(f, g, h)(x) := T_{ab}(f, g, h)(x)\varphi_n(x).$$

Suppose we can prove the estimate

$$\|T_{ab}^n(f, g, h)\|_r \lesssim \|f\tilde{\chi}_{I_n}\|_{p_1} \|g\tilde{\chi}_{I_n}\|_{p_2} \|h\tilde{\chi}_{I_n}\|_{p_3}, \tag{3.1}$$

where I_n is the interval $[n, n+1]$, and $\tilde{\chi}_{I_n}$ is defined as in (2.1).

Then, our main Theorem 1.3 can be proved by the following estimate:

$$\begin{aligned}
\|T_{ab}(f,g,h)\|_r &\lesssim \Big(\sum_{n\in\mathbb{Z}} \|T_{ab}^n(f,g,h)\|_r^r\Big)^{1/r} \\
&\lesssim \Big(\sum_{n\in\mathbb{Z}} \|f\tilde{\chi}_{I_n}\|_{p_1}^r \, \|g\tilde{\chi}_{I_n}\|_{p_2}^r \, \|h\tilde{\chi}_{I_n}\|_{p_3}^r\Big)^{1/r} \\
&\lesssim \Big(\sum_{n\in\mathbb{Z}} \|f\tilde{\chi}_{I_n}\|_{p_1}^{p_1}\Big)^{1/p_1} \Big(\sum_{n\in\mathbb{Z}} \|g\tilde{\chi}_{I_n}\|_{p_2}^{p_2}\Big)^{1/p_2} \Big(\sum_{n\in\mathbb{Z}} \|h\tilde{\chi}_{I_n}\|_{p_3}^{p_3}\Big)^{1/p_3} \\
&\lesssim \|f\|_{p_1} \, \|g\|_{p_2} \, \|h\|_{p_3}.
\end{aligned}$$

Thus, we only need to prove (3.1).

Consider that for a fixed $n_0 \in \mathbb{Z}$, we have

$$\begin{aligned}
T_{ab}^{n_0}(f,g,h)(x) = \int_{\mathbb{R}^3} & a(x,\xi,\eta)\tilde{\varphi}_{n_0}(x) b(x,\eta,\zeta)\tilde{\varphi}_{n_0}(x)\varphi_{n_0}(x) \\
&\times \hat{f}(\xi)\hat{g}(\eta)\hat{h}(\zeta) e^{2\pi i x(\xi+\eta+\zeta)} \, \mathrm{d}\xi \, \mathrm{d}\eta \, \mathrm{d}\zeta,
\end{aligned}$$

where $\tilde{\varphi}_{n_0}$ is a smooth function supported on the interval $[n_0 - 2, n_0 + 2]$, and equals 1 on the support of φ_{n_0}. Then, we rewrite the symbols $a(x,\xi,\eta)\tilde{\varphi}_{n_0}(x)$ and $b(x,\eta,\zeta)\tilde{\varphi}_{n_0}(x)$ by using Fourier series with respect to the x variable:

$$\begin{aligned}
a(x,\xi,\eta)\tilde{\varphi}_{n_0}(x) &= \sum_{\ell_1\in\mathbb{Z}} a_{\ell_1}(\xi,\eta) e^{2\pi i x \ell_1}, \\
b(x,\eta,\zeta)\tilde{\varphi}_{n_0}(x) &= \sum_{\ell_2\in\mathbb{Z}} b_{\ell_2}(\eta,\zeta) e^{2\pi i x \ell_2},
\end{aligned}$$

where by taking advantage of conditions (1.4), we can have

$$\begin{aligned}
|\partial_{\xi,\eta}^{\alpha,\beta} a_{\ell_1}(\xi,\eta)| &\lesssim \frac{1}{(1+|\ell_1|)^M} \frac{1}{(1+|\xi|+|\eta|)^{\alpha+\beta}}, \\
|\partial_{\eta,\zeta}^{\beta,\gamma} b_{\ell_2}(\eta,\zeta)| &\lesssim \frac{1}{(1+|\ell_2|)^M} \frac{1}{(1+|\eta|+|\zeta|)^{\beta+\gamma}},
\end{aligned}$$

for a large number M and sufficiently many indices α, β, γ. Note the decay in ℓ_1, ℓ_2 means we only need to consider the case for $\ell_1, \ell_2 = 0$, which is given by

$$\begin{aligned}
&T_{ab}^{n_0,0,0}(f,g,h)(x) \\
&\qquad = \Big(\int_{\mathbb{R}^3} a_0(\xi,\eta) b_0(\eta,\zeta) \hat{f}(\xi)\hat{g}(\eta)\hat{h}(\zeta) e^{2\pi i x(\xi+\eta+\zeta)} \, \mathrm{d}\xi \, \mathrm{d}\eta \, \mathrm{d}\zeta\Big) \varphi_{n_0}(x),
\end{aligned}$$

where symbols a_0, b_0 satisfy the following conditions:

$$|\partial_{\xi,\eta}^{\alpha,\beta} a_0(\xi,\eta)| \lesssim \frac{1}{(1+|\xi|+|\eta|)^{\alpha+\beta}}, \qquad |\partial_{\eta,\zeta}^{\beta,\gamma} b_0(\eta,\zeta)| \lesssim \frac{1}{(1+|\eta|+|\zeta|)^{\beta+\gamma}}. \tag{3.2}$$

Using the translation invariance, we only need to prove the following localized result for $n_0 = 0$.

Theorem 3.1. *The operator*

$$T_{ab}^{0,0}(f,g,h)(x) = \Big(\int_{\mathbb{R}^3} a_0(\xi,\eta) b_0(\eta,\zeta) \hat{f}(\xi)\hat{g}(\eta)\hat{h}(\zeta) e^{2\pi i x(\xi+\eta+\zeta)}\, \mathrm{d}\xi\, \mathrm{d}\eta\, \mathrm{d}\zeta \Big) \varphi_0(x) \tag{3.3}$$

has the boundedness property

$$\|T_{ab}^{0,0}(f,g,h)\|_r \lesssim \|f\tilde{\chi}_{I_0}\|_{p_1} \|g\tilde{\chi}_{I_0}\|_{p_2} \|h\tilde{\chi}_{I_0}\|_{p_3}$$

for $1 < p_1, p_2, p_3 < \infty$ *and* $1/p_1 + 1/p_2 + 1/p_3 = 1/r$, *where* φ_0 *is a smooth function supported within* $[-1,1]$, *and* a_0, b_0 *satisfy the conditions* (3.2).

In addition, this estimate also holds for the cases where at most one $p_i = \infty$ *for* $i = 1,2,3$ *or* $p_1, p_3 = \infty$, $1 < p_2 < \infty$.

We are now ready to do some decompositions to the operator in (3.3).

4. Reduction of the Localized Operator

In this section, we will mainly show that the problem can be reduced to some operators or paraproducts with which we are familiar.

Let $\varphi \in S(\mathbb{R})$ be a Schwartz function such that $\operatorname{supp}\hat{\varphi} \subseteq [-1,1]$ and $\hat{\varphi}(\xi) = 1$ on $[-\frac{1}{2}, \frac{1}{2}]$. Define $\psi \in S(\mathbb{R})$ to be the Schwartz function satisfying

$$\hat{\psi}(\xi) := \hat{\varphi}(\xi/2) - \hat{\varphi}(\xi),$$

and let $\widehat{\psi_k}(\cdot) = \hat{\psi}(\cdot/2^k)$ and $\widehat{\psi_{-1}}(\cdot) = \hat{\varphi}(\cdot)$. Note that $1 = \sum_{k\geq -1} \widehat{\psi_k}$, where $\operatorname{supp}\hat{\psi} \subseteq [-2^{k+1}, -2^{k-1}] \cup [2^{k-1}, 2^{k+1}]$ for $k \geq 0$. Then, for any $m, n \in \mathbb{Z}$, we use $m \gg n$ to denote $m - n > 100$, and $m \simeq n$ to denote $|m-n| \leq 100$. Consider the decomposition

$$1(\xi,\eta,\zeta) = \Big(\sum_{k_1' \geq -1} \sum_{k_1'' \geq -1} \widehat{\psi_{k_1'}}(\xi)\widehat{\psi_{k_1''}}(\eta) \Big) \times \Big(\sum_{k_2' \geq -1} \sum_{k_2'' \geq -1} \widehat{\psi_{k_2'}}(\eta)\widehat{\psi_{k_2''}}(\zeta) \Big). \tag{4.1}$$

Without loss of generality, we consider

$$
\begin{aligned}
(4.2)\qquad \Big(\sum_{k_1'\geq -1}\sum_{k_1''\geq -1}\widehat{\psi_{k_1'}}(\xi)\widehat{\psi_{k_1''}}(\eta)\Big)
&= \sum_{k_1'\gg k_1''\geq -1}\widehat{\psi_{k_1'}}(\xi)\widehat{\psi_{k_1''}}(\eta) + \sum_{-1\leq k_1'\ll k_1''}\widehat{\psi_{k_1'}}(\xi)\widehat{\psi_{k_1''}}(\eta)\\
&\quad + \sum_{\substack{k_1'\simeq k_1''\\ k_1'>100\\ \text{or } k_1''>100}}\widehat{\psi_{k_1'}}(\xi)\widehat{\psi_{k_1''}}(\eta) + \sum_{\substack{k_1'\simeq k_1''\\ k_1',k_1''\leq 100}}\widehat{\psi_{k_1'}}(\xi)\widehat{\psi_{k_1''}}(\eta)\\
&:= A+B+C+D,
\end{aligned}
$$

where the term D, containing a finite number of terms, can be written out specifically as $D = \hat{\varphi}(\xi)\hat{\varphi}(\eta) + \text{Others}$.

To estimate C, note that, in this case, actually both k_1' and k_1'' are at least 1. Suppose $k_1' > 100$; then, we have

$$
\sum_{\substack{k_1'\simeq k_1''\\ k_1'>100}}\widehat{\psi_{k_1'}}(\xi)\widehat{\psi_{k_1''}}(\eta) = \sum_{k>100}\widehat{\psi_k}(\xi)\widehat{\tilde{\psi}_k}(\eta),
$$

and then

$$
C = \sum_{k>100}\widehat{\psi_k}(\xi)\widehat{\tilde{\psi}_k}(\eta) + \sum_{k>100}\widehat{\tilde{\psi}_k}(\xi)\widehat{\psi_k}(\eta),
$$

where

$$
\operatorname{supp}\widehat{\tilde{\psi}_k} \subseteq [-2^{k+101},-2^{k-101}]\cup[2^{k-101},2^{k+101}].
$$

Estimates for A and B are quite similar:

$$
\begin{aligned}
A &= \sum_{k_1'}\Big(\sum_{-1\leq k_1''<k_1'-100}\widehat{\psi_{k_1''}}(\eta)\Big)\widehat{\psi_{k_1'}}(\xi) = \sum_{k\geq 100}\widehat{\psi_k}(\xi)\widehat{\varphi_k}(\eta),\\
B &= \sum_{k_1''}\Big(\sum_{-1\leq k_1'<k_1''-100}\widehat{\psi_{k_1'}}(\xi)\Big)\widehat{\psi_{k_1''}}(\eta) = \sum_{k\geq 100}\widehat{\varphi_k}(\xi)\widehat{\psi_k}(\eta),
\end{aligned}
$$

where φ_k is a Schwartz function with $\operatorname{supp}\widehat{\varphi_k}\subseteq[-2^{k-100},2^{k+100}]$. For $k\geq 0$, we refer to the families like $(\psi_k)_k$ as Ψ-*type* functions, whose Fourier transforms have almost disjoint supports for different scales; and we refer to the families like $(\varphi_k)_k$ as Φ-*type* functions, whose Fourier transforms have overlapping supports for different scales. In the rest of this paper, for convenience we do not distinguish between ψ_k and $\tilde{\psi}_k$, since they are of the same type and have comparative scales for the supports of their Fourier transforms, and we always use ψ_k to represent such Ψ-type functions. Similarly, we always use φ_k to represent a Φ-*type* function. With such notation, we can write (4.2) as

$$\Big(\sum_{k_1'\geq -1}\widehat{\psi_{k_1'}}(\xi)\Big)\Big(\sum_{k_1''\geq -1}\widehat{\psi_{k_1''}}(\eta)\Big) = \sum_{k\geq 100}\widehat{\psi_k}(\xi)\widehat{\varphi_k}(\eta)+\sum_{k\geq 100}\widehat{\varphi_k}(\xi)\widehat{\psi_k}(\eta)+\sum_{k>100}\widehat{\psi_k}(\xi)\widehat{\psi_k}(\eta)+D. \tag{4.3}$$

Later from the proof, we will see in (4.3) that the three summations work similarly, since what we really need is at least one lacunary family in each summation. Moreover, all the functions in D play the same role as $\hat{\varphi}(\xi)\hat{\varphi}(\eta)$, which means we actually can replace (4.3) by an equivalent version, which is

$$\sum_{k\geq 0}\widehat{\phi_k^1}(\xi)\widehat{\phi_k^2}(\eta)+\hat{\varphi}(\xi)\hat{\varphi}(\eta), \tag{4.4}$$

where at least one of the families $(\widehat{\phi_k^1}(\xi))_k$ and $(\widehat{\phi_k^2}(\xi))_k$ is Ψ-type.

Now, in dealing with (4.1), it is equivalent to consider

$$\begin{aligned}
1(\xi,\eta,\zeta) &= \Big(\sum_{k_1'\geq -1}\sum_{k_1''\geq -1}\widehat{\psi_{k_1'}}(\xi)\widehat{\psi_{k_1''}}(\eta)\Big)\Big(\sum_{k_2'\geq -1}\sum_{k_2''\geq -1}\widehat{\psi_{k_2'}}(\eta)\widehat{\psi_{k_2''}}(\zeta)\Big)\\
&\approx \Big(\sum_{k_1}\widehat{\phi_{k_1}^1}(\xi)\widehat{\phi_{k_1}^2}(\eta)+\hat{\varphi}(\xi)\hat{\varphi}(\eta)\Big)\Big(\sum_{k_2}\widehat{\phi_{k_2}^1}(\eta)\widehat{\phi_{k_2}^2}(\zeta)+\hat{\varphi}(\eta)\hat{\varphi}(\zeta)\Big)\\
&= \Big(\sum_{k_1}\widehat{\phi_{k_1}^1}(\xi)\widehat{\phi_{k_1}^2}(\eta)\sum_{k_2}\widehat{\phi_{k_2}^1}(\eta)\widehat{\phi_{k_2}^2}(\zeta)\Big)+\Big(\sum_{k_1}\widehat{\phi_{k_1}^1}(\xi)\widehat{\phi_{k_1}^2}(\eta)\Big)\hat{\varphi}(\eta)\hat{\varphi}(\zeta)\\
&\quad+\Big(\sum_{k_2}\widehat{\phi_{k_2}^1}(\eta)\widehat{\phi_{k_2}^2}(\zeta)\Big)\hat{\varphi}(\xi)\hat{\varphi}(\eta)+\hat{\varphi}(\xi)\hat{\varphi}(\eta)\hat{\varphi}(\eta)\hat{\varphi}(\zeta)\\
&:= E+F+G+H,
\end{aligned}$$

where, for convenience, the symbol $\approx$ is used to show the equivalence; we will simply treat $1(\xi,\eta,\zeta)=E+F+G+H$ in the rest of the paper.

Then, using the above and (3.3), we can decompose the localized operator as

$$\begin{aligned}
&T_{ab}^{0,0}(f,g,h)(x)\\
&\quad=\Big(\int_{\mathbb{R}^3}a_0(\xi,\eta)b_0(\eta,\zeta)\hat{f}(\xi)\hat{g}(\eta)\hat{h}(\zeta)e^{2\pi ix(\xi+\eta+\zeta)}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta\Big)\varphi_0(x)\\
&\quad=\Big(\int_{\mathbb{R}^3}a_0(\xi,\eta)b_0(\eta,\zeta)(E+F+G+H)\hat{f}(\xi)\hat{g}(\eta)\hat{h}(\zeta)\\
&\qquad\quad\times e^{2\pi ix(\xi+\eta+\zeta)}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta\Big)\varphi_0(x)\\
&\quad:= T_{ab}^{E,0,0}+T_{ab}^{F,0,0}+T_{ab}^{G,0,0}+T_{ab}^{H,0,0}.
\end{aligned} \tag{4.5}$$

4.1. Estimates for $T_{ab}^{H,0,0}$. Recall

$$T_{ab}^{H,0,0}(f,g,h)(x) = \Big(\int_{\mathbb{R}^3} a_0(\xi,\eta)b_0(\eta,\zeta)\hat{\varphi}(\xi)\hat{\varphi}(\eta)\hat{\varphi}(\eta)\hat{\varphi}(\zeta) \times \hat{f}(\xi)\hat{g}(\eta)\hat{h}(\zeta)e^{2\pi i x(\xi+\eta+\zeta)}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta\Big)\varphi_0(x),$$

and note that $m_H(\xi,\eta,\zeta) := a_0(\xi,\eta)b_0(\eta,\zeta)\hat{\varphi}(\xi)\hat{\varphi}(\eta)\hat{\varphi}(\eta)\hat{\phi}(\zeta)$ satisfies the condition

$$|\partial_\xi^\alpha\,\partial_\eta^\beta\,\partial_\zeta^\gamma m_H(\xi,\eta,\zeta)| \lesssim \frac{1}{(1+|\xi|+|\eta|+|\zeta|)^{\alpha+\beta+\gamma}}$$

for sufficiently many indices α,β,γ. Then, our desired localized estimate follows from Theorem 1.1, since we find that the operator $T_{ab}^{H,0,0}$ is just the localized operator used in the proof of Theorem 1.1 (see [6, 12]).

4.2. Estimates for $T_{ab}^{F,0,0} + T_{ab}^{G,0,0}$. Recall

$$F = \Big(\sum_{k_1}\widehat{\phi_{k_1}^1}(\xi)\widehat{\phi_{k_1}^2}(\eta)\Big)\hat{\varphi}(\eta)\hat{\varphi}(\zeta),$$

where at least one of the families $(\widehat{\phi_{k_1}^1})_{k_1}$ and $(\widehat{\phi_{k_1}^2})_{k_1}$ is Ψ-type.

When $(\widehat{\phi_{k_1}^2})_{k_1}$ is Ψ-type, note that, to make $\sum_{k_1}\widehat{\phi_{k_1}^2}(\eta)\hat{\varphi}(\eta) \neq 0$, k_1 will have an upper bound for the summation (say $k_1 \leq 100$). Then, the desired estimate under this situation can be done in the same way as in $T_{ab}^{H,0,0}$, since only a finite number of terms are involved.

When $(\widehat{\phi_{k_1}^2})_{k_1}$ is Φ-type, we must have $(\widehat{\phi_{k_1}^1})_{k_1}$ is Ψ-type. Recall

$$T_{ab}^{F,0,0}(f,g,h)(x) = \Big(\sum_{k_1}\int_{\mathbb{R}^3} a_0(\xi,\eta)\widehat{\phi_{k_1}^1}(\xi)\widehat{\phi_{k_1}^2}(\eta)b_0(\eta,\zeta)\hat{\varphi}(\eta)\hat{\varphi}(\zeta) \times \hat{f}(\xi)\hat{g}(\eta)\hat{h}(\zeta)e^{2\pi i x(\xi+\eta+\zeta)}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta\Big)\varphi_0(x). \tag{4.6}$$

Then, we can use Fourier series to write

$$a_0(\xi,\eta)\widehat{\phi_{k_1}^1}(\xi)\widehat{\phi_{k_1}^2}(\eta) = \sum_{n_1,n_2\in\mathbb{Z}} C_{n_1,n_2}^{k_1} e^{2\pi i n_1\xi/2^{k_1}}e^{2\pi i n_2\eta/2^{k_1}},$$

where the Fourier coefficients $C_{n_1,n_2}^{k_1}$ are given by

$$C_{n_1,n_2}^{k_1} = \frac{1}{2^{2k_1}}\int_{\mathbb{R}^2} a_0(\xi,\eta)\widehat{\phi_{k_1}^1}(\xi)\widehat{\phi_{k_1}^2}(\eta)e^{-2\pi i n_1\xi/2^{k_1}}e^{-2\pi i n_2\eta/2^{k_1}}.$$

By the decay condition (3.2) and the advantage that $(\widehat{\phi^1_{k_1}})_{k_1}$ is Ψ-type, we can get the following by integration by parts sufficiently many times:

$$|C^{k_1}_{n_1,n_2}| \lesssim \frac{1}{(1+|n_1|+|n_2|)^M}.$$

Note that, by the decay in n_1, n_2, we need only consider the case when $n_1, n_2 = 0$ (see [12] and the proof in Section 5 for more details), and similar things can be done for $b_0(\eta,\zeta)\hat{\varphi}(\eta)\hat{\varphi}(\zeta)$. Then, we can use Hölder's inequality and take advantage of the fact that φ is a bump function adapted to $[-1,1]$ to prove the localized result for (4.6), that is,

$$\begin{aligned}
&\Big\|\Big(\sum_{k_1}\int_{\mathbb{R}^3}\widehat{\phi^1_{k_1}}(\xi)\widehat{\phi^2_{k_1}}(\eta)\hat{\varphi}(\eta)\hat{\varphi}(\zeta)\hat{f}(\xi)\hat{g}(\eta)\hat{h}(\zeta)e^{2\pi i x(\xi+\eta+\zeta)}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta\Big)\varphi_0(x)\Big\|_r \\
&\quad\approx \Big\|\Big(\sum_{k_1}\int_{\mathbb{R}^3}\widehat{\phi^1_{k_1}}(\xi)\hat{\varphi}(\eta)\hat{\varphi}(\zeta)\hat{f}(\xi)\hat{g}(\eta)\hat{h}(\zeta)e^{2\pi i x(\xi+\eta+\zeta)}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta\Big)\varphi_0(x)\Big\|_r \\
&\quad= \Big\|\Big(\sum_{k_1}\phi^1_{k_1}*f\Big)(x)\varphi_0(x)(\varphi*g)(x)\tilde{\varphi}_0(x)(\varphi*h)(x)\tilde{\varphi}_0(x)\Big\|_r \\
&\quad\lesssim \Big\|\Big(\sum_{k_1}\phi^1_{k_1}*f\Big)(x)\varphi_0(x)\Big\|_{p_1}\|(\varphi*g)(x)\tilde{\varphi}_0(x)\|_{p_2}\|(\varphi*h)(x)\tilde{\varphi}_0(x)\|_{p_3} \\
&\quad\lesssim \|f\tilde{\chi}_{I_0}\|_{p_1}\|g\tilde{\chi}_{I_0}\|_{p_2}\|h\tilde{\chi}_{I_0}\|_{p_3},
\end{aligned}$$

where we take $\tilde{\phi}_0$ to be 1 on $\operatorname{supp}\phi_0$ and supported in a slightly larger interval containing $\operatorname{supp}\phi_0$. The last inequality is true since $(\varphi_{k_1})_{k_1}$ is Ψ-type. Also, we can simply write $\sum_{k_1}\widehat{\phi^2_{k_1}}(\eta)\hat{\varphi}(\eta) = \hat{\varphi}(\eta)$ in the above, since k_1 is positive.

4.3. Estimates for $T^{E,0,0}_{ab}$

Recall

$$E = \Big(\sum_{k_1\geq 0}\widehat{\phi^1_{k_1}}(\xi)\widehat{\phi^2_{k_1}}(\eta)\Big)\Big(\sum_{k_2\geq 0}\widehat{\phi^1_{k_2}}(\eta)\widehat{\phi^2_{k_2}}(\zeta)\Big),$$

where at least one of the families $(\widehat{\phi^1_{k_1}})_{k_1}$ and $(\widehat{\phi^2_{k_1}})_{k_1}$ is Ψ-type, and at least one of the families $(\widehat{\phi^1_{k_2}})_{k_2}$ and $(\widehat{\phi^2_{k_2}})_{k_2}$ is Ψ-type.

Also, we consider the corresponding localized operator

$$\begin{aligned}
&T^{E,0,0}_{ab}(f,g,h)(x) \\
&\quad= \Big(\int_{\mathbb{R}^3}\Big(\sum_{k_1}\widehat{\phi^1_{k_1}}(\xi)\widehat{\phi^2_{k_1}}(\eta)\Big)a_0(\xi,\eta)\Big(\sum_{k_2}\widehat{\phi^1_{k_2}}(\eta)\widehat{\phi^2_{k_2}}(\zeta)b_0(\eta,\zeta)\Big) \\
&\qquad\qquad\times\hat{f}(\xi)\hat{g}(\eta)\hat{h}(\zeta)e^{2\pi i x(\xi+\eta+\zeta)}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta\Big)\varphi_0(x).
\end{aligned}$$

By using Fourier series as before, we only need to consider the following operator:

$$\Big(\int_{\mathbb{R}^3}\Big(\sum_{k_1}\widehat{\phi^1_{k_1}}(\xi)\widehat{\phi^2_{k_1}}(\eta)\Big)\Big(\sum_{k_2}\widehat{\phi^1_{k_2}}(\eta)\widehat{\phi^2_{k_2}}(\zeta)\Big) \\ \times \hat{f}(\xi)\hat{g}(\eta)\hat{h}(\zeta)e^{2\pi ix(\xi+\eta+\zeta)}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta\Big)\varphi_0(x).$$

As usual, we consider three cases of E,

$$E=\Big(\sum_{k_1\gg k_2}+\sum_{k_1\ll k_2}+\sum_{k_1\simeq k_2}\Big)(\widehat{\phi^1_{k_1}}(\xi)\widehat{\phi^2_{k_1}}(\eta))(\widehat{\phi^1_{k_2}}(\eta)\widehat{\phi^2_{k_2}}(\zeta)):=I+J+K,$$

and decompose

$$T^{E,0,0}_{ab}:=T^{I,0,0}_{ab}+T^{J,0,0}_{ab}+T^{K,0,0}_{ab}.$$

Note that K is actually a symbol in $BS^0_{1,0}$, since k is positive. That is,

$$T^{K,0,0}_{ab}(f,g,h)(x) \\ =\Big(\int_{\mathbb{R}^3}m_K(\xi,\eta,\zeta)\hat{f}(\xi)\hat{g}(\eta)\hat{h}(\zeta)e^{2\pi ix(\xi+\eta+\zeta)}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta\Big)\varphi_0(x),$$

where $m_K(\xi,\eta,\zeta)$ satisfies the condition as (3.2). Thus, the desired localized estimate follows from the proof of Theorem 1.1, just as $T^{H,0,0}_{ab}$.

Since $T^{I,0,0}_{ab}$ and $T^{J,0,0}_{ab}$ are similar, we define T^{I}_{ab} by the following equality:

$$T^I_{ab}(f,g,h)(x)\cdot\varphi_0(x)=:T^{I,0,0}_{ab}(f,g,h)(x) \\ =\Big(\int_{\mathbb{R}^3}\Big(\sum_{k_1}\widehat{\phi^1_{k_1}}(\xi)\widehat{\phi^2_{k_1}}(\eta)\Big)\Big(\sum_{k_2}\widehat{\phi^1_{k_2}}(\eta)\widehat{\phi^2_{k_2}}(\zeta)\Big) \\ \times \hat{f}(\xi)\hat{g}(\eta)\hat{h}(\zeta)e^{2\pi ix(\xi+\eta+\zeta)}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta\Big)\varphi_0(x). \tag{4.7}$$

From [11, 12], we know T^I_{ab} can be written by using paraproducts, which is the following lemma.

Lemma 4.1. *Define T^I_{ab} as in (4.7); then, we can write*

$$T^I_{ab}(f,g,h)(x)=T_1(f,g,h)(x)+\sum_{\ell=1}^{M-1}\sum_{k_0=100}^{\infty}(2^{-k_0})^{\ell}T_{\ell,k_0}(f,g,h)(x) \\ +\sum_{k_0=100}^{\infty}(2^{-k_0})^{M}T_{M,k_0}(f,g,h)(x),$$

where

$$T_1(f,g,h) = \sum_{I\in\mathcal{I}} \frac{1}{|I|^{1/2}} \langle f, \phi_I^1\rangle \langle B_I^1(g,h), \phi_I^2\rangle \phi_I^3$$

with

$$B_I^1(g,h) = \sum_{\substack{J\in\mathcal{J}\\ |\omega_J^3|\le|w_I^2|}} \frac{1}{|J|^{1/2}} \langle g, \phi_J^1\rangle \langle h, \phi_J^2\rangle \phi_J^3,$$

and

$$T_{\ell,k_0}(f,g,h) = \sum_{I\in\mathcal{I}} \frac{1}{|I|^{1/2}} \langle f, \phi_I^1\rangle \langle B_{I,k_0}^\ell(g,h), \phi_I^2\rangle \phi_I^3$$

with

$$B_{I,k_0}^\ell(g,h) = \sum_{\substack{J\in\mathcal{J}\\ 2^{k_0}|\omega_J^3|\sim|w_I^2|}} \frac{1}{|J|^{1/2}} \langle g, \phi_J^1\rangle \langle h, \phi_J^2\rangle \phi_J^3.$$

In the above, note the following:

(a) $T_1(f,g,h)$ *and* $B_I^1(g,h)$ *are defined as* (2.2) *and* (2.3) *in definition* (2.5).

(b) *For each* ℓ, $T_\ell(f,g,h)$ *and* $B_I^\ell(g,h)$ *are of the type* (2.4) *and* (2.5) *in definition* 2.5; ℓ *here is actually involved in the families* $(\phi_I^2)_I$ *and* $(\phi_J^2)_J$, *but it would not affect our proof since it does not change the types of those functions.*

(c) M *is a large positive integer, and the multiplier* $m_{M,k_0}(\xi,\eta,\zeta)$ *in* T_{M,k_0} *satisfies the condition*

$$|\partial_\xi^\alpha \partial_\eta^\beta \partial_\zeta^\gamma m_{M,k_0}(\xi,\eta,\zeta)| \lesssim (2^{k_0})^{\alpha+\beta+\gamma} \frac{1}{(1+|\xi|+|\eta|+|\zeta|)^{\alpha+\beta+\gamma}} \tag{4.8}$$

for sufficiently many indices α,β,γ.

(d) *All of the dyadic intervals in* T_1 *and* T_{ℓ,k_0} *have lengths at most* 1 *for all* $k_0 \ge 100$, $1\le\ell\le M-1$.

Proof. Here, we follow closely the work [11], where the Fourier expansions of $\widehat{\phi_{k_1}^2}(\eta)$ are used to get the desired forms of paraproducts. The only two statements we need to show are that all the dyadic intervals there have lengths at most one, and that one can obtain a decay number 1 in the denominator from (4.8). Actually, both of these follow from the fact that $k_1, k_2 \ge 0$. □

So far, we have reduced Theorem 3.1 to the estimate of the operator $T_{ab}^{I,0,0}$.

5. PROOF OF THEOREM 3.1

In this section, by using the decomposition in Lemma 4.1, we are able to prove the localized estimate for $T_{ab}^{I,0,0}$, which will complete the proof of Theorem 3.1.

5.1. Estimates for $\sum_{k_0=100}^{\infty}(2^{-k_0})^M T_{M,k_0}(f,g,h)(x)$**.** For this part, note that the condition (4.8) is almost the classical case. Then, by repeating the work in [6, 12], we will see that this condition can provide an estimate

$$\|T_{M,k_0}(f,g,h)\varphi_0(x)\|_r \lesssim C2^{10k_0}\|f\tilde{\chi}_{I_0}\|_{p_1}\|g\tilde{\chi}_{I_0}\|_{p_2}\|h\tilde{\chi}_{I_0}\|_{p_3},$$

which is accepted since we can choose M large enough.

5.2. Estimates for $T_1(f,g,h)(x)$**.** Using the fact that $|I| \le 1$, we can split

$$\begin{aligned} T_1(f,g,h)(x) &= \sum_{I\subseteq 5I_0}\frac{1}{|I|^{1/2}}\langle f,\phi_I^1\rangle\langle B_I^1(g,h),\phi_I^2\rangle\phi_I^3 \\ &\quad+\sum_{I\subseteq(5I_0)^c}\frac{1}{|I|^{1/2}}\langle f,\phi_I^1\rangle\langle B_I^1(g,h),\phi_I^2\rangle\phi_I^3 \\ &= I+II. \end{aligned} \tag{5.1}$$

For Part I, we do the decompositions

$$f=\sum_{n_1}f\chi_{I_{n_1}},\quad g=\sum_{n_2}g\chi_{I_{n_2}},\quad h=\sum_{n_3}h\chi_{I_{n_3}}$$

first, where $I_{n_i}=[n_i,n_i+1]$, $i=1,2,3$, $n_i\in\mathbb{Z}$. Then, we can write

$$T_1(f,g,h)(x)=\sum_{n_1}\sum_{n_2}\sum_{n_3}T_1(f\chi_{I_{n_1}},g\chi_{I_{n_2}},h\chi_{I_{n_3}})(x).$$

When $|n_1|,|n_2|,|n_3|\le 10$, the desired estimate follows from Theorem 1.2:

$$\begin{aligned} &\Big\|\sum_{|n_1|\le 10}\sum_{|n_2|\le 10}\sum_{|n_3|\le 10}T_1(f\chi_{I_{n_1}},g\chi_{I_{n_2}},h\chi_{I_{n_3}})(x)\cdot\varphi_0(x)\Big\|_r \\ &\quad\lesssim\Big\|\sum_{|n_1|\le 10}f\chi_{I_{n_1}}\Big\|_{p_1}\Big\|\sum_{|n_2|\le 10}g\chi_{I_{n_2}}\Big\|_{p_2}\Big\|\sum_{|n_3|\le 10}h\chi_{I_{n_3}}\Big\|_{p_3} \\ &\quad\lesssim\|f\tilde{\chi}_{I_0}\|_{p_1}\|g\tilde{\chi}_{I_0}\|_{p_2}\|h\tilde{\chi}_{I_0}\|_{p_3}, \end{aligned}$$

where the last inequality holds from $\chi_{[-11,11]}\lesssim\tilde{\chi}_{I_0}(x)$.

When $|n_1|, |n_2|, |n_3| > 10$, we write

$$\begin{aligned}
&\|T_1(f\chi_{I_{n_1}}, g\chi_{I_{n_2}}, h\chi_{I_{n_3}})(x)\cdot\varphi_0(x)\|_r \\
&\quad= \Bigg\|\sum_{I\in\mathcal{I}}\sum_{\substack{J\in\mathcal{J}\\ |\omega_J^3|\le|\omega_I^2|}} \frac{1}{|I|^{1/2}}\frac{1}{|J|^{1/2}}\langle f\chi_{I_{n_1}}, \phi_I^1\rangle \\
&\qquad\times\langle g\chi_{I_{n_2}}, \phi_J^1\rangle\langle h\chi_{I_{n_3}}, \phi_J^3\rangle\langle\phi_I^2, \phi_J^3\rangle\phi_I^3(x)\varphi_0(x)\Bigg\|_r.
\end{aligned}$$

Then, we use Hölder's inequality to get

$$\begin{aligned}
&\Bigg\|\frac{1}{|I|^{1/2}}\frac{1}{|J|^{1/2}}\langle f\chi_{I_{n_1}}, \phi_I^1\rangle\langle g\chi_{I_{n_2}}, \phi_J^1\rangle \\
&\qquad\times\langle h\chi_{I_{n_3}}, \phi_J^3\rangle\langle\phi_I^2, \phi_J^3\rangle\phi_I^3(x)\varphi_0(x)\Bigg\|_r \\
&\lesssim \frac{1}{|I|^2}\frac{1}{|J|^2}\left(1+\frac{\operatorname{dist}(I_{n_1}, I)}{|I|}\right)^{-M_1}(\|f\chi_{I_{n_1}}\|_{p_1}|I|^{(p_1-1)/p_1}) \\
&\qquad\times\left(1+\frac{\operatorname{dist}(I_{n_2}, J)}{|J|}\right)^{-N_1}(\|g\chi_{I_{n_2}}\|_{p_2}|J|^{(p_2-1)/p_2}) \\
&\qquad\times\left(1+\frac{\operatorname{dist}(I_{n_3}, J)}{|J|}\right)^{-N_2}(\|h\chi_{I_{n_3}}\|_{p_3}|J|^{(p_3-1)/p_3})|I|^{1/r} \\
&\qquad\times\int_{\mathbb{R}}\left(1+\frac{\operatorname{dist}(x, I)}{|I|}\right)^{-M_2}\left(1+\frac{\operatorname{dist}(x, J)}{|J|}\right)^{-N_3}\,dx \\
&\lesssim \frac{1}{|I|}\left(\frac{|I|}{|J|}\right)^{1/p_2+1/p_3}\left(1+\frac{\operatorname{dist}(I_{n_1}, I)}{|I|}\right)^{-M_1} \\
&\qquad\times\left(1+\frac{\operatorname{dist}(I_{n_2}, J)}{|J|}\right)^{-N_1}\left(1+\frac{\operatorname{dist}(I_{n_3}, J)}{|J|}\right)^{-N_2} \\
&\qquad\times\int_{\mathbb{R}}\left(1+\frac{\operatorname{dist}(x, I)}{|I|}\right)^{-M_2}\left(1+\frac{\operatorname{dist}(x, J)}{|J|}\right)^{-N_3}\,dx \\
&\qquad\times\|f\chi_{I_{n_1}}\|_{p_1}\|g\chi_{I_{n_2}}\|_{p_2}\|h\chi_{I_{n_3}}\|_{p_3},
\end{aligned} \tag{5.2}$$

where M_j, N_j are sufficiently large integers, and ϕ_I^j, ϕ_J^j are L^2-normalized bump functions adapted to I, J for $j = 1, 2, 3$.

We first consider the case when $\operatorname{dist}(I, J) \le 3$. Recall we have the restriction that $|\omega_J^3| \le |\omega_I^2|$, which implies that $|I|/|J| \lesssim 1$. By using the subadditivity of $\|\cdot\|_r^r$, we have that

$$
\begin{aligned}
&\|T_1(f\chi_{I_{n_1}}, g\chi_{I_{n_2}}, h\chi_{I_{n_3}})(x)\cdot\varphi_0(x)\|_r^r \lesssim \\
&\quad\lesssim \sum_{i,j\geq 0}\ \sum_{\substack{I\subseteq 5I_0,\, J\subseteq 9I_0\\ |I|=2^{-i},\, |J|=2^{-j}}} \Bigg(\frac{1}{|I|}\left(1+\frac{\operatorname{dist}(I_{n_1},I)}{|I|}\right)^{-M_1} \\
&\qquad\times\left(1+\frac{\operatorname{dist}(I_{n_2},J)}{|J|}\right)^{-N_1}\left(1+\frac{\operatorname{dist}(I_{n_3},J)}{|J|}\right)^{-N_2} \\
&\qquad\times\int_{\mathbb{R}}\left(1+\frac{\operatorname{dist}(x,I)}{|I|}\right)^{-M_2}\left(1+\frac{\operatorname{dist}(x,J)}{|J|}\right)^{-N_3}\mathrm{d}x \\
&\qquad\times\|f\chi_{I_{n_1}}\|_{p_1}\|g\chi_{I_{n_2}}\|_{p_2}\|h\chi_{I_{n_3}}\|_{p_3}\Bigg)^r \\
&\quad\lesssim \sum_{i,j\geq 0}\ \sum_{\substack{I\subseteq 5I_0,\, J\subseteq 9I_0\\ |I|=2^{-i},\, |J|=2^{-j}}} \Big(2^i(1+2^i(|n_1|-6))^{-M_1}(1+2^j(|n_2|-9))^{-N_1} \\
&\qquad\times(1+2^j(|n_3|-9))^{-N_2}(\|f\chi_{I_{n_1}}\|_{p_1}\|g\chi_{I_{n_2}}\|_{p_2}\|h\chi_{I_{n_3}}\|_{p_3})\Big)^r \\
&\quad\lesssim \Big((|n_1|-6)^{-M_1}(|n_2|-9)^{-N_1}(|n_3|-9)^{-N_2} \\
&\qquad\times\|f\chi_{I_{n_1}}\|_{p_1}\|g\chi_{I_{n_2}}\|_{p_2}\|h\chi_{I_{n_3}}\|_{p_3}\Big)^r.
\end{aligned}
$$

Observe that, for large enough integers M_1, N_1, N_2, we have

$$
\begin{aligned}
\chi_{I_{n_1}}(|n_1|-6)^{-M_1/2} &\lesssim \tilde{\chi}_{I_0}, \\
\chi_{I_{n_2}}(|n_2|-9)^{-N_1/2} &\lesssim \tilde{\chi}_{I_0}, \\
\chi_{I_{n_3}}(|n_3|-9)^{-N_2/2} &\lesssim \tilde{\chi}_{I_0}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\Big\|\sum_{|n_1|>10}\sum_{|n_2|>10}\sum_{|n_3|>10} T_1(f\chi_{I_{n_1}}, g\chi_{I_{n_2}}, h\chi_{I_{n_3}})(x)\cdot\varphi_0(x)\Big\|_r^r \\
&\quad\lesssim \sum_{|n_1|>10}\sum_{|n_2|>10}\sum_{|n_3|>10}\Big((|n_1|-6)^{-M_1}(|n_2|-9)^{-N_1}(|n_3|-9)^{-N_2} \\
&\qquad\times\|f\chi_{I_{n_1}}\|_{p_1}\|g\chi_{I_{n_2}}\|_{p_2}\|h\chi_{I_{n_3}}\|_{p_3}\Big)^r \\
&\quad\lesssim \sum_{|n_1|>10}\sum_{|n_2|>10}\sum_{|n_3|>10}\Big((|n_1|-6)^{-M_1/2}(|n_2|-9)^{-N_1/2}(|n_3|-9)^{-N_2/2} \\
&\qquad\times\|f\tilde{\chi}_{I_0}\|_{p_1}\|g\tilde{\chi}_{I_0}\|_{p_2}\|h\tilde{\chi}_{I_0}\|_{p_3}\Big)^r \\
&\quad\lesssim (\|f\tilde{\chi}_{I_0}\|_{p_1}\|g\tilde{\chi}_{I_0}\|_{p_2}\|h\tilde{\chi}_{I_0}\|_{p_3})^r.
\end{aligned}
$$

For the other possibility, that is, when $\operatorname{dist}(I,J) > 3$, we consider whether J is close to I_{n_2} or I_{n_3}. Without loss of generality, we assume $\operatorname{dist}(J, I_{n_2}) \leq 2$,

$\operatorname{dist}(J, I_{n_3}) > 2$, and other cases will follow in the similar way. Using the notation $J_m = [m, m+1]$, $m \in \mathbb{Z}$, and (5.2), we can get

$$\begin{aligned}
&\|T_1(f\chi_{I_{n_1}}, g\chi_{I_{n_2}}, h\chi_{I_{n_3}})(x)\cdot\varphi_0(x)\|_r^r \\
&\lesssim \sum_{i,j\ge 0}\sum_{\substack{I\subseteq 5I_0\\ |I|=2^{-i}}}\sum_{|m|>3}\sum_{\substack{J\subseteq J_m,\,|J|=2^{-j}\\ \operatorname{dist}(J,I_{n_2})\le 2\\ \operatorname{dist}(J,I_{n_3})>2}}\Bigg(\frac{1}{|I|}\Big(1+\frac{\operatorname{dist}(I_{n_1},I)}{|I|}\Big)^{-M_1} \\
&\quad\times\Big(1+\frac{\operatorname{dist}(I_{n_2},J)}{|J|}\Big)^{-N_1}\Big(1+\frac{\operatorname{dist}(I_{n_3},J)}{|J|}\Big)^{-N_2} \\
&\quad\times\int_{\mathbb{R}}\Big(1+\frac{\operatorname{dist}(x,I)}{|I|}\Big)^{-M_2}\Big(1+\frac{\operatorname{dist}(x,J)}{|J|}\Big)^{-N_3}\mathrm{d}x \\
&\quad\times\|f\chi_{I_{n_1}}\|_{p_1}\|g\chi_{I_{n_2}}\|_{p_2}\|h\chi_{I_{n_3}}\|_{p_3}\Bigg)^r \\
&\lesssim \sum_{i,j\ge 0}\sum_{\substack{I\subseteq 5I_0\\ |I|=2^{-i}}}\sum_{|m|>3}\sum_{\substack{J\subseteq J_m |J|=2^{-j}\\ \operatorname{dist}(J,I_{n_2})\le 2\\ \operatorname{dist}(J,I_{n_3})>2}}\Big(2^i(1+2^i(|n_1|-6)^{-M_1}) \\
&\quad\times(1+2^j(|m-n_3|))^{-N_2}|m|^{-N_0}\|f\chi_{I_{n_1}}\|_{p_1}\|g\chi_{I_{n_2}}\|_{p_2}\|h\chi_{I_{n_3}}\|_{p_3}\Big)^r \\
&\lesssim \sum_{i,j\ge 0}\sum_{\substack{I\subseteq 5I_0\\ |I|=2^{-i}}}\sum_{|m|>3}\sum_{\substack{J\subseteq J_m |J|=2^{-j}\\ \operatorname{dist}(J,I_{n_2})\le 2\\ \operatorname{dist}(J,I_{n_3})>2}}\Big(2^i(1+2^i(|n_1|-6))^{-M_1} \\
&\quad\times(1+2^j(|m-n_3|))^{-N_2}|n_2|^{-N_0}\|f\chi_{I_{n_1}}\|_{p_1}\|g\chi_{I_{n_2}}\|_{p_2}\|h\chi_{I_{n_3}}\|_{p_3}\Big)^r,
\end{aligned}$$

where $N_0 = \min\{M_2, N_3\}$ is sufficiently large and we use $m \sim n_2$.

Now, we take the sum over n_1, n_2, n_3, and get

$$\begin{aligned}
&\Big\|\sum_{|n_1|>10}\sum_{|n_2|>10}\sum_{|n_3|>10}T_1(f\chi_{I_{n_1}}, g\chi_{I_{n_2}}, h\chi_{I_{n_3}})(x)\cdot\varphi_0(x)\Big\|_r^r \\
&\quad\lesssim \sum_{|n_1|>10}\sum_{|n_2|>10}\sum_{|n_3|>10}\Big((|n_1|-6)^{-M_1/2}|n_2|^{-N_0}(|n_3|-3)^{-N_2/2} \\
&\qquad\times\|f\chi_{I_{n_1}}\|_{p_1}\|g\chi_{I_{n_2}}\|_{p_2}\|h\chi_{I_{n_3}}\|_{p_3}\Big)^r \\
&\quad\lesssim \sum_{|n_1|>10}\sum_{|n_2|>10}\sum_{|n_3|>10}\Big((|n_1|-6)^{-M_1/4}|n_2|^{-N_0/2}(|n_3|-3)^{-N_2/4} \\
&\qquad\times\|f\tilde{\chi}_{I_0}\|_{p_1}\|g\tilde{\chi}_{I_0}\|_{p_2}\|h\tilde{\chi}_{I_0}\|_{p_3}\Big)^r \\
&\quad\lesssim (\|f\tilde{\chi}_{I_0}\|_{p_1}\|g\tilde{\chi}_{I_0}\|_{p_2}\|h\tilde{\chi}_{I_0}\|_{p_3})^r.
\end{aligned}$$

Other possible choices of n_1, n_2, and n_3 will be treated in different ways. Among these cases, when $|n_1| > 10$, we can do much as we did above, to get our desired estimate directly, by considering whether J is close to I or not. Note in this case we are free to take summation over J, since we have a decay on i and $j \leq i$.

When, however, $|n_1| \leq 10$ (say, $|n_1|, |n_2| \leq 10$, $|n_3| \geq 10$), the situation is different. In this situation, the term $(1 + \operatorname{dist}(I_{n_1}, I)/|I|)^{-M_1}$ in (5.2) would not give us a decay factor, which means we will have trouble when taking the summation over dyadic intervals I. Actually, the decay factors from other terms are with respect to j, which cannot help since $i > j$. Recall our desired estimate in this case:

$$\begin{aligned}\Big\| \sum_{|n_1|, |n_2| \leq 10} \sum_{|n_3| > 10} T_1(f\chi_{I_{n_1}}, g\chi_{I_{n_2}}, h\chi_{I_{n_3}})(x) \cdot \varphi_0(x) \Big\|_r &\lesssim \\ \lesssim \|f\tilde{\chi}_{I_0}\|_{p_1} \|g\tilde{\chi}_{I_0}\|_{p_2} \|h\tilde{\chi}_{I_0}\|_{p_3}.\end{aligned}$$

Suppose that from the proof of Theorem 1.2 (see [11, 12]) we can get an additional decay with respect to n_3, such as $1/|n_3|^M$ for sufficiently positive integer M; then, we only need to apply Theorem 1.2 to get

$$\begin{aligned}&\Big\| \sum_{|n_1|, |n_2| \leq 10} \sum_{|n_3| > 10} T_1(f\chi_{I_{n_1}}, g\chi_{I_{n_2}}, h\chi_{I_{n_3}})(x) \cdot \varphi_0(x) \Big\|_r \\ &\quad \lesssim \frac{1}{|n_3|^M} \|f\chi_{I_{n_1}}\|_{p_1} \|g\chi_{I_{n_2}}\|_{p_2} \|h\chi_{I_{n_3}}\|_{p_3} \\ &\quad \lesssim \|f\tilde{\chi}_{I_0}\|_{p_1} \|g\tilde{\chi}_{I_0}\|_{p_2} \|h\tilde{\chi}_{I_0}\|_{p_3}.\end{aligned}$$

Now, we will see how to get such a decay $1/|n_3|^M$. As before, we consider two possible cases, $\operatorname{dist}(I, J) \leq 3$ and $\operatorname{dist}(I, J) > 3$.

When $\operatorname{dist}(I, J) > 3$, as before consider the integral

$$\int_{\mathbb{R}} \left(1 + \frac{\operatorname{dist}(x, I)}{|I|}\right)^{-M_2} \left(1 + \frac{\operatorname{dist}(x, J)}{|J|}\right)^{-N_3} \mathrm{d}x.$$

We can get a decay about $|m|^{-M}$ for $J \subseteq J_m$, $m \in \mathbb{Z}$, and see whether J_m is close to n_3 or not. As before, by considering whether J is close to I_{n_3} or not, we will get an additional decay $1/|n_3|^M$.

When $\operatorname{dist}(I, J) \leq 3$, as before we have that J is near the origin $J \subseteq 9I_0$. In this case, our desired decay comes from the *size* and *energy* estimates used in the proof of Theorem 1.2 (see [11, 12]). Those *size* and *energy* terms corresponding to the function $h\chi_{n_3}$ would be defined based on the inner product terms like $|\langle h\chi_{I_{n_3}}, \phi_J^2\rangle|$. Now, since J is close to the origin, such an inner product will provide a decay about $1/|n_3|^M$. (Or, one can see the proof of Lemma 2.13 or Section 8.11 in [12] to see that clearly we can actually get such a decay factor for the *size* estimate.) This means we can get an additional decay from the result of Theorem 1.2, since the boundedness there is based on the *size* and *energy* estimates.

So far, we have proved Part I of (5.1).

For Part II, using the intervals $I_n = [n, n+1]$, $J_m = [m, m+1]$, $m, n \in \mathbb{Z}$, we can write

$$\begin{aligned}
&\|T_1(f,g,h)(x)\cdot\varphi_0(x)\|_r^r \\
&\quad = \Bigg\| \sum_{I\subseteq(5I_0)^c} \sum_{\substack{J\in\mathcal{J} \\ |\omega_J^3|\le|\omega_I^2|}} \frac{1}{|I|^{1/2}}\frac{1}{|J|^{1/2}} \langle f,\phi_I^1\rangle\langle g,\phi_J^1\rangle\langle h,\phi_J^3\rangle\langle \phi_I^2,\phi_J^3\rangle \phi_I^3(x)\varphi_0(x)\Bigg\|_r^r \\
&\quad \lesssim \sum_{|n|\ge5}\sum_{m\in\mathbb{Z}}\sum_{I\subseteq I_n}\sum_{\substack{J\subseteq J_m \\ |\omega_J^3|\le|\omega_I^2|}} \Bigg\| \frac{1}{|I|^{1/2}}\frac{1}{|J|^{1/2}} \langle f,\phi_I^1\rangle\langle g,\phi_J^1\rangle\langle h,\phi_J^3\rangle \\
&\qquad\qquad \times \langle \phi_I^2,\phi_J^3\rangle \phi_I^3(x)\varphi_0(x)\Bigg\|_r^r.
\end{aligned}$$

We will use Hölder's inequality and take advantage of the decay factors as before to write the above as

$$\begin{aligned}
(5.3)\quad &\sum_{|n|\ge5}\sum_{m\in\mathbb{Z}}\sum_{i,j\ge0}\sum_{\substack{I\subseteq I_n,\, J\subseteq J_m \\ |I|=2^{-i},\,|J|=2^{-j}}} \Bigg\| \frac{1}{|I|^{1/2}}\frac{1}{|J|^{1/2}} \langle f,\phi_I^1\rangle\langle g,\phi_J^1\rangle \\
&\qquad\qquad \times \langle h,\phi_J^3\rangle\langle \phi_I^2,\phi_J^3\rangle \phi_I^3(x)\varphi_0(x)\Bigg\|_r^r \\
&\lesssim \sum_{|n|\ge5}\sum_{m\in\mathbb{Z}}\sum_{i,j\ge0}\sum_{\substack{I\subseteq I_n,\, J\subseteq J_m \\ |I|=2^{-i},\,|J|=2^{-j}}} \Bigg(\frac{1}{|I|^2}\frac{1}{|J|^2} (\|f\tilde{\chi}_{I_n}\|_{p_1} |I|^{(p_1-1)/p_1}) \\
&\times (\|g\tilde{\chi}_{J_m}\|_{p_2} |J|^{(p_2-1)/p_2})(\|h\tilde{\chi}_{J_m}\|_{p_3} |J|^{(p_3-1)/p_3})|I|^{1/r}\left(1+\frac{\operatorname{dist}(I,I_0)}{|I|}\right)^{-M_3} \\
&\qquad \times \int_{\mathbb{R}} \left(1+\frac{\operatorname{dist}(x,I)}{|I|}\right)^{-M_2}\left(1+\frac{\operatorname{dist}(x,J)}{|J|}\right)^{-N_3} \mathrm{d}x\Bigg)^r \\
&\lesssim \sum_{|n|\ge5}\sum_{m\in\mathbb{Z}}\sum_{i,j\ge0}\sum_{\substack{I\subseteq I_n,\, J\subseteq J_m \\ |I|=2^{-i},\,|J|=2^{-j}}} \Bigg(2^i(1+2^i(|n|-2))^{-M_3}\|f\tilde{\chi}_{I_n}\|_{p_1}\|g\tilde{\chi}_{J_m}\|_{p_2} \\
&\qquad \times \|h\tilde{\chi}_{J_m}\|_{p_3} \int_{\mathbb{R}} \left(1+\frac{\operatorname{dist}(x,I)}{|I|}\right)^{-M_2}\left(1+\frac{\operatorname{dist}(x,J)}{|J|}\right)^{-N_3} \mathrm{d}x\Bigg)^r,
\end{aligned}$$

where again M_j, N_j are sufficiently large integers. Then, we consider two possible cases, $\operatorname{dist}(I_n, J_m) \le 5$ and $\operatorname{dist}(I_n, J_m) > 5$.

When $\operatorname{dist}(I_n, J_m) \le 5$, we use the same technique as before:

$$(|n|-2)^{-M/12}|\tilde{\chi}_{I_n}| \lesssim |\tilde{\chi}_{I_0}| \quad \text{and} \quad |\tilde{\chi}_{I_n}| \sim |\tilde{\chi}_{J_m}|,$$

for M sufficiently large. Note here that the decay factor for i actually implies a decay for the summation over dyadic intervals J, since $i \ge j$. Then, we can estimate (5.3) by

$$\begin{aligned}
&\lesssim \sum_{|n|\ge 5} ((|n-2|^{-M_3/2})\|f\tilde{\chi}_{I_n}\|_{p_1}\|g\tilde{\chi}_{J_m}\|_{p_2}\|h\tilde{\chi}_{J_m}\|_{p_3})^r \\
&\lesssim \sum_{|n|\ge 5} ((|n-2|^{-M_3/4})\|f\tilde{\chi}_0\|_{p_1}\|g\tilde{\chi}_0\|_{p_2}\|h\tilde{\chi}_0\|_{p_3})^r \\
&\lesssim (\|f\tilde{\chi}_{I_0}\|_{p_1}\|g\tilde{\chi}_{I_0}\|_{p_2}\|h\tilde{\chi}_{I_0}\|_{p_3})^r,
\end{aligned}$$

which is the desired estimate.

When $\operatorname{dist}(I_n, J_m) > 5$, we need to take advantage of the integral in (5.3). That is,

$$\int_{\mathbb{R}} \left(1+\frac{\operatorname{dist}(x,I)}{|I|}\right)^{-M_2}\left(1+\frac{\operatorname{dist}(x,J)}{|J|}\right)^{-N_3} \mathrm{d}x \lesssim |n-m|^{-L},$$

where $L = \min\{M_2, N_3\}$ is large enough. Now, (5.3) can be written by

$$\begin{aligned}
&\lesssim \sum_{|n|\ge 5}\sum_{|m-n|>5}\sum_{i,j\ge 0}\sum_{\substack{I\subseteq I_n,\, J\subseteq J_m\\ |I|=2^{-i},\, |J|=2^{-j}}} (2^i(1+2^i(|n|-2))^{-M_3} \\
&\qquad \times \|f\tilde{\chi}_{I_n}\|_{p_1}\|g\tilde{\chi}_{J_m}\|_{p_2}\|h\tilde{\chi}_{J_m}\|_{p_3}|m-n|^{-L})^r \\
&\lesssim \sum_{|n|\ge 5} ((|n-2|^{-M_3/2})\|f\tilde{\chi}_{I_n}\|_{p_1}\|g\tilde{\chi}_{J_n}\|_{p_2}\|h\tilde{\chi}_{J_n}\|_{p_3})^r \\
&\lesssim (\|f\tilde{\chi}_{I_0}\|_{p_1}\|g\tilde{\chi}_{I_0}\|_{p_2}\|h\tilde{\chi}_{I_0}\|_{p_3})^r,
\end{aligned}$$

where, as before, the decay factor for i allows us to take the summation over dyadic intervals J, since $i \ge j$.

We are now done with Part II, which means we have proved the desired estimate for $T_1(f,g,h)(x)$.

5.3. Estimates for $\sum_{k_0=100}^{\infty}(2^{-k_0})^{\ell}T_{\ell,k_0}(f,g,h)(x)$. There is nothing new in this case, since it will be almost the same as what we did for $T_1(f,g,h)(x)$. Note that, for $T_{\ell,k_0}(f,g,h)(x)$, the only difference is that we have

$$|I|^{-1} \sim |\omega_I^2| \sim 2^{k_0}|J|^{-1} \sim |\omega^3|_J$$

instead of $|I|^{-1} \sim |\omega_I^2| \ge |J|^{-1} \sim |\omega_J^3|$ in $T_1(f,g,h)(x)$. That is, given $|I| = 2^{-i}$, $|J| = 2^{-j}$, we will have $i - k_0 = j \ge 0$, $k_0 \ge 100$. Recall we only need $i \geq j$ in the proof for $T_1(f,g,h)(x)$, and the method obviously works for $T_{\ell,k_0}(f,g,h)(x)$ in the setting $i - k_0 = j \geq 0$, $k_0 \geq 100$, which will give us a bound uniformly with respect to k_0. Then, we will be able to take the summation over k_0 by using $\ell \geq 1$. In this way, we can get the estimate for $\sum_{k_0=100}^{\infty}(2^{-k_0})^{\ell} T_{\ell,k_0}(f,g,h)(x)$.

So far, we have proved the desired localized estimate for the operator

$$T_{ab}^{E,0,0}(f,g,h)(x)$$

in (4.5), which means Theorem 3.1 has been proved. Then, from this localized result, we can conclude that Theorem 1.3 is true.

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