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A sharpened Moser–Pohozaev–Trudinger inequality with mean value zero in $\mathbb{R}^2$

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ABSTRACT

Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain, and $q(x)$ be a polynomial with $q(0) \neq 0$. Then under some hypothesis on $q(x)$, there holds

$$\sup_{\int_\Omega |\nabla u|^2 dx=1, \int_\Omega u dx=0} \int_\Omega e^{\frac{2\pi u^2}{q(0)} q(\int_\Omega u^2 dx)} dx < +\infty.$$

A sufficient condition will be given to assure that the above inequality does not hold. Furthermore, the existence of the extremal functions will be derived.

1. Introduction and main results

Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain, $H_0^1(\Omega)$ be the completion of $C_0^\infty(\Omega)$ under the norm $\|u\|_{H_0^1(\Omega)} = (\int_\Omega |\nabla u|^2 dx)^{1/2}$, and $H^1(\Omega)$ be the completion of $C^\infty(\Omega)$ under the norm $\|u\|_{H^1(\Omega)} = (\int_\Omega (|u|^2 + |\nabla u|^2) dx)^{1/2}$. We state a special case of the Moser–Pohozaev–Trudinger inequality for functions with a mean value of zero.

Theorem A (*Chang–Yang [1]*). *Suppose Ω is a smooth bounded domain in $\mathbb{R}^2$. There exists a constant c_Ω such that for all $u \in H^1(\Omega)$ with $\int_\Omega |\nabla u|^2 dx = 1$ and $\int_\Omega u dx = 0$ we have $\int_\Omega e^{2\pi u^2} dx \leq c_\Omega$. If we replace 2π with any positive β, the integral is still finite, but if $\beta > 2\pi$ it can be made arbitrarily large by the appropriate choice of u.*

This should be compared with the following original Moser–Pohozaev–Trudinger inequality in dimension two.

Theorem B (*Pohozaev, Trudinger, Moser, [2–4]*). *Suppose Ω is a smooth bounded domain in $\mathbb{R}^2$. There exists a constant C such that if $u \in H_0^1(\Omega)$ such that $\int_\Omega |\nabla u|^2 dx = 1$, then*

$$\int_\Omega e^{4\pi u^2} dx \leq C|\Omega|.$$

If 4π is replaced by any $\alpha > 4\pi$, the integral on the left hand is still finite, but can be made arbitrarily large by an appropriate choice of u.

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In this note, we will establish a new Moser–Pohozaev–Trudinger inequality which is an extension of Theorem A. Let us introduce some notations before we state our main results. The first nonzero Neumann eigenvalue of the Laplacian operator reads

$$\lambda = \inf_{u\in H^1(\Omega),\int_\Omega u^2 dx=1,\int_\Omega u dx=0} \int_\Omega |\nabla u|^2 dx. \tag{1.1}$$

The basic variation principle together with the regularity theory for elliptic equations implies that λ can be attained by some smooth function u_0 satisfying

$$\begin{cases} -\triangle u_0 = \lambda u_0 \quad \text{in} \quad \Omega \\ \|u_0\|_2^2 = 1, \quad \dfrac{\partial u_0}{\partial \nu}|_{\partial\Omega} = 0, \end{cases} \tag{1.2}$$

where ν denotes the outward normal vector on $\partial\Omega$. Let

$$q(t) = a_0 + a_1 t + \cdots + a_k t^k \tag{1.3}$$

be a polynomial of degree k with nonnegative coefficients $a_0, a_1, \ldots, a_k$. We denote the derivative of $q(t)$ by $q'(t)$.

We state our main results as follows:

Theorem 1. *Let $q(t)$ and λ be defined by (1.1) and (1.3). Suppose $q(0) > 0$, $0 \le a_1 < \lambda a_0$, $0 \le a_2 \le \lambda a_1, \ldots, 0 \le a_k \le \lambda a_{k-1}$. Then we have*

$$\sup_{u\in H^1(\Omega),\int_\Omega |\nabla u|^2 dx=1,\int_\Omega u dx=0} \int_\Omega e^{\frac{2\pi u^2}{q(0)} q\left(\int_\Omega u^2 dx\right)} dx < +\infty.$$

Theorem 2. *Let $q(t)$ and λ be defined by (1.1) and (1.3). Suppose $q(0) > 0$, $a_1 \ge \lambda a_0$, $a_2 \ge 0, \ldots, a_k \ge 0$, then*

$$\sup_{u\in H^1(\Omega),\int_\Omega |\nabla u|^2 dx=1,\int_\Omega u dx=0} \int_\Omega e^{\frac{2\pi u^2}{q(0)} q\left(\int_\Omega u^2 dx\right)} dx = +\infty.$$

Theorem 3. *There exists $\epsilon_0 > 0$ such that if $0 \le q'(0) < \epsilon_0$, then there exists an extremal function for the inequality in Theorem 1.*

The proof of Theorems 1 and 3 is based on blow-up analysis, and the proof of Theorem 2 is based on test function computations. Earlier contributions in this direction similar to Theorems 1 and 2 are due to Adimurthi–Druet [5] when $q(t) = 1 + a_1 t$ for functions with boundary value zero. We should point out that the blow-up occurs on the boundary $\partial\Omega$ in our case, and therefore it is more difficult to deal with. For the existence of extremal functions for Moser–Pohozaev–Trudinger inequalities for functions with boundary value zero, we should mention Carleson–Chang [6], Flucher [7], Lin [8], Li [9] when $q(0) = 1$.

Before we end the introduction, we remark that a similar Moser–Pohozaev–Trudinger inequality for functions with boundary value zero can be established by the same idea. We only state but omit the proof of the following results in this note:

Theorem 4. *Let $q(t)$ be defined by (1.3), and λ_1 be defined by*

$$\lambda_1 = \inf_{u\in H_0^1(\Omega),\int_\Omega u^2 dx=1} \int_\Omega |\nabla u|^2 dx. \tag{1.4}$$

Suppose $q(0) > 0$, $0 \le a_1 < \lambda_1 a_0$, $0 \le a_2 \le \lambda_1 a_1, \ldots, 0 \le a_k \le \lambda_1 a_{k-1}$. Then we have

$$\sup_{u\in H_0^1(\Omega),\int_\Omega |\nabla u|^2 dx=1} \int_\Omega e^{\frac{4\pi u^2}{q(0)} q\left(\int_\Omega u^2 dx\right)} dx < +\infty.$$

Theorem 5. *Let $q(t)$ and λ_1 be defined by (1.3) and (1.4). Suppose $q(0) > 0$, $a_1 \ge \lambda_1 a_0$, $a_2 \ge 0, \ldots, a_k \ge 0$, then*

$$\sup_{u\in H_0^1(\Omega),\int_\Omega |\nabla u|^2 dx=1} \int_\Omega e^{\frac{4\pi u^2}{q(0)} q\left(\int_\Omega u^2 dx\right)} dx = +\infty.$$

Theorem 6. *There exists $\epsilon_0 > 0$ such that if $0 \le q'(0) < \epsilon_0$, then there exists a extremal function for the inequality in Theorem 4.*

A similar Moser–Pohozaev–Trudinger inequality holds on Riemannian manifolds with or without boundary. The method in this note can be adapted to prove results in those settings.

The organization of this paper is as follows: We give the proof of Theorems 1 and 3 in Section 2 and the proof of Theorem 2 is given in Section 3. In Section 4, we state some recent theorems derived in [10] for Adams' inequalities for bi-laplacian and extremal functions in dimension four which are considerably more difficult to prove.

2. Proof of Theorems 1 and 3

This section contributes to the proof of Theorem 1. Without loss of generality, we assume $q(0) = 1$, i.e., $q(x) = 1 + a_1x + \cdots + a_kx^k$, where $0 \le a_1 < \lambda$, $0 \le a_2 \le \lambda a_1, \ldots, 0 \le a_k \le \lambda a_{k-1}$. Our aim is to prove

$$\sup_{u\in H^1(\Omega),\int_\Omega |\nabla u|^2\mathrm{d}x=1,\int_\Omega u\mathrm{d}x=0} \int_\Omega \mathrm{e}^{2\pi u^2 q\left(\int_\Omega u^2\mathrm{d}x\right)}\mathrm{d}x < +\infty.$$

Since the proof is very long, we will divide it into several steps.

Step 1. *For any $\epsilon > 0$ there exists a $u_\epsilon \in H^1(\Omega)\cap C^\infty(\overline{\Omega})$ such that $\int_\Omega |\nabla u_\epsilon|^2\mathrm{d}x = 1$, $\int_\Omega u_\epsilon \mathrm{d}x = 0$, and the supremum*

$$\Lambda_\epsilon = \sup_{u\in H^1(\Omega),\int_\Omega |\nabla u|^2\mathrm{d}x=1,\int_\Omega u\mathrm{d}x=0} \int_\Omega \mathrm{e}^{(2\pi-\epsilon)u^2 q\left(\int_\Omega u^2\mathrm{d}x\right)}\mathrm{d}x$$

can be attained by u_ϵ. The Euler–Lagrange equation is

$$\begin{cases} -\triangle u_\epsilon = \dfrac{\beta_\epsilon}{\lambda_\epsilon} u_\epsilon \mathrm{e}^{\alpha_\epsilon u_\epsilon^2} + \gamma_\epsilon u_\epsilon - \dfrac{\mu_\epsilon}{\lambda_\epsilon} \quad \text{in} \quad \Omega \\ \dfrac{\partial u_\epsilon}{\partial \nu}|_{\partial\Omega} = 0, \quad \|\nabla u_\epsilon\|_2 = 1 \\ \alpha_\epsilon = (2\pi-\epsilon) q\left(\displaystyle\int_\Omega u_\epsilon^2 \mathrm{d}x\right) \\ \beta_\epsilon = \dfrac{q\left(\int_\Omega u_\epsilon^2\mathrm{d}x\right)}{q\left(\int_\Omega u_\epsilon^2\mathrm{d}x\right) + q'\left(\int_\Omega u_\epsilon^2\mathrm{d}x\right)\int_\Omega u_\epsilon^2\mathrm{d}x} \\ \gamma_\epsilon = \dfrac{q'\left(\int_\Omega u_\epsilon^2\mathrm{d}x\right)}{q\left(\int_\Omega u_\epsilon^2\mathrm{d}x\right) + q'\left(\int_\Omega u_\epsilon^2\mathrm{d}x\right)\int_\Omega u_\epsilon^2\mathrm{d}x} \\ \lambda_\epsilon = \displaystyle\int_\Omega u_\epsilon^2 \mathrm{e}^{\alpha_\epsilon u_\epsilon^2}\mathrm{d}x \\ \mu_\epsilon = \dfrac{\beta_\epsilon}{|\Omega|}\displaystyle\int_\Omega u_\epsilon \mathrm{e}^{\alpha_\epsilon u_\epsilon^2}\mathrm{d}x \end{cases} \tag{2.1}$$

Proof. Choosing $u_j \in H^1(\Omega)$ such that $\int_\Omega |\nabla u_j|^2\mathrm{d}x = 1$, $\int_\Omega u_j\mathrm{d}x = 0$, and

$$\lim_{j\to+\infty}\int_\Omega \mathrm{e}^{(2\pi-\epsilon)u_j^2 q\left(\int_\Omega u_j^2\mathrm{d}x\right)}\mathrm{d}x = \Lambda_\epsilon.$$

Since $\{u_j\}$ is bounded in $H^1(\Omega)$, passing to a subsequence (still denoted by u_j), we have for some u_ϵ,

$$u_j \rightharpoonup u_\epsilon \quad \text{weakly in } H^1(\Omega),$$
$$u_j \to u_\epsilon \quad \text{strongly in } L^2(\Omega).$$

Obviously $\int_\Omega u_\epsilon\mathrm{d}x = 0$, and $u_j \to u_\epsilon$ a.e. in Ω. A contradiction argument together with Theorem A implies that $u_\epsilon \not\equiv 0$.

We first claim a Lions' type result [11]: there holds for any $0 < r < 1/(1-\|\nabla u_\epsilon\|_2^2)$,

$$\limsup_{j\to+\infty}\int_\Omega \mathrm{e}^{2\pi r u_j^2}\mathrm{d}x < +\infty. \tag{2.2}$$

Notice that $\|\nabla(u_j - u_\epsilon)\|_2^2 \to 1 - \|\nabla u_\epsilon\|_2^2$ as $j\to+\infty$, the claim is an easy consequence of Theorem A and the inequality $ab \le \gamma a^2 + \frac{1}{4\gamma}b^2$ for any $\gamma > 0$.

Since $0 \le a_1 < \lambda$, $0 \le a_2 \le \lambda a_1, \ldots, 0 \le a_k \le \lambda a_{k-1}$, a direct calculation shows

$$\begin{aligned} q\left(\int_\Omega u_\epsilon^2\mathrm{d}x\right)(1-\|\nabla u_\epsilon\|^2) &= 1 + a_1\|u_\epsilon\|_2^2 + \cdots + a_k\|u_\epsilon\|_2^{2k} - \|\nabla u_\epsilon\|_2^2 \\ &\quad - a_1\|u_\epsilon\|_2^2\|\nabla u_\epsilon\|_2^2 - \cdots - a_k\|u_\epsilon\|_2^{2k}\|\nabla u_\epsilon\|_2^2 \\ &< 1 - a_k\|u_\epsilon\|_2^{2k}\|\nabla u_\epsilon\|_2^2. \end{aligned} \tag{2.3}$$

Hence

$$q\left(\int_\Omega u_j^2 \mathrm{d}x\right)(1-\|\nabla u_\epsilon\|^2) \to q\left(\int_\Omega u_\epsilon^2 \mathrm{d}x\right)(1-\|\nabla u_\epsilon\|^2) < 1 \quad \text{as} \quad j \to +\infty.$$

Again we have by our claim (2.2) that $\mathrm{e}^{(2\pi-\epsilon)u_j^2 q(\int_\Omega u_j^2 \mathrm{d}x)}$ is bounded in $L^r(\Omega)$ for some $r > 1$ provided that j is sufficiently large. Therefore

$$\mathrm{e}^{(2\pi-\epsilon)u_j^2 q\left(\int_\Omega u_j^2 \mathrm{d}x\right)} \to \mathrm{e}^{(2\pi-\epsilon)u_\epsilon^2 q(\int_\Omega u_\epsilon^2 \mathrm{d}x)} \quad \text{in } L^1(\Omega) \text{ as } j \to +\infty,$$

and the conclusion of Step 1 follows immediately. □

Step 2. *Energy concentration phenomenon of the maximizers u_ϵ: Precisely speaking, $u_\epsilon \rightharpoonup 0$ weakly in $H^1(\Omega)$, $u_\epsilon \to 0$ strongly in $L^2(\Omega)$, $|\nabla u_\epsilon|^2 \mathrm{d}x \rightharpoonup \delta_p$ in sense of measure, where δ_p is the Dirac measure at p, and p lies on the boundary $\partial\Omega$. Furthermore, $\alpha_\epsilon \to 2\pi$, $\beta_\epsilon \to 1$ and $\gamma_\epsilon \to q'(0)$.*

Proof. On one hand, we have by Step 1,

$$\begin{aligned}\int_\Omega \mathrm{e}^{\alpha_\epsilon u_\epsilon^2} \mathrm{d}x &= \sup_{\int_\Omega |\nabla u|^2 \mathrm{d}x=1, \int_\Omega u \mathrm{d}x=0} \int_\Omega \mathrm{e}^{(2\pi-\epsilon)u^2 q\left(\int_\Omega u^2 \mathrm{d}x\right)} \mathrm{d}x \\ &\le \sup_{\int_\Omega |\nabla u|^2 \mathrm{d}x=1, \int_\Omega u \mathrm{d}x=0} \int_\Omega \mathrm{e}^{2\pi u^2 q\left(\int_\Omega u^2 \mathrm{d}x\right)} \mathrm{d}x. \end{aligned} \tag{2.4}$$

One the other hand, we have for any u with $\int_\Omega |\nabla u|^2 \mathrm{d}x = 1$ and $\int_\Omega u \mathrm{d}x = 0$,

$$\begin{aligned}\int_\Omega \mathrm{e}^{2\pi u^2 q\left(\int_\Omega u^2 \mathrm{d}x\right)} \mathrm{d}x &\le \liminf_{\epsilon\to 0} \int_\Omega \mathrm{e}^{(2\pi-\epsilon)u^2 q\left(\int_\Omega u^2 \mathrm{d}x\right)} \mathrm{d}x \\ &\le \liminf_{\epsilon\to 0} \int_\Omega \mathrm{e}^{(2\pi-\epsilon)u_\epsilon^2 q\left(\int_\Omega u_\epsilon^2 \mathrm{d}x\right)} \mathrm{d}x.\end{aligned}$$

Hence we obtain

$$\lim_{\epsilon\to 0} \int_\Omega \mathrm{e}^{\alpha_\epsilon u_\epsilon^2} \mathrm{d}x = \sup_{\int_\Omega |\nabla u|^2 \mathrm{d}x=1, \int_\Omega u \mathrm{d}x=0} \int_\Omega \mathrm{e}^{2\pi u^2 q\left(\int_\Omega u^2 \mathrm{d}x\right)} \mathrm{d}x. \tag{2.5}$$

Using the inequality $\mathrm{e}^t \le 1 + t\mathrm{e}^t$, one has

$$\int_\Omega \mathrm{e}^{\alpha_\epsilon u_\epsilon^2} \mathrm{d}x \le |\Omega| + \alpha_\epsilon \lambda_\epsilon. \tag{2.6}$$

The Poincare inequality implies that α_ϵ is bounded, which together with (2.5) and (2.6) gives

$$\liminf_{\epsilon\to 0} \lambda_\epsilon > 0. \tag{2.7}$$

By the inequality $t\mathrm{e}^{t^2} \le e + t^2 \mathrm{e}^{t^2}$, there exists a constant c such that

$$|\mu_\epsilon/\lambda_\epsilon| \le c. \tag{2.8}$$

Let $c_\epsilon = |u_\epsilon|(x_\epsilon) = \max_\Omega |u_\epsilon|$. If c_ϵ is bounded, by (2.7), (2.8), the boundedness of β_ϵ and γ_ϵ, and the standard elliptic estimates (see for example [12], Chapter 9) with respect to (2.1), Theorem 1 holds. Hence we may assume without loss of generality that

$$x_\epsilon \to p \in \overline{\Omega}, \qquad c_\epsilon = u_\epsilon(x_\epsilon) \to +\infty \tag{2.9}$$

as $\epsilon \to 0$. Here and in the sequel, we do not distinguish sequence and subsequence, the reader can understand it from the context.

Since $\|\nabla u_\epsilon\|_2^2 = 1$ and $\int_\Omega u_\epsilon \mathrm{d}x = 0$, we have by Poincare inequality that u_ϵ is bounded in $H^1(\Omega)$. Hence we may assume $u_\epsilon \rightharpoonup u_0$ weakly in $H^1(\Omega)$, and $u_\epsilon \to u_0$ strongly in $L^2(\Omega)$. Suppose $u_0 \ne 0$, then the fact $\int_\Omega u_0 \mathrm{d}x = 0$ leads to $\|\nabla u_0\|_2^2 \ne 0$. Similar to (2.3), we have

$$q\left(\int_\Omega u_\epsilon^2 \mathrm{d}x\right) \to q\left(\int_\Omega u_0^2 \mathrm{d}x\right) < \frac{1}{1-\|\nabla u_0\|_2^2},$$

which together with (2.2) implies that $e^{\alpha_\epsilon u_\epsilon^2}$ is bounded in $L^r(\Omega)$ for some $r > 1$ provided that ϵ is sufficiently small. Applying the standard elliptic estimates to Eq. (2.1), one gets c_ϵ is bounded, and a contradiction. Hence $u_0 = 0$, and whence $\alpha_\epsilon \to 2\pi$, $\beta_\epsilon \to 1$ and $\gamma_\epsilon \to q'(0)$.

To prove the rest of Step 2, we consider two cases.

Case 1. *p lies in the interior of Ω.*

Pick $r > 0$ such that $B_r(p)$, the ball centered at p with radius r, is contained in the interior of Ω. Take a cut-off function $\phi \in C_0^1(B_r(p))$ satisfying $\phi \equiv 1$ on $B_{r/2}(p)$, and

$$\limsup_{\epsilon\to 0} \int_\Omega |\nabla(\phi u_\epsilon)|^2 dx < 1 + \eta \tag{2.10}$$

for some $0 < \eta < 1/2$. The existence of such ϕ is based on the facts $\int_\Omega |\nabla u_\epsilon|^2 dx = 1$ and $\int_\Omega u_\epsilon^2 dx \to 0$. Note that $\alpha_\epsilon \to 2\pi$, we have by the classical Moser–Pohozaev–Trudinger inequality (Theorem B), $e^{\alpha_\epsilon u_\epsilon^2}$ is bounded in $L^r(B_{r/2}(p))$ for some $r > 1$. Applying the interior elliptic estimates to (2.1), we obtain the boundedness of u_ϵ in $C^1(B_{r/2}(p))$, which contradicts (2.9). Hence this case can not occur.

Case 2. *$p \in \partial\Omega$.*

Assume $|\nabla u_\epsilon|^2 dx \rightharpoonup \mu$ in sense of measure. Note that $\int_\Omega |\nabla u_\epsilon|^2 dx = 1$, if $\mu \neq \delta_p$, there exists $r > 0$ such that

$$\lim_{\epsilon\to 0} \int_{B_r(p)\cap\Omega} |\nabla u_\epsilon|^2 dx = \mu(B_r(p)\cap\Omega) < 1.$$

Theorem A together with $\int_\Omega |u_\epsilon| dx \to 0$ gives $e^{\alpha_\epsilon u_\epsilon^2}$ is bounded in $L^s(\mathcal{U})$ for some $s > 1$ for sufficiently small ϵ, where $\mathcal{U}$ is some neighborhood of p such that $\partial\mathcal{U}$ is smooth, and $\overline{B_{r/2}(p)}\cap\partial\Omega \subset \overline{\mathcal{U}}\cap\partial\Omega \subset \overline{B_r(p)}\cap\partial\Omega$. Then, note that $\partial u_\epsilon/\partial\nu|_{\partial\Omega} = 0$, applying the boundary elliptic estimates to (2.1) we have the uniform boundedness of u_ϵ near p, which contradicts (2.9). Hence $\mu = \delta_p$. □

Step 3. *The blow-up behavior of u_ϵ near p.*

Take an isothermal coordinate system $(\mathcal{U}, \phi)$ near p such that $\phi(p) = 0$, $\phi : \mathcal{U}\cap\partial\Omega \to \partial\mathbb{R}^{2+}\cap\mathbb{B}_1$ and $\phi : \mathcal{U} \to \overline{\mathbb{B}_1^+} = \{y = (y_1, y_2) : y_1^2 + y_2^2 \le 1, y_2 \ge 0\}$. In such coordinates, the original metric $g = dx_1^2 + dx_2^2$ has the representation $g = e^{2f(y)}(dy_1^2 + dy_2^2)$ with $f(0) = 0$. Define a sequence of functions

$$\tilde{u}_\epsilon(y) = \begin{cases} u_\epsilon \circ \phi^{-1}(y_1, y_2) & \text{for } y_2 \ge 0 \\ u_\epsilon \circ \phi^{-1}(y_1, -y_2) & \text{for } y_2 < 0, \end{cases}$$

on $\mathbb{B}_1$. Let $r_\epsilon^2 = \frac{\lambda_\epsilon}{\beta_\epsilon c_\epsilon^2} e^{-\alpha_\epsilon c_\epsilon^2}$. By Hölder inequality and Moser–Pohozaev–Trudinger inequality, we have $r_\epsilon^2 e^{\beta c_\epsilon^2} \to 0$ for any fixed $\beta < 2\pi$, particularly $r_\epsilon \to 0$. Denote $\mathcal{U}_\epsilon = \{y \in \mathbb{R}^2 : \phi(x_\epsilon) + r_\epsilon y \in \mathbb{B}_1\}$. Let $\psi_\epsilon(y) = \tilde{u}(\phi(x_\epsilon) + r_\epsilon y)/c_\epsilon$ and

$$\varphi_\epsilon(y) = c_\epsilon(\tilde{u}_\epsilon(\phi(x_\epsilon) + r_\epsilon y) - c_\epsilon), \quad y \in \mathcal{U}_\epsilon.$$

Given $R > 0$, we have on $\mathbb{B}_R$,

$$\begin{cases} -\triangle_y \psi_\epsilon = \dfrac{1}{c_\epsilon^2}\psi_\epsilon e^{\alpha_\epsilon(u_\epsilon^2 - c_\epsilon^2)} + r_\epsilon^2\gamma_\epsilon\psi_\epsilon - r_\epsilon^2\dfrac{\mu_\epsilon}{c_\epsilon\lambda_\epsilon} \\ -\triangle_y \varphi_\epsilon = \psi_\epsilon e^{\alpha_\epsilon\varphi_\epsilon(1+\psi_\epsilon)} + c_\epsilon r_\epsilon^2\gamma_\epsilon u_\epsilon - c_\epsilon r_\epsilon^2\mu_\epsilon/\lambda_\epsilon. \end{cases}$$

Notice that $\varphi_\epsilon(0) = \sup_{\mathbb{B}_R}\varphi_\epsilon = 0$, applying Harnack inequality and the elliptic estimates to the above equations, we have $\psi_\epsilon \to 1$ in $C^1(\mathbb{B}_{R/2})$, and $\varphi_\epsilon \to \varphi$ in $C^1(\mathbb{B}_{R/4})$ with

$$\begin{cases} \triangle_{\mathbb{R}^2}\varphi = -e^{4\pi\varphi} \quad \text{in } \mathbb{B}_{R/4} \\ \varphi(0) = 0 = \sup\varphi \\ \displaystyle\int_{\mathbb{B}_{R/4}} e^{4\pi\varphi} dx \le 2. \end{cases}$$

In fact, we have $\varphi_\epsilon \to \varphi$ in $C^1_{loc}(\mathbb{R}^2)$, where φ satisfies the following equation

$$\begin{cases} \triangle_{\mathbb{R}^2}\varphi = -e^{4\pi\varphi} \quad \text{in } \mathbb{R}^2 \\ \varphi(0) = 0 = \sup\varphi \\ \displaystyle\int_{\mathbb{R}^2} e^{4\pi\varphi} dx \le 2. \end{cases} \tag{2.11}$$

The uniqueness theorem in [13] implies that

$$\varphi(x) = -\frac{1}{2\pi}\log\left(1 + \frac{\pi}{2}|x|^2\right), \qquad \int_{\mathbb{R}^2} e^{4\pi\varphi} dx = 2. \tag{2.12}$$

Denote $\mathcal{U}_\epsilon^+ = \{y \in \mathbb{R}^2 : \phi(x_\epsilon) + r_\epsilon y \in \mathbb{B}_1^+\}$, and $\mathcal{U}_\epsilon^- = \{y \in \mathbb{R}^2 : \phi(x_\epsilon) + r_\epsilon y \in \mathbb{B}_1^-\}$. For any fixed $R > 0$, let $\mathbb{B}_R' = \{y \in \mathbb{B}_R : \phi(x_\epsilon) + r_\epsilon y \in \mathbb{B}_1^+\}$, and $\mathbb{B}_R'' = \{y \in \mathbb{B}_R : \phi(x_\epsilon) + r_\epsilon y \in \mathbb{B}_1^-\}$ we have

$$\begin{aligned}\int_{\mathbb{B}_R} e^{4\pi\varphi} dy &= \lim_{\epsilon\to 0}\int_{\mathbb{B}_R} \frac{1}{\beta_\epsilon}\psi_\epsilon^2 e^{\alpha_\epsilon(1+\psi_\epsilon)\varphi_\epsilon} dy \\ &= \lim_{\epsilon\to 0}\int_{\mathbb{B}_{Rr_\epsilon}(\phi(x_\epsilon))} \frac{1}{\lambda_\epsilon}\tilde{u}_\epsilon^2 e^{\alpha_\epsilon \tilde{u}_\epsilon^2} dy \\ &\le \lim_{\epsilon\to 0}\int_{\mathbb{B}^+_{Rr_\epsilon}(\phi(x_\epsilon))} \frac{1}{\lambda_\epsilon}\tilde{u}_\epsilon^2 e^{\alpha_\epsilon \tilde{u}_\epsilon^2} dy + \lim_{\epsilon\to 0}\int_{\mathbb{B}^-_{Rr_\epsilon}(\phi(x_\epsilon))} \frac{1}{\lambda_\epsilon}\tilde{u}_\epsilon^2 e^{\alpha_\epsilon \tilde{u}_\epsilon^2} dy.\end{aligned}$$

This inequality together with $\int_{\mathcal{U}} \frac{1}{\lambda_\epsilon} u_\epsilon^2 e^{\alpha_\epsilon u_\epsilon^2} dx \le 1$ and (2.12) gives

$$\lim_{R\to+\infty}\lim_{\epsilon\to 0}\int_{\mathbb{B}^+_{Rr_\epsilon}(\phi(x_\epsilon))} \frac{1}{\lambda_\epsilon}\tilde{u}_\epsilon^2 e^{\alpha_\epsilon \tilde{u}_\epsilon^2} dy = 1, \tag{2.13}$$

$$\lim_{R\to+\infty}\lim_{\epsilon\to 0}\int_{\mathbb{B}^-_{Rr_\epsilon}(\phi(x_\epsilon))} \frac{1}{\lambda_\epsilon}\tilde{u}_\epsilon^2 e^{\alpha_\epsilon \tilde{u}_\epsilon^2} dy = 1. \tag{2.14}$$

Step 4. *The asymptotic behavior of u_ϵ away from p. We have for any $1 < q < 2$, $c_\epsilon u_\epsilon \rightharpoonup G$ weakly in $H^{1,q}(\Omega)$, where $G \in C^\infty(\overline{\Omega} \setminus \{p\})$ is a Green function satisfying the following*

$$\begin{cases} -\Delta G = \delta_p + q'(0)G - \dfrac{1}{|\Omega|} & \text{in } \Omega \\ \dfrac{\partial G}{\partial \nu} = 0 & \text{on } \partial\Omega \setminus \{p\} \\ \displaystyle\int_\Omega G dx = 0. \end{cases} \tag{2.15}$$

Furthermore, $\forall \tilde{\Omega} \subset\subset \Omega \setminus \{p\}$, $c_\epsilon u_\epsilon \to G$ in $C^\infty(\overline{\tilde{\Omega}})$.

Proof. Since the proof is similar to Lemma 4.9 in [14], only slight modification is needed, we omit the details. □

Step 5. *Completion of the proof of Theorem 1.*

By Step 4, we have $c_\epsilon u_\epsilon \to G$ strongly in $L^2(\Omega)$. Hence

$$\begin{aligned}\lim_{\epsilon\to 0}\int_\Omega e^{\alpha_\epsilon u_\epsilon^2} dx &\le \limsup_{\epsilon\to 0} e^{2\pi(a_1\|c_\epsilon u_\epsilon\|_2^2 + a_2\|c_\epsilon u_\epsilon\|_2^4 + \cdots + a_k\|c_\epsilon u_\epsilon\|_2^{2k})}\int_\Omega e^{(2\pi-\epsilon)u_\epsilon^2} dx \\ &\le e^{2\pi q\left(\int_\Omega G^2 dx\right)} \sup_{\int_\Omega |\nabla u|^2 dx = 1, \int_\Omega u dx = 0}\int_\Omega e^{2\pi u^2} dx.\end{aligned}$$

By Theorem A and (2.5), we have completed the proof of Theorem 1. □

Proof of Theorem 3. Combining Steps 3 and 4, we proceed as we did in the proof of Proposition 3.11 in [15]. We only give the outline of the proof here. For more details we refer the reader to [14,15]. Under the assumption that $c_\epsilon \to +\infty$ and $x_\epsilon \to p$, we obtain

$$\sup_{\int_\Omega |\nabla u|^2 dx = 1, \int_\Omega u dx = 0}\int_\Omega e^{\alpha_\epsilon u_\epsilon^2} dx \le |\Omega| + \frac{\pi}{2} e^{1+2\pi A_p}, \tag{2.16}$$

where

$$A_p = \lim_{x\to p}\left(G + \frac{1}{\pi}\log|x-p|\right).$$

Let $\beta = G(x) + \frac{1}{\pi}\log|x-p| - A_p$. Applying elliptic estimates to (2.15), we have $\beta = O(|x-p|)$. Set $r = |x-p|$ and

$$\phi_\epsilon = \begin{cases} \dfrac{c + \frac{1}{c}\left(-\frac{1}{2\pi}\log\left(1+\frac{\pi}{2}\frac{r^2}{\epsilon^2}\right)+B\right)}{\sqrt{q\left(\frac{1}{c^2}\int_\Omega G^2 dx\right)}}, & r \le R\epsilon, \\ \dfrac{G - \eta\beta}{\sqrt{c^2 q\left(\frac{1}{c^2}\int_\Omega G^2 dx\right)}}, & R\epsilon \le r \le 2R\epsilon, \\ \dfrac{G}{\sqrt{c^2 q\left(\frac{1}{c^2}\int_\Omega G^2 dx\right)}}, & r \ge 2R\epsilon, \end{cases} \tag{2.17}$$

where $\eta \in C_0^\infty(B_{2R\epsilon}(p))$ is a cutoff function, $\eta = 1$ in $B_{R\epsilon}(p)$, $\|\nabla\eta\|_{L^\infty} = O(\frac{1}{R\epsilon})$, B, R and c are constants depending on ϵ to be determined. The rest of the proof is almost the same as [14]. For sufficiently small positive $q'(0)$, we can choose ϕ_ϵ such that $\int_\Omega |\nabla\phi_\epsilon|^2 dx = 1$ and

$$\int_\Omega e^{2\pi(\phi_\epsilon - \int_\Omega \phi_\epsilon dx)^2 q\left(\int_\Omega(\phi_\epsilon - \int_\Omega \phi_\epsilon dx)^2 dx\right)} dx > |\Omega| + \frac{\pi}{2} e^{1+2\pi A_p},$$

which together with (2.16) implies that blow-up can not occur, and the extremal function does exist. □

3. Proof of Theorem 2

In this section, we choose test functions to prove Theorem 2. Let λ be the first nonzero neumann eigenvalue defined by (1.1), and u_0 be the corresponding eigenfunction satisfying (1.2). By elliptic estimates, $u_0 \in C^1(\overline{\Omega})$. We first claim the following:

Claim 3.1: *u_0 cannot be identically zero on the boundary $\partial\Omega$.*

Proof. Suppose not. Then u_0 satisfies

$$\begin{cases} -\triangle u_0 = \lambda u_0 & \text{in } \Omega \\ u_0 = 0, \dfrac{\partial u_0}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

Noting that $\Omega \subset \mathbb{R}^2$, according to Aviles ([16], Theorem 1), we know that Ω must be a ball, and u_0 is symmetric in the ball. It follows that $u_0 \equiv 0$, which contradicts $\int_\Omega u_0^2 dx = 1$. □

Once claim 3.1 is true, we can then adapt an argument similar to that of [5]. Without loss of generality we can assume there exists $p \in \partial\Omega$ such that $u_0(p) > 0$ for otherwise we consider $-u_0$ instead of u_0. Choose a neighborhood of p, say $\mathcal{U} \subset \overline{\Omega}$ such that $u_0 \ge u_0(p)/2$ in $\mathcal{U}$. Choose an isothermal coordinate system $(\mathcal{V}, \psi)$ around p such that $\mathcal{V} \subset \mathcal{U}$, $\psi : \mathcal{V} \to \overline{\mathbb{B}_\delta^+} = \{y = (y_1, y_2) \in \mathbb{R}^2 : y_1^2 + y_2^2 \le \delta^2, y_2 \ge 0\}$, $\psi(p) = 0$. In this coordinate system, the original metric $g = dx_1^2 + dx_2^2$ can be represented by $g = e^{2f(y)}(dy_1^2 + dy_2^2)$, where $f(y)$ is a smooth function with $f(0) = 0$.

On $\overline{\mathbb{B}_\delta^+}$, we define a sequence of functions

$$m_\epsilon(y) = \begin{cases} \sqrt{\dfrac{1}{2\pi}\log\dfrac{1}{\epsilon}}, & |y| \le \delta\sqrt{\epsilon}, \\ \dfrac{\sqrt{2}}{\sqrt{\pi\log\frac{1}{\epsilon}}}\log\dfrac{\delta}{|y|}, & \delta\sqrt{\epsilon} < |y| \le \delta. \end{cases}$$

We set

$$u_\epsilon = \begin{cases} m_\epsilon \circ \psi & \text{in } \psi^{-1}(\overline{\mathbb{B}_\delta^+}), \\ l_\epsilon \varphi & \text{in } \Sigma \setminus \psi^{-1}(\overline{\mathbb{B}_\delta^+}), \end{cases}$$

where $\varphi \in C_0^\infty(\overline{\Omega} \setminus \psi^{-1}(\overline{\mathbb{B}_\delta^+}))$ and l_ϵ is a real number such that $\int_\Omega u_\epsilon dx = 0$.

It is not difficult to check that

$$l_\epsilon = O\left(1/\left(\log\frac{1}{\epsilon}\right)^{1/2}\right), \qquad \|\nabla u_\epsilon\|_2^2 = 1 + O\left(1/\log\frac{1}{\epsilon}\right),$$

$$\|u_\epsilon\|_1 = O\left(1/\left(\log\frac{1}{\epsilon}\right)^{1/2}\right), \qquad \|u_\epsilon\|_2^2 = O\left(1/\log\frac{1}{\epsilon}\right).$$

Setting $v_\epsilon = u_\epsilon + t_\epsilon u_0$ with $t_\epsilon \to 0$, $t_\epsilon^2 \log \frac{1}{\epsilon} \to +\infty$ and $t_\epsilon^2 \left(\log \frac{1}{\epsilon}\right)^{1/2} \to 0$. Then we have

$$\begin{aligned}\|v_\epsilon\|_2^2 &= \|u_\epsilon\|_2^2 + t_\epsilon^2\|u_0\|_2^2 + 2t_\epsilon \int_\Omega u_\epsilon u_0 \mathrm{d}x \\ &= t_\epsilon^2 + 2t_\epsilon \int_\Omega u_\epsilon u_0 \mathrm{d}x + O\left(1/\log\frac{1}{\epsilon}\right).\end{aligned}$$

$$\begin{aligned}\|\nabla v_\epsilon\|_2^2 &= \|\nabla u_\epsilon\|_2^2 + t_\epsilon^2\|\nabla u_0\|_2^2 + 2t_\epsilon \int_\Omega \nabla u_\epsilon \nabla u_0 \mathrm{d}x \\ &= 1 + 2\lambda t_\epsilon \int_\Omega u_\epsilon u_0 \mathrm{d}x + \lambda t_\epsilon^2 + O\left(1/\log\frac{1}{\epsilon}\right).\end{aligned}$$

$$\frac{1}{\|\nabla v_\epsilon\|_2^2} q\left(\frac{\|v_\epsilon\|_2^2}{\|\nabla v_\epsilon\|_2^2}\right) = 1 + (q'(0)-\lambda)\left(t_\epsilon^2 + 2t_\epsilon \int_\Omega u_\epsilon u_0 \mathrm{d}x\right) + o\left(t_\epsilon \Big/ \left(\log\frac{1}{\epsilon}\right)^{1/2}\right).$$

We have for $q'(0) \geq \lambda$

$$\frac{1}{\|\nabla v_\epsilon\|_2^2} q\left(\frac{\|v_\epsilon\|_2^2}{\|\nabla v_\epsilon\|_2^2}\right) \geq 1 + o\left(t_\epsilon \Big/ \left(\log\frac{1}{\epsilon}\right)^{1/2}\right).$$

Note that on $\psi^{-1}(\overline{\mathbb{B}^+_{\delta\sqrt{\epsilon}}})$,

$$\begin{aligned}2\pi \frac{v_\epsilon^2}{\|\nabla v_\epsilon\|_2^2} q\left(\frac{\|v_\epsilon\|_2^2}{\|\nabla v_\epsilon\|_2^2}\right) &\geq 2\pi \left(t_\epsilon^2 u_0^2 + \frac{1}{2\pi}\log\frac{1}{\epsilon} + 2t_\epsilon\left(\frac{1}{2\pi}\log\frac{1}{\epsilon}\right)^{1/2} u_0\right)\left(1 + o\left(t_\epsilon \Big/ \left(\log\frac{1}{\epsilon}\right)^{1/2}\right)\right) \\ &\geq \log\frac{1}{\epsilon} + t_\epsilon\left(\log\frac{1}{\epsilon}\right)^{1/2}\left(\sqrt{8\pi}u_0 + o(1)\right).\end{aligned}$$

Hence

$$\begin{aligned}\int_\Omega \mathrm{e}^{2\pi \frac{v_\epsilon^2}{\|\nabla v_\epsilon\|_2^2} q\left(\frac{\|v_\epsilon\|_2^2}{\|\nabla v_\epsilon\|_2^2}\right)} \mathrm{d}x &\geq \int_{\psi^{-1}(\overline{\mathbb{B}^+_{\delta\sqrt{\epsilon}}})} \frac{1}{\epsilon} \mathrm{e}^{t_\epsilon \sqrt{\log\frac{1}{\epsilon}}(\sqrt{8\pi}u_0 + o(1))} \mathrm{d}x \\ &\geq C(\delta)\mathrm{e}^{t_\epsilon \sqrt{\log\frac{1}{\epsilon}}(\sqrt{2\pi}u_0(p) + o(1))}\end{aligned}$$

for some positive constant $C(\delta)$. Since $u_0(p) > 0$, then $\int_\Omega \mathrm{e}^{2\pi \frac{v_\epsilon^2}{\|\nabla v_\epsilon\|_2^2} q\left(\frac{\|v_\epsilon\|_2^2}{\|\nabla v_\epsilon\|_2^2}\right)} \mathrm{d}x \to +\infty$ as $\epsilon \to 0$. This completes the proof of Theorem 2. □

4. Adams' inequalities for bi-Laplacian and extremal functions in dimension four

This section reports some recent results on high order Moser's inequalities derived in [10], namely the Adams' inequalities. We refer the reader to [10] for proofs and more details.

Research on finding the sharp constants for higher order Moser's inequality started by the work of Adams [17]. To state Adams' result, we use the symbol $\nabla^m u$, m is a positive integer, to denote the m−th order gradient for $u \in C^m$, the class of m−th order differentiable functions:

$$\nabla^m u = \begin{cases} \triangle^{\frac{m}{2}} u & \text{for } m \text{ even} \\ \nabla \triangle^{\frac{m-1}{2}} u & \text{for } m \text{ odd.} \end{cases}$$

where ∇ is the usual gradient operator and $\triangle$ is the Laplacian. We use $\|\nabla^m u\|_p$ to denote the L^p norm $(1 \leq p \leq \infty)$ of the function $|\nabla^m u|$, the usual Euclidean length of the vector $\nabla^m u$. We also use $W_0^{k,p}(\Omega)$ to denote the Sobolev space which is a completion of $C_0^\infty(\Omega)$ under the norm of $\|u\|_{L^p(\Omega)} + \|\nabla^k u\|_{L^p(\Omega)}$. Then Adams proved the following

Theorem A. *Let Ω be an open and bounded set in $\mathbb{R}^n$. If m is a positive integer less than n, then there exists a constant $C_0 = C(n, m) > 0$ such that for any $u \in W_0^{m,\frac{n}{m}}(\Omega)$ and $\|\nabla^m u\|_{L^{\frac{n}{m}}(\Omega)} \leq 1$, then*

$$\frac{1}{|\Omega|}\int_\Omega \exp(\beta|u(x)|^{\frac{n}{n-m}})\mathrm{d}x \leq C_0$$

for all $\beta \le \beta(n, m)$ where

$$\beta(n,m)=\begin{cases}\dfrac{n}{w_{n-1}}\left[\dfrac{\pi^{n/2}2^m\Gamma(\frac{m+1}{2})}{\Gamma(\frac{n-m+1}{2})}\right]^{\frac{n}{n-m}} & \text{when } m \text{ is odd}\\ \dfrac{n}{w_{n-1}}\left[\dfrac{\pi^{n/2}2^m\Gamma(\frac{m}{2})}{\Gamma(\frac{n-m}{2})}\right]^{\frac{n}{n-m}} & \text{when } m \text{ is even.}\end{cases}$$

Furthermore, for any $\beta > \beta(n, m)$, the integral can be made as large as possible.

Note that $\beta(n, 1)$ coincides with Moser's value of β_0 and $\beta(2m, m) = 2^{2m}\pi^m\Gamma(m + 1)$ for both odd and even m. We are particularly interested in the case $n = 4$ and $m = 2$ in this paper where $\beta(4, 2) = 32\pi^2$.

It has remained an open question whether Adams' inequality has an extremal function, namely, whether the following supremum

$$\sup_{u\in W_0^{m,\frac{n}{m}}(\Omega),\|\nabla^m u\|_{L^{\frac{n}{m}}(\Omega)}\le 1}\frac{1}{|\Omega|}\int_\Omega \exp(\beta|u(x)|^{\frac{n}{n-m}})\mathrm{d}x$$

can be attained. Unlike in the Moser's inequality with first order derivatives, we are unable to adapt Carleson–Chang's idea [6] of symmetrization to establish the existence of extremal functions for inequalities involving high order derivatives. It is still a rather difficult problem to answer the above question in the most generality. Nevertheless, one of the main purposes of this paper is to address this issue and provide an affirmative answer in an important and particularly interesting case when $n = 4$ and $m = 2$, where considerable attention has been paid to the geometric analysis on fourth order differential operators on four manifolds (e.g., see the survey article [18]) and many references therein).

To state our results, let $\Omega \subset \mathbb{R}^n$ denote a smooth oriented bounded domain, $H_0^2(\Omega)$ denote the Sobolev space which is completion of space of smooth functions with compact support under the Dirichlet norm $\|u\|_{H_0^2(\Omega)} = \|\Delta u\|_2$, where $\|\cdot\|_2$ denotes the usual $L^2(\Omega)$-norm. Then Adams' inequality in the case of $n = 4$ and $m = 2$ can be stated as

$$\sup_{\|\Delta u\|_2\le 1}\int_\Omega e^{\gamma u^2}\mathrm{d}x < +\infty \quad \text{for all} \quad \gamma \le 32\pi^2. \tag{4.1}$$

This inequality is optimal in the sense that the corresponding supremum is infinite for any growth $e^{\gamma u^2}$ with $\gamma > 32\pi^2$.

Then we have strengthened in [10] the Adams inequality (4.1). Let

$$\lambda(\Omega) = \inf_{u\in H_0^2(\Omega),u\not\equiv 0}\frac{\|\Delta u\|_2^2}{\|u\|_2^2} \tag{4.2}$$

be the first eigenvalue of the bi-Laplacian operator Δ^2. By a direct method of variation, one can show that $\lambda(\Omega) > 0$. In [10] we have shown that replacing the best constant $32\pi^2$ by $32\pi^2(1 + \alpha\|u\|_2^2)$ for any α: $0 \le \alpha < \lambda(\Omega)$, (4.1) is still valid. More precisely, we proved in [10]

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^4$ be a smooth oriented bounded domain, $\lambda(\Omega)$ be defined by (4.2). Then for any α with $0 \le \alpha < \lambda(\Omega)$, we have*

$$\sup_{u\in H_0^2(\Omega),\|\Delta u\|_2^2=1}\int_\Omega e^{32\pi^2u^2(1+\alpha\|u\|_2^2)}\mathrm{d}x < +\infty. \tag{4.3}$$

The inequality is sharp in the sense that for any growth $e^{32\pi^2u^2(1+\alpha\|u\|_2^2)}$ with $\alpha \ge \lambda(\Omega)$ the supremum is infinite.

The special case of Theorem 4.1 when $\alpha = 0$ is exactly Adams' original inequality (4.1).

Next, we can further generalize Theorem 4.1 to the growth $e^{32\pi^2u^2q(\|u\|_2^2)}$ for some appropriate polynomial $q(t)$ defined on $\mathbb{R}$ with $q(0) = 1$, namely

Theorem 4.1*. *Let $\Omega \subset \mathbb{R}^4$ be a smooth oriented bounded domain, $\lambda(\Omega)$ be defined by (4.2), and $q(t) = 1 + a_1t + a_2t^2 + \cdots + a_kt^k (k \ge 1)$ be a polynomial of order k in $\mathbb{R}$. If $0 \le a_1 < \lambda(\Omega)$, $0 \le a_2 \le \lambda(\Omega)a_1, \ldots, 0 \le a_k \le \lambda(\Omega)a_{k-1}$, then there holds*

$$\sup_{u\in H_0^2(\Omega),\|\Delta u\|_2^2=1}\int_\Omega e^{32\pi^2u^2q(\|u\|_2^2)}\mathrm{d}x < +\infty.$$

If $a_1 \ge \lambda(\Omega)$, and $a_2, \ldots, a_k$ are arbitrary real numbers, then the supremum corresponding to the growth $e^{32\pi^2u^2q(\|u\|_2^2)}$ is infinite.

It is easy to see that Theorem 4.1 is a special case of Theorem 4.1* when $q(t) = 1 + \alpha t$. We have shown in [10] the existence of extremal function for the Adams inequality (4.1) in dimension four.

Theorem 4.2. *Let $\Omega \subset \mathbb{R}^4$ be a smooth oriented bounded domain. There exists $u^* \in H_0^2(\Omega) \cap C^4(\overline{\Omega})$ with $\|\Delta u^*\|_2^2 = 1$ such that*

$$\int_\Omega e^{32\pi^2 u^{*2}} dx = \sup_{u \in H_0^2(\Omega), \|\Delta u\|_2 \le 1} \int_\Omega e^{32\pi^2 u^2} dx.$$

In fact, we have proved the following more general result.

Theorem 4.2*. *Let $\Omega \subset \mathbb{R}^4$ be a smooth oriented bounded domain, $\lambda(\Omega)$ be defined by (4.2), and $q(t) = 1 + a_1 t + a_2 t^2 + \cdots + a_k t^k (k \ge 1)$ be a polynomial of order k in $\mathbb{R}$. If $0 \le a_1 < \lambda(\Omega)$, $0 \le a_2 \le \lambda(\Omega) a_1, \ldots, 0 \le a_k \le \lambda(\Omega) a_{k-1}$, then there exists a strictly positive constant $\epsilon_0 < \lambda(\Omega)$ depending only on Ω such that when $0 \le a_1 \le \epsilon_0$, $0 \le a_2 \le \lambda(\Omega) a_1, \ldots$, and $0 \le a_m \le \lambda(\Omega) a_{m-1}$, we can find $u^* \in H_0^2(\Omega) \cap C^4(\overline{\Omega})$ such that $\|\Delta u^*\|_2^2 = 1$ and*

$$\int_\Omega e^{32\pi^2 u^{*2} q(\|u^*\|_2^2)} dx = \sup_{u \in H_0^2(\Omega),\, \int_\Omega |\Delta u|^2 dx \le 1} \int_\Omega e^{32\pi^2 u^2 q(\|u\|_2^2)} dx.$$

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