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SHARP CONSTANT AND EXTREMAL FUNCTION FOR THE IMPROVED MOSER-TRUDINGER INEQUALITY INVOLVING L^p NORM IN TWO DIMENSION

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Abstract. Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain, and $H_0^1(\Omega)$ be the standard Sobolev space. Define for any $p > 1$,

$$\lambda_p(\Omega) = \inf_{u \in H_0^1(\Omega), u \not\equiv 0} \|\nabla u\|_2^2 / \|u\|_p^2,$$

where $\|\cdot\|_p$ denotes L^p norm. We derive in this paper a sharp form of the following improved Moser-Trudinger inequality involving the L^p-norm using the method of blow-up analysis:

$$\sup_{u \in H_0^1(\Omega), \|\nabla u\|_2 = 1} \int_\Omega e^{4\pi(1+\alpha\|u\|_p^2)u^2} dx < +\infty$$

for $0 \le \alpha < \lambda_p(\Omega)$, and the supremum is infinity for all $\alpha \ge \lambda_p(\Omega)$. We also prove the existence of the extremal functions for this inequality when α is sufficiently small.

1. **Introduction.** Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain, and $H_0^1(\Omega)$ be the completion of $C_0^\infty(\Omega)$ under the norm $\|u\|_{H_0^1(\Omega)} = (\int_\Omega (|u|^2 + |\nabla u|^2) dx)^{1/2}$. The classical Moser-Trudinger inequality [23, 26, 22] states:

$$\sup_{u \in H_0^1(\Omega), \|\nabla u\|_2 = 1} \int_\Omega e^{\alpha u^2} dx < +\infty \tag{1.1}$$

for any $\alpha \le 4\pi$. The supremum is infinity for any $\alpha > 4\pi$. Here and in the sequel, for any real number $q > 1$, $\|\cdot\|_q$ denotes the L^q-norm with respect to the Lebesgue measure.

On the other hand, P. L. Lions [18] proved the following:

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Theorem A. *Assume $u_\epsilon \in H_0^1(\Omega)$, $\|\nabla u_\epsilon\|_2 = 1$ and $u_\epsilon \rightharpoonup u_0$ weakly in $H_0^1(\Omega)$. Then for any $q < 1/(1-\|\nabla u_0\|_2^2)$, we have*

$$\limsup_{\epsilon\to 0} \int_\Omega e^{4\pi q u_\epsilon^2} dx < +\infty. \tag{1.2}$$

Clearly, this result is stronger than inequality (1.1) when $u_\epsilon \rightharpoonup u_0$ weakly in $H_0^1(\Omega)$ with $u_0 \not\equiv 0$. However, for the case $u_\epsilon \rightharpoonup 0$ weakly in $H_0^1(\Omega)$, Adimurthi and Druet [1] proved the following modified Moser-Trudinger inequality involving L^2-norm:

Theorem B. *Let Ω be a bounded smooth domain in $\mathbb{R}^2$ and let*

$$\lambda(\Omega) = \inf_{u\in H_0^1(\Omega), u\not\equiv 0} \|\nabla u\|_2^2/\|u\|_2^2$$

be the first eigenvalue of the Laplacian with Dirichlet boundary condition in Ω. Then we have

(1) *For any $0 \le \alpha < \lambda(\Omega)$,* $\displaystyle\sup_{u\in H_0^1(\Omega), \|\nabla u\|_2=1} \int_\Omega e^{4\pi u^2(1+\alpha\|u\|_2^2)} dx < +\infty$;

(2) *For any $\alpha \ge \lambda(\Omega)$,* $\displaystyle\sup_{u\in H_0^1(\Omega), \|\nabla u\|_2=1} \int_\Omega e^{4\pi u^2(1+\alpha\|u\|_2^2)} dx = +\infty$.

In this paper, we first extend L^2-norm in Theorem B to L^p-norm for any real number $p > 1$. For this purpose, we define

$$\lambda_p(\Omega) = \inf_{u\in H_0^1(\Omega), u\not\equiv 0} \frac{\|\nabla u\|_2^2}{\|u\|_p^2} \tag{1.3}$$

for any $p > 1$. The fact that $\lambda_p(\Omega)$ is attained and $\lambda_p(\Omega) > 0$ will be proved in the next section. Second, we prove the existence of extremal function for α sufficiently small. More precisely, one of the main results in this paper is the following:

Theorem 1.1. *Let Ω be a smooth bounded domain in $\mathbb{R}^2$, and $\lambda_p(\Omega)$ be defined by (1.3) for any $p > 1$. Then we have*

(1) *For any $0 \le \alpha < \lambda_p(\Omega)$,* $\displaystyle\sup_{u\in H_0^1(\Omega), \|\nabla u\|_2=1} \int_\Omega e^{4\pi u^2(1+\alpha\|u\|_p^2)} dx < +\infty$;

(2) *For any $\alpha \ge \lambda_p(\Omega)$,* $\displaystyle\sup_{u\in H_0^1(\Omega), \|\nabla u\|_2=1} \int_\Omega e^{4\pi u^2(1+\alpha\|u\|_p^2)} dx = +\infty$.

When $p = 2$, Theorem 1.1 is exactly Theorem B. Thus, our theorem extends that of [1]. To prove part (2) of Theorem 1.1, we first choose test functions to achieve the goal. Since the test function chosen in [1] does not meet our needs when $p \ne 2$, we will select a test function in our work which is quite different from that of [1], but more similar to that in [27]. We will make more precise comments on this at the end of the introduction. Next, we use blow-up analysis to prove part (1) of Theorem 1.1. The earlier blow-up scheme can be found in [13, 1].

Another fundamental question about Moser-Trudinger inequalities is whether extremal function exists or not. The first result in this direction is due to Carleson and Chang [2] in the case that Ω is a ball in $\mathbb{R}^n$ ($n \ge 2$). Then Flucher [8] extends this result when Ω is a general bounded smooth domain in $\mathbb{R}^2$. Later, Lin [16] generalized the existence result to a bounded smooth domain in $\mathbb{R}^n$. Recently, Li [13, 14], Li-Liu [15] obtained existence results for certain Moser-Trudinger inequalities on compact Riemannian manifolds with or without boundary. More recently, the authors of the current paper derived in [19] the sharpened Adams inequalities for bi-Laplacian and extremal functions in dimension four, and existence of extremal function for Moser-Trudinger inequality for functions with mean value zero in $\mathbb{R}^2$ in [20].

In this paper, we investigate the existence of extremal function for the modified Moser-Trudinger inequality involving L^p-norm, which is another main result of this paper.

Theorem 1.2. *For any fixed $p > 1$, for sufficiently small $\alpha > 0$, there exists $u_\alpha \in H_0^1(\Omega) \cap C^2(\Omega)$ with $\|\nabla u_\alpha\|_2 = 1$ such that*

$$\int_\Omega e^{4\pi(1+\alpha\|u_\alpha\|_p^2)u_\alpha^2}dx = \sup_{u\in H_0^1(\Omega), \|\nabla u\|_2=1} \int_\Omega e^{4\pi(1+\alpha\|u\|_p^2)u^2}dx \tag{1.4}$$

For the Moser-Trudinger inequalities and its extremal functions on Riemannian manifolds, we would like to mention the work by Fontana [9], Ding-Jost-Li-Wang [6], Druet-Hebey [7], and in the sub-Riemannian manifolds by Cohn-Lu [4, 5] and the references therein.

For simplicity, we introduce the notations

$$J_\beta^\alpha(u) = \int_\Omega e^{\beta(1+\alpha\|u\|_p^2)u^2}dx, \tag{1.5}$$

where $p > 1$, and

$$\mathcal{H} = \{u \in H_0^1(\Omega) : \|\nabla u\|_2 = 1\}.$$

Throughout this paper, we do not distinguish sequence and subsequence, the reader can recognize it easily from the context.

We mention in passing the substantial difference between our work and that of [1]. First, as pointed out earlier, our test function chosen to prove part (2) of Theorem 1.1 is significantly different from that of [1]. More precisely, let u_0 be a positive eigenfunction of the Laplacian,

$$\begin{cases} -\Delta u_0 = \lambda_1(\Omega)u_0 \quad \text{in} \quad \Omega \\ u_0 \in H_0^1(\Omega), \quad \|u_0\|_2^2 = 1 \quad u_0 > 0 \quad \text{in} \quad \Omega. \end{cases} \tag{1.6}$$

ϕ_ϵ is the cut-off Green function

$$\phi_\epsilon(x) = \begin{cases} \sqrt{\frac{1}{4\pi}\log\frac{1}{\epsilon}}, \quad |x| \le \sqrt{\epsilon}, \\ \frac{1}{\sqrt{\pi\log\frac{1}{\epsilon}}}\log\frac{1}{|x|}, \quad \sqrt{\epsilon} < |x| \le 1 \\ 0, \quad x \in \Omega\setminus\mathbb{B}_1(0). \end{cases}$$

Setting

$$v_\epsilon = \phi_\epsilon + t_\epsilon u_0 \tag{1.7}$$

with $t_\epsilon \to 0$, $t_\epsilon^2\log\frac{1}{\epsilon} \to +\infty$ and $t_\epsilon^2(\log\frac{1}{\epsilon})^{1/2} \to 0$. Adimurthi-Druet derived for $\alpha \ge \lambda_1(\Omega)$ that

$$\int_\Omega e^{4\pi\frac{v_\epsilon^2}{\|\nabla v_\epsilon\|_2^2}\left(1+\alpha\frac{\|v_\epsilon\|_2^2}{\|\nabla v_\epsilon\|_2^2}\right)}dx \to +\infty$$

as $\epsilon \to 0$. This completes the proof of (2) of theorem B.

However, our test function is more involved. To describe without loss of generality, we assume that $0 \in \Omega$ and $\mathbb{B}_1 \subset \Omega$. For any $\delta > 0$, we fix some $x_\delta \in \Omega$ such that $|x_\delta| = \delta$. Choose $t_\epsilon > 0$ as above and let

$$\phi_\epsilon(x) = \begin{cases} \sqrt{\frac{1}{2\pi}\log\frac{1}{\epsilon}}, & |x| < \epsilon \\ \frac{\sqrt{\frac{1}{2\pi}\log\frac{1}{\epsilon}}(\log\delta - \log|x|) - t_\epsilon\phi_0(x_\delta)(\log\epsilon - \log|x|)}{\log\delta - \log\epsilon}, & \epsilon \le |x| \le \delta \\ t_\epsilon\left[\phi_0(x_\delta) + \eta(x)\cdot(\phi_0(x) - \phi_0(x_\delta))\right], & |x| > \delta, \end{cases} \tag{1.8}$$

where ϕ_0 is the eigenfunction of the nonlinear equation (2.1) and $\eta \in C^\infty(\Omega)$ is a cut-off function (see Section 2 for more details).

The new idea to construct (1.8) is based on two facts: (i) ϕ_0 satisfies a nonlinear equation (2.1) (while (1.6) is a linear equation); (ii) The decomposition of $\|v_\epsilon\|_p^2$ in terms of t_ϵ and $\log\frac{1}{\epsilon}$ does not meet our needs when $p \neq 2$. The test function in [1] v_ϵ does not work here. To overcome these difficulties, we take cut-off Green function inside and eigenfunction outside. This enable us to decompose $\|v_\epsilon\|_p^2$ explicitly. Here we use the new $v_\epsilon = \phi_\epsilon/\|\nabla\phi_\epsilon\|_2$. Then by a delicate calculation, we have

$$\lambda_p(\Omega)\|v_\epsilon\|_p^2 \ge t_\epsilon^2(1 + O(t_\epsilon^2) + O(\delta^2)).$$

By a further careful choice of t_ϵ and δ, we arrive at the conclusion (2) of Theorem 1.1. We refer the reader to Section 2 for more details.

Second, We caution the reader that the method we use to prove the conclusion (1) of Theorem 1.1 is different from that used in [1] to handle the more complicated case of $p \neq 2$. To prove (1) of Theorem B, they considered in [1] the minimizers u_ϵ of the subcritical Moser-Trudinger functional

$$\begin{cases} -\Delta u_\epsilon = \frac{\beta_\epsilon}{\lambda_\epsilon}u_\epsilon e^{\alpha_\epsilon u_\epsilon^2} + \gamma_\epsilon u_\epsilon \\ u_\epsilon \in H_0^1(\Omega), \quad \|\nabla u_\epsilon\|_2 = 1, u > 0 \,\mathrm{in}\,\Omega \\ \alpha_\epsilon = (4\pi - \epsilon)(1 + \alpha\|u_\epsilon\|_2^2) \\ \beta_\epsilon = (1 + \alpha\|u_\epsilon\|_2^2)/(1 + 2\alpha\|u_\epsilon\|_2^2) \\ \gamma_\epsilon = \alpha/(1 + 2\alpha\|u_\epsilon\|_2^2) \\ \lambda_\epsilon = \int_\Omega u_\epsilon^2 e^{\alpha_\epsilon u_\epsilon^2}dx \end{cases} \tag{1.9}$$

By blowing up analysis, they prove that if $c_\epsilon = \max_\Omega u_\epsilon \to +\infty$, then $c_\epsilon u_\epsilon \to G$ in $L^2(\Omega)$ for some Green function (see earlier work in [13]). This leads to the conclusion (1) of Theorem B.

In our case, for any $p > 1$, u_ϵ satisfying (3.1) in Section 3. We derive an upper bound of the sharp Moser-Trudinger functional in case of blow-up (Step 1 of the proof of Theorem 1.1, see Section 3), which was not considered in [1]. It is known that such an upper bound together with another test function computation may lead to the existence result of the extremal functions.

Third, we derive the existence of extremal function of the Moser-Trudinger inequality for all $p > 1$. Thus, as a corollary of our existence result, we also establish the existence of extremal function for the inequality in Theorem B which was not considered in [1].

We finally remark here that results proved in this paper also hold for two dimensional Riemannian manifolds with or without boundaries by modifying the techniques given in this paper.

The organization of this paper is as follows: In Section 2, we construct test functions to prove part (2) of theorem 1.1. In Section 3, we consider the relevant Euler-Lagrange equation for the maximizers of the subcritical functional $J^{\alpha}_{4\pi-\epsilon}$ and deal with the asymptotic behavior of the maximizers through blow-up analysis. This leads to the proof of part (1) of Theorem 1.1. Section 4 gives the proof of the existence of extremal function for the modified Moser-Trudinger inequality, namely, Theorem 1.2.

2. **Proof of Part (2) of Theorem 1.1.** In this section, we select test functions to prove Part (2) of Theorem 1.1. The test functions we will construct here is quite different from that of [1]. Let $\lambda_p(\Omega)$ be defined by (1.3).

We begin with the following:

Lemma 2.1. *For any $p>1$, we have $\lambda_p(\Omega)>0$ and $\lambda_p(\Omega)$ is attained by a function $\phi_0\in H^1_0(\Omega)\cap C^\infty(\Omega)$ satisfying*

$$\begin{cases} -\Delta\phi_0=\lambda_p(\Omega)\|\phi_0\|_p^{2-p}\phi_0^{p-1} \quad \text{in} \quad \Omega \\ \|\nabla\phi_0\|_2=1, \quad \phi_0>0 \quad \text{in} \quad \Omega. \end{cases} \tag{2.1}$$

Proof. The proof is based on the direct method of variation. Given any $p>1$, choose a sequence of functions $u_k\in H^1_0(\Omega)$ such that $\|u_k\|_p=1$ and $\|\nabla u_k\|_2^2\to\lambda_p(\Omega)$. Hence u_k is bounded in $H^1_0(\Omega)$. Without loss of generality, we assume

$$\begin{aligned} u_k &\rightharpoonup u_0 &&\text{weakly in} && H^1_0(\Omega),\\ u_k &\to u_0 &&\text{strongly in} && L^p(\Omega). \end{aligned}$$

It follows that $\|u_0\|_p=1$. Since

$$\int_\Omega|\nabla u_0|^2dx=\int_\Omega\nabla u_0\nabla(u_0-u_k)dx+\int_\Omega\nabla u_0\nabla u_kdx,$$

we have

$$\int_\Omega|\nabla u_0|^2dx\le\limsup_{k\to+\infty}\int_\Omega|\nabla u_k|^2dx=\lambda_p(\Omega).$$

Thus $\lambda_p(\Omega)=\|\nabla u_0\|_2^2/\|u_0\|_p^2$, and whence $\lambda_p(\Omega)>0$ for otherwise $u_0\equiv 0$, which contradicts the fact $\|u_0\|_p=1$. Since $\|\nabla|u_0|\|_2^2\le\|\nabla u_0\|_2^2$, $|u_0|$ also attains

$$\inf_{u\in H^1_0(\Omega),u\not\equiv 0}\frac{\|\nabla u\|_2^2}{\|u\|_p^2}.$$

Set $\phi_0=|u_0|/\|\nabla|u_0|\|_2$. Then ϕ_0 attains the above infimum and satisfies the Euler-Lagrange equation (2.1). By the elliptic estimates ([11], Chapter 9), $\phi_0\in C^\infty(\Omega)$. The maximum principle implies that $\phi_0>0$ in Ω $\square$

Proof of Part (2) *of Theorem* 1.1. Here and in the sequel, we always assume $p>1$. Without loss of generality, we assume that $0\in\Omega$ and $\mathbb{B}_1\subset\Omega$. For any $\delta>0$, we fix some $x_\delta\in\Omega$ such that $|x_\delta|=\delta$. Choosing $t_\epsilon>0$ such that $t_\epsilon^2\log\frac{1}{\epsilon}\to+\infty$ and

$t_\epsilon^2\sqrt{\log\frac{1}{\epsilon}} \to 0$. Let

$$\phi_\epsilon(x) = \begin{cases} \sqrt{\frac{1}{2\pi}\log\frac{1}{\epsilon}}, & |x| < \epsilon \\ \frac{\sqrt{\frac{1}{2\pi}\log\frac{1}{\epsilon}}(\log\delta - \log|x|) - t_\epsilon\phi_0(x_\delta)(\log\epsilon - \log|x|)}{\log\delta - \log\epsilon}, & \epsilon \le |x| \le \delta \\ t_\epsilon\left[\phi_0(x_\delta) + \eta(x)\cdot(\phi_0(x) - \phi_0(x_\delta))\right], & |x| > \delta, \end{cases}$$

where ϕ_0 is described in Lemma 2.1, $\eta \in C^\infty(\Omega)$ satisfying $|\nabla\eta| \le 2/\delta$ and

$$\eta(x) = \begin{cases} 0, & |x| \le \delta \\ 0 < \eta < 1, & \delta < |x| < 2\delta \\ 1, & |x| \ge 2\delta. \end{cases}$$

Taking $\delta = \frac{1}{t_\epsilon\sqrt{\log\frac{1}{\epsilon}}}$, we have

$$\begin{aligned} \int_{\epsilon\le|x|\le\delta} |\nabla\phi_\epsilon|^2 dx &= \int_{\epsilon\le|x|\le\delta} \frac{|-\sqrt{\frac{1}{2\pi}\log\frac{1}{\epsilon}} + t_\epsilon\phi_0(x_\delta)|^2}{|x|^2(\log\delta - \log\epsilon)^2} dx \\ &= \frac{2\pi(t_\epsilon\phi_0(x_\delta) - \sqrt{\frac{1}{2\pi}\log\frac{1}{\epsilon}})^2}{\log\delta - \log\epsilon} \\ &= 1 - \frac{2t_\epsilon}{\sqrt{\frac{1}{2\pi}\log\frac{1}{\epsilon}}}\phi_0(x_\delta)(1 + o(1)), \end{aligned}$$

$$\int_{\delta\le|x|\le2\delta} |\nabla\phi_\epsilon|^2 dx = t_\epsilon^2 O(\delta^2),$$
$$\int_{|x|>2\delta} |\nabla\phi_\epsilon|^2 dx = t_\epsilon^2 \int_{|x|>2\delta} |\nabla\phi_0|^2 dx = t_\epsilon^2(1 + O(\delta^2)).$$

Hence

$$\int_\Omega |\nabla\phi_\epsilon|^2 dx = 1 - \frac{2t_\epsilon}{\sqrt{\frac{1}{2\pi}\log\frac{1}{\epsilon}}}\phi_0(x_\delta)(1 + o(1)) + t_\epsilon^2(1 + O(\delta^2)).$$

Let $v_\epsilon = \phi_\epsilon/\|\nabla\phi_\epsilon\|_2$. Then we have $\|\nabla v_\epsilon\|_2 = 1$, and

$$\begin{aligned} \lambda_p(\Omega)\|v_\epsilon\|_p^2 &\ge \frac{\lambda_p(\Omega)t_\epsilon^2}{\|\nabla\phi_\epsilon\|_2^2}\left(\int_{|x|\ge2\delta}\phi_0^p dx\right)^{2/p} \\ &= \lambda_p(\Omega)t_\epsilon^2(\|\phi_0\|_p^2 + O(\delta^2))\{1 + \frac{2t_\epsilon}{\sqrt{\frac{1}{2\pi}\log\frac{1}{\epsilon}}}\phi_0(x_\delta)(1 + o(1)) \\ &\quad -t_\epsilon^2(1 + O(\delta^2))\} \\ &= t_\epsilon^2(\lambda_p(\Omega)\|\phi_0\|_p^2 + O(\delta^2))(1 + O(t_\epsilon^2)) \\ &= t_\epsilon^2(1 + O(t_\epsilon^2) + O(\delta^2)). \end{aligned}$$

Here we have used the fact that $\lambda_p(\Omega)\|\phi_0\|_p^2 = \|\nabla\phi_0\|_2^2 = 1$.

On the domain $\{x \in \Omega : |x| < \epsilon\}$, we have

$$
\begin{aligned}
& 4\pi(1+\lambda_p(\Omega)\|v_\epsilon\|_p^2)v_\epsilon^2(x) \\
\geq\ & 2\log\frac{1}{\epsilon}(1+t_\epsilon^2(1+O(t_\epsilon^2)+O(\delta^2))) \\
& \times\{1+(2t_\epsilon/\sqrt{\frac{1}{2\pi}\log\frac{1}{\epsilon}})\phi_0(x_\delta)(1+o(1))-t_\epsilon^2(1+O(\delta^2))\} \\
=\ & 2\log\frac{1}{\epsilon}+4\sqrt{2\pi}\sqrt{\log\frac{1}{\epsilon}}\phi_0(x_\delta)(1+o(1)) \\
& +2t_\epsilon^2\log\frac{1}{\epsilon}(1+O(t_\epsilon^2)+O(\delta^2)).
\end{aligned}
$$

Note that $\phi_0(x_\delta)=\phi_0(0)+o(1)$, $t_\epsilon^2\log\frac{1}{\epsilon}O(\delta^2)=O(1)$, we obtain

$$\int_\Omega e^{4\pi(1+\lambda_p(\Omega)\|v_\epsilon\|_p^2)v_\epsilon^2}dx \geq Ce^{4\sqrt{2\pi\log\frac{1}{\epsilon}}\phi_0(0)(1+o(1))} \to +\infty$$

as $\epsilon\to 0$, where C is a positive constant independent of ϵ. Hence Part (2) of Theorem 1.1 follows. □

3. **Proof of Part (1) of Theorem 1.1.** In this section we prove Part (1) of Theorem 1.1. Let $\Omega\subset\mathbb{R}^2$ be a smooth bounded domain and $0\leq\alpha<\lambda_p(\Omega)$.

Step 1. *Existence of maximizer for $J^\alpha_{4\pi-\epsilon}$ and the Euler-Lagrange equation.*

Given any $\epsilon>0$, we take $u_\epsilon\in C^\infty(\Omega)\cap\mathcal{H}$ such that

$$J^\alpha_{4\pi-\epsilon}(u_\epsilon)=\sup_{u\in\mathcal{H}}J^\alpha_{4\pi-\epsilon}(u)$$

where we recall that

$$J^\alpha_\beta(u)=\int_\Omega e^{\beta(1+\alpha\|u\|_p^2)u^2}dx$$

for $p>1$ and

$$\mathcal{H}=\{u\in H_0^1(\Omega):\|\nabla u\|_2=1\}.$$

The existence of u_ϵ is based on the direct method of variation and the elliptic estimates. Thus the proof is similar to that in the case $p=2$ and we omit the proof here but refer the reader to [1] for details.

Furthermore, after a careful calculation one can check that the Euler-Lagrange equation of u_ϵ is

$$
\begin{cases}
-\Delta u_\epsilon=\frac{\beta_\epsilon}{\lambda_\epsilon}u_\epsilon e^{\alpha_\epsilon u_\epsilon^2}+\gamma_\epsilon\|u_\epsilon\|_p^{2-p}u_\epsilon^{p-1} \quad \text{in} \quad \Omega \\
\|\nabla u_\epsilon\|_2=1, \quad u_\epsilon>0 \quad \text{in} \quad \Omega \\
\alpha_\epsilon=(4\pi-\epsilon)(1+\alpha\|u_\epsilon\|_p^2) \\
\beta_\epsilon=(1+\alpha\|u_\epsilon\|_p^2)/(1+2\alpha\|u_\epsilon\|_p^2) \\
\gamma_\epsilon=\alpha/(1+2\alpha\|u_\epsilon\|_p^2) \\
\lambda_\epsilon=\int_\Omega u_\epsilon^2 e^{\alpha_\epsilon u_\epsilon^2}dx.
\end{cases}
\tag{3.1}
$$

Step 2. *The case when u_ϵ is uniformly bounded in ϵ.*

Let $c_\epsilon = u_\epsilon(x_\epsilon) = \max_\Omega u_\epsilon$. We first assume that $\{c_\epsilon\}$ is a bounded sequence as $\epsilon \to 0$. Since for any $1 < q < \frac{p}{p-1}$, Holder's inequality implies

$$\left(\int_\Omega (u_\epsilon^{p-1})^q dx\right)^{1/q} \le \|u_\epsilon\|_p^{p-1}|\Omega|^{\frac{1}{q}+\frac{1}{p}-1},$$

and thus

$$\left(\int_\Omega (\gamma_\epsilon \|u_\epsilon\|_p^{2-p} u_\epsilon^{p-1})^q dx\right)^{\frac{1}{q}} \le \gamma_\epsilon \|u_\epsilon\|_p |\Omega|^{\frac{1}{q}+\frac{1}{p}-1}, \tag{3.2}$$

which together with (3.1) implies that Δu_ϵ is bounded in $L^q(\Omega)$ for some $1 < q < p/(p-1)$ because c_ϵ is bounded. Hence $u_\epsilon \to u^*$ in $C^1(\Omega)$ for some $u^* \in \mathcal{H}$ by the standard elliptic estimates ([11], Chapter 9), and Theorem 1.1 follows immediately from the easy fact

$$\lim_{\epsilon \to 0} J^\alpha_{4\pi-\epsilon}(u_\epsilon) = \sup_{u \in \mathcal{H}} J^\alpha_{4\pi}(u). \tag{3.3}$$

Step 3. *Asymptotic behavior of the maximizers u_ϵ when u_ϵ is not uniformly bounded in ϵ.*

We will now use blow-up analysis to understand the asymptotic behavior of the maximizers u_ϵ. We proceed in the spirit of [13] and [1]. We assume

$$x_\epsilon \to x_0 \in \overline{\Omega}, \quad u_\epsilon(x_\epsilon) \to +\infty \tag{3.4}$$

as $\epsilon \to 0$.

We first claim that x_0 *can not lie on the boundary $\partial\Omega$.*

Using equation (3.1), we have

$$-\Delta u_\epsilon = \frac{\beta_\epsilon}{\lambda_\epsilon} u_\epsilon e^{\alpha_\epsilon u_\epsilon^2} + \gamma_\epsilon \|u_\epsilon\|_p^{2-p} u_\epsilon^{p-1},\ u_\epsilon > 0 \quad \text{in} \quad \Omega, \qquad u_\epsilon = 0 \quad \text{on} \quad \partial\Omega,$$

where $\alpha_\epsilon, \gamma_\epsilon, \beta_\epsilon$ are positive constants depending on ϵ as defined in (3.1). Thus, u_ϵ satisfies

$$-\Delta u = f_\epsilon(u)$$

where

$$f_\epsilon(u) = \frac{\beta_\epsilon}{\lambda_\epsilon} u e^{\alpha_\epsilon u^2} + \gamma_\epsilon \|u\|_p^{2-p} u^{p-1} > 0 \quad \text{in} \quad \Omega$$

Similar to the argument indicated in [1], we have by using the results of Gidas-Ni-Nirenberg [10] (see page 223 of [10]) that there is some $\delta > 0$ depending only on Ω (independent of f_ϵ and u_ϵ) such that u_ϵ has no stationary point in the $\delta-$neighbourhood of $\partial\Omega$. Therefore, x_0 can not lie on the boundary $\partial\Omega$. As a result, we have excluded the boundary blow-up[1].

From now on, we assume $x_0 \in \Omega$.

Sub-Step 3.1. *A Lions type Lemma of concentration compactness.*

[1]We thank the referee for pointing out this argument.

The following concentration phenomenon is crucial in our blow-up analysis:

Lemma 3.1. *For the sequence $\{u_\epsilon\}$ we have that $u_\epsilon \rightharpoonup 0$ weakly in $H_0^1(\Omega)$, $u_\epsilon \to 0$ strongly in $L^q(\Omega)$ for $q > 1$, $|\nabla u_\epsilon|^2 dx \rightharpoonup \delta_{x_0}$ in sense of measure, where δ_{x_0} is the Dirac measure at x_0. Furthermore, we have $\alpha_\epsilon \to 4\pi$, $\beta_\epsilon \to 1$ and $\gamma_\epsilon \to \alpha$.*

Proof of Lemma 3.1. Since $\|\nabla u_\epsilon\|_2 = 1$ and $u_\epsilon \in H_0^1(\Omega)$, we may assume

$$u_\epsilon \rightharpoonup u_0 \quad \text{weakly in} \quad H_0^1(\Omega),$$
$$u_\epsilon \to u_0 \quad \text{strongly in} \quad L^q(\Omega)$$

for any $q > 1$.

Suppose $u_0 \neq 0$, then we have for any $0 \leq \alpha < \lambda_p(\Omega)$,

$$1 + \alpha \|u_\epsilon\|_p^2 \to 1 + \alpha \|u_0\|_p^2 \leq 1 + \|\nabla u_0\|_2^2 < \frac{1}{1 - \|\nabla u_0\|_2^2}.$$

Thus by Theorem B of Lions we conclude that $e^{\alpha_\epsilon u_\epsilon^2}$ is bounded in $L^r(\Omega)$ for some $r > 1$ provided that ϵ is sufficiently small, which together with (3.2) implies that Δu_ϵ is bounded in $L^{q_0}(\Omega)$ for some $q_0 > 1$. Employing the elliptic estimates to (3.1), one gets u_ϵ is uniformly bounded, which contradicts (3.4). Therefore $u_0 = 0$, and consequently $\alpha_\epsilon \to 4\pi$, $\beta_\epsilon \to 1$ and $\gamma_\epsilon \to \alpha$.

Assume $|\nabla u_\epsilon|^2 dx \rightharpoonup \mu$ in the sense of measure. Note that $\|\nabla u_\epsilon\|_2^2 = 1$ and $u_\epsilon \to 0$ strongly in $L^q(\Omega)$ for any $q > 1$. If $\mu \neq \delta_{x_0}$, we can choose a cut-off function $\phi \in C_0^1(\Omega)$, which is supported in $\mathbb{B}_{r_0}(x_0) \subset \Omega$ and equal to 1 in $\mathbb{B}_{r_0/2}(x_0)$ for some small $r_0 > 0$ such that

$$\int_{\mathbb{B}_{r_0}(x_0)} |\nabla(\phi u_\epsilon)|^2 dx \leq 1 - \eta$$

for some $\eta > 0$ provided that ϵ is sufficiently small. By the classical Moser-Trudinger inequality (1.1), $e^{\alpha_\epsilon (\phi u_\epsilon)^2}$ is bounded in $L^s(\Omega)$ for some $s > 1$. Then the elliptic estimate on the Euler-Lagrange equation (3.1) implies that u_ϵ is bounded in $\mathbb{B}_{r_0/2}(x_0)$, which contradicts (3.4). Therefore, $|\nabla u_\epsilon|^2 dx \rightharpoonup \delta_{x_0}$. □

Sub-Step 3.2. *Asymptotic behavior of u_ϵ near the concentration point x_0.*

Let

$$r_\epsilon = \sqrt{\lambda_\epsilon \beta_\epsilon^{-1}} c_\epsilon^{-1} e^{-\frac{1}{2}\alpha_\epsilon c_\epsilon^2}. \tag{3.5}$$

Since $\alpha_\epsilon \to 4\pi$ and $\|\nabla u_\epsilon\|_2^2 = 1$, we have by Hölder inequality and Moser-Trudinger inequality (1.1),

$$\lambda_\epsilon = \int_\Omega u_\epsilon^2 e^{\alpha_\epsilon u_\epsilon^2} dx \leq e^{\frac{1}{4}\alpha_\epsilon c_\epsilon^2} \int_\Omega u_\epsilon^2 e^{\frac{3}{4}\alpha_\epsilon u_\epsilon^2} dx \leq C e^{\frac{1}{4}\alpha_\epsilon c_\epsilon^2}$$

for some constant C independent of ϵ. This together with (3.5) implies

$$r_\epsilon^2 e^{\frac{3}{4}\alpha_\epsilon c_\epsilon^2} \to 0 \quad \text{as} \quad \epsilon \to 0. \tag{3.6}$$

Denote

$$\Omega_\epsilon = \{x \in \mathbb{R}^2 : x_\epsilon + r_\epsilon x \in \Omega\}.$$

Define the blowing up functions

$$\psi_\epsilon(x) = c_\epsilon^{-1} u_\epsilon(x_\epsilon + r_\epsilon x), \tag{3.7}$$

$$\varphi_\epsilon(x) = c_\epsilon(u_\epsilon(x_\epsilon + r_\epsilon x) - c_\epsilon). \tag{3.8}$$

A direct computation gives

$$-\Delta\psi_\epsilon(x) = c_\epsilon^{-2}\psi_\epsilon e^{\alpha_\epsilon(u_\epsilon^2-c_\epsilon^2)} + r_\epsilon^2\gamma_\epsilon c_\epsilon^{p-2}\|u_\epsilon\|_p^{2-p}\psi_\epsilon^{p-1} \quad \text{in} \quad \Omega_\epsilon, \tag{3.9}$$

$$-\Delta\varphi_\epsilon(x) = \psi_\epsilon(x)e^{\alpha_\epsilon(1+\psi_\epsilon)\varphi_\epsilon} + r_\epsilon^2 c_\epsilon^p\gamma_\epsilon\|u_\epsilon\|_p^{2-p}\psi_\epsilon^{p-1} \quad \text{in} \quad \Omega_\epsilon. \tag{3.10}$$

By (3.6) and (3.7),

$$\begin{aligned}\left(\int_{B_R(0)}(r_\epsilon^2 c_\epsilon^p\gamma_\epsilon\|u_\epsilon\|_p^{2-p}\psi_\epsilon^{p-1})^{\frac{p}{p-1}}dx\right)^{\frac{p-1}{p}} &= r_\epsilon^{\frac{2}{p}}c_\epsilon^p\gamma_\epsilon\|u_\epsilon\|_p^{2-p}\|u_\epsilon\|_{L^p(B_{Rr_\epsilon}(x_\epsilon))}^{p-1}\\ &\le r_\epsilon^{\frac{2}{p}}\gamma_\epsilon c_\epsilon^p\|u_\epsilon\|_p \to 0.\end{aligned}$$

Applying the elliptic estimates ([11], Chapter 9) to (3.9) and (3.10), we have

$$\psi_\epsilon \to 1 \quad \text{in} \quad C^2_{loc}(\mathbb{R}^2), \tag{3.11}$$

$$\varphi_\epsilon \to \varphi \quad \text{in} \quad C^2_{loc}(\mathbb{R}^2), \tag{3.12}$$

where φ satisfies

$$\begin{cases} \Delta\varphi = -e^{8\pi\varphi} \quad \text{in} \quad \mathbb{R}^2\\ \varphi(0) = 0 = \sup\varphi\\ \int_{\mathbb{R}^2} e^{8\pi\varphi}dx \le 1. \end{cases} \tag{3.13}$$

Here we have used the definition of λ_ϵ, $\beta_\epsilon \to 1$ and the fact that for any fixed $R > 0$,

$$\int_{\mathbb{B}_R(0)} e^{8\pi\varphi}dx = \lim_{\epsilon\to 0}\frac{\beta_\epsilon}{\lambda_\epsilon}\int_{\mathbb{B}_{Rr_\epsilon}(x_\epsilon)} u_\epsilon^2(y)e^{\alpha_\epsilon u_\epsilon^2(y)}dy.$$

The uniqueness theorem obtained in [3] implies that

$$\varphi(x) = -\frac{1}{4\pi}\log(1+\pi|x|^2), \tag{3.14}$$

and

$$\int_{\mathbb{R}^2} e^{8\pi\varphi}dx = 1. \tag{3.15}$$

Sub-Step 3.3. *Convergence away from the concentration point.*

Similar to [13, 1], define $u_{\epsilon,\beta} = \min\{\beta c_\epsilon, u_\epsilon\}$, then we have

Lemma 3.2. For $0 < \beta < 1$, we have $\limsup_{\epsilon\to 0}\|\nabla u_{\epsilon,\beta}\|_2^2 \le \beta$.

Proof of Lemma 3.2. For any $0 < q < 1$, we have by the equation (3.1) and the divergence theorem,

$$\begin{aligned}\int_\Omega |\nabla(u_\epsilon - \beta c_\epsilon)^+|^2 dx &= \int_\Omega \nabla(u_\epsilon-\beta c_\epsilon)^+\nabla u_\epsilon dx\\ &= \int_\Omega (u_\epsilon-\beta c_\epsilon)^+\left(\frac{\beta_\epsilon}{\lambda_\epsilon}u_\epsilon e^{\alpha_\epsilon u_\epsilon^2} + \gamma_\epsilon\|u_\epsilon\|_p^{2-p}u_\epsilon^{p-1}\right)dx\\ &\ge \int_{B_{Rr_\epsilon}(x_\epsilon)}(u_\epsilon-\beta c_\epsilon)^+\left(\frac{\beta_\epsilon}{\lambda_\epsilon}u_\epsilon e^{\alpha_\epsilon u_\epsilon^2} + \gamma_\epsilon\|u_\epsilon\|_p^{2-p}u_\epsilon^{p-1}\right)dx.\end{aligned}$$

This inequality together with (3.11), (3.15) and the facts that $u_\epsilon > \beta c_\epsilon$ on $B_{Rr_\epsilon}(x_\epsilon)$, which is due to (3.8) and (3.12), and

$$\|(u_\epsilon-\beta c_\epsilon)^+\gamma_\epsilon\|u_\epsilon\|_p^{2-p}u_\epsilon^{p-1}\|_{L^1(B_{Rr_\epsilon(x_\epsilon)})} \le \gamma_\epsilon\|u_\epsilon\|_p^2 \to 0,$$

gives that

$$\int_\Omega |\nabla(u_\epsilon - \beta c_\epsilon)^+|^2 dx \geq (1-\beta)\int_{\mathbb{R}^2} e^{8\pi\varphi}dx + o_\epsilon(1) = 1-\beta+o_\epsilon(1).$$

Hence

$$\limsup_{\epsilon\to 0}\|\nabla u_{\epsilon,\beta}\|_2^2 = 1 - \liminf_{\epsilon\to 0}\|\nabla(u_\epsilon-\beta c_\epsilon)^+\|_2^2 \leq \beta.$$

□

Though the following estimate is not used in Step 3, it is a byproduct of Lemma 3.2 and will be employed in the next Section.

Lemma 3.3. $\lim_{\epsilon\to 0}\int_\Omega e^{\alpha_\epsilon u_\epsilon^2}dx \leq |\Omega| + \lim_{R\to+\infty}\limsup_{\epsilon\to 0}\int_{B_{Rr_\epsilon}(x_\epsilon)} e^{\alpha_\epsilon u_\epsilon^2}dx.$

Proof of Lemma 3.3. For any $0<\beta<1$,

$$\begin{aligned}\int_\Omega e^{\alpha_\epsilon u_\epsilon^2}dx &= \int_{u_\epsilon<\beta c_\epsilon} e^{\alpha_\epsilon u_\epsilon^2}dx + \int_{u_\epsilon\geq\beta c_\epsilon} e^{\alpha_\epsilon u_\epsilon^2}dx\\ &\leq \int_\Omega e^{\alpha_\epsilon u_{\epsilon,\beta}^2}dx + \frac{\lambda_\epsilon}{\beta^2 c_\epsilon^2}.\end{aligned}$$

By Lemma 3.2, $e^{\alpha_\epsilon u_{\epsilon,\beta}^2}$ is bounded in $L^q(\Omega)$ for some $q>1$. Through the proof of Lemma 3.1 one can see that $u_\epsilon\to 0$ in $C^1_{loc}(\Omega\setminus\{x_0\})$, and whence $\int_\Omega e^{\alpha_\epsilon u_{\epsilon,\beta}^2}dV_g \to |\Omega|$ as $\epsilon\to 0$. Therefore

$$\int_\Omega e^{\alpha_\epsilon u_\epsilon^2}dx \leq |\Omega| + \frac{\lambda_\epsilon}{\beta^2 c_\epsilon^2} + o_\epsilon(1),$$

where $o_\epsilon(1)\to 0$ as $\epsilon\to 0$. Letting $\epsilon\to 0$ first, then $\beta\to 1$, we obtain

$$\lim_{\epsilon\to 0}\int_\Omega e^{\alpha_\epsilon u_\epsilon^2}dx \leq |\Omega| + \limsup_{\epsilon\to 0}\frac{\lambda_\epsilon}{c_\epsilon^2}.$$

On the other hand, we have by (3.12) that

$$\int_{B_{Rr_\epsilon}(x_\epsilon)} e^{\alpha_\epsilon u_\epsilon^2}dx = \frac{\lambda_\epsilon}{\beta_\epsilon c_\epsilon^2}\left(\int_{B_R(0)} e^{8\pi\varphi}dx + o_\epsilon(R)\right),$$

where $o_\epsilon(R)\to 0$ as $\epsilon\to 0$ for any fixed $R>0$. By (3.15) and $\beta_\epsilon\to 1$,

$$\lim_{R\to+\infty}\limsup_{\epsilon\to 0}\int_{B_{Rr_\epsilon}(x_\epsilon)} e^{\alpha_\epsilon u_\epsilon^2}dx = \limsup_{\epsilon\to 0}\frac{\lambda_\epsilon}{c_\epsilon^2}.$$

Therefore

$$\lim_{\epsilon\to 0}\int_\Omega e^{\alpha_\epsilon u_\epsilon^2}dx \leq |\Omega| + \lim_{R\to+\infty}\limsup_{\epsilon\to 0}\int_{B_{Rr_\epsilon}(x_\epsilon)} e^{\alpha_\epsilon u_\epsilon^2}dx.$$

□

Using the similar idea of Lemma 3.7 in [13] and (3.26) in [1], one can prove without any difficulty that

$$\lim_{\epsilon\to 0}\int_\Omega \phi\frac{\beta_\epsilon}{\lambda_\epsilon}c_\epsilon u_\epsilon e^{\alpha_\epsilon u_\epsilon^2}dx = \phi(x_0), \quad \forall\,\phi\in C^1(\Omega). \tag{3.16}$$

The following result can be found in [25]:

Lemma 3.4 (Struwe). *If $f \in L^1(\Omega)$, and $u \in H_0^1(\Omega) \cap C^1(\Omega)$ is a positive solution of $-\Delta u = f$. Then for any $1 < q < 2$, $\|\nabla u\|_q \le C\|f\|_1$ for some constant C depending only on q and Ω.*

Using Lemma 3.4, we can prove the following:

Lemma 3.5. *For any $1 < q < 2$, $c_\epsilon u_\epsilon$ is bounded in $H_0^{1,q}(\Omega)$.*

Proof of Lemma 3.5. By (3.1),

$$-\Delta(c_\epsilon u_\epsilon) = \frac{\beta_\epsilon}{\lambda_\epsilon} c_\epsilon u_\epsilon e^{\alpha_\epsilon u_\epsilon^2} + \gamma_\epsilon \|c_\epsilon u_\epsilon\|_p^{2-p}(c_\epsilon u_\epsilon)^{p-1} \quad \text{in} \quad \Omega. \tag{3.17}$$

We claim that $\|c_\epsilon u_\epsilon\|_p$ is bounded. Suppose not, we can assume that $\|c_\epsilon u_\epsilon\|_p \to +\infty$ as $\epsilon \to 0$. Let $w_\epsilon = c_\epsilon u_\epsilon / \|c_\epsilon u_\epsilon\|_p$. Then we have $\|w_\epsilon\|_p = 1$ and

$$-\Delta w_\epsilon = \frac{1}{\|c_\epsilon u_\epsilon\|_p} \frac{\beta_\epsilon}{\lambda_\epsilon} c_\epsilon u_\epsilon e^{\alpha_\epsilon u_\epsilon^2} + \gamma_\epsilon \|w_\epsilon\|_p^{1-p} w_\epsilon^{p-1} \quad \text{in} \quad \Omega. \tag{3.18}$$

It can be deduced from (3.16) and the definition of w_ϵ that Δw_ϵ is bounded in $L^1(\Omega)$. By Lemma 3.4, w_ϵ is bounded in $H_0^{1,q}(\Omega)$ for any $1 < q < 2$. Assume $w_\epsilon \rightharpoonup w$ weakly in $H_0^{1,q}(\Omega)$, and $w_\epsilon \to w$ strongly in $L^p(\Omega)$. Testing (3.18) with $\phi \in C_0^1(\Omega)$ and letting $\epsilon \to 0$, we obtain by (3.16)

$$\int_\Omega \nabla\phi \nabla w dx = \alpha \int_\Omega \phi w^{p-1} dx, \tag{3.19}$$

here we have used $\|w\|_p = 1$. Since $\alpha < \lambda_p(\Omega)$, one can derive from (3.19) that $w \equiv 0$, which contradicts the fact that $\|w\|_p = 1$. Hence $\|c_\epsilon u_\epsilon\|_p$ is bounded. Again by Lemma 3.4, Lemma 3.5 follows. □

We now prove that $c_\epsilon u_\epsilon$ converges to the Green function for the operator $-\Delta G = \delta_{x_0} + \alpha\|G\|_p^{2-p} G^{p-1}$ in Ω when $\epsilon \to 0$ in a certain sense. More precisely, we have

Lemma 3.6. *We have for any $1 < q < 2$, $c_\epsilon u_\epsilon \rightharpoonup G$ weakly in $H^{1,q}(\Omega)$, where $G \in C^1(\Omega \setminus \{x_0\})$ is a Green function satisfying the following*

$$\begin{cases} -\Delta G = \delta_{x_0} + \alpha\|G\|_p^{2-p} G^{p-1} & \text{in} \quad \Omega \\ G = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{3.20}$$

Furthermore, $c_\epsilon u_\epsilon \to G$ in $C^1_{loc}(\overline{\Omega} \setminus \{x_0\})$.

Proof of Lemma 3.6. By Lemma 3.5, we can assume for any $1 < q < 2$ that

$$c_\epsilon u_\epsilon \rightharpoonup G \quad \text{weakly in} \quad H_0^{1,q}(\Omega)$$
$$c_\epsilon u_\epsilon \to G \quad \text{strongly in} \quad L^p(\Omega)$$

for some $G \in H_0^{1,q}(\Omega)$, where p is the same as that in Theorems 1.1 and 1.2. Testing (3.17) by $\phi \in C_0^1(\Omega)$, we have

$$\int_\Omega \nabla\phi\nabla(c_\epsilon u_\epsilon) dx \quad = \quad \int_\Omega \phi \frac{\beta_\epsilon}{\lambda_\epsilon} c_\epsilon u_\epsilon e^{\alpha_\epsilon u_\epsilon^2} dx + \gamma_\epsilon \|c_\epsilon u_\epsilon\|_p^{2-p} \int_\Omega \phi (c_\epsilon u_\epsilon)^{p-1} dx.$$

Letting $\epsilon \to 0$, we have by (3.16),

$$\int_\Omega \nabla\phi\nabla G dx = \phi(x_0) + \alpha\|G\|_p^{2-p} \int_\Omega \phi G^{p-1} dx.$$

Hence

$$-\Delta G = \delta_{x_0} + \alpha \|G\|_p^{2-p} G^{p-1}$$

in a distributional sense.

For any fixed $\delta > 0$, choose a cut-off function $\eta \in C_0^1(\Omega \setminus B_\delta(x_0))$ such that $\eta \equiv 1$ on $\Omega \setminus B_{2\delta}(x_0)$. By Lemma 3.1, we have $\|\nabla(\eta u_\epsilon)\|_2 \to 0$ as $\epsilon \to 0$. Hence $e^{\eta^2 u_\epsilon^2}$ is bounded in $L^r(\Omega \setminus B_\delta(x_0))$, and $e^{u_\epsilon^2}$ is bounded in $L^r(\Omega \setminus B_{2\delta}(x_0))$ for any $r > 1$. Notice that $\|c_\epsilon u_\epsilon\|_p^{2-p}(c_\epsilon u_\epsilon)^{p-1}$ is bounded in $L^{\frac{p}{p-1}}$, we derive from (3.17) that $\Delta(c_\epsilon u_\epsilon)$ is bounded in $L^{\frac{p}{p-1}}(\Omega \setminus B_{2\delta}(x_0))$. Applying the elliptic estimates to (3.17), we have $c_\epsilon u_\epsilon \to G$ in $C^1(\overline{\Omega} \setminus B_{4\delta}(x_0))$. Hence the second assertion holds. □

Therefore, we have completed Step 3.

Step 4. *Completion of Proof of Theorem 1.1.*

We will use the same notations as in the earlier steps. If blow-up occurs, i.e. $c_\epsilon \to +\infty$, we have

$$\begin{aligned}
\sup_{u\in\mathcal{H}} \int_\Omega e^{4\pi u^2(1+\alpha\|u\|_p^2)}dx &= \lim_{\epsilon\to 0}\int_\Omega e^{(4\pi-\epsilon)u_\epsilon^2(1+\alpha\|u_\epsilon\|_p^2)}dx \\
&\le \limsup_{\epsilon\to 0} e^{(4\pi-\epsilon)\alpha\|c_\epsilon u_\epsilon\|_p^2}\int_\Omega e^{(4\pi-\epsilon)u_\epsilon^2}dx \\
&\le e^{4\pi\alpha\|G\|_p^2}\sup_{u\in\mathcal{H}}\int_\Omega e^{4\pi u^2}dx \\
&< +\infty,
\end{aligned}$$

here we have used the Moser-Trudinger inequality (1.1). Thus, Part (1) of Theorem 1.1 follows. □

4. **Proof of Theorem 1.2.** In this section, we give the proof of Theorem 1.2 by dividing it into two steps.

Step 1. *Under the assumption that $c_\epsilon \to +\infty$, $x_\epsilon \to x_0 \in \Omega$,*

$$\sup_{u\in H_0^1(\Omega),\|\nabla u\|_2=1}\int_\Omega e^{4\pi u^2(1+\alpha\|u\|_p^2)}dx \le |\Omega| + \pi e^{1+4\pi A_{x_0}}, \tag{4.1}$$

where A_{x_0} is a constant defined in (4.3) below.

Similar to [16, 27], we need the following result belongs to Carleson and Chang [2]:

Lemma 4.1(Carleson-Chang). *Let $\mathbb{B}$ be the unit disc in $\mathbb{R}^2$. Assume $\{v_\epsilon\}_{\epsilon>0}$ is a sequence of functions in $H_0^1(\mathbb{B})$ with $\int_{\mathbb{B}}|\nabla v_\epsilon|^2dx = 1$. If $|\nabla v_\epsilon|^2dx \rightharpoonup \delta_0$ as $\epsilon \to 0$ weakly in sense of measure. Then $\limsup_{\epsilon\to 0}\int_{\mathbb{B}}(e^{4\pi v_\epsilon^2}-1)dx \le \pi e$.*

By Lemma 3.6, $c_\epsilon u_\epsilon \to G$ in $C^1(\overline{\Omega} \setminus B_\delta(x_0))$. Recall (3.20), we have

$$-\Delta(G + \frac{1}{2\pi}\log|x-x_0|) = \alpha\|G\|_p^{2-p}G^{p-1} \in L^r(\Omega), \quad \forall r > 1. \tag{4.2}$$

Hence $G + \frac{1}{2\pi}\log|x-x_0| \in C^1(\Omega)$ by the elliptic estimate ([11], Chapter 9). Hence the Green function G can be represented by

$$G = -\frac{1}{2\pi}\log|x-x_0| + A_{x_0} + \psi_\alpha(x), \tag{4.3}$$

where A_{x_0} is a constant depending on α, $\psi_\alpha \in C^1(\Omega)$ and $\psi_\alpha(x_0) = 0$. By (3.20),

$$\begin{aligned}\int_{\Omega\setminus B_\delta(x_0)} |\nabla G|^2 dx &= \alpha\|G\|_p^{2-p}\int_{\Omega\setminus B_\delta(x_0)} G^p dx + \int_{\partial(\Omega\setminus B_\delta(x_0))} G\frac{\partial G}{\partial n}ds \\ &= \frac{1}{2\pi}\log\frac{1}{\delta} + A_{x_0} + \alpha\|G\|_p^2 + o_\delta(1).\end{aligned}$$

Hence we obtain

$$\int_{\Omega\setminus B_\delta(x_0)} |\nabla u_\epsilon|^2 dx = \frac{1}{c_\epsilon^2}\left(\frac{1}{2\pi}\log\frac{1}{\delta} + A_{x_0} + \alpha\|G\|_p^2 + o_\delta(1) + o_\epsilon(1)\right) \tag{4.4}$$

Let $s_\epsilon = \sup_{\partial B_\delta(x_0)} u_\epsilon$ and $\overline{u}_\epsilon = (u_\epsilon - s_\epsilon)^+$. Then $\overline{u}_\epsilon \in H_0^1(B_\delta(x_0))$. By (4.4) and the fact that $\int_{B_\delta(x_0)} |\nabla u_\epsilon|^2 dx = 1 - \int_{\Omega\setminus B_\delta(x_0)} |\nabla u_\epsilon|^2 dx$, we have

$$\int_{B_\delta(x_0)} |\nabla \overline{u}_\epsilon|^2 dx \le \tau_\epsilon = 1 - \frac{1}{c_\epsilon^2}\left(\frac{1}{2\pi}\log\frac{1}{\delta} + A_{x_0} + \alpha\|G\|_p^2 + o_\delta(1) + o_\epsilon(1)\right). \tag{4.5}$$

Hence by Lemma 4.1,

$$\limsup_{\epsilon\to 0}\int_{B_\delta(x_0)} (e^{4\pi\overline{u}_\epsilon^2/\tau_\epsilon} - 1)dx \le \pi\delta^2 e. \tag{4.6}$$

By (3.12), we have on $B_{Rr_\epsilon}(x_\epsilon)$ that $u_\epsilon(x) = c_\epsilon + \frac{1}{c_\epsilon}\varphi(\frac{x-x_\epsilon}{r_\epsilon})$, which together with the fact that $c_\epsilon u_\epsilon \to G$ in $L^p(\Omega)$, gives on $B_{Rr_\epsilon}(x_\epsilon)$,

$$\begin{aligned}\alpha_\epsilon u_\epsilon^2 &\le 4\pi(1+\alpha\|u_\epsilon\|_p^2)(\overline{u}_\epsilon + s_\epsilon)^2 \\ &\le 4\pi\overline{u}_\epsilon^2 + 4\pi\alpha\|G_\alpha\|_p^2 + 8\pi s_\epsilon\overline{u}_\epsilon + o_\epsilon(1) \\ &\le 4\pi\overline{u}_\epsilon^2 + 4\pi\alpha\|G\|_p^2 - 4\log\delta + 8\pi A_{x_0} + o_\epsilon(1) + o_\delta(1) \\ &\le 4\pi\overline{u}_\epsilon^2/\tau_\epsilon - 2\log\delta + 4\pi A_{x_0} + o(1).\end{aligned}$$

Therefore

$$\begin{aligned}\int_{B_{Rr_\epsilon}(x_\epsilon)} e^{\alpha_\epsilon u_\epsilon^2}dx &\le \delta^{-2}e^{4\pi A_{x_0}+o(1)}\int_{B_{Rr_\epsilon}(x_\epsilon)} e^{4\pi\overline{u}_\epsilon^2/\tau_\epsilon}dx \\ &= \delta^{-2}e^{4\pi A_{x_0}+o(1)}\int_{B_{Rr_\epsilon}(x_\epsilon)} (e^{4\pi\overline{u}_\epsilon^2/\tau_\epsilon} - 1)dx + o(1) \\ &\le \delta^{-2}e^{4\pi A_{x_0}+o(1)}\int_{B_\delta(x_0)} (e^{4\pi\overline{u}_\epsilon^2/\tau_\epsilon} - 1)dx.\end{aligned}$$

It follows by (4.6) that

$$\limsup_{\epsilon\to 0}\int_{B_{Rr_\epsilon}(x_\epsilon)} e^{\alpha_\epsilon u_\epsilon^2}dx \le \pi e^{1+4\pi A_{x_0}}. \tag{4.7}$$

By Lemma 3.3, we obtain

$$\limsup_{\epsilon\to 0}\int_\Omega e^{\alpha_\epsilon u_\epsilon^2}dx \le |\Omega| + \pi e^{1+4\pi A_{x_0}}. \tag{4.8}$$

Hence we have (4.1).

Step 2. *Existence of extremal function.*

We will construct a blow-up sequence $\phi_\epsilon \in H_0^1(\Omega)$ such that $\|\nabla\phi_\epsilon\|_2 = 1$ and

$$\int_\Omega e^{4\pi\phi_\epsilon^2(1+\alpha\|\phi_\epsilon\|_p^2)}dx > |\Omega| + \pi e^{1+4\pi A_{x_0}} \tag{4.9}$$

for sufficiently small $\epsilon > 0$ and sufficiently small α. The contradiction between (4.9) and (4.1) implies that c_ϵ is bounded. Then elliptic estimate implies that Theorem 1.2 holds.

To prove (4.9), as we did in [28, 29], we set $\tilde\beta = G + \frac{1}{2\pi}\log r - A_{x_0}$, where x_0 is the concentration point as before, $r(x) = |x - x_0|$, and whence $\tilde\beta = O(r)$. Set

$$\phi_\epsilon = \begin{cases} \dfrac{c+\frac{1}{c}\left(-\frac{1}{4\pi}\log(1+\pi\frac{r^2}{\epsilon^2})+B\right)}{\sqrt{1+\frac{\alpha}{c^2}\|G\|_p^2}} & \text{for} \quad r \le R\epsilon \\ \dfrac{1}{\sqrt{c^2+\alpha\|G\|_p^2}}(G-\eta\tilde\beta) & \text{for} \quad R\epsilon < r < 2R\epsilon \\ \dfrac{1}{\sqrt{c^2+\alpha\|G\|_p^2}}G & \text{for} \quad r \ge 2R\epsilon \end{cases},$$

where $\eta \in C_0^\infty(B_{2R\epsilon}(x_0))$ is a cutoff function, $\eta = 1$ on $B_{R\epsilon}(x_0)$, $\|\nabla\eta\|_{L^\infty} = O(\frac{1}{R\epsilon})$, B is a constant to be determined later, and R, c depending on ϵ will also be chosen later such that $R\epsilon \to 0$ and $R \to +\infty$. In order to assure that $\phi_\epsilon \in H_0^1(\Omega)$, we set

$$c + \frac{1}{c}\left(-\frac{1}{4\pi}\log(1+\pi R^2) + B\right) = \frac{1}{c}(-\frac{1}{2\pi}\log(R\epsilon) + A_{x_0}),$$

which gives

$$2\pi c^2 = -\log\epsilon - 2\pi B + 2\pi A_{x_0} + \frac{1}{2}\log\pi + O(\frac{1}{R^2}). \tag{4.10}$$

A straightforward calculation shows

$$\begin{aligned}\int_\Omega |\nabla\phi_\epsilon|^2 dx \quad = \quad & \frac{1}{4\pi(c^2+\alpha\|G\|_p^2)}\left(2\log\frac{1}{\epsilon} + \log\pi - 1 + 4\pi A_p + 4\pi\alpha\|G\|_p^2\right. \\ & \left. +O(\frac{1}{R^2}) + O(R\epsilon\log(R\epsilon))\right).\end{aligned}$$

Let $\|\nabla\phi_\epsilon\|_2 = 1$, we have

$$c^2 = -\frac{\log\epsilon}{2\pi} + \frac{\log\pi}{4\pi} - \frac{1}{4\pi} + A_p + O(\frac{1}{R^2}) + O(R\epsilon\log(R\epsilon)). \tag{4.11}$$

By (4.10) and (4.11), we have

$$B = \frac{1}{4\pi} + O(\frac{1}{R^2}) + O(R\epsilon\log(R\epsilon)). \tag{4.12}$$

Set $R = -\log\epsilon$. It follows that on $B_{R\epsilon}(x_0)$

$$4\pi\phi_\epsilon^2(1+\alpha\|\phi_\epsilon\|_p^2) \ge 4\pi c^2 - 2\log(1+\pi\frac{r^2}{\epsilon^2}) + 8\pi B - \frac{8\pi\alpha^2\|G\|_p^4}{c^2} + O(\frac{\log R}{c^4}).$$

Hence we have by (4.11) and (4.12),

$$\int_{B_{R\epsilon}(x_0)} e^{4\pi(1+\alpha\|\phi_\epsilon\|_p^2)\phi_\epsilon^2}dx \ge \pi e^{1+4\pi A_{x_0}} - \frac{8\pi^2\alpha^2\|G\|_p^4}{c^2}e^{1+4\pi A_{x_0}} + O(\frac{\log\log\epsilon}{\log^2\epsilon}). \tag{4.13}$$

On the other hand,

$$\int_{\Omega\setminus B_{R\epsilon}(x_0)} e^{4\pi(1+\alpha\|\phi_\epsilon\|_p^2)\phi_\epsilon^2}dx \geq \int_{\Omega\setminus B_{2R\epsilon}(x_0)}(1+4\pi\phi_\epsilon^2)dx \tag{4.14}$$
$$\geq |\Omega|+4\pi\frac{\|G\|_p^2}{c^2}+O(\frac{1}{c^4}).$$

If the following hypothesis

$$2\pi\alpha^2\|G\|_p^2 e^{1+4\pi A_{x_0}}<1, \tag{4.15}$$

holds, we know from (4.13) and (4.14) that (4.9) holds for sufficiently small $\epsilon>0$. The elliptic estimate on the equation (4.2) implies that the hypothesis (4.15) can be satisfied for sufficiently small α.

Therefore the proof of Theorem 1.2 is completely finished. □

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