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Adams' inequalities for bi-Laplacian and extremal functions in dimension four ☆

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Dedicated to S.Y.A. Chang and Paul Yang on the occasion of their 60th birthdays with admiration and appreciation

Abstract

Let $\Omega \subset \mathbb{R}^4$ be a smooth oriented bounded domain, $H_0^2(\Omega)$ be the Sobolev space, and $\lambda(\Omega) = \inf_{u\in H_0^2(\Omega),\, \|u\|_2^2=1} \|\Delta u\|_2^2$ be the first eigenvalue of the bi-Laplacian operator Δ^2. Then for any α: $0 \leqslant \alpha < \lambda(\Omega)$, we have

$$\sup_{u\in H_0^2(\Omega),\, \|\Delta u\|_2^2=1} \int_\Omega e^{32\pi^2 u^2(1+\alpha\|u\|_2^2)}\, dx < +\infty$$

and the above supremum is infinity when $\alpha \geqslant \lambda(\Omega)$. This strengthens Adams' inequality in dimension 4 [D. Adams, A sharp inequality of J. Moser for high order derivatives, Ann. of Math. 128 (1988) 365–398] where he proved the above inequality holds for $\alpha = 0$. Moreover, we prove that for sufficiently small α an extremal function for the above inequality exists. As a special case of our results, we thus show that there exists $u^* \in H_0^2(\Omega) \cap C^4(\overline{\Omega})$ with $\|\Delta u^*\|_2^2 = 1$ such that

$$\int_\Omega e^{32\pi^2 u^{*2}}\, dx = \sup_{u\in H_0^2(\Omega),\, \int_\Omega |\Delta u|^2\, dx=1} \int_\Omega e^{32\pi^2 u^2}\, dx.$$

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This establishes the existence of an extremal function of the original Adams inequality in dimension 4.

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1. Introduction and main results

Sharp geometric inequalities and their extremal functions play an important role both in analysis and geometry. The investigation on the sharp constant for Moser–Trudinger's inequality dated back to 1960s to 70s. In 1971, J. Moser [30] sharpened the result of Pohozaev [34] and Trudinger [41] and found the largest positive constant $\beta_0 = n\omega_{n-1}^{\frac{1}{n-1}}$, where ω_{n-1} is the area of the surface of the unit n-ball, such that if Ω is an open subset of Euclidean space $\mathbb{R}^n$ $(n \geqslant 2)$ with finite Lebesgue measure, then there is a constant C_0 depending only on n such that

$$\frac{1}{|\Omega|}\int_{\Omega} \exp\bigl(\beta\bigl|f(x)\bigr|^{\frac{n}{n-1}}\bigr)\,dx \leqslant C_0$$

for any $\beta \leqslant \beta_0$, any f in the Sobolev space $W_0^{1,n}(\Omega)$, provided $\|\nabla f\|_{L^n(\Omega)} \leqslant 1$. Moser also proved that if β exceeds β_0, then the above inequality cannot hold with uniform C_0 independent of f.

In 1986, Carleson and Chang [7] proved that the following supremum

$$\sup_{f\in W_0^{1,n}(\Omega),\,\|\nabla f\|_{L^n(\Omega)}\leqslant 1}\left\{\frac{1}{|\Omega|}\int_{\Omega} \exp\bigl(n\omega_{n-1}^{\frac{1}{n-1}}\bigl|f(x)\bigr|^{\frac{n}{n-1}}\bigr)\,dx\right\}$$

has extremals for the case when Ω is a ball in $\mathbb{R}^n$ for $n \geqslant 2$. Carleson and Chang proved the existence of extremals by reduction to a one-dimensional problem using a symmetrization argument. Much work has been done since then, and we refer the reader to the sharp Moser–Onofri type inequality with extremal function for Paneitz operators on high dimensional spheres by Becker [4], Carlen and Loss [6], a sharp Moser inequality with mean value zero on domains in $\mathbb{R}^2$ by Chang and Yang [8] (see a recent extension to high dimension by Leckband [20]), the work on existence of extremal functions by Flucher [16] on smooth domains in $\mathbb{R}^n$ when $n = 2$, by Lin [27] for the case $n > 2$, and a Moser type inequality related to the mean field equation by Ding, Jost, Li and Wang [13,14], and more recently on existence of extremal functions on Riemannian manifolds by Y.X. Li [21] and Yang [43], and by Lu and Yang [29] for functions with mean value zero, and on unbounded domains by Ruf in $\mathbb{R}^2$ [35]. We should also mention that Tian and Zhu [40] proved a Moser–Trudinger type inequality for almost plurisubharmonic functions on any Kahler–Einstein manifolds with positive scalar curvature which generalizes the stronger version of the Moser–Onofri inequality on S^2 and also refines a weaker inequality found earlier by Tian in [39].

Research on finding the sharp constants for higher order Moser's inequality started by the work of D. Adams [1]. To state Adams' result, we use the symbol $\nabla^m u$, m is a positive integer, to denote the mth order gradient for $u \in C^m$, the class of mth order differentiable functions:

$$\nabla^m u = \begin{cases} \Delta^{\frac{m}{2}} u & \text{for } m \text{ even,} \\ \nabla \Delta^{\frac{m-1}{2}} u & \text{for } m \text{ odd,} \end{cases}$$

where ∇ is the usual gradient operator and Δ is the Laplacian. We use $||\nabla^m u||_p$ to denote the L^p norm ($1 \leqslant p \leqslant \infty$) of the function $|\nabla^m u|$, the usual Euclidean length of the vector $\nabla^m u$. We also use $W_0^{k,p}(\Omega)$ to denote the Sobolev space which is a completion of $C_0^\infty(\Omega)$ under the norm of $||u||_{L^p(\Omega)} + ||\nabla^k u||_{L^p(\Omega)}$. Then Adams proved the following

Theorem A. *Let Ω be an open and bounded set in $\mathbb{R}^n$. If m is a positive integer less than n, then there exists a constant $C_0 = C(n,m) > 0$ such that for any $u \in W_0^{m,\frac{n}{m}}(\Omega)$ and $||\nabla^m u||_{L^{\frac{n}{m}}(\Omega)} \leqslant 1$, then*

$$\frac{1}{|\Omega|} \int_\Omega \exp\bigl(\beta \bigl|u(x)\bigr|^{\frac{n}{n-m}}\bigr)\, dx \leqslant C_0$$

for all $\beta \leqslant \beta(n,m)$ where

$$\beta(n,m) = \begin{cases} \frac{n}{w_{n-1}} \bigl[\frac{\pi^{n/2} 2^m \Gamma(\frac{m+1}{2})}{\Gamma(\frac{n-m+1}{2})}\bigr]^{\frac{n}{n-m}} & \text{when } m \text{ is odd,} \\ \frac{n}{w_{n-1}} \bigl[\frac{\pi^{n/2} 2^m \Gamma(\frac{m}{2})}{\Gamma(\frac{n-m}{2})}\bigr]^{\frac{n}{n-m}} & \text{when } m \text{ is even.} \end{cases}$$

Furthermore, for any $\beta > \beta(n,m)$, the integral can be made as large as possible.

Note that $\beta(n,1)$ coincides with Moser's value of β_0 and $\beta(2m,m) = 2^{2m}\pi^m \Gamma(m+1)$ for both odd and even m. We are particularly interested in the case $n = 4$ and $m = 2$ in this paper where $\beta(4,2) = 32\pi^2$.

We remark here that both Moser and Carleson–Chang's works rely on a rearrangement argument. In order to adapt this symmetrization principle of Moser, one needs to establish the L^p-norm preserving properties of the high order gradient functions $\nabla^m u$, which is still not known to be true in general for $m \geqslant 2$. What Adams did was to represent the function u in terms of its gradient function $\nabla^m u$ using a convolution operator. Then he used the O'Neil's idea [33] of rearrangement of convolution of two functions together with the idea which originally goes back to Garcia. Such an argument avoids in dealing with the issue of $L^{\frac{n}{m}}$ norm preserving of the gradient of the rearranged function. This idea has also been developed to derive the sharp constants for Adams' inequality involving higher order derivatives on Riemannian manifolds without boundary by Fontana [17] and more recently in the subelliptic setting to derive the sharp Moser's inequality on the Heisenberg group and CR sphere by Cohn and Lu (see [10] and [11]).

It has remained an open question whether Adams' inequality has an extremal function, namely, whether the following supremum

$$\sup_{u \in W_0^{m,\frac{n}{m}}(\Omega),\, ||\nabla^m u||_{L^{\frac{n}{m}}(\Omega)} \leqslant 1} \frac{1}{|\Omega|} \int_\Omega \exp\bigl(\beta \bigl|u(x)\bigr|^{\frac{n}{n-m}}\bigr)\, dx$$

can be attained. Unlike in the Moser's inequality with first order derivatives, we are unable to adapt Carleson and Chang's idea [7] of symmetrization to establish the existence of extremal functions for inequalities involving high order derivatives. It is still a rather difficult problem to answer the above question in the most generality. Nevertheless, one of the main purposes of this paper is to address this issue and provide an affirmative answer in an important and particularly interesting case when $n = 4$ and $m = 2$, where considerable attention has been paid to the geometric analysis on fourth order differential operators on four manifolds (e.g., see the survey article [9] and many references therein). As it has been pointed out earlier that the sharp Moser–Onofri inequality and existence of extremal functions on high dimensional spheres $\mathbb{S}^n$ for high order derivatives were derived by Beckner using deep Fourier analysis techniques [4], see also Carlen and Loss [6] using elegant competing symmetry method, and the Beckner–Onofri inequality on CR sphere by Branson, Fontana and Morpurgo [5].

To state our results, let $\Omega \subset \mathbb{R}^n$ denote a smooth oriented bounded domain, $H_0^2(\Omega)$ denote the Sobolev space which is a completion of space of smooth functions with compact support under the Dirichlet norm $\|u\|_{H_0^2(\Omega)} = \|\Delta u\|_2$, where $\|\cdot\|_2$ denotes the usual $L^2(\Omega)$-norm. Then Adams' inequality in the case of $n = 4$ and $m = 2$ can be stated as

$$\sup_{\|\Delta u\|_2 \leqslant 1} \int_\Omega e^{\gamma u^2}\, dx < +\infty \quad \text{for all } \gamma \leqslant 32\pi^2. \tag{1.1}$$

This inequality is optimal in the sense that the corresponding supremum is infinite for any growth $e^{\gamma u^2}$ with $\gamma > 32\pi^2$.

The first aim of this paper is to strengthen the Adams inequality (1.1). Let

$$\lambda(\Omega) = \inf_{u \in H_0^2(\Omega),\, u \not\equiv 0} \frac{\|\Delta u\|_2^2}{\|u\|_2^2} \tag{1.2}$$

be the first eigenvalue of the bi-Laplacian operator Δ^2. By a direct method of variation, one can show that $\lambda(\Omega) > 0$. In this paper we show that replacing the best constant $32\pi^2$ by $32\pi^2(1 + \alpha\|u\|_2^2)$ for any α: $0 \leqslant \alpha < \lambda(\Omega)$, (1.1) is still valid. More precisely, we prove

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^4$ be a smooth oriented bounded domain, $\lambda(\Omega)$ be defined by* (1.2)*. Then for any α with $0 \leqslant \alpha < \lambda(\Omega)$, we have*

$$\sup_{u \in H_0^2(\Omega),\, \|\Delta u\|_2^2 = 1} \int_\Omega e^{32\pi^2 u^2 (1 + \alpha \|u\|_2^2)}\, dx < +\infty. \tag{1.3}$$

The inequality is sharp in the sense that for any growth $e^{32\pi^2 u^2(1+\alpha\|u\|_2^2)}$ with $\alpha \geqslant \lambda(\Omega)$ the supremum is infinite.

The special case of Theorem 1.1 when $\alpha = 0$ is exactly Adams' original inequality (1.1). We remark here that one can obtain a weaker version of the Adams inequality (1.1) for any $\gamma < 32\pi^2$ by using a sharp representation formula of the function u in terms of its higher order gradient $\nabla^m u$ and combining with Hedberg's idea [19]. However, this argument does not lead to the sharpest constant $\gamma = 32\pi^2$. Nevertheless, our argument of proving inequality (1.3) only requires to know that the weaker version of the inequality (1.1) holds. Namely, as long as we can show that (1.1) holds for any $\gamma < 32\pi^2$, we can derive the strengthened Adams inequality (1.3) for all $0 \leqslant \alpha < \lambda(\Omega)$. Thus, our method in this paper also provides an alternative way of deriving Adams' result for $\alpha = 0$ when $n = 4$ and $m = 2$.

Next, we can further generalize Theorem 1.1 to the growth $e^{32\pi^2 u^2 q(\|u\|_2^2)}$ for some appropriate polynomial $q(t)$ defined on $\mathbb{R}$ with $q(0) = 1$, namely

Theorem 1.1*. *Let $\Omega \subset \mathbb{R}^4$ be a smooth oriented bounded domain, $\lambda(\Omega)$ be defined by* (1.2), *and $q(t) = 1 + a_1 t + a_2 t^2 + \cdots + a_k t^k$ $(k \geqslant 1)$ be a polynomial of order k in $\mathbb{R}$. If $0 \leqslant a_1 < \lambda(\Omega)$, $0 \leqslant a_2 \leqslant \lambda(\Omega) a_1, \ldots, 0 \leqslant a_k \leqslant \lambda(\Omega) a_{k-1}$, then there holds*

$$\sup_{u \in H_0^2(\Omega),\, \|\Delta u\|_2^2 = 1} \int_\Omega e^{32\pi^2 u^2 q(\|u\|_2^2)}\, dx < +\infty.$$

If $a_1 \geqslant \lambda(\Omega)$, and $a_2, \ldots, a_k$ are arbitrary real numbers, then the supremum corresponding to the growth $e^{32\pi^2 u^2 q(\|u\|_2^2)}$ is infinite.

It is easy to see that Theorem 1.1 is a special case of Theorem 1.1* when $q(t) = 1 + \alpha t$.

Having obtained the sharpened version of the Adams inequality (1.3), we are naturally led to investigate the existence of extremal functions such that the supremum (1.3) is attained. This question is rather difficult and requires considerable efforts to accomplish when dealing with inequalities involving the high order derivatives.

The second aim of this paper is to show the existence of extremal function for the Adams inequality (1.1) in dimension four. We will prove

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^4$ be a smooth oriented bounded domain. There exists $u^* \in H_0^2(\Omega) \cap C^4(\overline{\Omega})$ with $\|\Delta u^*\|_2^2 = 1$ such that*

$$\int_\Omega e^{32\pi^2 u^{*2}}\, dx = \sup_{u \in H_0^2(\Omega),\, \|\Delta u\|_2 \leqslant 1} \int_\Omega e^{32\pi^2 u^2}\, dx.$$

In fact, we will prove the following more general result.

Theorem 1.2*. *Let $\Omega \subset \mathbb{R}^4$ be a smooth oriented bounded domain, $\lambda(\Omega)$ be defined by* (1.2), *and $q(t) = 1 + a_1 t + a_2 t^2 + \cdots + a_k t^k$ $(k \geqslant 1)$ be a polynomial of order k in $\mathbb{R}$. If $0 \leqslant a_1 < \lambda(\Omega)$, $0 \leqslant a_2 \leqslant \lambda(\Omega) a_1, \ldots, 0 \leqslant a_k \leqslant \lambda(\Omega) a_{k-1}$, then there exists a strictly positive constant $\epsilon_0 < \lambda(\Omega)$ depending only on Ω such that when $0 \leqslant a_1 \leqslant \epsilon_0$, $0 \leqslant a_2 \leqslant \lambda(\Omega) a_1, \ldots,$ and $0 \leqslant a_m \leqslant \lambda(\Omega) a_{m-1}$, we can find $u^* \in H_0^2(\Omega) \cap C^4(\overline{\Omega})$ such that $\|\Delta u^*\|_2^2 = 1$ and*

$$\int_\Omega e^{32\pi^2 u^{*2} q(\|u^*\|_2^2)}\, dx = \sup_{u \in H_0^2(\Omega),\, \int_\Omega |\Delta u|^2\, dx \leqslant 1} \int_\Omega e^{32\pi^2 u^2 q(\|u\|_2^2)}\, dx.$$

As a corollary of Theorem 1.2*, we have thus also shown the existence of extremal function of inequality (1.3) for sufficiently small $\alpha > 0$.

The following remarks are in order. First of all, to prove Theorems 1.1 and 1.2, we only need to prove Theorems 1.1* and 1.2*. Second, the proof of the second part of Theorem 1.1 (also

Theorem 1.1*) is based on the test function computations, while the first part is based on blow-up analysis. More precisely, using a Lions' type lemma, we can find $u_\epsilon \in H_0^2(\Omega) \cap C^4(\overline{\Omega})$ such that $\int_\Omega |\Delta u_\epsilon|^2\,dx = 1$, and

$$\int_\Omega e^{(32\pi^2-\epsilon)u_\epsilon^2(1+\alpha\|u_\epsilon\|_2^2)}\,dx = \sup_{u\in H_0^2(\Omega),\,\|\Delta u\|_2=1} \int_\Omega e^{(32\pi^2-\epsilon)u^2(1+\alpha\|u\|_2^2)}\,dx$$

for any $\epsilon > 0$. Denote $\alpha_\epsilon = (32\pi^2-\epsilon)(1+\alpha\|u_\epsilon\|_2^2)$, $\beta_\epsilon = (1+\alpha\|u_\epsilon\|_2^2)/(1+2\alpha\|u_\epsilon\|_2^2)$, $\gamma_\epsilon = \alpha/(1+2\alpha\|u_\epsilon\|_2^2)$, $\lambda_\epsilon = \int_\Omega u_\epsilon^2 e^{\alpha_\epsilon u_\epsilon^2}\,dx$. Then the Euler–Lagrange equation of u_ϵ is

$$\begin{cases} \Delta^2 u_\epsilon = \dfrac{\beta_\epsilon}{\lambda_\epsilon} u_\epsilon e^{\alpha_\epsilon u_\epsilon^2} + \gamma_\epsilon u_\epsilon & \text{in } \Omega, \\ u_\epsilon = \dfrac{\partial u_\epsilon}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

Write $c_\epsilon = u_\epsilon(x_\epsilon) = \max_{x\in\Omega}|u_\epsilon|$. Without loss of generality we assume $c_\epsilon \to +\infty$ (namely blow-up occurs) and $x_\epsilon \to p \in \overline{\Omega}$. Using the Pohozaev identity and elliptic estimates, we will exclude the scenario of the boundary blow-up. We also prove that $c_\epsilon u_\epsilon$ converges to some Green function weakly in $H_0^2(\Omega)$, which immediately leads to Theorem 1.1 (Theorem 1.1*). Third, for the proof of Theorem 1.2 (Theorem 1.2*), we will derive an upper bound of the functional $\int_\Omega e^{32\pi^2u^2}\,dx$ under the assumption that blow-up occurs by using a certain type of capacitary estimate, and then construct a sequence of functions to reach a contradiction. This leads to the existence of extremal function. Fourth, as we have pointed out earlier, throughout the paper we will not require the best Adams inequality (the best constant is $32\pi^2$), but only require the subcritical Adams inequality, i.e.

$$\sup_{\|\Delta u\|_2\leqslant 1} \int_\Omega e^{\gamma u^2}\,dx < +\infty \quad \text{for all } \gamma < 32\pi^2.$$

This is interesting in its own right. Fifth, we also caution the reader that α_ϵ is not necessarily approaching to $32\pi^2$ or bounded above by $32\pi^2$ when $\epsilon \to 0$. Thus, we cannot have the uniform boundedness with respect to $\epsilon > 0$ of the integral $\int_\Omega e^{\alpha_\epsilon u_\epsilon^2}\,dx$ in advance, which is obviously uniformly bounded for the case $q(t) \equiv 1$ in Theorem 1.1* (i.e., $\alpha = 0$ in Theorem 1.1), when we calculate the upper bound using the capacity estimates. This in turn creates considerably more difficulty in the proof of Theorem 1.2*. Sixth, an analogous case of Theorem 1.1 for first order derivatives in dimension two has been studied by Adimurthi and O. Druet in [2] using blow-up analysis, and existence of extremal function was considered in [43] in this case. A version of Theorem 1.2 on four dimensional Riemannian manifolds without boundary was recently considered by Y.X. Li and C. Ndiaye in [22] and existence of extremal functions was derived in [22]. Our results in this paper on bounded, open and orientable domains Ω in $\mathbb{R}^4$ can be generalized to the case on Riemannian manifolds of dimension four with boundary. We would also like to mention that blow-up techniques have been already employed by numerous authors in a relevant but quite different setting in dealing with Sobolev inequalities instead of Moser–Trudinger type ones. We refer the interested reader to the works in [3,15,24,26,36–38], etc.

The rest of the paper is arranged as follows. In Section 2, we construct test functions to prove the second part of Theorem 1.1 (Theorem 1.1*). In Section 3, we give the existence of maximizers of subcritical functionals. In Section 4, we analyze the asymptotic behavior of those maximizers. In Section 5, we obtain an upper bound of the critical functional under the assumption that blow-up occurs in the interior of Ω. We exclude the boundary bubble in Section 6 and finish the proof of Theorem 1.1*. In Section 7, we construct test functions to conclude the existence of extremals, and thus give the proof of Theorem 1.2*.

2. Proof of the second part of Theorem 1.1*

The main purpose of this section is to prove the second part of Theorem 1.1* by constructing test functions. Let $q(t) = 1 + a_1 t + \cdots + a_k t^k$ be the polynomial given in the assumption of Theorem 1.1*. We need to prove that for $a_1 \geqslant \lambda(\Omega)$ and arbitrary $a_2, \ldots, a_k$, there holds

$$\sup_{u \in H_0^2(\Omega),\, \|\Delta u\|_2^2 \leqslant 1} \int_\Omega e^{32\pi^2 u^2 q(\|u\|_2^2)}\, dx = +\infty. \tag{2.1}$$

Let $u_0 \in H_0^2(\Omega) \cap C^4(\overline{\Omega})$ be an eigenfunction of bi-Laplacian operator Δ^2 satisfying

$$\begin{cases} \Delta^2 u_0 = \lambda(\Omega) u_0 \quad \text{in } \Omega, \\ \|u_0\|_2 = 1, \quad u_0 = \dfrac{\partial u_0}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

The solvability of this equation is based on the direct method of variation. Without loss of generality we assume the unit ball $\mathbb{B} \Subset \Omega$ and $u_0 > C_0$ in $\mathbb{B}$ for some positive C_0, otherwise we consider $-u_0$ instead of u_0 and a ball in Ω with radius r and centered at some point x_0. Let

$$u_\epsilon = \begin{cases} \sqrt{\frac{1}{32\pi^2} \log \frac{1}{\epsilon}} - \frac{|x|^2}{\sqrt{8\pi^2 \epsilon \log \frac{1}{\epsilon}}} + \frac{1}{\sqrt{8\pi^2 \log \frac{1}{\epsilon}}}, & |x| \leqslant \epsilon^{\frac{1}{4}}, \\ \frac{1}{\sqrt{2\pi^2 \log \frac{1}{\epsilon}}} \log \frac{1}{|x|}, & \epsilon^{\frac{1}{4}} < |x| \leqslant 1, \\ \zeta_\epsilon, & |x| > 1, \end{cases}$$

where ζ_ϵ is a smooth function satisfying $\zeta_\epsilon|_{\partial\mathbb{B}} = \zeta_\epsilon|_{\partial\Omega} = 0$, $\frac{\partial \zeta_\epsilon}{\partial \nu}|_{\partial\mathbb{B}} = \frac{1}{\sqrt{2\pi^2 \log \frac{1}{\epsilon}}}$, $\frac{\partial \zeta_\epsilon}{\partial \nu}|_{\partial\Omega} = 0$, and $\zeta_\epsilon, \nabla\zeta_\epsilon, \Delta\zeta_\epsilon$ are all $O(\frac{1}{\sqrt{\log \frac{1}{\epsilon}}})$. One can check that $u_\epsilon \in H_0^2(\Omega)$, and

$$\|u_\epsilon\|_2^2 = O\left(1/\log \frac{1}{\epsilon}\right), \qquad \|\Delta u_\epsilon\|_2^2 = 1 + O\left(1/\log \frac{1}{\epsilon}\right).$$

Let $v_\epsilon = u_\epsilon + t_\epsilon u_0$ with $t_\epsilon \to 0$, $t_\epsilon^2 \log \frac{1}{\epsilon} \to +\infty$ and $t_\epsilon^2 (\log \frac{1}{\epsilon})^{1/2} \to 0$. Then we have

$$\begin{aligned} \|v_\epsilon\|_2^2 &= \|u_\epsilon\|_2^2 + t_\epsilon^2 \|u_0\|_2^2 + 2t_\epsilon \int_\Omega u_\epsilon u_0\, dx \\ &= t_\epsilon^2 + 2t_\epsilon \int_\Omega u_\epsilon u_0\, dx + O\left(1/\log \frac{1}{\epsilon}\right), \end{aligned}$$

$$\|\Delta v_\epsilon\|_2^2 = \|\Delta u_\epsilon\|_2^2 + t_\epsilon^2\|\Delta u_0\|_2^2 + 2t_\epsilon\int_\Omega \Delta u_\epsilon \Delta u_0\,dx$$
$$= 1 + 2\lambda(\Omega)t_\epsilon\int_\Omega u_\epsilon u_0\,dx + \lambda(\Omega)t_\epsilon^2 + O\left(1/\log\frac{1}{\epsilon}\right).$$

A straightforward calculation shows

$$\frac{1}{\|\Delta v_\epsilon\|_2^2}q\left(\frac{\|v_\epsilon\|_2^2}{\|\Delta v_\epsilon\|_2^2}\right) = 1 + \bigl(a_1 - \lambda(\Omega)\bigr)\left(t_\epsilon^2 + 2t_\epsilon\int_\Omega u_\epsilon u_0\,dx\right)$$
$$+ o\left(t_\epsilon\Big/\left(\log\frac{1}{\epsilon}\right)^{1/2}\right).$$

Noting that $\int_\Omega |u_\epsilon|\,dx = O(1/\sqrt{\log\frac{1}{\epsilon}})$, we have $t_\epsilon\int_\Omega u_\epsilon u_0\,dx = o(t_\epsilon^2)$. Hence for $a_1 \geqslant \lambda(\Omega)$,

$$\frac{1}{\|\Delta v_\epsilon\|_2^2}q\left(\frac{\|v_\epsilon\|_2^2}{\|\Delta v_\epsilon\|_2^2}\right) \geqslant 1 + o\left(t_\epsilon\Big/\left(\log\frac{1}{\epsilon}\right)^{1/2}\right).$$

Since $u_\epsilon \geqslant \sqrt{\frac{1}{32\pi^2}\log\frac{1}{\epsilon}}$ on $\mathbb{B}_{\epsilon^{1/4}}$, we obtain

$$\int_\Omega e^{32\pi^2\frac{v_\epsilon^2}{\|\Delta v_\epsilon\|_2^2}q(\frac{\|v_\epsilon\|_2^2}{\|\Delta v_\epsilon\|_2^2})}\,dx \geqslant \int_{\mathbb{B}_{\epsilon^{1/4}}}\frac{1}{\epsilon}e^{t_\epsilon\sqrt{\log\frac{1}{\epsilon}}(8\sqrt{2}\pi u_0+o(1))}\,dx$$
$$= e^{t_\epsilon\sqrt{\log\frac{1}{\epsilon}}(8\sqrt{2}\pi u_0(0)+o(1))}.$$

By our assumption $u_0(0) > C_0$ in $\mathbb{B}$,

$$\int_\Omega e^{32\pi^2\frac{v_\epsilon^2}{\|\Delta v_\epsilon\|_2^2}q(\frac{\|v_\epsilon\|_2^2}{\|\Delta v_\epsilon\|_2^2})}\,dx \to +\infty$$

as $\epsilon \to 0$. We get the desired result (2.1).

3. Extremals for the subcritical Adams inequality

In this section we mainly prove for any $\epsilon > 0$ the existence of maximizers of subcritical functionals

$$\int_\Omega e^{(32\pi^2-\epsilon)u^2 q(\|u\|_2^2)}\,dx \tag{3.1}$$

defined on space of functions satisfying $u \in H_0^2(\Omega)$ and $\|\Delta u\|_2 \leqslant 1$. Noting that $(32\pi^2 - \epsilon)q(\|u\|_2^2)$ is not necessarily less than the critical exponent $32\pi^2$, the existence of such maximizers is nontrivial.

We begin with proving the following Lions' type [28] concentration compactness result.

Proposition 3.1. *Let $\{u_\epsilon\}_{\epsilon>0} \subset H_0^2(\Omega)$ be a sequence of functions such that $\|\Delta u_\epsilon\|_2 = 1$ and $u_\epsilon \rightharpoonup u_0$ weakly in $H_0^2(\Omega)$. Then for any $p < 1/(1 - \|\Delta u_0\|_2^2)$,*

$$\limsup_{\epsilon \to 0} \int_\Omega e^{32\pi^2 p u_\epsilon^2}\, dx < +\infty.$$

Proof. If $u_0 = 0$, then nothing need to be proved because of the Adams inequality (1.1). If $u_0 \neq 0$, then one can see that

$$\left\|\Delta(u_\epsilon - u_0)\right\|_2^2 \to 1 - \|\Delta u_0\|_2^2 < 1.$$

Hence we have for $p < 1/(1 - \|\Delta u_0\|_2^2)$

$$\begin{aligned}\int_\Omega e^{32\pi^2 p u_\epsilon^2}\, dx &\leqslant \int_\Omega e^{32\pi^2 p(1+\delta)(u_\epsilon - u_0)^2 + 32\pi^2 p(1+1/\delta)u_0^2}\, dx \\ &\leqslant \left(\int_\Omega e^{32\pi^2 \frac{(u_\epsilon - u_0)^2}{\|\Delta(u_\epsilon - u_0)\|_2^2}}\, dx\right)^{1/r} \left(\int_\Omega e^{32\pi^2 p' u_0^2}\, dx\right)^{1/s}\end{aligned}$$

for some $\delta > 0$ and $p' > p$ provided that ϵ is sufficiently small, where $1/r + 1/s = 1$. By the Orlicz imbedding, $e^{u_0^2}$ is bounded in $L^s(\Omega)$ for any $s > 1$. The Adams inequality (1.1) implies

$$\limsup_{\epsilon \to 0} \int_\Omega e^{32\pi^2 p u_\epsilon^2}\, dx < +\infty. \qquad \Box$$

It is interesting to note that this concentration compactness estimate does not follow from the Adams inequality (1.1) and it is stronger than (1.1) when $\|\Delta u_0\|_2^2 \neq 0$. It is also remarkable that only the subcritical Adams inequality is required in the proof of Proposition 3.1, namely

$$\sup_{u \in H_0^2(\Omega),\, \|\Delta u\|_2 \leqslant 1} \int_\Omega e^{\gamma u^2}\, dx < +\infty \quad \text{for all } \gamma < 32\pi^2. \tag{3.2}$$

Next we prove the existence of maximizers for subcritical functionals (3.1).

Proposition 3.2. *Assume the assumptions of $q(t)$ in Theorem 1.1′ are satisfied. Then for any $\epsilon > 0$, there exists $u_\epsilon \in H_0^2(\Omega) \cap C^4(\overline{\Omega})$ such that $\int_\Omega |\Delta u_\epsilon|^2\, dx = 1$, and*

$$\int_\Omega e^{(32\pi^2 - \epsilon)u_\epsilon^2 q(\|u_\epsilon\|_2^2)}\, dx = \sup_{u \in H_0^2(\Omega),\, \|\Delta u\|_2 \leqslant 1} \int_\Omega e^{(32\pi^2 - \epsilon)u^2 q(\|u\|_2^2)}\, dx.$$

Here $32\pi^2 - \epsilon$ can be replaced by any sequence $\rho_\epsilon \uparrow 32\pi^2$ as $\epsilon \downarrow 0$.

Proof. We first note that the supremum is invariant if one replaces the condition $\|\Delta u\|_2 \leqslant 1$ by $\|\Delta u\|_2 = 1$. Hence, for any fixed $\epsilon > 0$, we can choose a maximizing sequence $\{u_j\} \subset H_0^2(\Omega)$ such that $\int_\Omega |\Delta u_j|^2\, dx = 1$, and

$$\int_\Omega e^{(32\pi^2-\epsilon)u_j^2 q(\|u_j\|_2^2)}\, dx \to \sup_{u\in H_0^2(\Omega),\, \|\Delta u\|_2\leqslant 1} \int_\Omega e^{(32\pi^2-\epsilon)u^2 q(\|u\|_2^2)}\, dx \tag{3.3}$$

as $j \to +\infty$. Since $\{u_j\}$ is bounded in $H_0^2(\Omega)$, we have

$$\begin{aligned} &u_j \rightharpoonup u_\epsilon \quad \text{weakly in } H_0^2(\Omega),\\ &u_j \to u_\epsilon \quad \text{strongly in } L^2(\Omega),\\ &u_j \to u_\epsilon \quad \text{a.e. in } \Omega. \end{aligned}$$

Hence

$$f_j = e^{(32\pi^2-\epsilon)u_j^2 q(\|u_j\|_2^2)} \to f_\epsilon = e^{(32\pi^2-\epsilon)u_\epsilon^2 q(\|u_\epsilon\|_2^2)} \quad \text{a.e. in } \Omega.$$

Suppose $u_\epsilon = 0$, then $1 + \alpha\|u_j\|_2^2 \to 1$. The subcritical Adams inequality (3.2) implies that

$$\int_\Omega e^{(32\pi^2-\frac{\epsilon}{4})u_j^2}\, dx < +\infty \quad \text{for all } j.$$

Thus f_j is bounded in $L^s(\Omega)$ for some $s > 1$ and $f_j \to 1$ in $L^1(\Omega)$. Here s depends only on ϵ. Passing to the limit $j \to +\infty$ in (3.3), one has

$$|\Omega| = \sup_{u\in H_0^2(\Omega),\, \|\Delta u\|_2 = 1} \int_\Omega e^{(32\pi^2-\epsilon)u^2 q(\|u\|_2^2)}\, dx,$$

which is impossible. Therefore $u_\epsilon \neq 0$. By Proposition 3.1, we have for any $p < 1/(1 - \|\Delta u_\epsilon\|_2^2)$

$$\limsup_{j\to+\infty} \int_\Omega e^{32\pi^2 p u_j^2}\, dx < +\infty. \tag{3.4}$$

By our assumption on $q(t)$, $0 \leqslant a_1 < \lambda(\Omega)$, $0 \leqslant a_2 \leqslant \lambda(\Omega)a_1, \ldots, 0 \leqslant a_k \leqslant \lambda(\Omega)a_{k-1}$, there holds for any $u \in H_0^2(\Omega)$ with $\|u\|_2 \neq 0$,

$$\begin{aligned} q\big(\|u\|_2^2\big)\big(1 - \|\Delta u\|_2^2\big) &= 1 + a_1\|u\|_2^2 + \cdots + a_k\|u\|_2^{2k} - \|\Delta u\|_2^2 \\ &\quad - a_1\|u\|_2^2\|\Delta u\|_2^2 - \cdots - a_k\|u\|_2^{2k}\|\Delta u\|_2^2 \\ &< 1 - a_k\|u\|_2^{2k}\|\Delta u\|_2^2. \end{aligned} \tag{3.5}$$

This leads to

$$q\big(\|u_j\|_2^2\big) \to q\big(\|u_\epsilon\|_2^2\big) < \frac{1}{1 - \|\nabla u_\epsilon\|_2^2}. \tag{3.6}$$

Combining (3.4) and (3.6), we conclude that f_j is bounded in $L^r(\Omega)$ for some $r > 1$. It follows that $f_i \to f_\epsilon$ strongly in $L^1(\Omega)$. We get the desired result immediately. $\square$

In the rest of the paper, we would mostly analyze the asymptotic behavior of maximizers u_ϵ described in Proposition 3.2. To do this, we consider the corresponding Euler–Lagrange equation of u_ϵ, namely

$$\begin{cases} \Delta^2 u_\epsilon = \dfrac{\beta_\epsilon}{\lambda_\epsilon} u_\epsilon e^{\alpha_\epsilon u_\epsilon^2} + \gamma_\epsilon u_\epsilon \quad \text{in } \Omega, \\ \|\Delta u_\epsilon\|_2 = 1, \quad u_\epsilon = \dfrac{\partial u_\epsilon}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \\ \alpha_\epsilon = (32\pi^2 - \epsilon) q\big(\|u_\epsilon\|_2^2\big), \\ \beta_\epsilon = \dfrac{q(\|u_\epsilon\|_2^2)}{q(\|u_\epsilon\|_2^2) + q'(\|u_\epsilon\|_2^2)\int_\Omega u_\epsilon^2\,dx}, \\ \gamma_\epsilon = \dfrac{q'(\|u_\epsilon\|_2^2)}{q(\|u_\epsilon\|_2^2) + q'(\|u_\epsilon\|_2^2)\int_\Omega u_\epsilon^2\,dx}, \\ \lambda_\epsilon = \displaystyle\int_\Omega u_\epsilon^2 e^{\alpha_\epsilon u_\epsilon^2}\,dx. \end{cases} \tag{3.7}$$

Here and in the sequel we denote the derivative of $q(t)$ by $q'(t)$. It is of significance to estimates the constants appeared in (3.7). Firstly, we have the following

Lemma 3.3. *α_ϵ, β_ϵ and γ_ϵ are all bounded sequences. Moreover α_ϵ has positive lower bound if all coefficients $a_1, \dots, a_k$ of $q(t)$ are nonnegative.*

Proof. Since $u_\epsilon \in H_0^2(\Omega)$ satisfies $\|\Delta u_\epsilon\| = 1$, we have $\|u_\epsilon\|_2 \leqslant C$ by the elliptic estimate (see [18] for example). Then the desired result is an easy consequence of the definitions of α_ϵ, β_ϵ and γ_ϵ. $\square$

Secondly, we have the following property of λ_ϵ.

Lemma 3.4. *There holds* $\liminf_{\epsilon\to 0} \lambda_\epsilon > 0$.

Proof. Using the inequality $e^t \leqslant 1 + te^t$ for $t \geqslant 0$, we obtain

$$\int_\Omega e^{\alpha_\epsilon u_\epsilon^2}\,dx \leqslant |\Omega| + \alpha_\epsilon \lambda_\epsilon. \tag{3.8}$$

By Proposition 3.2, one gets for any $\epsilon > 0$

$$\int_\Omega e^{\alpha_\epsilon u_\epsilon^2}\,dx \leqslant \sup_{u \in H_0^2(\Omega),\, \|\Delta u\|_2 = 1} \int_\Omega e^{32\pi^2 u^2 q(\|u\|_2^2)}\,dx.$$

On the other hand, for any fixed $u \in H_0^2(\Omega)$ with $\|\Delta u\|_2 = 1$, Fatou's Lemma together with Proposition 3.2 implies

$$\int_\Omega e^{32\pi^2 u^2 q(\|u\|_2^2)}\,dx \leqslant \lim_{\epsilon\to 0}\int_\Omega e^{(32\pi^2-\epsilon)u^2 q(\|u\|_2^2)}\,dx \leqslant \lim_{\epsilon\to 0}\int_\Omega e^{\alpha_\epsilon u_\epsilon^2}\,dx.$$

Combining the above two inequalities, we have

$$\lim_{\epsilon\to 0}\int_\Omega e^{\alpha_\epsilon u_\epsilon^2}\,dx = \sup_{u\in H_0^2(\Omega),\ \|\Delta u\|_2^2=1}\int_\Omega e^{2\pi u^2 q(\|u\|_2^2)}\,dx, \tag{3.9}$$

which together with the inequality (3.8) and Lemma 3.3 gives the desired result. □

4. Asymptotic behavior of extremals for subcritical functionals

We will analyze in this section the asymptotic behavior of u_ϵ. We will prove the uniqueness of the blow-up point, and understand the behavior of u_ϵ near the blow-up point and away from the blow-up point.

The crucial tool to study the regularity of high order equations is the Green representation formula. Recall that the Green function $G(x,y)$ for Δ^2 under the Dirichlet condition is defined by

$$\Delta^2 G(x,y) = \delta_x(y) \quad \text{in } \Omega, \qquad G(x,y) = \frac{\partial G(x,y)}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \tag{4.1}$$

All functions $u \in H_0^2(\Omega)\cap C^4(\overline{\Omega})$ satisfying $\Delta^2 u = f$ can be represented by

$$u(x) = \int_\Omega G(x,y) f(y)\,dy.$$

Several useful estimates of $G(x,y)$ are listed here for future reference, see for example [12], namely there exists $C>0$ such that for all $x, y\in\Omega$, $x\neq y$, we have

$$\left|G(x,y)\right| \leqslant C\log\left(2+\frac{1}{|x-y|}\right), \qquad \left|\nabla^i G(x,y)\right| \leqslant C|x-y|^{-i}, \quad i\geqslant 1. \tag{4.2}$$

Denote $c_\epsilon = |u_\epsilon|(x_\epsilon) = \max_\Omega |u_\epsilon|$. If c_ϵ is bounded, then applying the *standard regularity theory* to (3.7) we obtain $u_\epsilon \to u^*$ in $C^4(\overline{\Omega})$ for some $u^* \in H_0^2(\Omega)\cap C^4(\overline{\Omega})$ with $\|\Delta u^*\|_2 = 1$. This together with (3.9) leads to the conclusions of both Theorem 1.1* and Theorem 1.2*.

Without loss of generality we assume there exists some point $p\in\overline{\Omega}$ such that

$$x_\epsilon \to p, \quad c_\epsilon = u_\epsilon(x_\epsilon) = \max_\Omega u_\epsilon \to +\infty \quad \text{as } \epsilon\to 0, \tag{4.3}$$

for otherwise we consider $-u_\epsilon$ instead. We call p as the blow-up point. Here and in the sequel, we do *not* distinguish sequence and subsequence, the reader can understand it from the context.

Since u_ϵ is bounded in $H_0^2(\Omega)$, we may assume $u_\epsilon \rightharpoonup u_0$ weakly in $H_0^2(\Omega)$, and $u_\epsilon \to u_0$ strongly in $L^s(\Omega)$ for any $s>1$. Suppose $u_0\neq 0$, then $\|\Delta u_0\|_2 \neq 0$. We have by (3.5)

$$q\big(\|u_\epsilon\|\big) \to q\big(\|u_0\|\big) < \frac{1}{1-\|\Delta u_0\|_2^2},$$

which together with Proposition 3.1 implies that $e^{\alpha_\epsilon u_\epsilon^2}$ is bounded in $L^r(\Omega)$ for some $r > 1$ provided that ϵ is sufficiently small. Applying the standard regularity theory to Eq. (3.7), one gets c_ϵ is bounded and a contradiction with (4.3). Hence

$$\begin{cases} u_\epsilon \rightharpoonup 0 \quad \text{weakly in } H_0^2(\Omega), \\ u_\epsilon \to 0 \quad \text{strongly in } L^s(\Omega),\ \forall s > 1, \\ \alpha_\epsilon \to 32\pi^2, \quad \beta_\epsilon \to 1, \quad \gamma_\epsilon \to a_1. \end{cases} \tag{4.4}$$

In the rest of this section we focus on the case $p \in \Omega$, and leave the case $p \in \partial\Omega$ to Section 6. When $p \in \Omega$, we *claim* a Lions type energy concentration result, i.e.

$$|\Delta u_\epsilon|^2\,dx \rightharpoonup \delta_p \quad \text{in sense of measure,} \tag{4.5}$$

where δ_p is the usual Dirac measure supported at p. Suppose (4.5) is not true. Noting that $\|\Delta u_\epsilon\|_2 = 1$, we can find $r > 0$ and $\eta > 0$ such that

$$\limsup_{\epsilon \to 0} \int_{B_r(p)} |\Delta u_\epsilon|^2\,dx \leqslant 1 - \eta.$$

Sobolev imbedding theorem together with (4.4) leads to $\nabla u_\epsilon \to 0$ strongly in $L^2(\Omega)$. Hence for any cut-off function $\phi \in C_0^2(B_r(p))$ with $0 \leqslant \phi \leqslant 1$ on $B_r(p)$ and $\phi \equiv 1$ on $B_{r/2}(p)$, there holds

$$\limsup_{\epsilon \to 0} \int_{B_r(p)} \left|\Delta(\phi u_\epsilon)\right|^2 dx \leqslant 1 - \eta.$$

The Adams inequality (1.1) together with (4.4) implies that $e^{\alpha_\epsilon \phi^2 u_\epsilon^2}$ is bounded in $L^{\frac{2}{2-\eta}}(\Omega)$, and thus $e^{\alpha_\epsilon u_\epsilon^2}$ is bounded in $L^{\frac{2}{2-\eta}}(B_{r/2}(p))$ provided that ϵ is sufficiently small. Applying the standard regularity theory to (3.7), we have u_ϵ is bounded in $C^1(\overline{B_{r/4}(p)})$. This contradicts our assumption (4.3). Hence we conclude (4.5). In fact we have proved that there is no other blow-up point if p lies in the interior of Ω due to the fact that $\|\Delta u_\epsilon\|_2 = 1$.

To proceed, we introduce the following quantities

$$b_\epsilon = \lambda_\epsilon \Big/ \int_\Omega |u_\epsilon| e^{\alpha_\epsilon u_\epsilon^2}\,dx, \qquad \tau = \lim_{\epsilon \to 0} \frac{c_\epsilon}{b_\epsilon}, \qquad \sigma = \lim_{\epsilon \to 0} \frac{\int_\Omega u_\epsilon e^{\alpha_\epsilon u_\epsilon^2}\,dx}{\int_\Omega |u_\epsilon| e^{\alpha_\epsilon u_\epsilon^2}\,dx}. \tag{4.6}$$

By the definition of λ_ϵ (see (3.7)), we have $\tau \geqslant 1$ or $\tau = +\infty$. Obviously $|\sigma| \leqslant 1$. We will prove $\sigma = 1$ at the end of this section.

Let $r_\epsilon^4 = \frac{\lambda_\epsilon}{\beta_\epsilon c_\epsilon^2} e^{-\alpha_\epsilon c_\epsilon^2}$ and $\Omega_\epsilon = \{x \in \mathbb{R}^4\colon\ x_\epsilon + r_\epsilon x \in \Omega\}$. We *claim* that r_ϵ converges to zero rapidly. Indeed we have for any $\gamma\colon 0 < \gamma < 32\pi^2$,

$$r_\epsilon^4 c_\epsilon^2 e^{\gamma c_\epsilon^2} = \frac{1}{\beta_\epsilon} e^{(\gamma - \alpha_\epsilon) c_\epsilon^2} \int_\Omega u_\epsilon^2 e^{\alpha_\epsilon u_\epsilon^2}\,dx \leqslant \frac{1}{\beta_\epsilon} \int_\Omega u_\epsilon^2 e^{\gamma u_\epsilon^2}\,dx \to 0. \tag{4.7}$$

Here we have used the Hölder inequality and (4.4). In particular $r_\epsilon \to 0$ and $\Omega_\epsilon \to \mathbb{R}^4$ as $\epsilon \to 0$. To understand the asymptotic behavior of u_ϵ near the blow-up point p, we define two sequences of functions on Ω_ϵ, namely

$$\psi_\epsilon(x) = \frac{u_\epsilon(x_\epsilon + r_\epsilon x)}{c_\epsilon}, \qquad \varphi_\epsilon(x) = b_\epsilon\big(u_\epsilon(x_\epsilon + r_\epsilon x) - c_\epsilon\big). \tag{4.8}$$

Firstly the asymptotic behavior of ψ_ϵ will be considered by proving the following

Lemma 4.1. $\psi_\epsilon(x) \to 1$ *in* $C^4_{loc}(\mathbb{R}^4)$.

Proof. Obviously, $|\psi_\epsilon| \leqslant 1$. Since for any fixed $R > 0$, $x \in B_R(0)$,

$$\left|\Delta^2 \psi_\epsilon(x)\right| = \left| r_\epsilon^4 \left(\frac{\beta_\epsilon}{\lambda_\epsilon} \psi_\epsilon(x) e^{\alpha_\epsilon u_\epsilon^2(x_\epsilon + r_\epsilon x)} + \gamma_\epsilon \psi_\epsilon \right) \right| \leqslant \frac{1}{c_\epsilon^2} + r_\epsilon^4 \gamma_\epsilon \to 0,$$

and

$$\int_{B_R(0)} |\Delta \psi_\epsilon|^2 \, dx = \frac{1}{c_\epsilon^2} \int_{B_{Rr_\epsilon}(x_\epsilon)} |\Delta u_\epsilon|^2(y) \, dy \to 0. \tag{4.9}$$

The standard regularity theory and (4.9) give $\psi_\epsilon \to \psi$ in $C^4_{loc}(\mathbb{R}^4)$ with $\Delta\psi(x) = 0$ in $\mathbb{R}^4$. Noting that $\psi_\epsilon(0) = 1$, one gets by using the Liouville Theorem $\psi \equiv 1$ in $\mathbb{R}^4$. $\square$

Now we investigate the convergence of φ_ϵ.

Lemma 4.2. *Let τ be defined in* (4.6)*. Then $\varphi_\epsilon \to \varphi$ in $C^4_{loc}(\mathbb{R}^4)$, where*

$$\varphi(x) = \begin{cases} \frac{1}{16\pi^2\tau} \log \frac{1}{1 + \frac{\pi}{\sqrt{6}}|x|^2}, & x \in \mathbb{R}^4 \textit{ if } \tau < +\infty, \\ 0, & x \in \mathbb{R}^4 \textit{ if } \tau < +\infty. \end{cases}$$

Proof. Using the Green representation formula and the estimates (4.2), we have for $i = 1, 2$ and $x \in B_R(0)$

$$\begin{aligned} \left|\nabla^i \varphi_\epsilon(x)\right| &= b_\epsilon r_\epsilon^i \left| \int_\Omega \nabla_x^i G(x_\epsilon + r_\epsilon x, y) \Delta^2 u_\epsilon(y) \, dy \right| \\ &\leqslant C b_\epsilon r_\epsilon^i \left(\int_\Omega \frac{\frac{\beta_\epsilon}{\lambda_\epsilon} |u_\epsilon(y)| e^{\alpha_\epsilon u_\epsilon^2(y)}}{|x_\epsilon + r_\epsilon x - y|^i} \, dy + \int_\Omega \frac{\gamma_\epsilon |u_\epsilon(y)|}{|x_\epsilon + r_\epsilon x - y|^i} \, dy \right) \\ &\leqslant C b_\epsilon r_\epsilon^i \left(\int_{B_{2R}(x_\epsilon)} \frac{\frac{\beta_\epsilon}{\lambda_\epsilon} |u_\epsilon(y)| e^{\alpha_\epsilon u_\epsilon^2(y)}}{|x_\epsilon + r_\epsilon x - y|^i} \, dy + \int_\Omega \frac{\gamma_\epsilon |u_\epsilon(y)|}{|x_\epsilon + r_\epsilon x - y|^i} \, dy \right. \\ &\quad \left. + \int_{\Omega \setminus B_{2R}(x_\epsilon)} \frac{\frac{\beta_\epsilon}{\lambda_\epsilon} |u_\epsilon(y)| e^{\alpha_\epsilon u_\epsilon^2(y)}}{|x_\epsilon + r_\epsilon x - y|^i} \, dy \right) \end{aligned}$$

$$\leqslant C\left(\frac{b_\epsilon}{c_\epsilon}\int\limits_{B_{2R}(0)}\frac{dz}{|x-z|^i}+r_\epsilon^i\gamma_\epsilon b_\epsilon c_\epsilon\int\limits_{\Omega}\frac{dy}{|x_\epsilon+r_\epsilon x-y|^i}+\frac{1}{R^i}\right)$$
$$\leqslant C(R). \tag{4.10}$$

Here we have used (4.7). Note that φ_ϵ satisfies the following equation:

$$\Delta^2\varphi_\epsilon(x)=\frac{b_\epsilon}{c_\epsilon}\psi_\epsilon(x)e^{\alpha_\epsilon\frac{c_\epsilon}{b_\epsilon}(1+\psi_\epsilon(x))\varphi_\epsilon(x)}+\gamma_\epsilon b_\epsilon c_\epsilon r_\epsilon^4\psi_\epsilon(x),\quad x\in\Omega_\epsilon. \tag{4.11}$$

Because of (4.10), applying the standard regularity theory to (4.11), we have $\varphi_\epsilon\to\varphi$ in $C^4_{loc}(\mathbb{R}^4)$. If $\tau=\lim_{\epsilon\to 0}c_\epsilon/b_\epsilon<+\infty$, then one can see from (4.11) and Lemma 4.1 that φ satisfies

$$\Delta^2\varphi(x)=\frac{1}{\tau}e^{64\pi^2\tau\varphi(x)},\quad \varphi(x)\leqslant\varphi(0)=0,\quad \int\limits_{\mathbb{R}^4}e^{64\pi^2\tau\varphi(x)}\,dx<+\infty. \tag{4.12}$$

To understand φ further, we calculate

$$\Delta\varphi_\epsilon(x)=b_\epsilon r_\epsilon^2\int\limits_{\Omega}\Delta_x G(x_\epsilon+r_\epsilon x,y)\left(\frac{\beta_\epsilon}{\lambda_\epsilon}u_\epsilon(y)e^{\alpha_\epsilon u_\epsilon^2(y)}+\gamma_\epsilon u_\epsilon(y)\right)dy$$

and for any $R>0$,

$$\int\limits_{B_R(0)}|\Delta\varphi_\epsilon|\,dx\leqslant Cb_\epsilon r_\epsilon^2\int\limits_{\Omega}\frac{\beta_\epsilon}{\lambda_\epsilon}\big|u_\epsilon(y)\big|e^{\alpha_\epsilon u_\epsilon^2(y)}\left(\int\limits_{B_R(0)}\frac{dx}{|x_\epsilon+r_\epsilon x-y|^2}\right)dy$$
$$+Cb_\epsilon\gamma_\epsilon r_\epsilon^2\int\limits_{\Omega}\big|u_\epsilon(y)\big|\left(\int\limits_{B_R(0)}\frac{dx}{|x_\epsilon+r_\epsilon x-y|^2}\right)dy$$
$$\leqslant CR^2.$$

Hence, for any $R>0$, we have $\int_{B_R(0)}|\Delta\varphi|\,dx\leqslant CR^2$, which together with (4.12) and results of [25,42] gives that

$$\varphi(x)=\frac{1}{16\pi^2\tau}\log\frac{1}{1+\frac{\pi}{\sqrt{6}}|x|^2},\quad x\in\mathbb{R}^4. \tag{4.13}$$

If $\tau=+\infty$, we have by (4.10), $|\Delta\varphi(x)|\leqslant CR^{-2}$ for all $x\in B_R(0)$. Letting $R\to+\infty$ we know that φ is a harmonic function in $\mathbb{R}^4$. Noting that $\varphi(x)\leqslant\varphi(0)=0$, Liouville Theorem leads to $\varphi\equiv 0$. □

Next we consider the asymptotic behavior of u_ϵ away from the blow-up point p. We first have the following

Lemma 4.3. *$b_\epsilon u_\epsilon$ is bounded in $H_0^{2,r}(\Omega)$ for any $1 < r < 2$. In particular there exists a constant C depending only on Ω, $\lambda(\Omega)$ and α_0 such that $\|b_\epsilon u_\epsilon\|_{H_0^{2,r}(\Omega)} \leqslant C$ uniformly for $a_1 \in [0, \alpha_0]$ with $\alpha_0 < \lambda(\Omega)$, where $a_1 = q'(0)$.*

Proof. Let v_ϵ be a solution of the following equation:

$$\begin{cases} \Delta^2 v_\epsilon(x) = \dfrac{\beta_\epsilon}{\lambda_\epsilon} b_\epsilon u_\epsilon(x) e^{\alpha_\epsilon u_\epsilon^2(x)} & \text{in } \Omega, \\ v_\epsilon = \dfrac{\partial v_\epsilon}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.14}$$

By the Green representation formula, we calculate for $i = 1, 2$,

$$\left|\nabla^i v_\epsilon(x)\right| \leqslant C \int_\Omega |x-y|^{-i} \frac{\beta_\epsilon}{\lambda_\epsilon} b_\epsilon \left|u_\epsilon(y)\right| e^{\alpha_\epsilon u_\epsilon^2(y)}\, dy.$$

For any $1 < r < 2$, we have by Hölder inequality and definition of b_ϵ (see (4.6) above),

$$\left|\nabla^i v_\epsilon(x)\right|^r \leqslant C \int_\Omega |x-y|^{-ir} \frac{\beta_\epsilon}{\lambda_\epsilon} b_\epsilon \left|u_\epsilon(y)\right| e^{\alpha_\epsilon u_\epsilon^2(y)}\, dy.$$

Hence Fubini's Theorem implies $\|\nabla^i v_\epsilon\|_r \leqslant C$, $i = 1, 2$, whence

$$\|v_\epsilon\|_{H_0^{2,r}} \leqslant C. \tag{4.15}$$

Denote $w_\epsilon = b_\epsilon u_\epsilon - v_\epsilon$. Then, by (3.7) and (4.14), w_ϵ satisfies

$$\begin{cases} \Delta^2 w_\epsilon(x) = \gamma_\epsilon w_\epsilon + \gamma_\epsilon v_\epsilon & \text{in } \Omega, \\ w_\epsilon = \dfrac{\partial w_\epsilon}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.16}$$

Noting that $\gamma_\epsilon \to a_1 < \lambda(\Omega)$, testing (4.16) by w_ϵ, we have by the definition of $\lambda(\Omega)$ and the Hölder inequality,

$$\begin{aligned} \int_\Omega |\Delta w_\epsilon|^2\, dx &= \gamma_\epsilon \int_\Omega w_\epsilon^2\, dx + \gamma_\epsilon \int_\Omega v_\epsilon w_\epsilon\, dx \\ &\leqslant \frac{\gamma_\epsilon}{\lambda(\Omega)} \int_\Omega |\Delta w_\epsilon|^2\, dx + \frac{\gamma_\epsilon}{\sqrt{\lambda(\Omega)}} \|v_\epsilon\|_2 \|\Delta w_\epsilon\|_2, \end{aligned}$$

which together with (4.15) gives $\|\Delta w_\epsilon\|_2 \leqslant C$ for sufficiently small ϵ. This constant C depends only on Ω, $\lambda(\Omega)$ and α_0. Hence $\|w_\epsilon\|_{H_0^2(\Omega)} \leqslant C$, which together with (4.15) and Sobolev imbedding Theorem implies that $b_\epsilon u_\epsilon$ is bounded in $H_0^{2,r}$ for any $1 < r < 2$. $\square$

Secondly we will prove that $c_\epsilon u_\epsilon$ converges to some Green function. Noting that both b_ϵ and σ are defined in (4.6), we will prove the following

Lemma 4.4. *$b_\epsilon u_\epsilon \rightharpoonup G_{a_1}(\cdot, p)$ weakly in $H_0^{2,r}(\Omega)$ for any $1 < r < 2$ with*

$$\begin{cases} \Delta^2 G_{a_1} = \sigma \delta_p + a_1 G_{a_1} & \text{in } \Omega, \\ G_{a_1} = \dfrac{\partial G_{a_1}}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.17}$$

Furthermore, $b_\epsilon u_\epsilon \to G_{a_1}(\cdot, p)$ in $C^4_{loc}(\overline{\Omega} \setminus \{p\})$. Also we have

$$G_{a_1} = -\frac{\sigma}{8\pi^2} \log|x - p| + A_p + \psi(x), \tag{4.18}$$

where A_p is a constant depending on p and α, $\psi \in C^3(\overline{\Omega})$ and $\psi(p) = 0$.

Proof. By Lemma 4.3, there exists some function $G_{a_1}(\cdot, p) \in H_0^{2,s}$ such that $b_\epsilon u_\epsilon \rightharpoonup G_{a_1}(\cdot, p)$ weakly in $H_0^{2,s}(\Omega)$ for any $1 < s < 2$. For any fixed $r > 0$, by (4.5), $e^{\alpha_\epsilon u_\epsilon^2}$ is bounded in $L^s(\Omega \setminus B_r(p))$ for some $s > 1$. This is based on the subcritical Adams inequality and a cut-off function argument. By the standard regularity theory $b_\epsilon u_\epsilon \to G$ in $C^4_{loc}(\overline{\Omega} \setminus \{p\})$. It is easy to see form (3.7) that $b_\epsilon u_\epsilon$ satisfies

$$\begin{cases} \Delta^2(b_\epsilon u_\epsilon) = \dfrac{\beta_\epsilon}{\lambda_\epsilon} b_\epsilon u_\epsilon e^{\alpha_\epsilon u_\epsilon^2} + \gamma_\epsilon b_\epsilon u_\epsilon & \text{in } \Omega, \\ b_\epsilon u_\epsilon = \partial(b_\epsilon u_\epsilon)/\partial\nu = 0 & \text{on } \partial\Omega. \end{cases}$$

For any $\phi \in C^\infty(\overline{\Omega})$, we have

$$\begin{aligned} \int_\Omega \phi\left(\frac{\beta_\epsilon}{\lambda_\epsilon} b_\epsilon u_\epsilon e^{\alpha_\epsilon u_\epsilon^2} + \gamma_\epsilon b_\epsilon u_\epsilon\right) dx &= \int_\Omega \big(\phi(x) - \phi(p)\big) \frac{\beta_\epsilon}{\lambda_\epsilon} b_\epsilon u_\epsilon(x) e^{\alpha_\epsilon u_\epsilon^2(x)}\, dx \\ &\quad + \phi(p) \int_\Omega \frac{\beta_\epsilon}{\lambda_\epsilon} b_\epsilon u_\epsilon e^{\alpha_\epsilon u_\epsilon^2}\, dx + \int_\Omega \phi \gamma_\epsilon b_\epsilon u_\epsilon\, dx. \end{aligned}$$

Lebesgue dominated convergence theorem implies that $\int_{\{x \in \Omega:\ |u_\epsilon| < 1\}} e^{\alpha_\epsilon u_\epsilon^2}\, dx \to |\Omega|$. We have by (3.9) that

$$\int_\Omega |u_\epsilon| e^{\alpha_\epsilon u_\epsilon^2}\, dx \geqslant \int_{\{x \in \Omega:\ |u_\epsilon| \geqslant 1\}} e^{\alpha_\epsilon u_\epsilon^2}\, dx \to \sup_{u \in H_0^2(\Omega),\ \|\Delta u\|_2 \leqslant 1} \int_\Omega e^{32\pi^2 u^2 q(\|u\|_2^2)}\, dx - |\Omega|.$$

This leads to the fact that $b_\epsilon/\lambda_\epsilon$ is bounded. Hence we obtain by using Hölder inequality together with (4.4) and that $e^{\alpha_\epsilon u_\epsilon^2}$ is bounded in $L^s(\Omega \setminus B_r(p))$ for some $s > 1$,

$$\lim_{\epsilon \to 0} \int_{\Omega \setminus B_r(p)} \big(\phi - \phi(p)\big) \frac{\beta_\epsilon}{\lambda_\epsilon} b_\epsilon u_\epsilon e^{\alpha_\epsilon u_\epsilon^2}\, dx = 0.$$

Since $\int_{B_r(p)} \frac{1}{\lambda_\epsilon} b_\epsilon |u_\epsilon| e^{\alpha_\epsilon u_\epsilon^2}\, dx \leqslant 1$, $\beta_\epsilon \to 1$, and

$$\left|\int_{B_r(p)} \big(\phi-\phi(p)\big)\frac{\beta_\epsilon}{\lambda_\epsilon} b_\epsilon u_\epsilon e^{\alpha_\epsilon u_\epsilon^2}\,dx\right| \leqslant \int_{B_r(p)} \big|\phi-\phi(0)\big|\frac{\beta_\epsilon}{\lambda_\epsilon} b_\epsilon |u_\epsilon| e^{\alpha_\epsilon u_\epsilon^2}\,dx.$$

We immediately have

$$\lim_{r\to 0}\lim_{\epsilon\to 0}\int_{B_r(p)} \big(\phi-\phi(p)\big)\frac{\beta_\epsilon}{\lambda_\epsilon} b_\epsilon u_\epsilon e^{\alpha_\epsilon u_\epsilon^2}\,dx = 0.$$

It is obvious that

$$\lim_{\epsilon\to 0}\int_\Omega \phi\gamma_\epsilon b_\epsilon u_\epsilon\,dx = a_1\int_\Omega \phi G_{a_1}\,dx, \qquad \lim_{\epsilon\to 0}\int_\Omega \frac{\beta_\epsilon}{\lambda_\epsilon} b_\epsilon u_\epsilon e^{\alpha_\epsilon u_\epsilon^2}\,dx = \sigma.$$

Combing all above estimates, we obtain

$$\lim_{\epsilon\to 0}\int_\Omega \phi\left(\frac{\beta_\epsilon}{\lambda_\epsilon} b_\epsilon u_\epsilon e^{\alpha_\epsilon u_\epsilon^2} + \gamma_\epsilon b_\epsilon u_\epsilon\right)dx = \sigma\phi(p) + a_1\int_\Omega \phi G_{a_1}\,dx.$$

This proves the first part of the lemma.

To prove the second part, we define a cut-off function $\eta \in C_0^4(\Omega)$ such that $0 \leqslant \eta \leqslant 1$, $\eta \equiv 1$ on $B_r(p)$ and $\eta \equiv 0$ on $\Omega \setminus B_{2r}(p)$, where $B_{2r}(p) \Subset \Omega$. Let

$$g(x) = G_{a_1}(x,p) + \frac{\sigma}{8\pi^2}\eta(x)\log|x-p|.$$

It can be checked that g is a solution of

$$\begin{cases} \Delta^2 g = f & \text{in } \Omega, \\ g = \dfrac{\partial g}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

in a distributional sense, where

$$\begin{aligned} f(x) = -\frac{\sigma}{8\pi^2}\big(&\Delta^2\eta\log|x-p| + 2\nabla\Delta\eta\nabla\log|x-p| + 2\Delta\eta\Delta\log|x-p| \\ &+ 2\Delta\big(\nabla\eta\nabla\log|x-p|\big) + 2\nabla\eta\nabla\Delta\log|x-p|\big) + a_1 G_{a_1}(x,p). \end{aligned}$$

Noting that r is a fixed positive number, we can see from Lemma 4.3 that $f \in L^s(\Omega)$ for any $s > 1$. Standard regularity theory implies that $g \in C^3(\overline{\Omega})$. Let $A_p = g(p)$ and

$$\psi(x) = g(x) - g(p) + \frac{\sigma}{8\pi^2}(1-\eta)\log|x-p|.$$

We get the desired result. $\square$

Now we are in a position to derive an upper bound of $\int_\Omega e^{\alpha_\epsilon u_\epsilon^2}\,dx$ by using a Pohozaev identity, namely

Lemma 4.5. *(See [31,32].) Assume $\Omega' \subset \mathbb{R}^4$ is a smooth bounded domain. Let $u \in C^4(\overline{\Omega'})$ be a solution of $\Delta^2 u = f(u)$ in Ω'. Then we have for any $y \in \mathbb{R}^4$,*

$$4\int_{\Omega'} F(u)\,dx = \int_{\partial\Omega'} \langle x-y,\nu\rangle F(u)\,d\omega + \frac{1}{2}\int_{\partial\Omega'} v^2\langle x-y,\nu\rangle\,d\omega + 2\int_{\partial\Omega'} \frac{\partial u}{\partial\nu} v\,d\omega$$
$$+ \int_{\partial\Omega'} \left(\frac{\partial v}{\partial\nu}\langle x-y,\nabla u\rangle + \frac{\partial u}{\partial\nu}\langle x-y,\nabla v\rangle - \langle\nabla v,\nabla u\rangle\langle x-y,\nu\rangle\right) d\omega,$$

where $F(u) = \int_0^u f(s)\,ds$, $-\Delta u = v$ and $\nu(x)$ is the normal outward derivative of x on $\partial\Omega'$.

Choosing $\Omega' = B_r(x_\epsilon)$, $y = x_\epsilon$, $u = u_\epsilon$, $f(u_\epsilon) = \frac{\beta_\epsilon}{\lambda_\epsilon} u_\epsilon e^{\alpha_\epsilon u_\epsilon^2} + \gamma_\epsilon u_\epsilon$, noting that $v = -\Delta u_\epsilon$, $F(u_\epsilon) = \frac{\beta_\epsilon}{2\alpha_\epsilon\lambda_\epsilon} e^{\alpha_\epsilon u_\epsilon^2} + \frac{1}{2}\gamma_\epsilon u_\epsilon^2$, we obtain by Lemma 4.5,

$$\begin{aligned}\int_{B_r(x_\epsilon)} e^{\alpha_\epsilon u_\epsilon^2}\,dx = &-\frac{\alpha_\epsilon\lambda_\epsilon}{\beta_\epsilon b_\epsilon^2}\gamma_\epsilon \int_{B_r(x_\epsilon)} (b_\epsilon u_\epsilon)^2\,dx + \frac{r}{4}\int_{\partial B_r(x_\epsilon)} e^{\alpha_\epsilon u_\epsilon^2}\,d\omega \\ &+ \frac{\alpha_\epsilon\lambda_\epsilon\gamma_\epsilon}{4\beta_\epsilon b_\epsilon^2} r \int_{\partial B_r(x_\epsilon)} (b_\epsilon u_\epsilon)^2\,dx + \frac{\alpha_\epsilon\lambda_\epsilon}{4\beta_\epsilon b_\epsilon^2} r \int_{\partial B_r(x_\epsilon)} \left|\Delta(b_\epsilon u_\epsilon)\right|^2 d\omega \\ &- \frac{\alpha_\epsilon\lambda_\epsilon}{\beta_\epsilon b_\epsilon^2} \int_{\partial B_r(x_\epsilon)} \frac{\partial(b_\epsilon u_\epsilon)}{\partial\nu}\Delta(b_\epsilon u_\epsilon)\,d\omega \\ &- \frac{\alpha_\epsilon\lambda_\epsilon}{2\beta_\epsilon b_\epsilon^2} r \int_{\partial B_r(x_\epsilon)} \left(2\frac{\partial\Delta(b_\epsilon u_\epsilon)}{\partial r}\frac{\partial(b_\epsilon u_\epsilon)}{\partial r} - \nabla\Delta(b_\epsilon u_\epsilon)\nabla(b_\epsilon u_\epsilon)\right) d\omega.\end{aligned} \tag{4.19}$$

By the representation of $G_{a_1}(\cdot, p)$ (see Lemma 4.4), we have for any fixed $r > 0$

$$\int_{B_r(p)} G_{a_1}^2\,dx = o(1), \qquad \int_{\partial B_r(p)} G_{a_1}^2\,d\omega = o(1), \qquad r\int_{\partial B_r(p)} |\Delta G_{a_1}|^2\,dx = \frac{\sigma^2}{8\pi^2} + o(1),$$
$$\int_{\partial B_r(p)} \frac{\partial G_{a_1}}{\partial r}\Delta G_{a_1}\,d\omega = \frac{\sigma^2}{16\pi^2} + o(1), \qquad r\int_{\partial B_r(p)} \frac{\partial\Delta G_{a_1}}{\partial r}\frac{\partial G_{a_1}}{\partial r}\,d\omega = -\frac{\sigma^2}{8\pi^2} + o(1),$$
$$r\int_{\partial B_r(p)} \nabla\Delta G_{a_1}\nabla G_{a_1}\,d\omega = -\frac{\sigma^2}{8\pi^2} + o(1),$$

where $o(1) \to 0$ as $r \to 0$. Note that $\int_{\partial B_r(x_\epsilon)} e^{\alpha_\epsilon u_\epsilon^2}\,d\omega \to |\partial B_r(p)|$ as $\epsilon \to 0$. Fixing $r > 0$ first and letting $\epsilon \to 0$ in (4.19), then letting $r \to 0$, we have by Lemma 4.4

$$\lim_{r\to 0}\lim_{\epsilon\to 0}\int\limits_{B_r(x_\epsilon)} e^{\alpha_\epsilon u_\epsilon^2}\,dx=\sigma^2\lim_{\epsilon\to 0}\frac{\lambda_\epsilon}{b_\epsilon^2}.$$

Since $|\Delta u_\epsilon|^2\,dx\rightharpoonup\delta_p$ in sense of measure, we immediately have

$$\lim_{r\to 0}\lim_{\epsilon\to 0}\int\limits_{\Omega\setminus B_r(p)} e^{\alpha_\epsilon u_\epsilon^2}\,dx=|\Omega|.$$

Combining these two identities we obtain

$$\lim_{\epsilon\to 0}\int\limits_{\Omega} e^{\alpha_\epsilon u_\epsilon^2}\,dx=|\Omega|+\sigma^2\lim_{\epsilon\to 0}\frac{\lambda_\epsilon}{b_\epsilon^2},\tag{4.20}$$

provided that $\sigma\neq 0$ in case $\lim_{\epsilon\to 0}\frac{\lambda_\epsilon}{b_\epsilon^2}=+\infty$. It is remarkable that we do not know whether or not the limits in (4.20) are finite at this stage. Keeping in mind that $|\sigma|\leqslant 1$ (see (4.6) above), we *claim* that $\sigma\neq 0$. For otherwise, if $\lambda_\epsilon/b_\epsilon^2\to+\infty$, noting that Hölder inequality gives

$$\begin{aligned}\frac{\lambda_\epsilon}{b_\epsilon^2}&=\frac{1}{\lambda_\epsilon}\biggl(\int\limits_{\Omega}|u_\epsilon|e^{\alpha_\epsilon u_\epsilon^2}\,dx\biggr)^2\\&\leqslant\frac{1}{\lambda_\epsilon}\int\limits_{\Omega}u_\epsilon^2e^{\alpha_\epsilon u_\epsilon^2}\,dx\int\limits_{\Omega}e^{\alpha_\epsilon u_\epsilon^2}\,dx=\int\limits_{\Omega}e^{\alpha_\epsilon u_\epsilon^2}\,dx,\end{aligned}\tag{4.21}$$

we can easily get a contradiction with (4.19) for sufficiently small ϵ; if $\lambda_\epsilon/b_\epsilon^2$ is bounded, $\sigma=0$ leads to $\int_\Omega e^{\alpha_\epsilon u_\epsilon^2}dx\to|\Omega|$ because of (4.20). This contradicts (3.9) and confirms our claim. We can further locate σ as follows.

Lemma 4.6. *There holds* $\sigma=1$.

Proof. By Lemma 4.4, $b_\epsilon u_\epsilon\to G_{a_1}$ in $C^4_{loc}(\overline{\Omega}\setminus\{p\})$, and $G_{a_1}=-\frac{\sigma}{8\pi^2}\log|x-p|$. We have known that $\sigma\neq 0$. Suppose $\sigma<0$, then $G_{a_1}(x,p)\leqslant -C<0$ in $B_\rho(p)$ for some $\rho>0$ and positive constant C. Whence $u_\epsilon<0$ in $B_\rho(p)\setminus\{p\}$ for sufficiently small ϵ. On the other hand we have by Lemma 4.2, $u_\epsilon>0$ in $B_{Rr_\epsilon}(x_\epsilon)$ for any fixed $R>0$ and sufficiently small ϵ. Note that $B_{Rr_\epsilon}(x_\epsilon)\subset B_\rho(p)$ for sufficiently small $\epsilon>0$. We get a contradiction with the sign of u_ϵ on $B_{Rr_\epsilon}(x_\epsilon)\setminus\{p\}$. Hence $\sigma>0$. Then $G_{a_1}(x,p)\geqslant C_1>0$ in $B_r(p)\setminus\{p\}$ for some $r>0$ and positive constant C_1. Whence $u_\epsilon>0$ in $B_r(p)\setminus\{p\}$ for sufficiently small $\epsilon>0$. Hence

$$\int\limits_{B_\rho(p)}|u_\epsilon|e^{\alpha_\epsilon u_\epsilon^2}\,dx=\int\limits_{B_\rho(p)}u_\epsilon e^{\alpha_\epsilon u_\epsilon^2}\,dx.$$

Since $|\Delta u_\epsilon|^2\,dx\rightharpoonup\delta_p$ and $u_\epsilon\to 0$ in $L^s(\Omega)$ for all $s>1$, Hölder inequality leads to

$$\lim_{\epsilon\to 0}\int\limits_{\Omega\setminus B_r(p)}|u_\epsilon|e^{\alpha_\epsilon u_\epsilon^2}\,dx=0.$$

It can be easily derived from Lebesgue dominated convergence theorem and (3.9) that

$$\lim_{\epsilon\to 0}\int\limits_{\{x\in\Omega:\,u_\epsilon\leqslant 1\}}e^{\alpha_\epsilon u_\epsilon^2}\,dx=|\Omega|,\qquad \lim_{\epsilon\to 0}\int\limits_{\{x\in\Omega:\,u_\epsilon>1\}}e^{\alpha_\epsilon u_\epsilon^2}\,dx>0.$$

Noting that $u_\epsilon\to 0$, a.e. in Ω, we have

$$\lim_{\epsilon\to 0}\int\limits_{B_r(p)}|u_\epsilon|e^{\alpha_\epsilon u_\epsilon^2}\,dx\geqslant\lim_{\epsilon\to 0}\int\limits_{\{x\in\Omega:\,u_\epsilon>1\}}e^{\alpha_\epsilon u_\epsilon^2}\,dx>0.$$

By definition of σ, we calculate

$$\begin{aligned}\sigma&=\lim_{\epsilon\to 0}\frac{\int_{\Omega\setminus B_r(p)}u_\epsilon e^{\alpha_\epsilon u_\epsilon^2}\,dx+\int_{B_r(p)}u_\epsilon e^{\alpha_\epsilon u_\epsilon^2}\,dx}{\int_{\Omega\setminus B_r(p)}|u_\epsilon|e^{\alpha_\epsilon u_\epsilon^2}\,dx+\int_{B_r(p)}|u_\epsilon|e^{\alpha_\epsilon u_\epsilon^2}\,dx}\\&=\frac{\lim_{\epsilon\to 0}\int_{B_r(p)}u_\epsilon e^{\alpha_\epsilon u_\epsilon^2}\,dx}{\lim_{\epsilon\to 0}\int_{B_r(p)}u_\epsilon e^{\alpha_\epsilon u_\epsilon^2}\,dx}=1.\qquad\square\end{aligned}$$

Thus Lemma 4.4 can be restated as

Lemma 4.7. *$b_\epsilon u_\epsilon\rightharpoonup G_{a_1}(\cdot,p)$ weakly in $H_0^{2,r}(\Omega)$ for any $1<r<2$ with*

$$\begin{cases}\Delta^2 G_{a_1}=\delta_p+a_1G_{a_1} & \text{in } \Omega,\\ G_{a_1}=\dfrac{\partial G_{a_1}}{\partial\nu}=0 & \text{on } \partial\Omega.\end{cases}$$

Furthermore, $b_\epsilon u_\epsilon\to G_{a_1}(\cdot,p)$ in $C^4_{loc}(\overline{\Omega}\setminus\{p\})$. Also we have

$$G_{a_1}=-\frac{1}{8\pi^2}\log|x-p|+A_p+\psi(x),$$

where A_p is a constant depending on p and a_1, $\psi\in C^3(\overline{\Omega})$ and $\psi(p)=0$.

5. Neck analysis

In this section, we still assume u_ϵ blows up and the blow-up point $p\in\Omega$. We will use capacity estimates to calculate the limit of $\lambda_\epsilon/b_\epsilon^2$, which together with (4.20) gives the supremum of the functional $\int_\Omega e^{32\pi^2u^2q(\|u\|_2^2)}\,dx$ under the assumption that u_ϵ blows up. The technique of capacity estimates applied to this kind of problems was first used in [21] in dealing with Moser's inequality of first order derivatives.

Let $\chi(t):(0,+\infty)\to\mathbb{R}$ be a smooth function satisfying $0\leqslant\chi\leqslant 1$, $|\chi'(t)|\leqslant 5$, $|\chi''(t)|\leqslant 10$, $\chi\equiv 1$ when $t\leqslant 4/3$, and $\chi\equiv 0$ when $t\geqslant 5/3$. Define a function g_ϵ on domain $B_{2Rr_\epsilon}(x_\epsilon)$ by

$$g_\epsilon(x) = u_\epsilon(x) - \chi\left(\frac{|x - x_\epsilon|}{Rr_\epsilon}\right)\left(u_\epsilon - c_\epsilon - \frac{1}{b_\epsilon}\varphi\left(\frac{x - x_\epsilon}{r_\epsilon}\right)\right),$$

where φ is given by Lemma 4.3. It is easy to check that g_ϵ satisfies the following boundary conditions

$$\begin{cases} g_\epsilon(x) = c_\epsilon + \dfrac{1}{b_\epsilon}\varphi\left(\dfrac{x - x_\epsilon}{r_\epsilon}\right) \quad \text{on } \partial B_{Rr_\epsilon}(x_\epsilon), \qquad g_\epsilon = u_\epsilon \quad \text{on } \partial B_{2Rr_\epsilon}(x_\epsilon), \\ \dfrac{\partial g_\epsilon}{\partial \nu} = \dfrac{1}{b_\epsilon r_\epsilon}\dfrac{\partial \varphi}{\partial \nu}\left(\dfrac{x - x_\epsilon}{r_\epsilon}\right) \quad \text{on } \partial B_{Rr_\epsilon}(x_\epsilon), \qquad \dfrac{\partial g_\epsilon}{\partial \nu} = \dfrac{\partial u_\epsilon}{\partial \nu} \quad \text{on } \partial B_{2Rr_\epsilon}(x_\epsilon), \end{cases} \tag{5.1}$$

where ν is the outward unit vector on the boundary of $B_{2Rr_\epsilon}(x_\epsilon) \setminus B_{Rr_\epsilon}(x_\epsilon)$. By Lemma 4.3, we get on $B_{2Rr_\epsilon}(x_\epsilon)$

$$\begin{cases} u_\epsilon(x) = c_\epsilon + \dfrac{1}{b_\epsilon}\varphi\left(\dfrac{x - x_\epsilon}{r_\epsilon}\right) + o\left(\dfrac{1}{b_\epsilon}\right), \\ r_\epsilon \nabla u_\epsilon(x) = \dfrac{1}{b_\epsilon}\nabla\varphi\left(\dfrac{x - x_\epsilon}{r_\epsilon}\right) + o\left(\dfrac{1}{b_\epsilon}\right), \\ r_\epsilon^2 \Delta u_\epsilon(x) = \dfrac{1}{b_\epsilon}\Delta\varphi\left(\dfrac{x - x_\epsilon}{r_\epsilon}\right) + o\left(\dfrac{1}{b_\epsilon}\right). \end{cases}$$

Whence we obtain on $B_{2Rr_\epsilon}(x_\epsilon)$

$$\Delta(u_\epsilon - g_\epsilon) = o\left(\frac{1}{b_\epsilon}\right) r_\epsilon^{-2}, \qquad r_\epsilon^2 |\Delta u_\epsilon| \leqslant C(R) b_\epsilon^{-1}$$

for some constant $C(R)$ depending only on R. This immediately leads to

$$\int\limits_{B_{2Rr_\epsilon}(x_\epsilon)\setminus B_{Rr_\epsilon}(x_\epsilon)} \left|\Delta g_\epsilon(x)\right|^2 dx = \int\limits_{B_{2Rr_\epsilon}(x_\epsilon)\setminus B_{Rr_\epsilon}(x_\epsilon)} \left|\Delta u_\epsilon(x)\right|^2 dx + o\left(\frac{1}{b_\epsilon^2}\right) C(R). \tag{5.2}$$

We also define a sequence of functions h_ϵ on $B_\delta(x_\epsilon)$ by

$$h_\epsilon(x) = u_\epsilon - \left(1 - \chi\left(2\delta^{-1}|x - x_\epsilon|\right)\right)\left(u_\epsilon + \frac{1}{8\pi^2 b_\epsilon}\log|x - x_\epsilon| - \frac{A_p}{b_\epsilon}\right).$$

One can check that h_ϵ satisfies the following boundary conditions

$$\begin{cases} h_\epsilon(x) = \dfrac{1}{b_\epsilon}\left(\dfrac{1}{8\pi^2}\log\dfrac{1}{\delta} + A_p\right) \quad \text{on } \partial B_\delta(x_\epsilon), \qquad h_\epsilon = u_\epsilon \quad \text{on } \partial B_{\delta/2}(x_\epsilon), \\ \dfrac{\partial h_\epsilon}{\partial \nu} = -\dfrac{1}{8\pi^2 \delta b_\epsilon} \quad \text{on } \partial B_\delta(x_\epsilon), \qquad \dfrac{\partial h_\epsilon}{\partial \nu} = \dfrac{\partial u_\epsilon}{\partial \nu} \quad \text{on } \partial B_{\delta/2}(x_\epsilon), \end{cases} \tag{5.3}$$

where ν denotes the outward unit vector on $\partial(B_\delta(x_\epsilon) \setminus B_{\delta/2}(x_\epsilon))$. According to Lemma 4.7, we have for $x \in B_\delta(x_\epsilon) \setminus B_{\delta/2}(x_\epsilon)$

$$\begin{cases} u_\epsilon(x) = -\dfrac{1}{8\pi^2 b_\epsilon}\log|x-x_\epsilon| + \dfrac{A_p + o(1)}{b_\epsilon}, \\ b_\epsilon \nabla u_\epsilon(x) = -\dfrac{1}{8\pi^2}\dfrac{x-p}{|x-p|^2} + \nabla\psi(x) + o(1), \\ b_\epsilon \Delta u_\epsilon(x) = -\dfrac{1}{4\pi^2|x-p|^2} + \Delta\psi(x) + o(1). \end{cases}$$

Here and in the sequel, $o(1) \to 0$ as $\epsilon \to 0$ first and then $\delta \to 0$. A straightforward computation shows

$$\int\limits_{B_\delta(x_\epsilon)\setminus B_{\delta/2}(x_\epsilon)} \left|\Delta(u_\epsilon - h_\epsilon)\right|^2 dx = \frac{o(1)}{b_\epsilon^2}.$$

This, together with the fact that $|b_\epsilon \Delta u_\epsilon| \leqslant C$ on $B_\delta(x_\epsilon) \setminus B_{\delta/2}(x_\epsilon)$ for some constant C depending only on δ, gives

$$\int\limits_{B_\delta(x_\epsilon)\setminus B_{\delta/2}(x_\epsilon)} |\Delta h_\epsilon|^2 dx = \int\limits_{B_\delta(x_\epsilon)\setminus B_{\delta/2}(x_\epsilon)} |\Delta u_\epsilon|^2 dx + \frac{o(1)}{b_\epsilon^2}. \tag{5.4}$$

Define a sequence of functions

$$u_\epsilon^\star(x) = \begin{cases} g_\epsilon(x), & x \in B_{2Rr_\epsilon}(x_\epsilon) \setminus B_{Rr_\epsilon}(x_\epsilon), \\ u_\epsilon(x), & x \in B_{\delta/2}(x_\epsilon) \setminus B_{2Rr_\epsilon}(x_\epsilon), \\ h_\epsilon, & x \in B_\delta(x_\epsilon) \setminus B_{\delta/2}(x_\epsilon). \end{cases}$$

Combining (5.1)–(5.4), we conclude $u_\epsilon^\star \in H^2(B_\delta(x_\epsilon) \setminus B_{Rr_\epsilon}(x_\epsilon))$ and satisfies boundary conditions

$$\begin{cases} u_\epsilon^\star(x) = \dfrac{1}{b_\epsilon}\left(\dfrac{1}{8\pi^2}\log\dfrac{1}{\delta} + A_p\right) & \text{on } \partial B_\delta(x_\epsilon), \\ u_\epsilon^\star(x) = c_\epsilon + \dfrac{1}{b_\epsilon}\varphi\left(\dfrac{x-x_\epsilon}{r_\epsilon}\right) & \text{on } \partial B_{Rr_\epsilon}(x_\epsilon), \\ \dfrac{\partial u_\epsilon^\star}{\partial\nu} = -\dfrac{1}{8\pi^2\delta b_\epsilon} & \text{on } \partial B_\delta(x_\epsilon), \\ \dfrac{\partial u_\epsilon^\star}{\partial\nu} = \dfrac{1}{b_\epsilon r_\epsilon}\dfrac{\partial\varphi}{\partial\nu}\left(\dfrac{x-x_\epsilon}{r_\epsilon}\right) & \text{on } \partial B_{Rr_\epsilon}(x_\epsilon), \end{cases} \tag{5.5}$$

and energy identity

$$\int\limits_{B_\delta(x_\epsilon)\setminus B_{Rr_\epsilon}(x_\epsilon)} \left|\Delta u_\epsilon^\star(x)\right|^2 dx = \int\limits_{B_\delta(x_\epsilon)\setminus B_{Rr_\epsilon}(x_\epsilon)} \left|\Delta u_\epsilon(x)\right|^2 dx + \frac{o(1)}{b_\epsilon^2}. \tag{5.6}$$

Now we start to derive the capacity estimates. Write for simplicity

$$i_{\delta,R,\epsilon} = \inf \int_{B_\delta(x_\epsilon)\setminus B_{Rr_\epsilon}(x_\epsilon)} |\Delta u|^2\,dx,$$

here the infimum takes through all functions belonging to $H^2(B_\delta(x_\epsilon) \setminus B_{Rr_\epsilon}(x_\epsilon))$ with the same boundary conditions as $u^\star$, namely (5.5). Then, noting that $\|\Delta u_\epsilon\|_2 = 1$ and (5.6), we have by Lemmas 4.3 and 4.7,

$$\begin{aligned} i_{\delta,R,\epsilon} &\leqslant \int_{B_\delta(x_\epsilon)\setminus B_{Rr_\epsilon}(x_\epsilon)} |\Delta u^\star|^2\,dx \\ &= \int_{B_\delta(x_\epsilon)\setminus B_{Rr_\epsilon}(x_\epsilon)} \left|\Delta u_\epsilon(x)\right|^2 dx + \frac{o(1)}{b_\epsilon^2} \\ &\leqslant 1 - \int_{\Omega\setminus B_\delta(x_\epsilon)} \left|\Delta u_\epsilon(x)\right|^2 dx - \int_{B_{Rr_\epsilon}(x_\epsilon)} \left|\Delta u_\epsilon(x)\right|^2 dx + \frac{o(1)}{b_\epsilon^2} \\ &= 1 - \frac{1}{b_\epsilon^2}\left(\int_{\Omega\setminus B_\delta(p)} |\Delta G_{a_1}|^2\,dx + \int_{B_R(0)} |\Delta\varphi|^2\,dx\right) + \frac{o(1)}{b_\epsilon^2}. \end{aligned} \tag{5.7}$$

It is known (see for example [22,23]) that the infimum $i_{\delta,R,\epsilon}$ can be attained by a bi-harmonic function $\mathcal{T}$ which is defined in the annular domain $B_\delta(x_\epsilon) \setminus B_{Rr_\epsilon}(x_\epsilon)$ with the same boundary condition as $u^\star$. Moreover $\mathcal{T}$ takes the form

$$\mathcal{T}(x) = \mathcal{A}\log|x - x_\epsilon| + \mathcal{B}|x - x_\epsilon|^2 + \mathcal{C}\frac{1}{|x - x_\epsilon|^2} + \mathcal{D}$$

for some constants $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$. Hence

$$\begin{aligned} \int_{B_\delta(x_\epsilon)\setminus B_{Rr_\epsilon}(x_\epsilon)} |\Delta\mathcal{T}|^2\,dx &= 8\pi^2\mathcal{A}^2\log\frac{\delta}{Rr_\epsilon} + 32\pi^2\mathcal{A}\mathcal{B}\big(\delta^2 - R^2r_\epsilon^2\big) \\ &\quad + 32\pi^2\mathcal{B}^2\big(\delta^4 - R^4r_\epsilon^4\big). \end{aligned} \tag{5.8}$$

Substituting $\mathcal{T}(x)$ into the boundary conditions (5.5), one gets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ by solving a linear system, in particular

$$\begin{cases} \mathcal{A} = \dfrac{\mathcal{P}_1 - \mathcal{P}_2 + \frac{Rr_\epsilon(\delta^2 - Rr_\epsilon^2)}{2(\delta^2 + Rr_\epsilon^2)}\mathcal{Q}_1 + \frac{\delta(\delta^2 - Rr_\epsilon^2)}{2(\delta^2 + Rr_\epsilon^2)}\mathcal{Q}_2}{\frac{\delta^2 - Rr_\epsilon^2}{\delta^2 + Rr_\epsilon^2} + \log\frac{Rr_\epsilon}{\delta}}, \\ \mathcal{B} = \dfrac{2(\mathcal{P}_2 - \mathcal{P}_1) - (Rr_\epsilon + \frac{2R^3r_\epsilon^3}{\delta^2 - R^2r_\epsilon^2}\log\frac{Rr_\epsilon}{\delta})\mathcal{Q}_1 + (\delta + \frac{2\delta^3}{\delta^2 - R^2r_\epsilon^2}\log\frac{Rr_\epsilon}{\delta})\mathcal{Q}_2}{4(\delta^2 - R^2r_\epsilon^2) + 4(\delta^2 + R^2r_\epsilon^2)\log\frac{Rr_\epsilon}{\delta}}. \end{cases}$$

Here $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are boundary values of $\mathcal{T}$, precisely

$$\mathcal{P}_1 = \mathcal{T}|_{\partial B_{Rr_\epsilon}(x_\epsilon)} = c_\epsilon + \frac{\varphi(R)}{b_\epsilon}, \qquad \mathcal{P}_2 = \mathcal{T}|_{\partial B_\delta(x_\epsilon)} = \frac{\frac{1}{8\pi^2}\log\frac{1}{\delta} + A_p}{b_\epsilon},$$
$$\mathcal{Q}_1 = -\frac{\partial \mathcal{T}}{\partial \nu}|_{\partial B_{Rr_\epsilon}(x_\epsilon)} = \frac{\varphi'(R)}{b_\epsilon r_\epsilon}, \qquad \mathcal{Q}_2 = \frac{\partial \mathcal{T}}{\partial \nu}|_{\partial B_\delta(x_\epsilon)} = -\frac{1}{8\pi^2 \delta b_\epsilon}.$$

Noting that $\varphi(x)$ is radially symmetric, we have denoted $\varphi(x)$ by $\varphi(|x|)$ without any confusion. Now we can calculate the capacity $i_{\delta,R,\epsilon}$, i.e. $\|\Delta\mathcal{T}\|_2^2$ precisely. Our aim to calculate the capacity is to derive the limit of $\lambda_\epsilon/c_\epsilon^2$. Though we have no idea on whether or not $\lambda_\epsilon/c_\epsilon^2$ is bounded at this stage, we can control its possible divergence speed, say

Lemma 5.1. *We have* $\frac{1}{c_\epsilon^2}\log\frac{\lambda_\epsilon}{\beta_\epsilon c_\epsilon^2} \to 0$ *as* $\epsilon \to 0$.

Proof. For any fixed $s > 0$, by definition of λ_ϵ, we have

$$\frac{\lambda_\epsilon}{c_\epsilon^2} \leqslant \int\limits_\Omega e^{\alpha_\epsilon u_\epsilon^2}\,dx \leqslant e^{sc_\epsilon^2}\int\limits_\Omega e^{(\alpha_\epsilon - s)u_\epsilon^2}\,dx.$$

Noting that $\int_\Omega e^{(\alpha_\epsilon - s)u_\epsilon^2}\,dx \to |\Omega|$ and $\beta_\epsilon \to 1$, we immediately obtain

$$\limsup_{\epsilon\to 0}\frac{1}{c_\epsilon^2}\log\frac{\lambda_\epsilon}{\beta_\epsilon c_\epsilon^2} \leqslant s.$$

Letting $s \to 0$, one has

$$\limsup_{\epsilon\to 0}\frac{1}{c_\epsilon^2}\log\frac{\lambda_\epsilon}{\beta_\epsilon c_\epsilon^2} \leqslant 0.$$

By $\beta_\epsilon \to 1$ and Lemma 3.4, there exists a constant $\kappa > 0$ such that $\frac{\lambda_\epsilon}{\beta_\epsilon c_\epsilon^2} \geqslant \kappa\frac{1}{c_\epsilon^2}$, whence

$$\liminf_{\epsilon\to 0}\frac{1}{c_\epsilon^2}\log\frac{\lambda_\epsilon}{\beta_\epsilon c_\epsilon^2} \geqslant 0.$$

Hence we get the desired result. $\square$

Remark 5.2. We caution the reader that when $q(t) \equiv 1$, $\lambda_\epsilon/c_\epsilon^2$ is certainly bounded due to the Adams inequality. A serious difficulty in the case $q(t) \not\equiv 1$ would arise when we aim to prove that $\lambda_\epsilon/c_\epsilon^2$ is bounded.

Recalling that $r_\epsilon^4 = \frac{\lambda_\epsilon}{\beta_\epsilon c_\epsilon^2}e^{-\alpha_\epsilon c_\epsilon^2}$, one has

$$\log\frac{Rr_\epsilon}{\delta} = \log\frac{R}{\delta} + \frac{\log\frac{\lambda_\epsilon}{\beta_\epsilon c_\epsilon^2} - \alpha_\epsilon c_\epsilon^2}{4}. \tag{5.9}$$

This together with Lemma 5.1 gives

$$\mathcal{A}=\frac{c_\epsilon+\frac{\varphi(R)+2R\varphi'(R)+\frac{1}{8\pi^2}\log\delta-A_p-\frac{1}{16\pi^2}+o(1)}{b_\epsilon}}{1+\log\frac{R}{\delta}+\frac{\log\frac{\lambda_\epsilon}{\beta_\epsilon c_\epsilon^2}-\alpha_\epsilon c_\epsilon^2}{4}+o(1)}=O\left(\frac{1}{c_\epsilon}\right). \tag{5.10}$$

Similarly we calculate

$$\mathcal{B}=\frac{-2c_\epsilon+\frac{\alpha_\epsilon c_\epsilon^2+\log\frac{\lambda_\epsilon}{\beta_\epsilon c_\epsilon^2}}{16\pi^2 b_\epsilon}+O(\frac{\varphi(R)}{b_\epsilon})}{\delta^2(\log\frac{\lambda_\epsilon}{\beta_\epsilon c_\epsilon^2}-\alpha_\epsilon c_\epsilon^2+4\log\frac{R}{\delta})+o(1)}=\frac{1}{\delta^2}O\left(\frac{1}{b_\epsilon}\right). \tag{5.11}$$

According to (5.10), a straightforward calculation shows

$$\begin{aligned}\mathcal{A}^2=\frac{16}{\alpha_\epsilon^2 c_\epsilon^2}\biggl(1&+\frac{2\varphi(R)+4R\varphi'(R)+\frac{1}{4\pi^2}\log\delta-2A_p-\frac{1}{8\pi^2}}{b_\epsilon c_\epsilon}\\&+\frac{2\log\frac{\lambda_\epsilon}{\beta_\epsilon c_\epsilon^2}+8+8\log\frac{R}{\delta}}{\alpha_\epsilon c_\epsilon^2}+O\left(\frac{\log^2\frac{\lambda_\epsilon}{\beta_\epsilon c_\epsilon^2}}{c_\epsilon^4}\right)+o\left(\frac{1}{b_\epsilon c_\epsilon}\right)\biggr).\end{aligned}$$

Multiplying (5.9) by $8\pi^2\mathcal{A}^2$, one has

$$\begin{aligned}8\pi^2\mathcal{A}^2\log\frac{\delta}{Rr_\epsilon}=\frac{32\pi^2}{\alpha_\epsilon}\biggl(1&+\frac{2\varphi(R)+4R\varphi'(R)+\frac{1}{4\pi^2}\log\delta-2A_p-\frac{1}{8\pi^2}}{b_\epsilon c_\epsilon}\\&+\frac{\log\frac{\lambda_\epsilon}{\beta_\epsilon c_\epsilon^2}+8+4\log\frac{R}{\delta}}{\alpha_\epsilon c_\epsilon^2}+O\left(\frac{\log^2\frac{\lambda_\epsilon}{\beta_\epsilon c_\epsilon^2}}{c_\epsilon^4}\right)+o\left(\frac{1}{b_\epsilon c_\epsilon}\right)\biggr).\end{aligned} \tag{5.12}$$

It follows from (5.10) and (5.11) that

$$32\pi^2\mathcal{A}\mathcal{B}\bigl(R_2^2-R_1^2\bigr)=O\left(\frac{1}{b_\epsilon c_\epsilon}\right),\qquad 32\pi^2\mathcal{B}^2\bigl(R_2^4-R_1^4\bigr)=O\left(\frac{1}{b_\epsilon^2}\right). \tag{5.13}$$

Integrating by parts, we have by Lemma 4.7

$$\begin{aligned}\int_{\Omega\setminus B_\delta(p)}|\Delta G_{a_1}|^2\,dx&=-\int_{\partial B_\delta(p)}\Delta G\frac{\partial G}{\partial r}\,d\omega+\int_{\partial B_\delta(p)}G\frac{\partial\Delta G}{\partial r}\,d\omega+a_1\int_{\Omega\setminus B_\delta(p)}G_{a_1}^2\,dx\\&=-\frac{1}{16\pi^2}-\frac{1}{8\pi^2}\log\delta+A_p+a_1\|G_{a_1}\|_2^2+O(\delta\log\delta).\end{aligned} \tag{5.14}$$

Note also that

$$\alpha_\epsilon=\bigl(32\pi^2-\epsilon\bigr)q\bigl(\|u_\epsilon\|_2^2\bigr)=\bigl(32\pi^2-\epsilon\bigr)\left(1+\frac{a_1\|G_{a_1}\|_2^2}{b_\epsilon^2}+o\left(\frac{1}{b_\epsilon^2}\right)\right). \tag{5.15}$$

Inserting (5.12) and (5.13) into (5.8), then inserting (5.14), (5.15) and (5.8) into (5.7), we obtain

$$\frac{32\pi^2}{32\pi^2-\epsilon}\left(1-\frac{a_1\|G_{a_1}\|_2^2}{b_\epsilon^2}+\frac{2\varphi(R)+4R\varphi'(R)+\frac{1}{4\pi^2}\log\delta-2A_p-\frac{1}{8\pi^2}}{b_\epsilon c_\epsilon}\right.$$
$$\left.+\frac{\log\frac{\lambda_\epsilon}{\beta_\epsilon c_\epsilon^2}+8+4\log\frac{R}{\delta}}{\alpha_\epsilon c_\epsilon^2}+O\left(\frac{\log^2\frac{\lambda_\epsilon}{\beta_\epsilon c_\epsilon^2}}{c_\epsilon^4}\right)+O\left(\frac{1}{b_\epsilon^2}\right)\right)$$
$$\leqslant 1-\frac{\int_{B_R(0)}|\Delta\varphi|^2\,dx-\frac{1}{16\pi^2}-\frac{1}{8\pi^2}\log\delta+A_p+a_1\|G_{a_1}\|_2^2+O(\delta\log\delta)}{b_\epsilon^2}.$$

Noting that $\frac{32\pi^2}{32\pi^2-\epsilon}>1$, we have

$$\frac{32\pi^2}{32\pi^2-\epsilon}\left(-\frac{a_1\|G_{a_1}\|_2^2}{b_\epsilon^2}+\frac{2\varphi(R)+4R\varphi'(R)+\frac{1}{4\pi^2}\log\delta-2A_p-\frac{1}{8\pi^2}}{b_\epsilon c_\epsilon}\right.$$
$$\left.+\frac{\log\frac{\lambda_\epsilon}{\beta_\epsilon c_\epsilon^2}+8+4\log\frac{R}{\delta}}{\alpha_\epsilon c_\epsilon^2}+O\left(\frac{\log^2\frac{\lambda_\epsilon}{\beta_\epsilon c_\epsilon^2}}{c_\epsilon^4}\right)+O\left(\frac{1}{b_\epsilon^2}\right)\right)$$
$$\leqslant -\frac{\int_{B_R(0)}|\Delta\varphi|^2\,dx-\frac{1}{16\pi^2}-\frac{1}{8\pi^2}\log\delta+A_p+a_1|G_{a_1}\|_2^2+O(\delta\log\delta)}{b_\epsilon^2}.$$

Using $\frac{32\pi^2}{32\pi^2-\epsilon}=1+O(\epsilon)$ and multiplying both sides of the above inequality by $\alpha_\epsilon c_\epsilon^2$, we obtain

$$\left(1+O(\epsilon)+O\left(\frac{\log\frac{\lambda_\epsilon}{\beta_\epsilon c_\epsilon^2}}{c_\epsilon^2}\right)\right)\log\frac{\lambda_\epsilon}{\beta_\epsilon c_\epsilon^2}$$
$$\leqslant -\alpha_\epsilon\frac{c_\epsilon^2}{b_\epsilon^2}\left(\int\limits_{B_R(0)}|\Delta\varphi|^2\,dx-\frac{1}{8\pi^2}\log\delta\right)$$
$$-4\log\frac{R}{\delta}-\big(1+O(\epsilon)\big)\alpha_\epsilon\frac{c_\epsilon}{b_\epsilon}\left(2\varphi(R)+4R\varphi'(R)+\frac{1}{4\pi^2}\log\delta\right)+O\left(\frac{c_\epsilon^2}{b_\epsilon^2}\right).$$

Since $\frac{1}{c_\epsilon^2}\log\frac{\lambda_\epsilon}{\beta_\epsilon c_\epsilon^2}\to 0$ (Lemma 5.1) and $\log\frac{\lambda_\epsilon}{\beta_\epsilon c_\epsilon^2}=\log\frac{\lambda_\epsilon}{\beta_\epsilon b_\epsilon^2}+\log\frac{b_\epsilon^2}{c_\epsilon^2}$, the above inequality gives

$$\log\frac{\lambda_\epsilon}{\beta_\epsilon b_\epsilon^2}\leqslant -\big(1+o(1)\big)\alpha_\epsilon\frac{c_\epsilon^2}{b_\epsilon^2}\left(\int\limits_{B_R(0)}|\Delta\varphi|^2\,dx-\frac{1}{8\pi^2}\log\delta\right)$$
$$-\big(1+o(1)\big)\alpha_\epsilon\frac{c_\epsilon}{b_\epsilon}\left(2\varphi(R)+4R\varphi'(R)+\frac{1}{4\pi^2}\log\delta\right)$$
$$+\big(1+o(1)\big)\log\frac{c_\epsilon^2}{b_\epsilon^2}-\big(4+o(1)\big)\log\frac{R}{\delta}+O\left(\frac{c_\epsilon^2}{b_\epsilon^2}\right). \tag{5.16}$$

From (3.9) and (4.20) we know that $\lim_{\epsilon\to 0}\lambda_\epsilon/b_\epsilon^2>0$, whence $\log\frac{\lambda_\epsilon}{\beta_\epsilon b_\epsilon^2}\geqslant -C_0$ for some constant $C_0>0$. We now *claim* that c_ϵ/b_ϵ is bounded. Suppose not, $c_\epsilon/b_\epsilon\to+\infty$. By Lemma 4.3, $\varphi\equiv 0$ and (5.16) becomes

$$\log\frac{\lambda_\epsilon}{\beta_\epsilon b_\epsilon^2}\leqslant\big(4+o(1)\big)\frac{c_\epsilon^2}{b_\epsilon^2}\log\delta-\big(8+o(1)\big)\frac{c_\epsilon}{b_\epsilon}\log\delta$$
$$-\big(4+o(1)\big)\log\frac{R}{\delta}+O\left(\frac{c_\epsilon^2}{b_\epsilon^2}\right).\tag{5.17}$$

When ϵ and δ are sufficiently small, the right-hand side of (5.17) is less than $-2C_0$. This contradicts with $\log\frac{\lambda_\epsilon}{\beta_\epsilon b_\epsilon^2}\geqslant -C_0$. Hence our claim holds. In other words, $\tau=\lim_{\epsilon\to 0}\frac{c_\epsilon}{b_\epsilon}$ is a positive real number. Whence $\log\frac{\lambda_\epsilon}{\beta_\epsilon b_\epsilon^2}$ is also bounded from above according to (5.16). Then it follows from (5.16) that

$$\log\frac{\lambda_\epsilon}{\beta_\epsilon b_\epsilon^2}\leqslant\left(4\frac{c_\epsilon^2}{b_\epsilon^2}-8\frac{c_\epsilon}{b_\epsilon}+4+o(1)\right)\log\delta-\big(4+o(1)\big)\log R$$
$$-\big(64\pi^2+o(1)\big)\frac{c_\epsilon}{b_\epsilon}\big(\varphi(R)+2R\varphi'(R)\big)+O\left(\frac{c_\epsilon^2}{b_\epsilon^2}\right).\tag{5.18}$$

The power of (5.18) is evident. Noting that

$$4\frac{c_\epsilon^2}{b_\epsilon^2}-8\frac{c_\epsilon}{b_\epsilon}+4+o(1)\to 4\tau^2-8\tau+4=4(\tau-1)^2,$$

we *conclude* $\tau=1$ for otherwise we can reach a contradiction with (5.18) by taking ϵ and δ sufficiently small. Therefore the convergence in Lemma 4.3 becomes

$$c_\epsilon\big(u_\epsilon(x_\epsilon+r_\epsilon x)-c_\epsilon\big)\to\varphi(x)=\frac{1}{16\pi^2}\log\frac{1}{1+\frac{\pi}{\sqrt{6}}|x|^2}\quad\text{in } C^4_{loc}\big(\mathbb{R}^4\big).$$

Also b_ϵ can be replaced by c_ϵ in Lemma 4.7, in particular

$$c_\epsilon u_\epsilon\to G_{a_1}(\cdot,p)\quad\text{in } C^4_{loc}\big(\Omega\setminus\{p\}\big),\qquad c_\epsilon u_\epsilon\to G_{a_1}(\cdot,p)\quad\text{in } L^s(\Omega),\ \forall s>1.$$

Now we come back to (5.7). Now there holds

$$\int\limits_{B_\delta(x_\epsilon)\setminus B_{Rr_\epsilon}(x_\epsilon)}|\Delta\mathcal{T}|^2\,dx\leqslant 1-\frac{\int_{B_R(0)}|\Delta\varphi|^2\,dx+\int_{\Omega\setminus B_\delta(p)}|\Delta G_{a_1}|^2\,dx+o(1)}{c_\epsilon^2}.\tag{5.19}$$

We estimate further the energy of $\int_{B_\delta(x_\epsilon)\setminus B_{Rr_\epsilon}(x_\epsilon)}|\Delta\mathcal{T}|^2\,dx$. (5.12) can be re-estimated as follows:

$$8\pi^2\mathcal{A}^2\log\frac{\delta}{Rr_\epsilon}\geqslant 1+\frac{\log\frac{\lambda_\epsilon}{\beta_\epsilon c_\epsilon^2}-\log\frac{\pi^4}{36}-\log R^4+\log\delta^4-64\pi^2A_p}{\alpha_\epsilon c_\epsilon^2}$$
$$-\frac{a_1\|G_{a_1}\|_2^2}{c_\epsilon^2}+o\left(\frac{1}{c_\epsilon^2}\right). \tag{5.20}$$

Also replacing the estimate (5.11) with

$$\mathcal{B}=\frac{-2c_\epsilon+(2+o(1))c_\epsilon}{-\alpha_\epsilon\delta^2c_\epsilon^2+o(c_\epsilon^2)}=o\left(\frac{1}{c_\epsilon}\right)\frac{1}{\delta^2},$$

we obtain instead of (5.13),

$$32\pi^2\mathcal{A}\mathcal{B}\left(\delta^2-R^2r_\epsilon^2\right)=o\left(\frac{1}{c_\epsilon^2}\right),\qquad 32\pi^2\mathcal{B}^2\left(\delta^4-R^4r_\epsilon^4\right)=o\left(\frac{1}{c_\epsilon^2}\right). \tag{5.21}$$

A direct calculation shows

$$\left|\Delta\varphi(x)\right|^2=\frac{1}{96\pi^2}\left(\frac{1}{(1+\frac{\pi}{\sqrt{6}}|x|^2)^2}+\frac{2}{(1+\frac{\pi}{\sqrt{6}}|x|^2)^3}+\frac{1}{(1+\frac{\pi}{\sqrt{6}}|x|^2)^4}\right).$$

We compute

$$\int\limits_{B_R(0)}\frac{1}{(1+\frac{\pi}{\sqrt{6}}|x|^2)^2}\,dx=6\log\left(1+\frac{\pi}{\sqrt{6}}R^2\right)-6+O\left(\frac{1}{R^2}\right),$$
$$\int\limits_{B_R(0)}\frac{1}{(1+\frac{\pi}{\sqrt{6}}|x|^2)^3}\,dx=3+O\left(\frac{1}{R^2}\right),\qquad \int\limits_{B_R(0)}\frac{1}{(1+\frac{\pi}{\sqrt{6}}|x|^2)^4}\,dx=1+O\left(\frac{1}{R^4}\right).$$

Hence

$$\int\limits_{B_R(0)}\left|\Delta\varphi(x)\right|^2dx=\frac{1}{16\pi^2}\log\left(1+\frac{\pi}{\sqrt{6}}R^2\right)+\frac{1}{96\pi^2}+O\left(\frac{1}{R^2}\right). \tag{5.22}$$

Combining (5.19)–(5.22) and (5.14), we obtain

$$\begin{aligned}\log\frac{\lambda_\epsilon}{\beta_\epsilon c_\epsilon^2}&\leqslant\log\frac{\pi^4}{36}+\log R^4-\log\delta^4+64\pi^2A_p+\alpha_\epsilon a_1\|G_{a_1}\|_2^2\\&\quad-\frac{\alpha_\epsilon}{16\pi^2}\log\left(1+\frac{\pi}{\sqrt{6}}R^2\right)-\frac{\alpha_\epsilon}{96\pi^2}+O\left(\frac{1}{R^2}\right)+O(\delta\log\delta)\\&\quad+\frac{\alpha_\epsilon}{16\pi^2}+\frac{\alpha_\epsilon}{8\pi^2}\log\delta-\alpha_\epsilon A_p-\alpha_\epsilon a_1\|G_{a_1}\|_2^2+o(1)\\&=\frac{5}{3}+32\pi^2A_p+\log\frac{\pi^2}{6}+o(1)+O\left(\frac{1}{R^2}\right)+O(\delta\log\delta).\end{aligned}$$

Letting $\epsilon \to 0$ first, then $R \to +\infty$ and finally $\delta \to 0$, we obtain

$$\lim_{\epsilon \to 0} \frac{\lambda_\epsilon}{c_\epsilon^2} \leqslant \frac{\pi^2}{6} e^{\frac{5}{3}+32\pi^2 A_p}.$$

This together with (4.20) and Lemma 4.6 gives

$$\sup_{u \in H_0^2(\Omega),\, \|\Delta u\|_2^2 \leqslant 1} \int_\Omega e^{32\pi^2 u^2 q(\|u\|_2^2)}\, dx = \lim_{\epsilon \to 0} \int_\Omega e^{\alpha_\epsilon u_\epsilon^2}\, dx \leqslant \frac{\pi^2}{6} e^{\frac{5}{3}+32\pi^2 A_p}. \tag{5.23}$$

It is remarkable that this supremum is estimated under the assumption that u_ϵ blows up and the blow-up point p lies in the interior of Ω.

6. Nonexistence of boundary bubbles

The main goal of this section is to exclude boundary bubbles. Suppose without loss of generality $c_\epsilon = u(x_\epsilon) = \max_{x \in \Omega} u_\epsilon \to +\infty$ and $x_\epsilon \to p \in \partial\Omega$. As in the case $p \in \Omega$, $u_\epsilon \rightharpoonup 0$ weakly in $H_0^2(\Omega)$ and strongly in $H^1(\Omega)$. Moreover we have

Lemma 6.1. *There holds $|\Delta u_\epsilon|^2 dx \rightharpoonup \delta_p$ in sense of measure.*

Proof. Suppose not. There exists sufficiently small $r > 0$ such that

$$\lim_{\epsilon \to 0} \int_{B_r(p) \cap \Omega} |\Delta u_\epsilon|^2\, dx = \theta < 1.$$

Choosing a cut-off function $\eta \in C^4(\overline{\Omega})$ such that $0 \leqslant \eta \leqslant 1$, $\eta \equiv 1$ on $\Omega \cap B_{r/2}(p)$, $\eta \equiv 0$ on $\Omega \setminus B_r(p)$, $|\nabla \eta| \leqslant 4/r$. Since $u_\epsilon \to 0$ weakly in $H_0^2(\Omega)$ and strongly in $H^1(\Omega)$, whence

$$\lim_{\epsilon \to 0} \int_{B_r(p) \cap \Omega} \left|\Delta(\eta u_\epsilon)\right|^2 dx = \theta.$$

This together with (3.7) implies that ηu_ϵ is a weak solution of $\Delta^2(\eta u_\epsilon) = \tilde{f}_\epsilon$ for some $\tilde{f}_\epsilon$ which is bounded in $L^r(\Omega)$ for some $r > 1$. Thus regularity theory implies ηu_ϵ is bounded in $C^3(\overline{\Omega})$, in particular, c_ϵ is bounded. This is a contradiction and we get the desired result. □

Lemma 6.1 implies that if there is a blow-up point on the boundary $\partial\Omega$, then this is the unique blow-up point in $\overline{\Omega}$. Let b_ϵ be defined in (4.6). Comparing with Lemma 4.4, we have

Lemma 6.2. *$b_\epsilon u_\epsilon \rightharpoonup 0$ weakly in $H_0^{2,q}(\Omega)$ for all $1 < q < 2$.*

Proof. By the same proof of Lemma 4.3 we have $b_\epsilon u_\epsilon$ is bounded in $H_0^{2,q}(\Omega)$ for any $1 < q < 2$. Hence there exists $F \in H_0^{2,q}$ such that $b_\epsilon u_\epsilon \to F$ weakly in $H_0^{2,q}(\Omega)$ and strongly in $H_0^1(\Omega)$. Using the same method in the proof of Lemma 4.4 we conclude that F satisfies $\Delta^2 F = a_1 F$ in Ω. Regularity theory gives $F \in C^3(\overline{\Omega})$. Since $a_1 < \lambda(\Omega)$, we obtain $F \equiv 0$. □

Applying the Pohozaev identity (Lemma 4.5) to Eq. (3.7) on the domain $\Omega \cap B_\delta(p)$, we have by the same way to drive (4.20) that

$$\lim_{\epsilon\to 0}\int_\Omega e^{\alpha_\epsilon u_\epsilon^2}\,dx = |\Omega|.$$

This is impossible according to (3.9). Therefore we exclude the boundary bubble, i.e., p cannot lie on $\partial\Omega$.

Getting back to Section 5, we have in fact proved the following:

Proposition 6.3. *Let c_ϵ, x_ϵ, p and A_p be as before. If blow-up occurs, i.e. $c_\epsilon \to +\infty$, then the blow-up point p must lie in the interior of Ω, and the supremum*

$$\sup_{u\in H_0^2(\Omega),\,\|\Delta u\|_2=1}\int_\Omega e^{32\pi^2u^2q(\|u\|_2^2)}\,dx \leqslant \frac{\pi^2}{6}e^{\frac{5}{3}+32\pi^2A_p}.$$

We end up this section with proving Theorem 1.1*.

Proof of Theorem 1.1*. If there exists an extremal function u_0 such that

$$\int_\Omega e^{32\pi^2u_0^2q(\|u_0\|_2^2)}\,dx = \sup_{u\in H_0^2(\Omega),\,\|\Delta u\|_2^2\leqslant 1}\int_\Omega e^{32\pi^2u^2q(\|u\|_2^2)}\,dx,$$

then Theorem 1.1* already holds. Otherwise, in case blow-up occurs, Theorem 1.1* is an easy consequence of Proposition 6.3. □

7. Proof of Theorem 1.2*

In this section we shall construct functions $\phi_\epsilon \in H_0^2(\Omega)$ such that $\|\Delta\phi_\epsilon\|_2^2 = 1$ and

$$\int_\Omega e^{32\pi^2\phi_\epsilon^2q(\|\phi_\epsilon\|_2^2)}\,dx > |\Omega| + \frac{\pi^2}{6}e^{\frac{5}{3}+32\pi^2A_p}$$

provided that $a_1 \geqslant 0$ is sufficiently small, where $q(t)$ satisfies the assumptions in Theorem 1.2* and A_p is defined by (4.22). Since $a_i \geqslant 0$ ($i = 1, 2, \ldots, k$), we only need to require

$$\int_\Omega e^{32\pi^2\phi_\epsilon^2(1+a_1\|\phi_\epsilon\|_2^2)}\,dx > |\Omega| + \frac{\pi^2}{6}e^{\frac{5}{3}+32\pi^2A_p}. \tag{7.1}$$

This together with Proposition 6.3 and regularity theory leads to the conclusion of Theorem 1.2*.

Denote $r = |x-p|$. Recall that $G_{a_1}(x,p) = -\frac{1}{8\pi^2}\log r + A_p + \psi(x)$. We write

$$\phi_\epsilon = \begin{cases} c + \frac{a-\frac{1}{16\pi^2}\log(1+\frac{\pi}{\sqrt{6}}\frac{r^2}{\epsilon^2})}{c} + \frac{A_p+\psi}{c} + \frac{b}{c}r^2, & r \leqslant R\epsilon,\\ \frac{1}{c}G_{a_1}, & r > R\epsilon, \end{cases} \tag{7.2}$$

where a, b, c are constants to be determined later. We choose $R = \log \frac{1}{\epsilon}$. To ensure $\phi_\epsilon \in H_0^2(\Omega)$, we require

$$\lim_{r \to R\epsilon - 0} \phi_\epsilon = \lim_{r \to R\epsilon + 0} \phi_\epsilon, \qquad \lim_{r \to R\epsilon - 0} \nabla\phi_\epsilon = \lim_{r \to R\epsilon + 0} \nabla\phi_\epsilon.$$

That is exactly

$$\begin{cases} a = -c^2 + \dfrac{\log(1 + \frac{\pi}{\sqrt{6}} R^2)}{16\pi^2} - \dfrac{\log(R\epsilon)}{8\pi^2} + bR^2\epsilon^2, \\ b = -\dfrac{1}{16\pi^2 R^2 \epsilon^2 (1 + \frac{\pi}{\sqrt{6}} R^2)}. \end{cases} \tag{7.3}$$

To ensure $\|\Delta\phi_\epsilon\|_2^2 = 1$, we calculate

$$\begin{aligned} \int\limits_{B_{R\epsilon}(p)} |\Delta\phi_\epsilon|^2\,dx &= \int\limits_{B_{R\epsilon}(p)} \left(-\frac{1}{16\pi^2 c} \frac{\frac{8\pi}{\sqrt{6}}\epsilon^2 + \frac{2\pi^2}{3} r^2}{(\epsilon^2 + \frac{\pi}{\sqrt{6}} r^2)^2} + \frac{\Delta\psi}{c} + \frac{8b}{c}\right)^2 dx \\ &= \frac{1}{96\pi^2 c^2}\left(6\log\left(1 + \frac{\pi}{\sqrt{6}} R^2\right) + 1 + O\left(\frac{1}{\log^2\epsilon}\right)\right). \end{aligned} \tag{7.4}$$

Recalling that $\Delta^2 G_{a_1}(x,p) = \alpha G_{a_1}(x,p)$ in $\Omega \setminus B_{R\epsilon}(p)$ and integrating by parts, we have

$$\begin{aligned} \int\limits_{\Omega\setminus B_{R\epsilon}(p)} |\Delta\phi_\epsilon|^2\,dx &= \frac{1}{c^2} \int\limits_{\Omega\setminus B_{R\epsilon}(p)} |\Delta G_{a_1}|^2\,dx \\ &= \frac{1}{c^2}\left(a_1 \int\limits_{\Omega\setminus B_{R\epsilon}(p)} G_{a_1}^2\,dx + \int\limits_{\partial B_{R\epsilon}(p)} G_{a_1} \frac{\partial}{\partial\nu}\Delta G_{a_1}\,d\omega \right. \\ &\quad \left. - \int\limits_{\partial B_{R\epsilon}(p)} \Delta G_{a_1} \frac{\partial G_{a_1}}{\partial\nu}\,d\omega\right) \\ &= \frac{\log\frac{1}{R^2\epsilon^2} + 16\pi^2 A_p - 1}{16\pi^2 c^2} + a_1 \frac{\|G_{a_1}\|_2^2}{c^2} + O\left(\frac{\epsilon\log^2\epsilon}{c^2}\right). \end{aligned} \tag{7.5}$$

Putting (7.4) and (7.5) together, we have

$$\|\Delta\phi_\epsilon\|_2^2 = \frac{\log\frac{\pi}{\sqrt{6}\epsilon^2} + 16\pi^2 A_p - \frac{5}{6}}{16\pi^2 c^2} + a_1 \frac{\|G_{a_1}\|_2^2}{c^2} + O\left(\frac{\epsilon\log^2\epsilon}{c^2}\right).$$

Let $\|\Delta\phi_\epsilon\|_2^2 = 1$, then we have

$$c^2 = \frac{\log\frac{\pi}{\sqrt{6}\epsilon^2}}{16\pi^2} + A_p - \frac{5}{96\pi^2} + a_1\|G_{a_1}\|_2^2 + O\left(\frac{1}{\log^2\epsilon}\right).$$

By (7.2) and (7.3), we calculate on $B_{R\epsilon}(p)$,

$$32\pi^2\phi_\epsilon^2\big(1+\alpha\|\phi_\epsilon\|_2^2\big) \geqslant \log\frac{\pi^2}{6\epsilon^4} + 32\pi^2 A_p + \frac{5}{3}\big(1+2a_1\|G_{a_1}\|_2^2\big) - \frac{64\pi^2 a_1^2\|G_{a_1}\|_2^4}{c^2} - \left(1+\frac{a_1\|G_{a_1}\|_2^2}{c^2}\right)\log\left(1+\frac{\pi r^2}{\sqrt{6}\epsilon^2}\right)^4 + O\left(\frac{1}{\log^2\epsilon}\right).$$

Hence

$$\begin{aligned}\int\limits_{B_{R\epsilon}(p)} e^{32\pi^2\phi_\epsilon^2(1+a_1\|\phi_\epsilon\|_2^2)}\,dx &= \frac{\pi^2}{6\epsilon^4}e^{\frac{5}{3}+32\pi^2A_p+\frac{10a_1\|G_{a_1}\|_2^2}{3c^2}-\frac{64\pi^2a_1^2\|G_{a_1}\|_2^4}{c^2}} \\ &\quad\times \int\limits_{B_{R\epsilon}(p)}\left(1+\frac{\pi r^2}{\sqrt{6}\epsilon^2}\right)^{-4-\frac{4a_1\|G_{a_1}\|_2^2}{c^2}}dx \\ &= \frac{\pi^2}{6\epsilon^4}e^{\frac{5}{3}+32\pi^2A_p+\frac{10a_1\|G_{a_1}\|_2^2}{3c^2}-\frac{64\pi^2a_1^2\|G_{a_1}\|_2^4}{c^2}} \\ &\quad\times\epsilon^4\left(1-\frac{10a_1\|G_{a_1}\|_2^2}{3c^2}+O\left(\frac{1}{\log^2\epsilon}\right)\right) \\ &= \frac{\pi^2}{6}e^{\frac{5}{3}+32\pi^2A_p}\left(1-\frac{64\pi^2a_1^2\|G_{a_1}\|_2^4}{c^2}+O\left(\frac{1}{\log^2\epsilon}\right)\right).\end{aligned}$$

On the other hand,

$$\begin{aligned}\int\limits_{\Omega\setminus B_{R\epsilon}(p)} e^{32\pi^2\phi_\epsilon^2(1+a_1\|\phi_\epsilon\|_2^2)}\,dx &\geqslant \int\limits_{\Omega\setminus B_{R\epsilon}(p)}\big(1+32\pi^2\phi_\epsilon^2\big(1+a_1\|\phi_\epsilon\|_2^2\big)\big)\,dx \\ &= |\Omega| + 32\pi^2\frac{\|G_{a_1}\|_2^2}{c^2} + O\left(\frac{1}{\log^2\epsilon}\right).\end{aligned}$$

Combing the above two integral estimates, we obtain

$$\begin{aligned}\int\limits_{\Omega} e^{32\pi^2\phi_\epsilon^2(1+a_1\|\phi_\epsilon\|_2^2)}\,dx &\geqslant 32\pi^2\frac{\|G_{a_1}\|_2^2}{c^2} - \frac{64\pi^4a_1^2\|G_{a_1}\|_2^4}{6c^2}e^{\frac{5}{3}+32\pi^2A_p} \\ &\quad + |\Omega| + \frac{\pi^2}{6}e^{\frac{5}{3}+32\pi^2A_p} + O\left(\frac{1}{\log^2\epsilon}\right).\end{aligned}\tag{7.6}$$

Noting that $c^2 = O(\log\epsilon)$, we have obtained the desired estimate when $a_1 = 0$. While in the case $a_1 \neq 0$ it is rather difficult to determine the sign of

$$32\pi^2\frac{\|G_{a_1}\|_2^2}{c^2} - \frac{64\pi^4a_1^2\|G_{a_1}\|_2^4}{6c^2}e^{\frac{5}{3}+32\pi^2A_p} + O\left(\frac{1}{\log^2\epsilon}\right)$$

because both $\|G_{a_1}\|_2$ and A_p depend on a_1. However we *claim* the following

Proposition 7.1. *There exists a constant C depending only on Ω, $\lambda(\Omega)$ and α_0: $0<\alpha_0<\lambda(\Omega)$ such that*

$$A_p \leqslant C, \qquad \|G_{a_1}\|_2^2 \leqslant C \quad \text{for all } a_1 \in [0, \alpha_0].$$

Proof. By Lemma 4.4, $b_\epsilon u_\epsilon$ is bounded in $H_0^{1,q}$ ($\forall 1<q<2$) uniformly for $a_1 \in [0,\alpha_0]$ with $\alpha_0 < \lambda(\Omega)$. Hence G_{a_1} is uniformly bounded in $L^2(\Omega)$. Employing the Green function $G(x,y)$ defined by (4.1), in particular

$$\begin{cases} \Delta^2 G(\cdot, p) = \delta_p & \text{in } \Omega, \\ G(\cdot, p) = \dfrac{\partial G(\cdot, p)}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

we define a function $w_{a_1}(x) = G_{a_1}(x) - G(x,p)$ on $\Omega \setminus \{p\}$. Then $w_{a_1}(x)$ is a distributional solution of the equation

$$\begin{cases} \Delta^2 w_{a_1} = a_1 G_{a_1} & \text{in } \Omega, \\ w_{a_1} = \dfrac{\partial w_{a_1}}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{7.7}$$

Using the Green representation formula and the Hölder inequality, we have for all $x \in \Omega$,

$$\left|w_{a_1}(x)\right| = \left|\int_\Omega G(x,y) a_1 G_{a_1}(y)\,dy\right| \leqslant \sup_{x\in\Omega} \left\|G(x,\cdot)\right\|_2 \|a_1 G_{a_1}\|_2 \leqslant C \tag{7.8}$$

for some constant C depending only on Ω, $\lambda(\Omega)$ and α_0. Let

$$R(x,y) = G(x,y) + \frac{1}{8\pi^2}\log|x-y|.$$

It is known that $\sup_{x\in\Omega} R(x,x)$ is a real number depending only on Ω, see [31] for example. This together with (7.8) gives

$$\begin{aligned} A_p &= \lim_{x\to p}\left(G_{a_1}(x,p) + \frac{1}{8\pi^2}\log|x-p|\right) \\ &= \lim_{x\to p}\left(w_{a_1}(x) + G(x,p) + \frac{1}{8\pi^2}\log|x-p|\right) \\ &\leqslant \sup_{x\in\Omega} w_{a_1}(x) + \sup_{p\in\Omega} R(p,p) \leqslant C \end{aligned}$$

for some constant C depending only on Ω, $\lambda(\Omega)$ and α_0. Hence we conclude the proposition. □

Therefore we have by (7.6) and Proposition 7.1,

$$\int_\Omega e^{32\pi^2\phi_\epsilon^2(1+a_1\|\phi_\epsilon\|_2^2)}\,dx > |\Omega| + \frac{\pi^2}{6}e^{\frac{5}{3}+32\pi^2 A_p},$$

provided that a_1 and ϵ are sufficiently small, i.e. (7.1) holds. This completes the proof of Theorem 1.2*.

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