

# A SHARP TRUDINGER-MOSER INEQUALITY ON ANY BOUNDED AND CONVEX PLANAR DOMAIN

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ABSTRACT. Wang and Ye conjectured in [22]:

Let  $\Omega$  be a regular, bounded and convex domain in  $\mathbb{R}^2$ . There exists a finite constant  $C(\Omega) > 0$  such that

$$\int_{\Omega} e^{\frac{4\pi u^2}{H_d(u)}} dx dy \leq C(\Omega), \quad \forall u \in C_0^\infty(\Omega),$$

where  $H_d = \int_{\Omega} |\nabla u|^2 dx dy - \frac{1}{4} \int_{\Omega} \frac{u^2}{d(z, \partial\Omega)^2} dx dy$  and  $d(z, \partial\Omega) = \min_{z_1 \in \partial\Omega} |z - z_1|$ .

The main purpose of this paper is to confirm that this conjecture indeed holds for any bounded and convex domain in  $\mathbb{R}^2$  via the Riemann mapping theorem (the smoothness of the boundary of the domain is thus irrelevant).

We also give a rearrangement-free argument for the following Trudinger-Moser inequality on the hyperbolic space  $\mathbb{B} = \{z = x + iy : |z| = \sqrt{x^2 + y^2} < 1\}$ :

$$\sup_{\|u\|_{\mathcal{H}} \leq 1} \int_{\mathbb{B}} (e^{4\pi u^2} - 1 - 4\pi u^2) dV = \sup_{\|u\|_{\mathcal{H}} \leq 1} \int_{\mathbb{B}} \frac{(e^{4\pi u^2} - 1 - 4\pi u^2)}{(1 - |z|^2)^2} dx dy < \infty,$$

by using the method employed earlier by Lam and the first author [9, 10], where  $\mathcal{H}$  denotes the closure of  $C_0^\infty(\mathbb{B})$  with respect to the norm

$$\|u\|_{\mathcal{H}} = \int_{\mathbb{B}} |\nabla u|^2 dx dy - \int_{\mathbb{B}} \frac{u^2}{(1 - |z|^2)^2} dx dy.$$

Using this strengthened Trudinger-Moser inequality, we also give a simpler proof of the Hardy-Moser-Trudinger inequality obtained by Wang and Ye [22].

## 1. INTRODUCTION

As a borderline case of the Sobolev embedding  $W_0^{1,p}(\Omega) \subset L^q(\Omega)$  where  $p < N$  when  $\Omega \subset \mathbb{R}^N$  ( $N \geq 2$ ) is a bounded domain with  $1 \leq q \leq \frac{Np}{N-p}$ , Trudinger [19] proved that  $W_0^{1,N}(\Omega) \subset L_{\varphi_N}(\Omega)$ , where  $L_{\varphi_N}(\Omega)$  is the Orlicz space associated with the Young function  $\varphi_N(t) = \exp(\beta|t|^{N/(N-1)}) - 1$  for some  $\beta > 0$  (see also Yudovich [20], Pohozaev [21]). J. Moser proved the following sharp result in his 1971 paper [16]:

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**Theorem A.** *Let  $\Omega$  be a domain with finite measure in Euclidean  $N$ -space  $\mathbb{R}^N$ ,  $N \geq 2$ . Then there exists a sharp constant  $\alpha_N = N \left( \frac{N\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2}+1)} \right)^{\frac{1}{N-1}}$  such that*

$$\frac{1}{|\Omega|} \int_{\Omega} \exp(\beta |u|^{\frac{N}{N-1}}) dx \leq c_0$$

for any  $\beta \leq \alpha_N$ , any  $u \in W_0^{1,N}(\Omega)$  with  $\int_{\Omega} |\nabla u|^N dx \leq 1$ . This constant  $\alpha_N$  is sharp in the sense that if  $\beta > \alpha_N$ , then the above inequality can no longer hold with some  $c_0$  independent of  $u$ .

There have been many generalizations related to the Trudinger-Moser inequality on hyperbolic spaces (see [14], [12], [13], [15]). For instance, Mancini and Sandeep [14] (see also [3]) proved the following improved Trudinger-Moser inequalities on  $\mathbb{B} = \{z = x + iy : |z| = \sqrt{x^2 + y^2} < 1\}$ :

$$\sup_{u \in W_0^{1,2}(\mathbb{B}), \int_{\mathbb{B}} |\nabla u|^2 dx dy \leq 1} \int_{\mathbb{B}} \frac{e^{4\pi u^2} - 1}{(1 - |z|^2)^2} dx dy < \infty.$$

In [12, 13], the first author and Tang established independently different type of sharp Trudinger-Moser inequalities from [15] on the high dimensional hyperbolic spaces. Since our main focus of this paper is on the Trudinger-Moser inequality on two dimensional case, we will not discuss further here.

Wang and Ye [22] proved, among other results, an improved Trudinger-Moser inequality by combining the Hardy inequality. Their result is the following

**Theorem 1.1.** *There exists a finite constant  $C_1 > 0$  such that*

$$\int_{\mathbb{B}} e^{\frac{4\pi u^2}{\|u\|_{\mathcal{H}}}} dx dy \leq C_1, \quad \forall u \in C_0^\infty(\mathbb{B}),$$

where  $\|u\|_{\mathcal{H}} = \int_{\mathbb{B}} |\nabla u|^2 dx dy - \int_{\mathbb{B}} \frac{u^2}{(1 - |z|^2)^2} dx dy$ .

We note that the proof of Theorem 1.1 in [22] depends on Schwartz rearrangement argument. In the same paper, they conjecture that such Hardy-Moser-Trudinger inequality holds for bounded and convex domains with smooth boundary:

**Conjecture** ([22]) *Let  $\Omega$  be a regular, bounded and convex domain in  $\mathbb{R}^2$ . There exists a finite constant  $C(\Omega) > 0$  such that*

$$\int_{\Omega} e^{\frac{4\pi u^2}{H_d(u)}} dx dy \leq C(\Omega), \quad \forall u \in C_0^\infty(\Omega),$$

where  $H_d = \int_{\Omega} |\nabla u|^2 dx dy - \frac{1}{4} \int_{\Omega} \frac{u^2}{d(z, \partial\Omega)^2} dx dy$  and  $d(z, \partial\Omega) = \min_{z_1 \in \partial\Omega} |z - z_1|$ .

Using Theorem 1.1, Mancini, Sandeep and Tintarev [15] proved, among other results, the following strengthened Trudinger-moser inequality on  $\mathbb{B}$  and their proof also depends on symmetrization argument.

**Theorem 1.2.** *There exists a constant  $C_2$  such that for all  $u \in C_0^\infty(\mathbb{B})$  with*

$$\|u\|_{\mathcal{H}} = \int_{\mathbb{B}} |\nabla u|^2 dx dy - \int_{\mathbb{B}} \frac{u^2}{(1 - |z|^2)^2} dx dy \leq 1,$$

*there holds*

$$\int_{\mathbb{B}} \frac{(e^{4\pi u^2} - 1 - 4\pi u^2)}{(1 - |z|^2)^2} dx dy \leq C_2.$$

Recently, Lam and the first author [9] develop a new approach to establish sharp Trudinger-Moser inequalities in unbounded domains in the settings (e.g. Heisenberg groups) where the classical symmetrization argument does not work. Such an approach avoids using the rearrangement argument which is not available in an optimal way on the Heisenberg group and can be used in other settings such as high order Sobolev spaces, hyperbolic spaces, non-compact and complete Riemannian manifolds, etc (see e.g. [11], [10], [13], [23]).

One of the aims of this paper is that, in the spirit of [9, 10], we give a new approach to establish the strengthened Trudinger-Moser inequality and Hardy-Moser-Trudinger inequality on  $\mathbb{B}$ . Our approach is much simpler and also avoids using the rearrangement argument.

The second and main aim of this paper is that, using the strengthened Trudinger-Moser inequality, we give an affirmative answer to the conjecture given by Wang and Ye via Riemann mapping theorem. The main result of our paper is the following

**Theorem 1.3.** *Let  $\Omega$  be a proper and convex domain in  $\mathbb{R}^2$  and  $u \in C_0^\infty(\Omega)$  be such that*

$$\int_{\Omega} |\nabla u|^2 dx dy - \frac{1}{4} \int_{\Omega} \frac{u^2}{d(z, \partial\Omega)^2} dx dy \leq 1.$$

*Then there exists a constant  $C_3$  which is independent of  $u$  such that*

$$\int_{\Omega} \frac{e^{4\pi u^2} - 1 - 4\pi u^2}{d(z, \partial\Omega)^2} dx dy \leq C_3.$$

*Furthermore, if  $\Omega$  is bounded, then there exists a constant  $C_4$  which is independent of  $u$  such that*

$$\int_{\Omega} e^{4\pi u^2} dx dy \leq C_4.$$

## 2. CONFORMAL MAP

Let  $U$  and  $V$  be two open sets in  $\mathbb{C}$ . A bijection holomorphic function  $f : U \rightarrow V$  is called a conformal map or a biholomorphism. We say that  $U$  and  $V$  are conformally equivalent or simply biholomorphic if such a conformal map  $f$  exists. Here we give a number of special examples of conformal mappings.

**Example 2.1.** Denote by  $\mathbb{H}$  the upper half-plane which consists of those complex numbers with positive imaginary parts; that is

$$\mathbb{H} = \{z = x + iy \in \mathbb{C} : y > 0\}.$$

It is known that  $\mathbb{H}$  is conformally equivalent to the unit disc  $\mathbb{B}$ . In fact,  $f : \mathbb{H} \rightarrow \mathbb{B}$  defined by

$$f(z) = \frac{z - i}{z + i}.$$

is a conformal map.

**Example 2.2.** For each  $\alpha \in \mathbb{C}$  with  $|\alpha| < 1$ , denote by

$$\psi_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha}z}.$$

Then  $\psi_\alpha : \mathbb{B} \rightarrow \mathbb{B}$  is a conformal map. It is easy to check  $\psi_\alpha(\alpha) = 0$  and  $\psi_\alpha(0) = \alpha$ . Furthermore, if  $f$  is an automorphism of  $\mathbb{B}$ , then there exists  $\theta \in \mathbb{R}$  and  $\alpha \in \mathbb{B}$  such that  $f = e^{i\theta}\psi_\alpha$ .

Now we recall the Riemann mapping theorem, which states that any non-empty open simply connected proper subset of  $\mathbb{C}$  admits a bijective conformal map to the open unit disk  $\mathbb{B}$ . We state it as follows:

**Theorem 2.3.** *Suppose  $\Omega$  is proper and simply connected. If  $z_0 \in \Omega$ , then there exists a unique conformal map  $F : \Omega \rightarrow \mathbb{B}$  such that*

$$F(z_0) = 0, \quad \text{and} \quad F'(z_0) > 0.$$

*Remark 2.4.* It has been shown in [1] that if  $f : \Omega \rightarrow \mathbb{B}$  holomorphic, injective,  $f(z_0) = 0$  and  $f'(z_0) > 0$ , then  $F'(z_0) \geq f'(z_0)$ .

### 3. HYPERBOLIC SPACE OF DIMENSION TWO

Recall that the Poincaré conformal disc model of dimension two is the unit ball

$$\mathbb{B} = \{z = x + iy \in \mathbb{C} : |z| < 1\}$$

equipped with the usual Poincaré metric

$$ds^2 = \frac{4(dx^2 + dy^2)}{(1 - |z|^2)^2}.$$

The hyperbolic volume element is

$$dV = \left( \frac{2}{1 - |z|^2} \right)^2 dx dy.$$

The associated Laplace-Beltrami operator is given by

$$\Delta_{\mathbb{H}} = \frac{(1 - |z|^2)^2}{4} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

and the corresponding hyperbolic gradient is

$$\nabla_{\mathbb{H}} = \frac{1 - |z|^2}{2} \nabla = \frac{1 - |z|^2}{2} \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right).$$

It is easy to check that

$$\int_{\mathbb{B}} |\nabla u|^2 dx dy - \int_{\mathbb{B}} \frac{u^2}{(1-|z|^2)^2} dx dy = \int_{\mathbb{B}} |\nabla_{\mathbb{H}} u|^2 dV - \frac{1}{4} \int_{\mathbb{B}} u^2 dV.$$

For  $z_1, z_2 \in \mathbb{B}^n$ , we denote by  $\rho(z_1, z_2)$  the associated distance from  $z_1$  to  $z_2$  in  $\mathbb{B}^n$ . It is well known that

$$(3.1) \quad \rho(z_1, z_2) = 2 \tanh^{-1} \left( \frac{|z_1 - z_2|}{\sqrt{1 - 2\operatorname{Re}(z_1 \bar{z}_2) + |z_1|^2 |z_2|^2}} \right)^{\frac{1}{2}}.$$

In particular, if  $z_2 = 0$ , then  $\rho(z_1, 0) = \frac{1}{2} \log \frac{1+|z_1|}{1-|z_1|}$ . Furthermore, the polar coordinates associated with  $\rho$  is

$$\int_{\mathbb{B}} f dx dy = \int_0^\infty \int_0^{2\pi} f \sinh \rho d\rho d\theta, \quad f \in L^1(\mathbb{B}).$$

Let  $e^{t\Delta_{\mathbb{H}}}$  be the heat kernel on  $\mathbb{B}$ . It is known that  $e^{t\Delta_{\mathbb{H}}}$  depends only on  $t$  and  $\rho$ . The explicit formula is (see e.g. [6])

$$h(t, \rho) := e^{t\Delta_{\mathbb{H}}} = \frac{\sqrt{2}}{8\pi^{\frac{3}{2}}} t^{-\frac{3}{2}} e^{-\frac{t}{4}} \int_{\rho}^{+\infty} \frac{r e^{-\frac{r^2}{4t}}}{\sqrt{\cosh r - \cosh \rho}} dr.$$

Via the heat kernel, the fractional power

$$(3.2) \quad \begin{aligned} (-\Delta_{\mathbb{H}} - 1/4)^{-\frac{1}{2}} &= \frac{1}{\Gamma(1/2)} \int_0^{+\infty} t^{-\frac{1}{2}} e^{t(\Delta_{\mathbb{H}} + 1/4)} dt \\ &= \frac{\sqrt{2}}{8\pi^2} \int_{\rho}^{+\infty} \frac{r}{\sqrt{\cosh r - \cosh \rho}} dr \int_0^{+\infty} t^{-2} e^{-\frac{r^2}{4t}} dt \\ &= \frac{\sqrt{2}}{2\pi^2} \int_{\rho}^{+\infty} \frac{1}{r \sqrt{\cosh r - \cosh \rho}} dr. \end{aligned}$$

*Remark 3.1.* The operator  $-\Delta_{\mathbb{H}} - 1/4$  has been studied earlier by Beckner [4]. In fact, it has been shown by Beckner that for  $F \in C_0^\infty(M)$ ,

$$(3.3) \quad [\|F\|_{L^6(M)}]^2 \leq 4\pi^{-2/3} \left[ \int_M |DF|^2 d\nu - \frac{3}{16} \int_M F^2 d\nu \right];$$

$$(3.4) \quad [\|F\|_{L^6(M)}]^2 \leq \frac{4}{3} \pi^{-2/3} \left[ \int_M |DF|^2 d\nu - \frac{1}{4} \int_M F^2 d\nu \right];$$

$$(3.5) \quad [\|F\|_{L^4(M)}]^2 \leq 2\pi^{-1/2} \left[ \int_M |DF|^2 d\nu - \frac{1}{4} \int_M F^2 d\nu \right],$$

where  $M$  is the half space model of two-dimensional hyperbolic space,  $DF = \frac{1}{y} \nabla F$  and  $d\nu = \frac{1}{y^2} dx dy$ . It seems that inequalities (3.3) and (3.4) would be contradictory since both estimates are sharp as limiting forms. However, as pointed out by Beckner, the right-hand side of inequality (3.4) is to be evaluated as limiting forms for functions that may not be in  $L^2(M)$ .

**Lemma 3.2.** Set  $\phi(\rho) = (-\Delta_{\mathbb{H}} - 1/4)^{-\frac{1}{2}}$ . Then for  $\rho > 0$ ,

$$(3.6) \quad \phi(\rho) \leq \frac{1}{4\pi \sinh \frac{\rho}{2}};$$

$$(3.7) \quad \phi(\rho) \leq \frac{1}{2\pi\rho \sinh \frac{\rho}{2}}.$$

*Proof.* Since for  $\rho > 0$ ,  $\sinh \frac{\rho}{2} \leq \frac{\rho}{2} \cosh \frac{\rho}{2}$ , we have, for  $\rho > 0$ ,

$$\begin{aligned} & \int_{\rho}^{+\infty} \frac{1}{r\sqrt{\cosh r - \cosh \rho}} dr \leq \int_{\rho}^{+\infty} \frac{\cosh \frac{r}{2}}{2 \sinh \frac{r}{2} \sqrt{\cosh r - \cosh \rho}} dr \\ (3.8) \quad &= \int_{\rho}^{+\infty} \frac{\cosh \frac{r}{2}}{2 \sinh \frac{r}{2} \sqrt{2 \sinh^2 \frac{r}{2} - 2 \sinh^2 \frac{\rho}{2}}} dr \\ &= \frac{1}{\sqrt{2} \sinh \frac{\rho}{2}} \arctan \left( \sinh^2 \frac{r}{2} - \sinh^2 \frac{\rho}{2} \right) \Big|_{\rho}^{+\infty} \\ &= \frac{\pi}{2\sqrt{2} \sinh \frac{\rho}{2}}. \end{aligned}$$

Combing (3.2) and (3.8) yields (3.6).

Similarly, using the fact  $\sinh \frac{\rho}{2} \leq \cosh \frac{\rho}{2}$ , we have

$$\begin{aligned} & \int_{\rho}^{+\infty} \frac{1}{r\sqrt{\cosh r - \cosh \rho}} dr \leq \frac{1}{\rho} \int_{\rho}^{+\infty} \frac{1}{\sqrt{\cosh r - \cosh \rho}} dr \\ & \leq \frac{1}{\rho} \int_{\rho}^{+\infty} \frac{\cosh \frac{r}{2}}{\sinh \frac{r}{2} \sqrt{\cosh r - \cosh \rho}} dr \\ (3.9) \quad &= \frac{1}{\rho} \int_{\rho}^{+\infty} \frac{\cosh \frac{r}{2}}{\sinh \frac{r}{2} \sqrt{2 \sinh^2 \frac{r}{2} - 2 \sinh^2 \frac{\rho}{2}}} dr \\ &= \frac{\sqrt{2}}{\rho \sinh \frac{\rho}{2}} \arctan \left( \sinh^2 \frac{r}{2} - \sinh^2 \frac{\rho}{2} \right) \Big|_{\rho}^{+\infty} \\ &= \frac{\sqrt{2}\pi}{2\rho \sinh \frac{\rho}{2}}. \end{aligned}$$

Combing (3.2) and (3.9) yields (3.7). □

We now recall the rearrangement of a real functions on  $\mathbb{B}$ . Suppose  $f$  is a real function on  $\mathbb{B}$ . The non-increasing rearrangement of  $f$  is defined by

$$(3.10) \quad f^*(t) = \inf\{s > 0 : \lambda_f(s) \leq t\},$$

where

$$\lambda_f(s) = |\{z \in \mathbb{B} : |f(z)| > s\}| = \int_{\{z \in \mathbb{B} : |f(z)| > s\}} \left( \frac{2}{1 - |z|^2} \right)^2 dx dy.$$

Here we use the notation  $|\Sigma|$  for the measure of a measurable set  $\Sigma \subset \mathbb{B}$ .

**Lemma 3.3.** Set  $\phi(\rho) = (-\Delta_{\mathbb{H}} - 1/4)^{-\frac{1}{2}}$ . Then, for  $t > 0$ ,

$$(3.11) \quad \phi^*(t) \leq \frac{1}{\sqrt{4\pi t}}$$

and, for each  $a > 0$ ,

$$(3.12) \quad \int_a^\infty |\phi^*(t)|^2 dt < \infty.$$

*Proof.* Define, for any  $s > 0$ ,

$$(3.13) \quad \lambda_\phi(s) = \int_{\{\phi(\rho) > s\}} dV = 2\pi \int_0^{\rho_s} \sinh \rho d\rho,$$

where  $\rho_s$  is the solution of equation

$$(3.14) \quad \phi(\rho) = s.$$

Therefore, since  $\phi^*(t) = \inf\{s > 0 : \lambda_\phi(s) \leq t\}$ , we have

$$(3.15) \quad t = \lambda_\phi(\phi^*(t)) = 2\pi \int_0^{\rho_{\phi^*(t)}} \sinh \rho d\rho = 2\pi(\cosh \rho_{\phi^*(t)} - 1),$$

where  $\rho_{\phi^*(t)}$  satisfies

$$(3.16) \quad \phi(\rho_{\phi^*(t)}) = \phi^*(t).$$

By (3.6)

$$(3.17) \quad \phi^*(t) = \phi(\rho_{\phi^*(t)}) \leq \frac{1}{4\pi \sinh \frac{\rho_{\phi^*(t)}}{2}}.$$

Combing (3.15) and (3.17) yields

$$(3.18) \quad t|\phi^*(t)|^2 \leq \frac{2\pi(\cosh \rho_{\phi^*(t)} - 1)}{16\pi^2 \sinh^2 \frac{\rho_{\phi^*(t)}}{2}} = \frac{1}{4\pi}.$$

This proves inequality (3.11).

Now we prove (3.12). Using the substitution  $t = 2\pi(\cosh \rho_{\phi^*(t)} - 1)$ , we have, by (3.7) and (3.16),

$$(3.19) \quad \begin{aligned} \int_a^\infty |\phi^*(t)|^2 dt &= \int_b^\infty |\phi(\rho_{\phi^*(t)})|^2 2\pi \sinh \rho_{\phi^*(t)} d\rho_{\phi^*(t)} \\ &\leq \int_b^\infty \frac{\sinh \rho_{\phi^*(t)}}{\rho_{\phi^*(t)}^2 \sinh^2 \frac{\rho_{\phi^*(t)}}{2}} d\rho_{\phi^*(t)} = \int_b^\infty \frac{\sinh s}{s^2 \sinh^2 \frac{s}{2}} ds \\ &= 2 \int_b^\infty \frac{\cosh \frac{s}{2}}{s^2 \sinh \frac{s}{2}} ds, \end{aligned}$$

where  $b > 0$  satisfies  $a = 2\pi(\cosh b - 1)$ . Since  $\lim_{s \rightarrow +\infty} \frac{\cosh \frac{s}{2}}{\sinh \frac{s}{2}} = 1$ , there exists a positive constant  $C_b$  such that  $\frac{\cosh \frac{s}{2}}{\sinh \frac{s}{2}} \leq C_b$  for all  $s \in [b, +\infty)$ . Therefore, by (3.19),

$$\int_a^\infty |\phi^*(t)|^2 dt \leq 2C_b \int_b^\infty \frac{1}{s^2} ds = \frac{2C_b}{b}.$$

The desired result follows.  $\square$

#### 4. PROOF OF THEOREM 1.1 AND THEOREM 1.2

Before we prove the theorems, we need the following lemma from Adams' paper [2].

**Lemma 4.1.** *Let  $a(s, t)$  be a non-negative measurable function on  $(-\infty, +\infty) \times [0, +\infty)$  such that (a.e.)*

$$a(s, t) \leq 1, \text{ when } 0 < s < t,$$

$$\sup_{t>0} \left( \int_{-\infty}^0 a(s, t)^{n'} ds + \int_t^\infty a(s, t)^{n'} ds \right)^{1/n'} = b < \infty,$$

where  $n' = \frac{n}{n-1}$ . Then there is a constant  $c_0 = c_0(n, b)$  such that if for  $\phi \geq 0$  with  $\int_{-\infty}^\infty \phi(s)^n ds \leq 1$ , then

$$\int_0^\infty e^{-F(t)} dt \leq c_0,$$

where

$$F(t) = t - \left( \int_{-\infty}^\infty a(s, t) \phi(s) ds \right)^{n'}.$$

Now we prove Theorem 1.2. The main idea is to adapt the level set developed by Lam and the first author to derive a global Trudinger-Moser inequality from a local one (see [9, 10]). We firstly prove Theorem 1.2.

**Proof of Theorem 1.2** Let  $u \in C_0^\infty(\mathbb{B})$  be such that

$$\int_{\mathbb{B}} |\nabla u|^2 dx dy - \int_{\mathbb{B}} \frac{u^2}{(1 - |z|^2)^2} dx dy = \int_{\mathbb{B}} |\nabla_{\mathbb{H}} u|^2 dV - \frac{1}{4} \int_{\mathbb{B}} u^2 dV \leq 1.$$

Set  $\Omega(u) = \{z \in \mathbb{B} : |u(z)| \geq 1\}$ . By inequality (3.5),

$$\begin{aligned} |\Omega(u)| &= \int_{\Omega(u)} dV \leq \int_{\mathbb{B}} |u(z)|^4 dV \\ (4.1) \quad &\leq 4\pi^{-1} \left( \int_{\mathbb{B}} |\nabla_{\mathbb{H}} u|^2 dV - \frac{1}{4} \int_{\mathbb{B}} u^2 dV \right)^2 \\ &\leq 4\pi^{-1}. \end{aligned}$$

we write

$$\begin{aligned}
(4.2) \quad & \int_{\mathbb{B}} (e^{4\pi u^2} - 1 - 4\pi u^2) dV \\
&= \int_{\Omega(u)} (e^{4\pi u^2} - 1 - 4\pi u^2) dV + \int_{\mathbb{B} \setminus \Omega(u)} (e^{4\pi u^2} - 1 - 4\pi u^2) dV \\
&\leq \int_{\Omega(u)} e^{4\pi u^2} dV + \int_{\mathbb{B} \setminus \Omega(u)} (e^{4\pi u^2} - 1 - 4\pi u^2) dV.
\end{aligned}$$

Notice that on the domain  $\mathbb{B} \setminus \Omega(u)$ , we have  $|u(z)| < 1$ . Thus, by (4.1) and (3.5),

$$\begin{aligned}
(4.3) \quad & \int_{\mathbb{B} \setminus \Omega(u)} (e^{4\pi u^2} - 1 - 4\pi u^2) dV = \int_{\mathbb{B} \setminus \Omega(u)} \sum_{n=2}^{\infty} \frac{(4\pi u^2)^n}{n!} dV \\
&\leq \int_{\mathbb{B} \setminus \Omega(u)} \sum_{n=2}^{\infty} \frac{(4\pi)^n u^4}{n!} dV \\
&\leq \sum_{n=2}^{\infty} \frac{(4\pi)^n}{n!} \int_{\mathbb{B}} |u(z)|^4 dV \\
&\leq e^{4\pi} 4\pi^{-1} \left( \int_{\mathbb{B}} |\nabla_{\mathbb{H}} u|^2 dV - \frac{1}{4} \int_{\mathbb{B}} u^2 dV \right)^2 \\
&\leq 4\pi^{-1} e^{4\pi}.
\end{aligned}$$

To finish the proof, it is enough to show  $\int_{\Omega(u)} e^{4\pi u^2} dV$  is bounded by some universal constant. We rewrite

$$(4.4) \quad \int_{\mathbb{B}} |\nabla_{\mathbb{H}} u|^2 dV - \frac{1}{4} \int_{\mathbb{B}} u^2 dV = \int_{\mathbb{B}} \left| \left( -\Delta_{\mathbb{H}} - \frac{1}{4} \right)^{\frac{1}{2}} u \right|^2 dV \leq 1.$$

Therefore, we can write  $u$  as a potential via (3.2)

$$\begin{aligned}
(4.5) \quad & u(z) = \int_{\mathbb{B}} v(\omega) \left( \frac{\sqrt{2}}{2\pi^2} \int_{\rho(\omega, z)}^{+\infty} \frac{1}{r \sqrt{\cosh r - \cosh \rho(\omega, z)}} dr \right) dV_{\omega} \\
&= \int_{\mathbb{B}} v(\omega) \phi(\rho(\omega, z)) dV_{\omega},
\end{aligned}$$

where

$$\phi(\rho(\omega, z)) = \frac{\sqrt{2}}{2\pi^2} \int_{\rho(\omega, z)}^{+\infty} \frac{1}{r \sqrt{\cosh r - \cosh \rho(\omega, z)}} dr.$$

Furthermore, by (4.4),  $v \in L^2(\mathbb{B})$  with  $\int_{\mathbb{B}} v^2 dV \leq 1$ .

By O'Neil's lemma and (4.5), we have, for all  $t > 0$ ,

$$(4.6) \quad u^*(t) \leq \frac{1}{t} \int_0^t v^*(s) ds \int_0^t \phi^*(s) ds + \int_t^\infty v^*(s) \phi^*(s) ds.$$

Then, since  $|\Omega(u)| \leq 4\pi^{-1} < 4$ , we have, by (4.6)

$$\begin{aligned}
& \int_{\Omega(u)} e^{4\pi u^2} dV = \int_0^{|\Omega(u)|} \exp(4\pi |u^*(t)|^2) dt \\
& \leq \int_0^4 \exp(4\pi |u^*(t)|^2) dt \\
(4.7) \quad & \leq \int_0^4 \exp \left( 4\pi \left| \frac{1}{t} \int_0^t v^*(s) ds \int_0^t \phi^*(s) ds + \int_t^\infty v^*(s) \phi^*(s) ds \right|^2 \right) dt \\
& = 4 \int_0^\infty \exp \left( -t + \frac{\pi}{4} \left| e^t \int_0^{4e^{-t}} v^*(s) ds \int_0^{4e^{-t}} \phi^*(s) ds + 4 \int_{4e^{-t}}^\infty v^*(s) \phi^*(s) ds \right|^2 \right) dt.
\end{aligned}$$

To get the last equation, we use the substitution  $t := 4e^{-t}$ . Next, we change the variables

$$\begin{aligned}
\psi(t) &= 2e^{-t/2} v^*(4e^{-t}); \\
\varphi(t) &= 4\sqrt{\pi} e^{-t/2} \phi^*(4e^{-t}).
\end{aligned}$$

it is easy to check

$$\begin{aligned}
(4.8) \quad & e^t \int_t^\infty e^{-s/2} \psi(s) ds \int_t^\infty e^{-s/2} \varphi(s) ds = \frac{\sqrt{\pi}}{2} e^t \int_0^{4e^{-t}} v^*(s) ds \int_0^{4e^{-t}} \phi^*(s) ds; \\
& \int_{-\infty}^t \psi(s) \varphi(s) ds = 2\sqrt{\pi} \int_{4e^{-t}}^\infty v^*(s) \phi^*(s) ds.
\end{aligned}$$

Combing (4.7) and (4.8) yields

$$\begin{aligned}
& \int_{\Omega(u)} e^{4\pi u^2} dV \leq \int_0^4 \exp(4\pi |u^*(t)|^2) dt \\
(4.9) \quad & = 4 \int_0^\infty \exp \left( -t + \frac{\pi}{4} \left| e^t \int_0^{4e^{-t}} v^*(s) ds \int_0^{4e^{-t}} \phi^*(s) ds + 4 \int_{4e^{-t}}^\infty v^*(s) \phi^*(s) ds \right|^2 \right) dt \\
& = 4 \int_0^\infty e^{-F(t)} dt,
\end{aligned}$$

where

$$F(t) = t - \left( e^t \int_t^\infty e^{-s/2} \psi(s) ds \int_t^\infty e^{-s/2} \varphi(s) ds + \int_{-\infty}^t \psi(s) \varphi(s) ds \right)^2.$$

Using Lemma 3.2, we have

$$\begin{aligned}
& \sup_{s>0} \varphi(s) = 4 \sup_{s>0} \{ \sqrt{\pi} e^{-s/2} \phi^*(4e^{-s}) \} \leq 4 \sup_{s>0} \left\{ \sqrt{\pi} e^{-s/2} \frac{1}{\sqrt{4\pi 4e^{-s}}} \right\} = 1; \\
& \int_{-\infty}^0 |\psi(s)|^2 ds = 16\pi \int_{-\infty}^0 e^{-t} |\phi^*(4e^{-t})|^2 dt = 4\pi \int_4^\infty |\phi^*(t)|^2 dt < \infty.
\end{aligned}$$

Furthermore,

$$\int_{-\infty}^{+\infty} |\psi(s)|^2 ds = \int_0^\infty |v^*(s)|^2 ds = \int_{\mathbb{B}} |v(z)|^2 dV \leq 1.$$

Thus, if we set

$$a(s, t) = \begin{cases} \varphi(s), & s < t; \\ e^t \left( \int_t^\infty e^{-s/2} \varphi(s) ds \right) e^{-s/2}, & s > t, \end{cases}$$

then by Lemma 4.1,  $\int_{\Omega(u)} e^{4\pi u^2} dV$  is bounded by some constant which is independent of  $u$  and  $\Omega(u)$ . The proof of Theorem 1.2 is thereby completed.

Now we can prove Theorem 1.1 via Theorem 1.2.

**Proof of Theorem 1.1** Let  $u \in C_0^\infty(\mathbb{B})$  be such that

$$\int_{\mathbb{B}} |\nabla u|^2 dxdy - \int_{\mathbb{B}} \frac{u^2}{(1 - |z|^2)^2} dxdy = \int_{\mathbb{B}} |\nabla_{\mathbb{H}} u|^2 dV - \frac{1}{4} \int_{\mathbb{B}} u^2 dV \leq 1.$$

By Theorem 1.2, there exist a positive constant  $C_2$  which is independent of  $u$  such that

$$\int_{\mathbb{B}} \frac{(e^{4\pi u^2} - 1 - 4\pi u^2)}{(1 - |z|^2)^2} dxdy \leq C_2.$$

Therefore,

$$\begin{aligned} \int_{\mathbb{B}} e^{4\pi u^2} dxdy &= \int_{\mathbb{B}} (e^{4\pi u^2} - 1 - 4\pi u^2) dxdy + \int_{\mathbb{B}} dxdy + 4\pi \int_{\mathbb{B}} u^2 dxdy \\ &\leq \int_{\mathbb{B}} \frac{(e^{4\pi u^2} - 1 - 4\pi u^2)}{(1 - |z|^2)^2} dxdy + \int_{\mathbb{B}} dxdy + 4\pi \int_{\mathbb{B}} u^2 dxdy \\ &\leq C_2 + |\mathbb{B}| + 4\pi \cdot C^{-1}. \end{aligned}$$

To get the last inequality, we use the improved Hardy inequality (see e.g. [5, 22])

$$\int_{\mathbb{B}} |\nabla u|^2 dxdy - \int_{\mathbb{B}} \frac{u^2}{(1 - |z|^2)^2} dxdy \geq C \int_{\mathbb{B}} u^2 dxdy.$$

The desired result follows.

## 5. PROOF OF THEOREM 1.3

Since a convex domain in  $\mathbb{R}^2$  is also simply connected, we have, by Riemann mapping theorem, there exists a conformal map  $F : \Omega \rightarrow \mathbb{B}$ . Therefore, by Theorem 1.2, Then there exists a constant  $C_3$  such that for all  $u \in C_0^\infty(\Omega)$  with

$$(5.1) \quad \int_{\Omega} |\nabla u|^2 dxdy - \int_{\Omega} u^2 \frac{|F'(z)|^2}{(1 - |F(z)|^2)^2} dxdy \leq 1,$$

there holds

$$(5.2) \quad \int_{\Omega} (e^{4\pi u^2} - 1 - 4\pi u^2) \frac{|F'(z)|^2}{(1 - |F(z)|^2)^2} dxdy \leq C_3.$$

Next we shall show that for each  $z_0 \in \Omega$ ,

$$(5.3) \quad \frac{|F'(z_0)|^2}{(1 - |F(z_0)|^2)^2} \geq \frac{1}{4d(z_0, \partial\Omega)^2}.$$

Since  $\Omega$  is proper, there exists  $z_1 \in \partial\Omega$  such that  $d(z_0, \partial\Omega) = |z_0 - z_1|$ . Furthermore, since  $\Omega$  is convex,  $\Omega$  lies in the half-plane (see e.g. [18])

$$H_{z_0} := \left\{ z \in \mathbb{C} : \operatorname{Re} \frac{z - z_1}{z_0 - z_1} > 0 \right\}.$$

Now we construct a holomorphic and injective from  $H_{z_0}$  into  $\mathbb{B}$ . Via Example 2.1, it is easy to check

$$f_{z_0}(z) := \frac{\frac{z-z_1}{z_0-z_1} - 1}{\frac{z-z_1}{z_0-z_1} + 1} \cdot \frac{z_0 - z_1}{|z_0 - z_1|} = \frac{z - z_0}{z + z_0 - 2z_1} \cdot \frac{z_0 - z_1}{|z_0 - z_1|}$$

is such a function. Furthermore,

$$(5.4) \quad f_{z_0}(z_0) = 0, \quad f'_{z_0}(z_0) = \frac{1}{2|z_0 - z_1|} = \frac{1}{2d(z_0, \partial\Omega)}.$$

Set

$$G(z) = -\frac{|F'(z_0)|}{F'(z_0)} \psi_{F(z_0)}(F(z)) = -\frac{|F'(z_0)|}{F'(z_0)} \frac{F(z_0) - F(z)}{1 - \overline{F(z_0)}F(z)},$$

where  $\psi_{F(z_0)}$  is defined in Example 2.2. Since  $F : \Omega \rightarrow \mathbb{B}$  be a conformal map, so does  $G$ . Furthermore,

$$G(z_0) = 0, \quad G'(z_0) = \frac{|F'(z_0)|}{1 - |F(z_0)|^2} > 0.$$

Therefore, by Remark 2.4,

$$\frac{|F'(z_0)|}{1 - |F(z_0)|^2} = G'(z_0) \geq f'_{z_0}(z_0) = \frac{1}{2d(z_0, \partial\Omega)}.$$

Let  $u \in C_0^\infty(\Omega)$  be such that

$$\int_{\Omega} |\nabla u|^2 dx dy - \frac{1}{4} \int_{\Omega} \frac{u^2}{d(z, \partial\Omega)^2} dx dy \leq 1.$$

Then

$$\int_{\Omega} |\nabla u|^2 dx dy - \int_{\Omega} u^2 \frac{|F'(z)|^2}{(1 - |F(z)|^2)^2} dx dy \leq \int_{\Omega} |\nabla u|^2 dx dy - \frac{1}{4} \int_{\Omega} \frac{u^2}{d(z, \partial\Omega)^2} dx dy \leq 1.$$

By (5.1) and (5.2), we have

$$\int_{\Omega} \frac{e^{4\pi u^2} - 1 - 4\pi u^2}{d(z, \partial\Omega)^2} dx dy \leq 4 \int_{\Omega} (e^{4\pi u^2} - 1 - 4\pi u^2) \frac{|F'(z)|^2}{(1 - |F(z)|^2)^2} dx dy \leq 4C_3.$$

Furthermore, if  $\Omega$  is bounded, there exists a positive constant  $M$  such that for all  $z \in \Omega$ ,  $d(z, \partial\Omega) \leq M$ . Therefore, for each  $u \in C_0^\infty(\Omega)$  with

$$\int_{\Omega} |\nabla u|^2 dx dy - \frac{1}{4} \int_{\Omega} \frac{u^2}{d(z, \partial\Omega)^2} dx dy \leq 1,$$

we have

$$\begin{aligned} \int_{\Omega} e^{4\pi u^2} dx dy &= \int_{\Omega} (e^{4\pi u^2} - 1 - 4\pi u^2) dx dy + \int_{\Omega} dx dy + 4\pi \int_{\Omega} u^2 dx dy \\ &\leq M^2 \int_{\Omega} \frac{(e^{4\pi u^2} - 1 - 4\pi u^2)}{d(z, \partial\Omega)^2} dx dy + \int_{\Omega} dx dy + 4\pi \int_{\Omega} u^2 dx dy \\ &\leq M^2 C_4 + |\Omega| + 4\pi \cdot C_6^{-1}. \end{aligned}$$

To get the last inequality, we use the improved Hardy inequality (see e.g. [5, 8])

$$\int_{\Omega} |\nabla u|^2 dx dy - \frac{1}{4} \int_{\Omega} \frac{u^2}{d(z, \partial\Omega)^2} dx dy \geq C_6 \int_{\Omega} u^2 dx dy.$$

The desired result follows.

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