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On the uniqueness of a solution of a two-phase free boundary problem

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Abstract

In this paper, we study the uniqueness problem of a two-phase elliptic free boundary problem arising from the phase transition problem subject to given boundary data. We show that in general the comparison principle between the sub- and super-solutions does not hold, and there is no uniqueness of either a viscosity solution or a minimizer of this free boundary problem by constructing counter-examples in various cases in any dimension. In one-dimension, a bifurcation phenomenon presents and the uniqueness problem has been completely analyzed. In fact, the critical case signifies the change from uniqueness to non-uniqueness of a solution of the free boundary problem. Non-uniqueness of a solution of the free boundary problem suggests different physical stationary states caused by different processes, such as melting of ice or solidification of water, even with the same prescribed boundary data. However, we prove that a uniqueness theorem is true for the initial–boundary value problem of an ε-evolutionary problem which is the smoothed two-phase parabolic free boundary problem.

Keywords: Two-phase; Free boundary problems; Uniqueness; Bifurcation

1. Introduction

The two-phase free boundary problem about phase transition has been under study for a long time. The free boundary problem for the Laplace equation has been studied extensively by Caf-

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farelli, in [5–7], and by others, e.g. [1], in the 1980s. In [5–7], Caffarelli proved the existence and regularity of a solution, together with the regularity of its free boundary, given the boundary data. His main tools are the method of variable radii and a boundary Harnack inequality across the free boundary. His results and methods have influenced many researchers working in the subject and been generalized in many directions including to fully non-linear elliptic partial differential equations [10–13], etc., and to the parabolic heat equation [2–4], etc. On the other hand, Caffarelli, Jerison and Kenig proved some new monotonicity theorems where the boundedness, instead of the monotonicity, of the 'monotone' function holds so that the regularity of a weak solution of the Prandtl–Batchelor equation, the inhomogeneous two-phase free boundary problem for the Laplacian, follows (see [8]). Nevertheless, the uniqueness of a solution of the two-phase free boundary problem even for the Laplace equation with given boundary data is, however, untouched. This paper provides an attempt to answer the uniqueness question about a solution of the two-phase free boundary problem for the Laplace equation. Contrary to initially believed by the authors, the uniqueness of a viscosity solution or a minimizer is in general false. Instead, we have found an interesting bifurcation phenomenon about the uniqueness of a solution of the free boundary problem in 1D. On the other hand, we have proved uniqueness of a viscosity solution of an ε-evolutionary problem holds. This evidence together with the counter-examples in the stationary case lead us to believe the non-uniqueness arises from evolutions with different initial states and is inevitable even if more stringent topological or boundary conditions are imposed.

We start out with introduction of concepts and notations. Suppose Ω is a bounded domain in $\mathbf{R}^n$ with sufficiently smooth boundary, say C^1 boundary. Let $\sigma \in C(\partial\Omega)$ and $g:[0,\infty) \to (0,\infty)$, where g is strictly increasing Lipschitz continuous function with polynomial growth at ∞. Note $g(0) > 0$. Typical examples of such functions g are $g(s) = \sqrt{s^2+1}$ and $g(s) = s+1$.

For a continuous function $u : \Omega \to \Re$, we define $\Omega^+(u) = \{x \in \Omega\colon\ u(x) > 0\}$, $\Omega^-(u) = \Omega \backslash \overline{\Omega^+(u)}$, and $\mathcal{F}(u) = \partial\Omega^+(u) \cap \Omega$ which is called **the free boundary** of u. $\Omega^+(u)$ and $\Omega^-(u)$ are the positive and negative phases. A free boundary point $x_0 \in \mathcal{F}(u)$ is said to be **regular** if there is a ball $B_\rho \subset \Omega^+(u)$ or $B_\rho \subset \Omega^-(u)$ that touches $\mathcal{F}(u)$ at x_0. If this is the case, we denote by ν the radial direction at the tangent point x_0 that points inward of $\Omega^+(u)$.

The free boundary problem of phase transition we consider is formulated in a PDE form as

$$\begin{cases} \Delta u = 0 & \text{in } \Omega^+(u) \cup \Omega^-(u), \\ u_\nu^+ = g(u_\nu^-) & \text{along } \mathcal{F}(u), \\ u = \sigma & \text{on } \partial\Omega, \end{cases}$$

where $u \in C(\bar{\Omega})$, or variationally as minimizing the functional

$$J(u) = \int_\Omega |\nabla u|^2 + \lambda^2(u)\,dx,$$

where $u \in W^{1,2}(\Omega) \cap C(\bar{\Omega})$ such that $\lim_{x\in\Omega\to a} u(x) = \sigma(a)$ for every $a \in \partial\Omega$,

$$\lambda^2(u) = \begin{cases} \lambda_1^2 & \text{if } u \leqslant 0, \\ \lambda_2^2 & \text{if } u > 0, \end{cases}$$

and $\lambda_2^2 - \lambda_1^2 > 0$.

We define a viscosity solution of the free boundary problem as follows.

Definition 1.1. A continuous function u is called **a viscosity sub-solution** of the elliptic two-phase free boundary problem, if it verifies the following conditions.

1. $\Delta u \geqslant 0$ in $\Omega^+(u) \cup \Omega^-(u)$ in the viscosity sense.
2. $\forall x_0 \in \mathcal{F}(u) := \partial\Omega \cap \Omega$, if there exists a ball $B_\rho \subset \Omega^+(u)$ that touches $\mathcal{F}(u)$ at x_0, then there exists $\beta > 0$ such that

$$u(x) \geqslant \alpha\langle x - x_0, \nu\rangle^+ - \beta\langle x - x_0, \nu\rangle^- + \circ\big(|x - x_0|\big)$$

for all x near x_0, where $\alpha = g(\beta)$ and ν is the radial direction of ∂B_ρ at x_0 pointing to $\Omega^+(u)$.

A continuous function v is **a viscosity super-solution** of the elliptic two-phase free boundary problem in Ω, if it verifies the following conditions.

1. $\Delta v \leqslant 0$ in $\Omega^+(v) \cup \Omega^-(v)$ in the viscosity sense.
2. $\forall x_0 \in \mathcal{F}(v) := \partial\Omega^+(v) \cap \Omega$, if there exists a ball $B_\rho \subset \Omega^-(v)$ that touches $\mathcal{F}(v)$ at x_0, then there exists $\beta > 0$ such that

$$v(x) \leqslant \alpha\langle x - x_0, \nu\rangle^+ - \beta\langle x - x_0, \nu\rangle^- + \circ\big(|x - x_0|\big)$$

for all x near x_0, where $\alpha = g(\beta)$ and ν is the radial direction of ∂B_ρ at x_0 pointing to $\Omega^+(v)$.

A continuous function u is **a viscosity solution** of the elliptic two-phase free boundary problem if it is both a viscosity sub-solution and viscosity super-solution.

Remark 1.1. According to Caffarelli's theory [5–7], a viscosity solution of the free boundary problem is indeed an as classical as possible solution of the free boundary problem. In particular, the set of singular free boundary points is of $(n-1)$-Hausdorff measure 0. Nevertheless, in the following we still adopt the term "viscosity solutions" instead of "classical solutions" to distinguish them from minimizers.

Contrary to the properties of viscosity solutions of the Dirichlet problem for the Laplace equation, the following facts about a viscosity solution of the free boundary problem deserve mentioning.

That u is a viscosity solution does not imply $-u$ is also a viscosity solution.

That u is a viscosity solution does not imply $u + C$ is also a viscosity solution for a constant C.

That u and v are both viscosity solutions does not imply $u + v$ or $u - v$ is also a viscosity solution.

The uniqueness problem about the phase transition is formulated either in a PDE way as "Is there a unique viscosity solution of the free boundary problem, given a continuous boundary date σ?" or variationally as "Is there a unique minimizer of the functional $J(u)$, given a continuous boundary data σ?" This paper answers these questions with counter-examples. On the

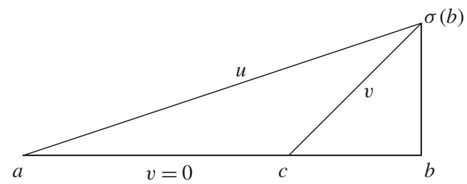

Fig. 1. The basic picture.

other hand, a uniqueness theorem of an evolutionary problem is proved. We propose a plausible explanation of non-uniqueness in the elliptic problem. However, many detailed questions about non-uniqueness in the elliptic free boundary problem are still open. Some of them are summarized in the last section.

We organize the paper in the following order. In the next three sections, we provide counter-examples to the uniqueness question in various cases in 1D and multi-dimensions, followed by a section devoted to the 1D bifurcation phenomenon. In the last section, we prove the uniqueness theorem for the ε-evolutionary problem. We conclude the paper with a list of open questions about uniqueness in the elliptic case.

2. Counter-examples in 1D

In this section, we provide counter-examples to the uniqueness problem in 1D of various kinds of free boundaries and boundary data.

We start with the basic picture. Take $\Omega = (a,b)$, and the boundary data $\sigma(a) = 0$ and $\sigma(b) > 0$, where a is taken so that $a < c := b - \frac{\sigma(b)}{g(0)}$. Recall that $g : [0,\infty) \to (0,\infty)$ is the function that prescribes the free boundary condition.

Define $u : \Omega \to \mathbf{R}$ by $u(x) = \frac{x-a}{b-a}\sigma(b)$, and $v : \Omega \to \mathbf{R}$ by

$$v(x) = \begin{cases} 0 & \text{if } x \in [a,c], \\ g(0)(x-c) & \text{if } x \in (c,b]. \end{cases}$$

Then u is harmonic on Ω with no free boundary point. Thus it is a viscosity solution of the free boundary problem. v has exactly one free boundary point c at which $v_\nu^+ = g(0)$ and $v_\nu^- = 0$. So the free boundary condition $v_\nu^+ = g(v_\nu^-)$ is verified at the free boundary point c. Therefore v is also a viscosity solution and $v = u$ on $\partial\Omega$. u and v are two viscosity solutions of the free boundary problem with equal boundary condition. Fig. 1 illustrates the counter-example.

We now modify the basic picture to obtain a counter-example in which both u and v have free boundary points. In fact, we glue two pieces of the basic picture with the roles of u and v switched in the two cases. More precisely, let $\Omega = (a,b) \cup (b,c)$, $\sigma(a) > 0$, $\sigma(b) = 0$, and $\sigma(c) > 0$, where $d := a + \frac{\sigma(a)}{g(0)} < b < e := c - \frac{\sigma(x)}{g(0)}$ by taking a small enough and c large enough. Define $u, v : \Omega \to \mathbf{R}$ by

$$u(x) = \begin{cases} -g(0)(x-d) & \text{if } x \in [a,d], \\ 0 & \text{if } x \in (d,b], \\ \frac{x-b}{c-b}\sigma(c) & \text{if } x \in (b,c], \end{cases}$$

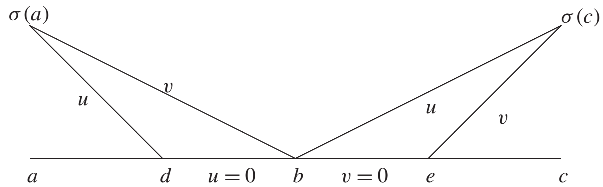

Fig. 2. Both with free boundary.

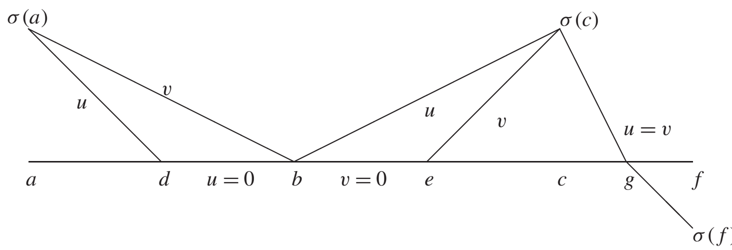

Fig. 3. With changing sign.

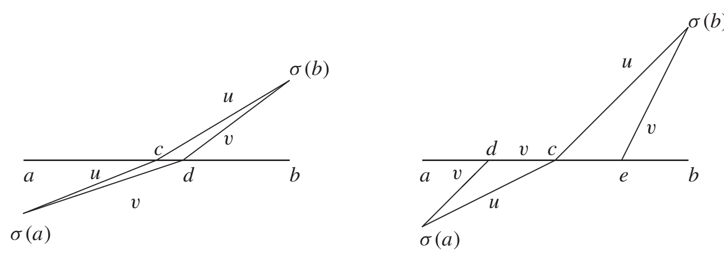

Fig. 4. Impossible pictures.

and

$$v(x) = \begin{cases} \frac{x-b}{a-b}\sigma(a) & \text{if } x \in [a, b], \\ 0 & \text{if } x \in (b, e], \\ g(0)(x - e) & \text{if } x \in (e, c]. \end{cases}$$

Then u and v have both exactly one free boundary point, namely d and e respectively, at which the free boundary condition is readily verified. So u and v are two different viscosity solutions satisfying the same boundary condition which have free boundary points. See Fig. 2.

At last, we give counter-examples in case the boundary data change sign. We simply attach a viscosity solution to the two distinct viscosity solutions obtained in the preceding case.

Take $\Omega = (a, b) \cup (b, c) \cup (c, f)$, $\sigma(a) > 0$, $\sigma(b) = 0$, $\sigma(c) > 0$, $\sigma(f) < 0$, and take d, e, and g as in the previous case so that the free boundary condition is verified. So u and v are distinct viscosity solutions of the free boundary problem with the same boundary data as illustrated in Fig. 3.

On the other hand, the pictures in Fig. 4 are impossible due to the monotonicity of the free boundary condition $u_\nu^+ = g(u_\nu^-)$, where $\Omega = (a, b)$.

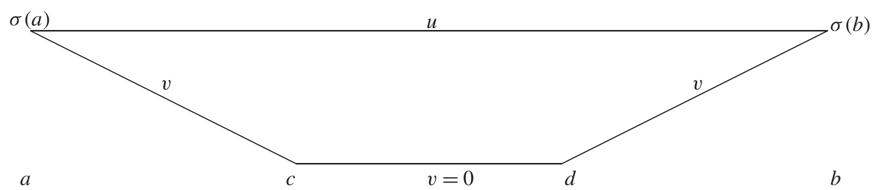

Fig. 5. Minimum principle fails.

The picture of two viscosity solutions on the left is impossible as $u_\nu^+(c) < v_\nu^+(d)$, $u_\nu^-(c) > v_\nu^-(d)$, $u_\nu^+(c) = g(u_\nu^-(c))$, $v_\nu^+(d) = g(v_\nu^-(d))$, and g is strictly increasing. For similar reasons, the picture of two viscosity solutions on the right is not possible, either.

Another counter-example is worth mentioning. If $\sigma(a) > 0$, $\sigma(b) > 0$, and $b - a > \frac{\sigma(a)}{g(0)} + \frac{\sigma(b)}{g(0)}$, then we have the counter-example in Fig. 5.

In words,

$$u(x) = \frac{b-x}{b-a}\sigma(a) + \frac{x-a}{b-a}\sigma(b), \quad \text{for } x \in [a,b],$$

and

$$v(x) = \begin{cases} g(0)(c-x) & \text{if } a \leqslant x < c, \\ 0 & \text{if } c \leqslant x < d, \\ g(0)(x-d) & \text{if } d \leqslant x \leqslant b, \end{cases}$$

where $c = a + \frac{\sigma(a)}{g(0)}$ and $d = b - \frac{\sigma(b)}{g(0)}$. Notice that $d > c$ as $b - a > \frac{\sigma(a)}{g(0)} + \frac{\sigma(b)}{g(0)}$.

On account of this counter-example, the minimum principle does not hold if $\sigma > 0$ on $\partial\Omega$.

The following two sections give counter-examples in multi-dimensions. Using these counter-examples and attach more annuli or shells in the same way as in 1D, we may have counter-examples of various cases as above. In fact, we can construct counter-examples in any dimension in this way. One should be convinced that the non-uniqueness is a physical phenomenon instead of a problem arising from mathematical modeling.

3. A counter-example to the uniqueness of a viscosity solution in multi-dimensions

Similar to the 1D case, even in the simplest form of a two-phase free boundary problem in multi-dimensions, the uniqueness of a viscosity solution is false, as shown by the following example. Indeed, we consider the uniqueness of a viscosity solution of the following two-phase free boundary problem.

$$\begin{cases} \Delta u^+ = 0 & \text{in } \Omega^+(u), \\ \Delta u^- = 0 & \text{in } \Omega^-(u), \\ \left(u_\nu^+\right)^2 - \left(u_\nu^-\right)^2 = 1 & \text{along } \mathcal{F}(u) := \partial\Omega^+(u) \cap \Omega, \\ u = \sigma & \text{on } \partial\Omega. \end{cases}$$

We take $\Omega = B_1(0)$, the unit ball of $\mathbf{R}^n$. Here we assume $n > 2$ for simplicity. The example also works in dimension two with proper modification in the formula of the function constructed.

Pick any value in $(0, 1)$ for a number s. We take a constant function $\sigma(x) = \frac{n-2}{s-s^{n-1}} > 0$ on $\partial\Omega$. Define a function $u_0 \in C(\bar{\Omega})$ by

$$u_0(x) \equiv \frac{n-2}{s-s^{n-1}}$$

on $\bar{\Omega}$. Clearly u_0 is a viscosity solution of the two-phase free boundary problem, as it does not even have a free boundary.

The second function $u_1 \in C(\bar{\Omega})$ is defined by the formula

$$u_1(x) = a|x|^{2-n} + b$$

with $a = -\frac{s^{n-1}}{n-2} < 0$ and $b = \frac{s}{n-2} > 0$.

Then $u_1 = a + b = \frac{s-s^{n-1}}{n-2}$ on $\partial\Omega$ and $u_1 = as^{2-n} + b = 0$ on $\partial B_s(0)$. Clearly $\Delta u_1 = 0$ in $B_1 \backslash \bar{B}_s$ as u_1 is basically the fundamental solution of the Laplacian.

Furthermore, along $\mathcal{F}(u_1) = \partial B_s(0)$,

$$\partial_r u_1^+ = a(2-n)s^{1-n} = 1$$

while $\partial_r u_1^- = 0$. So the free boundary condition

$$\left(\partial_r u_1^+\right)^2 - \left(\partial_r u_1^-\right)^2 = 1$$

is verified in the classical sense and hence in the viscosity sense.

So, for the same boundary data $\sigma \in C(\partial\Omega)$, one obtains two distinct viscosity solutions u_0 and u_1, for any s with $0 < s < 1$, of the same two-phase free boundary problem.

4. A counter-example of the uniqueness of a minimizer in multi-dimensions

One might think though there are more than one viscosity solution of a two-phase free boundary problem, there is probably only one minimizer of a corresponding variational problem. Well, we give a counter-example to the uniqueness of a minimizer of the following simplest variational problem. For simplicity, we assume again the dimension $n > 2$. A similar counter-example may be constructed in two dimensions or in 1D.

Let $\Omega = B_1$, the unit ball of $\mathbf{R}^n$ as in the previous section. We take $\lambda_1 > 0$ and $\lambda_2 > 0$ such that $\Lambda = \lambda_2^2 - \lambda_1^2 > 0$, but otherwise the values of λ_1 and λ_2 are free to pick.

Define $h(s) = \frac{n-2}{n}s^n + s^{n-2} - 1$, for $s \in [0, 1]$. Then $h(0) = -1 < 0 < \frac{n-2}{n} = h(1)$. So there exists an $s_0 \in (0, 1)$ such that $h(s_0) = 0$, i.e.

$$\frac{n-2}{n}s_0^n + s_0^{n-2} - 1 = 0.$$

Then

$$\frac{n-2}{n} = \frac{1-s_0^{n-2}}{s_0^n} = \frac{s_0^{2-n}-1}{s_0^2}.$$

Take $\Lambda = \frac{(n-2)n}{s_0^n(s_0^{2-n}-1)} > 0$. It follows that $\frac{(n-2)n}{s_0^{2-n}-1} = \Lambda s_0^n$. This equality combined with $\frac{n-2}{n} = \frac{s_0^{2-n}-1}{s_0^2}$ implies that

$$\frac{(n-2)^2}{\Lambda} = s_0^{n-2}\bigl(s_0^{2-n}-1\bigr)^2.$$

Now let $g(s) = (n-2)^2 - \Lambda s^{n-2}(s^{2-n}-1)^2$, $s \in (0,1]$. For $s \in (0,1)$, $g'(s) = -(n-2)\Lambda s^{n-3}(s^{2-n}-1)^2 + 2(n-2)\Lambda s^{-1}(s^{2-n}-1) = (n-2)\Lambda s^{-1}(s^{2-n}-1)(1+s^{n-2}) > 0$. So g is an increasing function, and $g(1) = (n-2)^2 > 0$ and $\lim_{s\to 0+} g(s) = -\infty$. In addition, the choice of s_0 implies that $g(s_0) = 0$ as $\frac{(n-2)^2}{\Lambda} = s_0^{n-2}(s_0^{2-n}-1)^2$. So

$$g(s)\begin{cases} <0 & \text{for } 0 < s < s_0,\\ =0 & \text{for } s = s_0,\\ >0 & \text{for } s_0 < s < 1.\end{cases}$$

Define $f(s) = \frac{(n-2)n}{s^{2-n}-1} + \lambda_2^2 - \Lambda s^n$, where $0 \leqslant s < 1$. For $0 < s < 1$, $f'(s) = \frac{ns^{1-n}}{(s^{2-n}-1)^2} g(s)$. So f attains its absolute minimum at $s = s_0$, according to our analysis of the function g. Note that $f(0) = \lambda_2^2 - (n-2)n$ and $\lim_{s\to 1-} f(s) = +\infty$.

We minimize the functional

$$J(u) = \int_\Omega |\nabla u|^2 + \lambda^2(u)\,dx,$$

with $u = 1$ on $\partial\Omega$, where $\lambda^2(s) = \lambda_2^2$, if $s > 0$, and $\lambda^2(s) = \lambda_1^2$, if $s \leqslant 0$.

If there were only one minimizer u under the condition $u = 1$ on $\partial\Omega$, then u must be radial as all the rotation of u around the origin are minimizers of the same boundary data.

Now suppose $u(x) = 0$ for $|x| = s$ for some $s \in [0,1]$. As a result of the maximum principle of harmonic functions, $u(x) = 0$ for all $x \in B_s$. Therefore, for some $s \in [0,1]$, $u(x) > 0$ and $\Delta u = 0$ in $s < |x| < 1$, while $u(x) = 0$ in $|x| \leqslant s$. This forces u to take the form

$$u(x) = a|x|^{2-n} + b.$$

The boundary data give conditions on a and b, i.e.

$$\begin{cases} a + b = 1,\\ as^{2-n} + b = 0.\end{cases}$$

So $a = \frac{1}{1-s^{2-n}} < 0$ and $b = \frac{s^{2-n}}{s^{2-n}-1} \geqslant 0$. And $\nabla u = a(2-n)r^{1-n}\hat{x}$, where $\hat{x} = \frac{x}{|x|}$.

If we denote the measure of the unit ball by $\sigma = |B_1|$, then

$$\int_{B\setminus B_s} |\nabla u|^2\,dx = \int_s^1 a^2(n-2)^2\rho^{1-n}\,d\rho\, n\sigma = a^2(n-2)n\sigma\bigl(s^{2-n}-1\bigr) = \frac{(n-2)n}{s^{2-n}-1}\sigma$$

and

$$\lambda_2^2|B\backslash B_s| + \lambda_1^2|B_s| = \lambda_2^2\sigma - \Lambda s^n\sigma.$$

Therefore

$$J(u) = \frac{(n-2)n}{s^{2-n}-1}\sigma + \lambda_2^2\sigma - \Lambda s^n\sigma = f(s)\sigma.$$

The minimizer u_0 of $J(u)$ should be the one corresponding to $s = s_0 \in (0,1)$. Then $J(u_0) = f(s_0)\sigma = \frac{(n-2)n}{s_0^{2-n}-1}\sigma + \lambda_2^2\sigma - \Lambda s_0^n\sigma$.

On the other hand, if we define $u_1(x) \equiv 1$ in $\bar{\Omega}$. In this case, $J(u_1) = \lambda_2^2\sigma$.

Then

$$J(u_0) - J(u_1) = \left(\frac{(n-2)n}{s_0^{2-n}-1} - \Lambda s_0^n\right)\sigma = 0$$

as a result of $\frac{(n-2)n}{s_0^{2-n}-1} = \Lambda s_0^n$. So both u_0 and u_1 are minimizers of the functional $J(u)$ with the equal boundary data. Of course, as $0 < s_0 < 1$, they are distinct minimizers, under the assumption there is a unique minimizer. We are done.

5. A bifurcation phenomenon in 1D

In 1D, an open set is the disjoint union of at most countably many open intervals. Thus in 1D, we write $\Omega = \bigcup_{j\in\Lambda} I_j$, where $I_j = (a_j, b_j)$ is an interval.

Lemma 5.1 *(Maximum–minimum principle for the free boundary problem). Let Ω be a bounded domain in $\mathbf{R}^n$, and u a continuous viscosity solution of the two-phase free boundary problem in Ω.*

(a)
$$\sup_{\Omega} u = \max_{\partial\Omega} u$$

holds, while $\inf_{\Omega} u$ *may be smaller than* $\min_{\partial\Omega} u$.

(b) *If, in addition,* $\min_{\partial\Omega} u \leqslant 0$*, then*

$$\inf_{\Omega} u = \min_{\partial\Omega} u$$

holds.

Proof. Both (a) and (b) follow from a simple argument by contradiction and the maximum–minimum principle for the Laplacian in either phase. □

Now we show, in 1D, a bifurcation phenomenon. We may restrict to every component $I_j = (a_j, b_j)$ of Ω. We also omit the subscript j.

First assume $\sigma(a) > 0$ and $\sigma(b) > 0$. An obvious solution is the one without any free boundary point, namely $u(x) = \frac{b-x}{b-a}\sigma(a) + \frac{x-a}{b-a}\sigma(b)$, $x \in I$. If $b - a \leqslant \frac{\sigma(a)+\sigma(b)}{g(0)}$, then there cannot

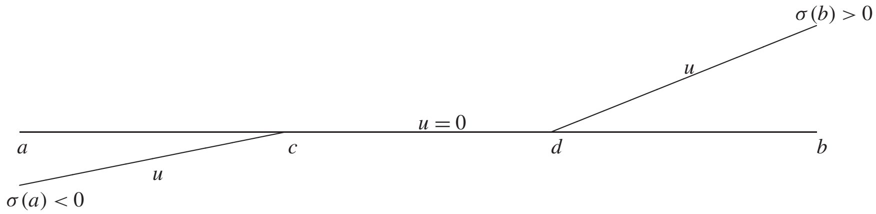

Fig. 6. Picture for $c < d$.

be a free boundary point for a solution. So the affine function just found is the only viscosity solution of the free boundary problem. Otherwise, assume c is the least value in (a,b) at which a viscosity solution $u = 0$ and d is the largest value in (a,b) at which $u = 0$. Then u is a viscosity solution of the two-phase free boundary problem in (c,d) with zero boundary data. The maximum–minimum principle implies that $u = 0$ on (c,d). So there is only one more viscosity solution of the free boundary problem on (a,b) other than the affine solution, namely the solution v defined by

$$v(x) = \begin{cases} g(0)(c-x) & \text{if } a \leqslant x < c, \\ 0 & \text{if } c \leqslant x < d, \\ g(0)(x-d) & \text{if } d \leqslant x \leqslant b, \end{cases}$$

where $c = a + \frac{\sigma(a)}{g(0)}$ and $d = b - \frac{\sigma(b)}{g(0)}$.

If $\sigma(a)\sigma(b) < 0$, say $\sigma(a) < 0$ and $\sigma(b) > 0$, then u has at least one free boundary point. (Just keep in mind that there might not even be a viscosity solution u if (a,b) is too short with respect to $\sigma(b)$.) Suppose there exist two (or more) points x_1 and x_2 such that $u(x_1) = u(x_2) = 0$. Then the maximum–minimum principle implies $u = 0$ on $[x_1, x_2]$.

Define $c := \inf\{x \in (a,b)\colon\ u(x) = 0\}$ and $d := \sup\{x \in (a,b)\colon\ u(x) = 0\}$. Clearly $u(c) = u(d) = 0$.

Step 1. We claim $c = d$.

Suppose $c < d$. We then have the picture in Fig. 6.

At the free boundary point c, $u_\nu^+ = 0$ and $u_\nu^- = -\frac{\sigma(a)}{c-a}$. Then $0 = g(-\frac{\sigma(a)}{c-a}) > g(0) > 0$, which is impossible. So $c = d$.

Step 2. u is unique.

Suppose there are two viscosity solutions u and v, and $u = v$ on $\partial\Omega$, as shown in Fig. 7. Without loss of generality, we assume $c < d$.

At c, $u_\nu^+ = g(u_\nu^-)$ where $u_\nu^+ = \frac{\sigma(b)}{b-c}$ and $u_\nu^- = -\frac{\sigma(a)}{c-a}$.

At d, $v_\nu^+ = g(v_\nu^-)$ where $v_\nu^+ = \frac{\sigma(b)}{b-d}$ and $v_\nu^- = -\frac{\sigma(a)}{d-a}$.

Note $u_\nu^+ < v_\nu^+$ and $u_\nu^- > v_\nu^-$, while the monotonicity of g implies that $u_\nu^+ = g(u_\nu^-) > g(v_\nu^-) = v_\nu^+$, which is a contradiction. So u is unique if $\sigma(a)\sigma(b) < 0$.

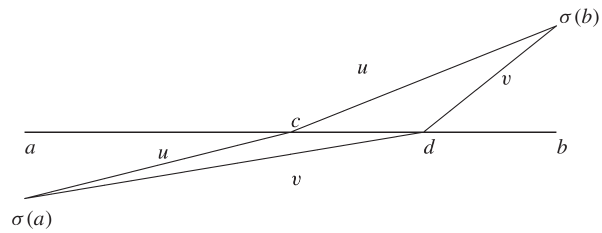

Fig. 7. u is unique.

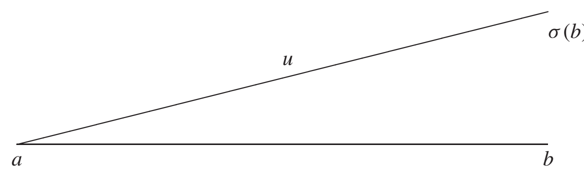

Fig. 8. The unique solution when (a, b) is short.

The critical case, $\sigma(a)\sigma(b) = 0$.

If $\sigma(a) = \sigma(b) = 0$, then $u = 0$ everywhere. Otherwise, we may assume $\sigma(a) = 0$ and $\sigma(b) > 0$.

If $\frac{\sigma(b)}{b-a} > g(0)$, then there cannot be any free boundary point. The only viscosity solution is $u(x) = \frac{x-a}{b-a}\sigma(b)$ as shown in Fig. 8.

If $\frac{\sigma(b)}{b-a} = g(0)$, still there is no free boundary point strictly between a and b. The unique viscosity solution is $u(x) = g(0)(x - a)$.

If $\frac{\sigma(b)}{b-a} < g(0)$, we have seen the counter-example in Section 2 that declines the uniqueness of a viscosity solution. In fact, let $u(x) = \frac{x-a}{b-a}\sigma(b)$ for $x \in [a, b]$ and v be the function defined by

$$v(x) = \begin{cases} 0 & \text{if } x \in [a, c], \\ g(0)(x - c) & \text{if } x \in (c, b], \end{cases}$$

where $c = b - \frac{\sigma(b)}{g(0)}$. Both u and v are viscosity solutions of the free boundary problem on $[a, b]$. We show that v is stable under perturbations of boundary data from below, and the perturbations of boundary data from above cause two perturbed solutions which converge to u and v respectively.

Indeed, as $\frac{\sigma(b)}{b-a} < g(0)$, $\exists! c$ such that $a < c < b$ and $\frac{\sigma(b)}{b-c} = g(0)$.

Let $\sigma_\varepsilon(a) = -\varepsilon$ and $\sigma_\varepsilon(b) \to \sigma(b)$ as $\varepsilon \to 0+$. Let u_ε be the unique solution of the free boundary problem with $u_\varepsilon = \sigma_\varepsilon$ on the boundary. The u_ε has a unique free boundary point c_ε. Obviously, $c_\varepsilon > c$. So it is easy to see that u_ε converges to v uniformly on $[a, b]$, as $\varepsilon \to 0$.

Let $\sigma^\varepsilon(a) = \varepsilon$ and $\sigma^\varepsilon(b) \to \sigma(b)$ as $\varepsilon \to 0+$. Then there are two solutions of the free boundary with boundary data σ^ε. Let u_1^ε be the solution without a free boundary and u_2^ε the solution with a free boundary. Clearly, $u_1^\varepsilon \to u$ as $u_1^\varepsilon(x) = \frac{b-x}{b-a}\varepsilon + \frac{x-a}{b-a}\sigma^\varepsilon(b)$. Also,

$$u_2^\varepsilon(x) = \begin{cases} g(0)(c^\varepsilon - x) & \text{if } a \leqslant x < c^\varepsilon, \\ 0 & \text{if } c^\varepsilon \leqslant x < d^\varepsilon, \\ g(0)(x - d^\varepsilon) & \text{if } d^\varepsilon \leqslant x \leqslant b, \end{cases}$$

where $c^\varepsilon = a + \frac{\varepsilon}{g(0)}$ and $d^\varepsilon = b - \frac{\sigma^\varepsilon(b)}{g(0)}$. Clearly, $c^\varepsilon \to a$ and $d^\varepsilon \to c$ as $\varepsilon \to 0$. So $u_2^\varepsilon \to u$ uniformly.

In case $\sigma(a) < 0$ and $\sigma(b) < 0$, the maximum–minimum principle implies $\sup_\Omega u < 0$. So a solution has only one phase. Therefore there is only one viscosity solution $u(x) = \frac{b-x}{b-a}\sigma(a) + \frac{x-a}{b-a}\sigma(b)$, $x \in (a,b)$.

6. Uniqueness for the ε-evolutionary problem

Heuristically, the elliptic free boundary problem describes the limiting stationary state of the corresponding evolutionary free boundary problem. Unlike the elliptic free boundary problem, the evolutionary problem seems to enjoy the uniqueness of a viscosity solution with prescribed initial–boundary data. In fact, if we smooth the free boundary condition in a very small scale, we can prove the uniqueness of a viscosity solution of the smoothed parabolic ε-evolutionary problem.

In the following, we prove the uniqueness of a viscosity solution of the following ε-evolutionary problem,

$$\begin{cases} H^\varepsilon w = w_t - Lw + \beta_\varepsilon(w) = 0 & \text{in } \Omega \times \Re_T, \\ w(x,t) = \sigma(x) & \text{on } \partial\Omega \times \Re_T, \\ w(x,0) = w_0(x) & \text{on } \bar{\Omega}, \end{cases}$$

where $Lw = F(\nabla w, D^2 w)$ is a degenerate linear or non-linear elliptic operator such that $F(p, O_{n\times n}) = 0$ such as the Laplacian or p-Laplacian, $\beta_\varepsilon(w) = \frac{1}{\varepsilon}\beta(\frac{w}{\varepsilon})$, $\beta : \Re \to [0,\infty)$ is a compactly supported around origin, smooth non-negative function with $\beta(0) > 0$, Ω is a bounded domain in $\Re^n$, $\Re_T = (0,T)$ with T possibly being infinity, and the compatibility condition $\sigma = w_0$ on $\partial\Omega$ is verified. Here the partial differential equation is verified in the viscosity sense, namely if a smooth function φ satisfies $\varphi \geqslant w$ (or $\varphi \leqslant w$) in a neighborhood of (x_0,t_0) and $\varphi(x_0,t_0) = w(x_0,t_0)$, which is usually denoted by $w \prec_{(x_0,t_0)} \varphi$, then $H^\varepsilon\varphi(x_0,t_0) \leqslant 0$ (or $H^\varepsilon\varphi(x_0,t_0) \geqslant 0$). The parabolic sub- and super-jets $\mathcal{P}^{2,-}w(x_0,t_0)$ and $\mathcal{P}^{2,+}$ are defined by

$$\mathcal{P}^{2,-}w(x_0,t_0) = \{(\varphi_t(x_0,t_0), \nabla\varphi(x_0,t_0), D^2\varphi(x_0,t_0)) \mid \varphi \prec_{(x_0,t_0)} w\} \tag{6.1}$$

and

$$\mathcal{P}^{2,+}w(x_0,t_0) = \{(\varphi_t(x_0,t_0), \nabla\varphi(x_0,t_0), D^2\varphi(x_0,t_0)) \mid w \prec_{(x_0,t_0)} \varphi\}. \tag{6.2}$$

The "closures" of the semi-jets are defined by

$$\bar{\mathcal{P}}^{2,-}w(x_0,t_0) = \{(b,p,X) \in \Re \times \Re^n \times \mathcal{S}_{n\times n} \mid \exists (x_k,t_k,b_k,p_k,X_k) \tag{6.3}$$

$$\in \Omega \times \Re_T \times \Re \times \Re^n \times \mathcal{S}_{n\times n} \text{ such that } (b_k,p_k,X_k) \in \mathcal{P}^{2,-}w(x_k,t_k) \tag{6.4}$$

$$\text{and } (x_k,t_k,b_k,p_k,X_k) \to (x_0,t_0,b,p,X)\} \tag{6.5}$$

and

$$\bar{\mathcal{P}}^{2,+}w(x_0,t_0) = \{(b,p,X) \in \Re \times \Re^n \times \mathcal{S}_{n\times n} \mid \exists (x_k,t_k,b_k,p_k,X_k) \tag{6.6}$$

$$\in \Omega \times \Re_T \times \Re \times \Re^n \times \mathcal{S}_{n\times n} \text{ such that } (b_k, p_k, X_k) \in \mathcal{P}^{2,+} w(x_k, t_k) \tag{6.7}$$

$$\text{and } (x_k, t_k, b_k, p_k, X_k) \to (x_0, t_0, b, p, X)\big\}, \tag{6.8}$$

where $\mathcal{S}_{n\times n}$ is the set of symmetric $n \times n$ matrices.

We also require σ and w_0 to be continuous on $\partial\Omega$ and $\bar{\Omega}$ respectively.

Note that $w \mapsto -Lw + \beta_\varepsilon(w)$ is not a proper elliptic operator in the sense of Crandall–Ishii–Lions.

As there is no confusion, we will skip the superscript and subscript ε, and write H for H^ε and β for β_ε.

Lemma 6.1. *For $T > 0$ small enough, if $Hw \leqslant 0 \leqslant Hw_2$ in $\Omega \times \Re_T$ and $w < w_2$ on $\partial_p(\Omega \times \Re_T)$, then $w \leqslant w_2$ in $\Omega \times \Re_T$.*

Proof. As β is compactly supported and smooth, it is globally Lipschitz continuous for some Lipschitz constant K.

For any given small number $\delta > 0$, we define a new function w_1 by

$$w_1(x,t) = w(x,t) - \frac{\delta}{T-t},$$

where $x \in \bar{\Omega}$ and $0 \leqslant t < T$. In order to prove $w \leqslant w_2$ in $\Omega \times \Re_T$, it suffices to prove $w_1 \leqslant w_2$ in $\Omega \times \Re_T$ for all small $\delta > 0$. Clearly, $w_1 < w_2$ on $\partial_p(\Omega \times \Re_T)$, and $\lim_{t\to T} w_1(x,t) = -\infty$ uniformly on Ω. Moreover,

$$\begin{aligned}
Hw_1 &= w_t - \frac{\delta}{(T-t)^2} - F\big(\nabla w, D^2 w\big) + \beta\left(w - \frac{\delta}{T-t}\right)\\
&= Hw - \frac{\delta}{(T-t)^2} + \beta\left(w - \frac{\delta}{T-t}\right) - \beta(w)\\
&\leqslant Hw - \frac{\delta}{(T-t)^2} + K\frac{\delta}{T-t} \quad \text{due to the Lipschitz continuity of } \beta\\
&\leqslant Hw - \frac{\delta}{(T-t)^2} + \frac{\delta}{2(T-t)^2} \quad \text{for } T \leqslant \frac{1}{2K} \text{ so that } 2K \leqslant \frac{1}{T-t}\\
&= Hw - \frac{\delta}{2(T-t)^2} \leqslant -\frac{\delta}{2(T-t)^2}\\
&\leqslant -\frac{\delta}{2T^2} < 0.
\end{aligned}$$

The above differential equalities and inequalities are all in the viscosity sense. Every step can be made rigorous in the viscosity sense. We leave the work to the reader.

Define, for $j = 1, 2$, $v_j(x,t) = e^{-\lambda t} w_j(x,t)$, where $\lambda > 2K$. So $w_j(x,t) = e^{\lambda t} v_j(x,t)$.

Obviously, $w_1 \leqslant w_2$ in $\Omega \times \Re_T$ is equivalent to $v_1 \leqslant v_2$ in $\Omega \times \Re_T$. A simple computation shows that in the viscosity sense, $Hw_j = e^{\lambda t}\tilde{H}v_j$, where $\tilde{H}v = v_t - e^{-\lambda t}F(e^{\lambda t}\nabla v, e^{\lambda t}D^2 v) + \lambda v + e^{-\lambda t}\beta(e^{\lambda t}v)$. Then, in the viscosity sense, $\tilde{H}v_1 \leqslant -\frac{\delta}{2T^2}e^{-\lambda t} \leqslant -\frac{\delta}{2T^2}e^{-\lambda T} < 0$ and $\tilde{H}v_2 \geqslant 0$. Furthermore, $v_1 < v_2$ on $\partial_p(\Omega \times \Re_T)$, and $\lim_{t\to T-} v_1(x,t) = -\infty$ uniformly on $\bar{\Omega}$.

Suppose $\sup_{\Omega\times\Re_T}(v_1 - v_2) > 0$. Then $\sup_{\Omega\times\Re_T}(v_1 - v_2)$ is a maximum and is assumed exclusively in $\Omega \times (0, T)$, due to the last two conditions on v_1 and v_2.

Let $M_0 = \sup_{\Omega\times\Re_T}(v_1 - v_2) = \max_{\overline{\Omega\times\Re_T}}(v_1 - v_2)$.

For any small $\varepsilon > 0$, we define

$$u^\varepsilon(x, y, t) = v_1(x, t) - v_2(y, t) - \frac{1}{2\varepsilon}|x - y|^2, \quad x, y \in \bar{\Omega},\ t \in [0, T). \tag{6.9}$$

We observe first that $\max_{\bar{\Omega}\times\bar{\Omega}\times[0,T)} u^\varepsilon(x, y, t)$ exists as $\lim_{t\to T} v_1(x, t) = -\infty$ uniformly on $\bar{\Omega}$.

Let $M_\varepsilon = u^\varepsilon(x^\varepsilon, y^\varepsilon, t^\varepsilon) = \max_{\bar{\Omega}\times\bar{\Omega}\times[0,T)} u^\varepsilon$, where $x^\varepsilon, y^\varepsilon \in \bar{\Omega}$ and $t^\varepsilon \in [0, T') \subset [0, T)$ for some $T' < T$ independent of ε. Clearly, $M_\varepsilon \geqslant M_0 > 0$. According to Proposition 3.7 in [9], a generalization of Lemma 3.1 in [9], $\lim_{\varepsilon\downarrow 0} M_\varepsilon = M_0$ and $\lim_{\varepsilon\downarrow 0} \frac{1}{2\varepsilon}|x^\varepsilon - y^\varepsilon|^2 = 0$ hold.

We claim that $x^\varepsilon, y^\varepsilon \in \Omega$ and $t^\varepsilon > 0$ for all sufficiently small ε.

Suppose not. There exists a sequence $\varepsilon_j \to 0$ such that either $(x^{\varepsilon_j}, t^{\varepsilon_j}) \in \partial_p(\Omega \times \Re_T)$ or $(y^{\varepsilon_j}, t^{\varepsilon_j}) \in \partial_p(\Omega \times \Re_T)$, and without loss of generality $\{x^{\varepsilon_j}\}$, $\{y^{\varepsilon_j}\}$, $\{t^{\varepsilon_j}\}$ converge. As $\frac{1}{2\varepsilon_j}|x^{\varepsilon_j} - y^{\varepsilon_j}|^2 \to 0$ implies $|x^{\varepsilon_j} - y^{\varepsilon_j}| \to 0$, we may assume $x^{\varepsilon_j} \to x_0$, $y^{\varepsilon_j} \to x_0$, $t^{\varepsilon_j} \to t_0$, where $(x_0, t_0) \in \partial_p(\Omega \times \Re_T)$, and $t_0 \leqslant T' < T$. So

$$0 < M_0 \leqslant \limsup_j M_{\varepsilon_j} = v_1(x_0, t_0) - v_2(x_0, t_0) < 0$$

as $(x_0, t_0) \in \partial_p(\Omega \times \Re_T)$, which is an obvious contradiction.

For any small $\varepsilon > 0$, Theorem 8.3 in [9] implies that there exist $X, Y \in \mathcal{S}_{n\times n}$, and $b \in \Re$ such that $(b, \frac{x^\varepsilon - y^\varepsilon}{\varepsilon}, X) \in \bar{\mathcal{P}}^{2,+} v_1(x^\varepsilon, t^\varepsilon)$, $(b, \frac{x^\varepsilon - y^\varepsilon}{\varepsilon}, Y) \in \bar{\mathcal{P}}^{2,-} v_2(y^\varepsilon, t^\varepsilon)$, and

$$-\frac{3}{\varepsilon} I \leqslant \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leqslant \frac{3}{\varepsilon} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

The last inequality implies that $X \leqslant Y$, while the first two inclusion conditions imply that

$$b - e^{-\lambda t^\varepsilon} F\left(e^{\lambda t^\varepsilon} \frac{x^\varepsilon - y^\varepsilon}{\varepsilon}, e^{\lambda t^\varepsilon} X\right) + \lambda v_1\big(x^\varepsilon, t^\varepsilon\big) + e^{-\lambda t^\varepsilon} \beta\big(e^{\lambda t^\varepsilon} v_1\big(x^\varepsilon, t^\varepsilon\big)\big) \leqslant -\frac{\delta}{2T^2} e^{-\lambda T} < 0 \tag{6.10}$$

and

$$b - e^{-\lambda t^\varepsilon} F\left(e^{\lambda t^\varepsilon} \frac{x^\varepsilon - y^\varepsilon}{\varepsilon}, e^{\lambda t^\varepsilon} Y\right) + \lambda v_2\big(y^\varepsilon, t^\varepsilon\big) + e^{-\lambda t^\varepsilon} \beta\big(e^{\lambda t^\varepsilon} v_2\big(y^\varepsilon, t^\varepsilon\big)\big) \geqslant 0. \tag{6.11}$$

That F is degenerate elliptic implies that $F(e^{\lambda t^\varepsilon} \frac{x^\varepsilon - y^\varepsilon}{\varepsilon}, e^{\lambda t^\varepsilon} X) \leqslant F(e^{\lambda t^\varepsilon} \frac{x^\varepsilon - y^\varepsilon}{\varepsilon}, e^{\lambda t^\varepsilon} Y)$.

As a result of the three preceding inequalities,

$$\begin{aligned}
0 &> \lambda\big(v_1\big(x^\varepsilon, t^\varepsilon\big) - v_2\big(x^\varepsilon, t^\varepsilon\big)\big) + e^{-\lambda t^\varepsilon}\big\{\beta\big(e^{\lambda t^\varepsilon} v_1\big(x^\varepsilon, t^\varepsilon\big)\big) - \beta\big(e^{\lambda t^\varepsilon} v_2\big(y^\varepsilon, t^\varepsilon\big)\big)\big\} \\
&\geqslant \lambda\big(v_1\big(x^\varepsilon, t^\varepsilon\big) - v_2\big(x^\varepsilon, t^\varepsilon\big)\big) - K\big|v_1\big(x^\varepsilon, t^\varepsilon\big) - v_2\big(y^\varepsilon, t^\varepsilon\big)\big| \\
&\qquad \text{as } \beta \text{ is Lipschitz continuous with Lipschitz constant } K \\
&\geqslant \lambda\big(v_1\big(x^\varepsilon, t^\varepsilon\big) - v_2\big(x^\varepsilon, t^\varepsilon\big)\big) - \frac{\lambda}{2}\big|v_1\big(x^\varepsilon, t^\varepsilon\big) - v_2\big(y^\varepsilon, t^\varepsilon\big)\big| \quad \text{as } \lambda > 2K.
\end{aligned}$$

On account of the reasons that justify the preceding claim, we know that there exists a sequence $\varepsilon_j \to 0$ such that $x^{\varepsilon_j} \to x_0$, $y^{\varepsilon_j} \to x_0$, $t^{\varepsilon_j} \to t_0$, and $x_0 \in \Omega$, $0 < t_0 \leqslant T' < T$. In addition, Proposition 3.7 in [9] implies $v_1(x_0, t_0) - v_2(x_0, t_0) = M_0$. Taking limits in $0 \geqslant \lambda(v_1(x^{\varepsilon_j}, t^{\varepsilon_j}) - v_2(y^{\varepsilon_j}, t^{\varepsilon_j})) - \frac{\lambda}{2}|v_1(x^{\varepsilon_j}, t^{\varepsilon_j}) - v_2(y^{\varepsilon_j}, t^{\varepsilon_j})|$, we obtain, since $v_1(x_0, t_0) - v_2(x_0, t_0) = M_0 > 0$, that

$$0 \geqslant \frac{\lambda}{2}\big(v_1(x_0, t_0) - v_2(x_0, t_0)\big) > 0,$$

which is an obvious contradiction. We are done. □

We now loose the strict inequality restriction to a non-strict one.

Lemma 6.2. *For $T > 0$ sufficiently small, if $Hw_1 \leqslant 0 \leqslant Hw_2$ in $\Omega \times \Re_T$ and $w_1 \leqslant w_2$ on $\partial_p(\Omega \times \Re_T)$, then $w_1 \leqslant w_2$ on $\overline{\Omega \times \Re_T}$.*

Proof. For any $\delta > 0$, let $w = w_1 - \delta t - \tilde{\delta}$, where the value of $\tilde{\delta} > 0$ will be taken in the following. Then $w < w_1 \leqslant w_2$ on $\partial_p(\Omega \times \Re_T)$, and

$$\begin{aligned} Hw &= Hw_1 - \delta - \beta_\varepsilon(w_1) + \beta_\varepsilon(w_1 + \delta t + \tilde{\delta}) \\ &\leqslant -\delta + K\delta t + K\tilde{\delta} \leqslant -\delta + K\delta T + K\tilde{\delta} \\ &< -\delta + \frac{1}{2}\delta + \frac{1}{4}\delta \quad \text{for } T \text{ small and } \tilde{\delta} \leqslant \frac{\delta}{4K} \\ &= -\frac{1}{4}\delta < 0. \end{aligned}$$

Again, the above differential equality and inequalities are in the viscosity sense and can be made rigorous.

The preceding lemma implies $w \leqslant w_2$ on $\overline{\Omega \times \Re_T}$ for small T, for any $\delta > 0$. Therefore $w_1 \leqslant w_2$ on $\overline{\Omega \times \Re_T}$. □

The following comparison principle for the ε-evolutionary problem follows quite easily.

Lemma 6.3. *For any $T > 0$ including ∞, if $H^\varepsilon w_1 \leqslant 0 \leqslant H^\varepsilon w_2$ in $\Omega \times \Re_T$ and $w_1 \leqslant w_2$ on $\partial_p(\Omega \times \Re_T)$, then $w_1 \leqslant w_2$ on $\overline{\Omega \times \Re_T}$.*

Proof. Let $T_0 > 0$ be any small value of T in the preceding lemma so that the conclusion of the preceding lemma holds. Then $w_1 \leqslant w_2$ on $\overline{\Omega \times (0, T_0)}$. In particular, $w_1 \leqslant w_2$ on $\partial_p(\Omega \times (T_0, 2T_0))$. The preceding lemma may be applied again to conclude that $w_1 \leqslant w_2$ on $\overline{\Omega \times (T_0, 2T_0)}$. And so on. In the end, we see that $w_1 \leqslant w_2$ on $\overline{\Omega \times \Re_T}$. □

The uniqueness of a viscosity solution of the ε-evolutionary problem is the straightforward corollary of the preceding comparison result.

Theorem 6.1. *The ε-evolutionary problem*

$$\begin{cases} H^{\varepsilon} w = w_t - Lw + \beta_{\varepsilon}(w) = 0 & \text{in } \Omega \times \Re_T, \\ w(x,t) = \sigma(x) & \text{on } \partial\Omega \times \Re_T, \\ w(x,0) = w_0(x) & \text{on } \bar{\Omega}, \end{cases}$$

possesses at most one viscosity solution.

A feasible explanation of the non-uniqueness of a viscosity solution of the elliptic free boundary problem versus the uniqueness of a viscosity solution of the ε-evolutionary problem is that different physical evolutionary processes with the same boundary condition may end up with different steady states. For example, if the melting of ice and solidification of water observe the physical laws described by the mathematical model so far discussed, we may have the following phenomenon. We manage to keep the temperature distribution on the surface of a closed container fixed as time goes by (however, the distribution in general is non-constant, somewhere above the freezing point and somewhere below). If ice or water is put in the container, after a long time, the temperature distribution inside the container reaches a steady state. Even though the boundary temperature distribution is the same for either case, the steady states resulted may differ from each other depending on the initial state. It needs rigorous mathematical justification and is the subject of the authors' following study. For now, we content ourselves with some questions about the elliptic free boundary problem for which the uniqueness of a solution fails.

Let $S(\sigma)$ be the set of solutions of the elliptic free boundary problem with continuous initial and boundary data

$$\begin{cases} \Delta u = 0 & \text{in } \Omega^{+}(u) \cup \Omega^{-}(u), \\ u_{\nu}^{+} = g(u_{\nu}^{-}) & \text{along } \mathcal{F}(u), \\ u = \sigma & \text{on } \partial\Omega. \end{cases}$$

We ask the following questions about the set of solutions $S(\sigma)$.

How to determine, from the initial value, to which viscosity solution in $S(\sigma)$ do viscosity solutions of the evolutionary free boundary problem converge as time goes to infinity?

Is $S(\sigma)$ a finite set?

Are there a largest element and a least element of $S(\sigma)$ in general?

Does $S(\sigma)$ contain only two solutions in general, which model the stationary states resulting from the melting of ice and the solidification of water respectively? And under what condition do they coincide with each other?

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