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Best Constants for Adams' Inequalities with the Exact Growth Condition in $\mathbb{R}^n$

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Abstract

In this paper, we establish the following sharp Adams inequality with exact growth condition in the entire space $\mathbb{R}^n$ $(n \geq 3)$: There exists a constant $C(n) > 0$ such that for all $f \in W^{2,\frac{n}{2}}(\mathbb{R}^n)$ $(n \geq 3)$ with $\|\Delta f\|_{\frac{n}{2}} \leq 1$,

$$\int_{\mathbb{R}^n} \frac{\Phi(\beta_n |f|^{\frac{n}{n-2}})}{(1+|f|)^{\frac{n}{n-2}}} dx \leq C(n)\|f\|_{\frac{n}{2}}^{\frac{n}{2}},$$

where $\Phi(t) = \exp(t) - \sum_{j=0}^{j_{\frac{n}{2}}-2} \frac{t^j}{j!}$, $j_{\frac{n}{2}} = \min\{j \in \mathbb{R} : j \geq \frac{n}{2}\} \geq n/2$ and $\beta_n = \beta(n,2) = \frac{n}{\omega_{n-1}}[\frac{\pi^{\frac{n}{2}}4}{\Gamma(n/2-1)}]^{\frac{n}{n-2}}$. This extends the main result in [27] when $n = 4$ to all dimensions $n \geq 3$. A crucial technical lemma we need is Lemma 4.2 for all $p > 1$ (corresponding to the Adams inequality for all $n \geq 3$) whose proof is quite involved. As an application, we obtain the best constant for Ozawa's inequality of Adams type in the Sobolev space $W^{2,\frac{n}{2}}(\mathbb{R}^n)$ in [29]: For any $\alpha < \beta_n$, there exists a constant $C(\alpha, n) > 0$ such that for all $f \in W^{2,\frac{n}{2}}(\mathbb{R}^n)$ $(n \geq 3)$ satisfying $\|\Delta f\|_{\frac{n}{2}} \leq 1$, we have $\int_{\mathbb{R}^n} \Phi(\alpha|f|^{\frac{n}{n-2}})dx \leq C(\alpha, n)\|f\|_{\frac{n}{2}}^{\frac{n}{2}}$. Moreover, if $\alpha \geq \beta_n$ then the inequality cannot hold with a uniform constant $C(\alpha, n)$.

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1 The Moser-Trudinger inequality

The Moser-Trudinger inequalities can be considered as the limiting case of Sobolev inequalities. They were established independently by V. Yudovič [36], S. Pohožaev [30] and N. Trudinger [35]. In 1971, J. Moser [28], sharpening Trudinger's inequality, proved that

Theorem A *Let Ω be a domain with finite measure in Euclidean $n-$space $\mathbb{R}^n$, $n \geq 2$. Then there exist a positive constant $C(n) > 0$ and a sharp constant $\alpha_n = n\omega_{n-1}^{\frac{1}{n-1}}$ (where ω_{n-1} is the area of the surface of the unit $n-$ball) such that*

$$\frac{1}{|\Omega|}\int_\Omega \exp\left(\alpha\,|f|^{\frac{n}{n-1}}\right)dx \leq C(n) < \infty \tag{1.1}$$

for any $\alpha \leq \alpha_n$, any $f \in C_0^\infty(\Omega)$ with $\int_\Omega |\nabla f|^n\,dx \leq 1$. This constant α_n is sharp in the sense that if $\alpha > \alpha_n$, then the above inequality can no longer hold with some $C(n)$ independent of f.

This result has been generalized in many directions. For instance, the best constants for Moser-Trudinger inequalities on domains of finite measure on the Heisenberg group were established in [7, 17]. There has also been substantial progress for the Moser-Trudinger inequalities on spheres, CR spheres, or compact Riemannian manifolds, hyperbolic spaces, etc. We refer the interested reader to [3], [4], [8], [10], [21], and the references therein. Moser-Trudinger inequalities have found many applications in geometric analysis and PDEs, see e.g., [6], [28], [31], [22], [37], etc.

When Ω has infinite volume, sharp versions of Moser-Trudinger type inequalities with best constants on unbounded domains were obtained by S. Adachi and K. Tanaka [1]. They proved that

Theorem B *Let $0 < \alpha < \alpha_n$. There exists a constant $C(\alpha, n) > 0$ such that*

$$\sup_{u\in W^{1,n}(\mathbb{R}^n),\int_{\mathbb{R}^n}|\nabla u|^n dx\leq 1}\int_{\mathbb{R}^n}\Phi_n(\alpha|u|^{\frac{n}{n-1}})dx \leq C(\alpha,n)\|u\|_n^n,$$

where $\Phi_n(t) := e^t - \sum_{i=0}^{n-2}\frac{t^i}{i!}$. Moreover, the constant α_n is sharp in the sense that if $\alpha \geq \alpha_n$, the supremum will become infinite.

B. Ruf [31] (for the case $n = 2$), Y. X. Li and B. Ruf [22] (for the general case $n \geq 2$) established a critical Moser-Trudinger type inequality for unbounded domains in Euclidean spaces. They obtain the following theorem.

Theorem C *There exists a constant $C(n) > 0$ such that for any domain $\Omega \subset \mathbb{R}^n$,*

$$\sup_{u\in W_0^{1,n}(\Omega),\|u\|_{W^{1,n}(\Omega)}\leq 1}\int_\Omega \Phi_n(\alpha_n|u|^{\frac{n}{n-1}})dx \leq C(n).$$

Moreover, the constant α_n is sharp in the sense that if α_n is replaced by any $\alpha > \alpha_n$, the supremum will become infinite.

The existence of extremals of the Moser-Trudinger inequality (1.1) on bounded domains was first established by Carleson and Chang in their celebrated work [5] on balls, and then extended to arbitrary smooth domains in [9] and [23]. The existence of extremal functions for the Moser-Trudinger inequality on the entire space was dealt with in [31], [22] and [12]. More recently, Such a sharp Moser-Trudinger inequality at the critical case has also been established on the entire Heisenberg group in [14] and at the subcritical case in [18] where symmetrization argument is not available.

We note that there is a sharp difference between the inequalities in Theorems B and C. In Theorem B, the inequality only holds for $\alpha < \alpha_n$ while the inequality in Theorem C holds for all $\alpha \leq \alpha_n$. The reason behind is that the restriction on the class of functions in Theorem B is for all with the L^n norm of their gradients being less than or equal to 1 while the function class in Theorem C is for those with the full Sobolev $W^{1,n}$ norm less than or equal to 1. Though there are subtle differences between these two type of inequalities, surprisingly, Lam, Zhang and the first author proved in [19] that these critical and subcritical Moser-Trudinger inequalities are actually equivalent. Moreover, we also establish the asymptotic behavior of the supremum for the subcritical Moser-Trudinger inequalities on the entire Euclidean spaces and provide a precise relationship between the supremums for the critical and subcritical Moser-Trudinger inequalities. Since the critical Moser-Trudinger inequalities can be easier to prove than subcritical ones in some occasions (for instance for Moser-Trudinger inequalities on complete and noncompact Riemannian manifolds [20]), and more difficult to establish in other occasions, our results and the method in [19] suggest a new approach to both the critical and subcritical Moser-Trudinger and Adams type inequalities.

In particular, we establish in [19] the asymptotic estimates when α goes to α_n for the following supremum:

$$\sup_{\|\nabla u\|_n \leq 1} \frac{1}{\|u\|_n^{n-\beta}} \int_{\mathbb{R}^n} \Phi_n\left(\alpha\left(1-\frac{\beta}{n}\right)|u|^{\frac{n}{n-1}}\right)\frac{dx}{|x|^\beta}.$$

The following theorem provides the lower and upper bounds asymptotically for the supremum.

Theorem D *Let $n \geq 2$, $\alpha_n = n\left(\frac{n\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}\right)^{\frac{1}{n-1}}$, $0 \leq \beta < n$ and $0 \leq \alpha < \alpha_n$. Denote*

$$AT(\alpha,\beta) = \sup_{\|\nabla u\|_n \leq 1} \frac{1}{\|u\|_n^{n-\beta}} \int_{\mathbb{R}^n} \Phi_n\left(\alpha\left(1-\frac{\beta}{n}\right)|u|^{\frac{n}{n-1}}\right)\frac{dx}{|x|^\beta}.$$

Then there exist positive constants $c = c(n,\beta)$ and $C = C(n,\beta)$ such that when α is close enough to α_n :

$$\frac{c(n,\beta)}{\left(1-\left(\frac{\alpha}{\alpha_n}\right)^{n-1}\right)^{(n-\beta)/n}} \leq AT(\alpha,\beta) \leq \frac{C(n,\beta)}{\left(1-\left(\frac{\alpha}{\alpha_n}\right)^{n-1}\right)^{(n-\beta)/n}}. \tag{1.2}$$

Moreover, the constant α_n is sharp in the sense that $AT(\alpha_n,\beta) = \infty$.

We note that we do not assume a priori the validity of the critical Moser-Trudinger inequality with the restriction on the full norm in order to derive the above asymptotic behavior of the supremum $AT(\alpha,\beta)$.

Next, we like to know how the supremum $AT(\alpha,\beta)$ we established in Theorem D will provide a proof to the sharp critical Moser-Trudinger inequality. Thus, this gives a new proof of the sharp critical Moser-Trudinger inequality in all dimension n. We also answer the question under for which a and b the critical Moser-Trudinger inequality holds under the restriction of the full norm $\|\nabla u\|_n^a + \|u\|_n^b \le 1$. Moreover, Lam, Zhang and the first author establish the precise relationship between the supremums for the critical and subcritical Moser-Trudinger inequalities.

Theorem E *Let $n \ge 2,\ 0 \le \beta < n,\ a,\ b > 0$. Denote*

$$MT_{a,b}(\beta) = \sup_{\|\nabla u\|_n^a + \|u\|_n^b \le 1} \int_{\mathbb{R}^n} \Phi_n\left(\alpha_n\left(1 - \frac{\beta}{n}\right)|u|^{\frac{n}{n-1}}\right)\frac{dx}{|x|^\beta},$$

and

$$MT(\beta) = MT_{n,n}(\beta).$$

Then $MT_{a,b}(\beta) < \infty$ if and only if $b \le n$. The constant α_n is sharp. Moreover, we have the following identity:

$$MT_{a,b}(\beta) = \sup_{\alpha \in (0,\alpha_n)} \left(\frac{1 - \left(\frac{\alpha}{\alpha_n}\right)^{\frac{n-1}{n}a}}{\left(\frac{\alpha}{\alpha_n}\right)^{\frac{n-1}{n}b}}\right)^{\frac{n-\beta}{b}} AT(\alpha,\beta). \tag{1.3}$$

In particular, $MT(\beta) < \infty$ and

$$MT(\beta) = \sup_{\alpha \in (0,\alpha_n)} \left(\frac{1 - \left(\frac{\alpha}{\alpha_n}\right)^{n-1}}{\left(\frac{\alpha}{\alpha_n}\right)^{n-1}}\right)^{\frac{n-\beta}{n}} AT(\alpha,\beta).$$

As we have discussed earlier, the failure of the original Moser-Trudinger inequality (1.1) on the entire $\mathbb{R}^n$ can be recovered either by weakening the exponent $\alpha = n\omega_{n-1}^{\frac{1}{n-1}}$ or by strengthening the Dirichlet norm $\|\nabla u\|_{L^n}$. Then a natural question arises:

Can we still achieve the best constant $\alpha_n = n\omega_{n-1}^{\frac{1}{n-1}}$ when we only require the restriction on the norm $\|\nabla u\|_{L^n(\mathbb{R}^n)} \le 1$?

Ibrahim, Masmoudi and Nakanishi [11] answered the question in the two dimensional case. They established the following theorem.

Theorem F *There exists a constant $C > 0$, such that*

$$\int_{\mathbb{R}^2} \frac{e^{4\pi u^2} - 1}{(1 + |u|)^2}dx \le C\|u\|_2^2,$$

for any $u \in W^{1,2}(\mathbb{R}^2)$ *with* $\|\nabla u\|_{L^2(\mathbb{R}^2)} \le 1$*. Moreover, this fails if the power* 2 *in the denominator is replaced with any* $p < 2$*.*

Recently, the first two authors [25] of the paper have established the Moser-Trudinger inequality with exact growth condition on the hyperbolic space $\mathbb{H}^n$ for all $n \ge 2$ which improves the earlier result by the same authors in [24].

2 Adams' inequality and statement of our main results

In 1988, D. Adams [2] extended the original Moser-Trudinger inequality to the high order space $W_0^{m,\frac{n}{m}}(\Omega)$ on any domain Ω with finite measure. In fact, Adams proved the following inequality:

Theorem G *Let* Ω *be a bounded domain. There exists a constant* $C_0 = C(m,n)$ *such that for any* $f \in W_0^{m,\frac{n}{m}}(\Omega)$ *with* $\|\nabla^m f\|_{\frac{n}{m}} \le 1$*, then*

$$\frac{1}{|\Omega|}\int_\Omega \exp(\beta|f(x)|^{\frac{n}{n-m}})dx \le C_0,$$

for all $\beta \le \beta(n,m)$ *where*

$$\beta(n,m) = \begin{cases} \frac{n}{\omega_{n-1}}[\frac{\pi^{n/2}2^m\Gamma(\frac{m+1}{2})}{\Gamma(\frac{n-m+1}{2})}]^{\frac{n}{n-m}}, \text{when } m \text{ is odd.} \\ \frac{n}{\omega_{n-1}}[\frac{\pi^{n/2}2^m\Gamma(\frac{m}{2})}{\Gamma(\frac{n-m}{2})}]^{\frac{n}{n-m}}, \text{when } m \text{ is even.} \end{cases}$$

Moreover, for any $\beta > \beta(n,m)$*, the integral can be made as large as possible.*

Adams inequality on domains of finite volume has been extended by Tarsi [34] to the case where the functions satisfy the Navier boundary condition. In 1995, T. Ozawa [29] established some version of the Adams inequality in Sobolev space $W^{m,\frac{n}{m}}(\mathbb{R}^n)$ on the entire Euclidean space $\mathbb{R}^n$ only using the restriction $\|\nabla^m f\|_{\frac{n}{m}} \le 1$. T. Ozawa proved the following

Theorem H *There exist positive constants* β *and* $C(n) > 0$ *such that for all* $f \in W^{m,\frac{n}{m}}(\mathbb{R}^n)$ *satisfying* $\|\nabla^m f\|_{\frac{n}{m}} \le 1$*, then*

$$\int_{\mathbb{R}^n} \Phi_{\frac{n}{m}}(\beta|f|^{\frac{n}{n-m}})dx \le C(n)\|f\|_{\frac{n}{m}}^{\frac{n}{m}},$$

where $\Phi_{\frac{n}{m}}(t) = \exp(t) - \sum_{j=0}^{j_{\frac{n}{m}}-2}\frac{t^j}{j!}$, $j_{\frac{n}{m}} = \min\{j \in \mathbb{R} : j \ge \frac{n}{m}\} \ge n/m$.

However, with the argument in [29], one can't obtain the best possible exponent β for this type of inequality.

The Adams type inequality on Sobolev spaces $W_0^{m,\frac{n}{m}}(\Omega)$ when Ω has infinite volume (e.g., $\Omega = \mathbb{R}^n$) with the restrictions on full Sobolev norms has also been studied. In fact, Kozono, Sato and Wadade obtained such inequalities with non-optimal constants [13]. When m is an even integer, sharp Adams' inequality was established by B. Ruf and F. Sani [32] using the comparison principle of solutions to elliptic polyharmonic operators, while N. Lam and the first author [15] established the Adams type inequalities on unbounded domain when m is odd. Furthermore, N. Lam and the first author [16] developed a new approach to establish sharp Adams inequalities in Sobolev spaces $W^{\gamma,p}(\mathbb{R}^n)$ for any fractional order $\gamma > 0$ in $\mathbb{R}^n$ without using the standard symmetrization argument which is not available for high order Sobolev spaces.

More precisely, the following sharp Adams type inequalities with best constants on Sobolev spaces $W^{\gamma,\frac{n}{\gamma}}(\mathbb{R}^n)$ of arbitrary positive fractional order $\gamma < n$ have been established by Lam and Lu in [16].

Theorem I *Let $0 < \gamma < n$ be an arbitrary real positive number, $p = \frac{n}{\gamma}$ and $\tau > 0$. There holds*

$$\sup_{u\in W^{\gamma,p}(\mathbb{R}^n),\left\|(\tau I-\Delta)^{\frac{\gamma}{2}}u\right\|_p\leq 1}\int_{\mathbb{R}^n}\phi\left(\beta_0(n,\gamma)|u|^{p'}\right)dx<\infty$$

where

$$\phi(t) = e^t - \sum_{j=0}^{j_p-2}\frac{t^j}{j!},$$

$$j_p = \min\{j\in\mathbb{N} : j\geq p\}\geq p.$$

Furthermore this inequality is sharp, i.e., if $\beta_0(n,\gamma)$ is replaced by any $\beta > \beta_0(n,\gamma)$, then the supremum is infinite.

We remark here that the approach developed in [16] is to use the level set for the functions under consideration and derive the global Adams inequalities on unbounded domains through the local ones on the level set of the function. This local to global argument is rather general and can be used in many other settings such as the Heisenberg group [14, 18] and Riemannian manifolds [20] where symmetrization principle is not valid. In [14, 18, 20], both critical and subcritical Moser-Trudinger type inequalities are established through this local to global principle using the ideas of the level sets.

From the works [32, 15, 16], we can see that, in order to get the sharp Adams inequalities in unbounded domains, one needs to strength the restriction on the norm $\|\nabla^m f\|_{\frac{n}{m}} \leq 1$ to $\|(I-\Delta)^{\frac{m}{2}}f\|_{L^{\frac{n}{m}}(\mathbb{R}^n)} \leq 1$. We note that the norm $\|(I-\Delta)^{\frac{m}{2}}\|_{L^{\frac{n}{m}}(\mathbb{R}^n)}$ is equivalent to the Sobolev norm $\|f\|_{W^{m,\frac{n}{m}}(\mathbb{R}^n)} = \sum_{i=1}^m \|\nabla^i f\|_{L^{\frac{n}{m}}(\mathbb{R}^n)}$ and is much larger than $\|\nabla^m f\|_{\frac{n}{m}}$.

Recently, Masmoudi and Sani [27], only imposing the restriction $\|\Delta u\|_2 \leq 1$, have established the following second order Adams' inequality with the exact growth condition in $\mathbb{R}^4$.

Theorem J *There exists a constant $C > 0$, such that*

$$\int_{\mathbb{R}^4}\frac{e^{32\pi^2u^2}-1}{(1+|u|)^2}dx \leq C\|u\|^2_{L^2(\mathbb{R}^2)},$$

for any $u\in W^{2,2}(\mathbb{R}^4)$ with $\|\Delta u\|_{L^2(\mathbb{R}^4)} \leq 1$. Moreover, this fails if the power 2 in the denominator is replaced with any $p < 2$.

Nevertheless, Adams' inequality with the exact growth condition for general n remains open. In this paper, we will give an answer to this question in $\mathbb{R}^n$ with all the dimension $n \geq 3$. Our first main result is the following

Theorem 2.1 *There exists a constant $C(n) > 0$ such that for all $f \in W^{2,\frac{n}{2}}(\mathbb{R}^n)$ $(n \geq 3)$ satisfying $\|\Delta f\|_{\frac{n}{2}} \leq 1$,*

$$\int_{\mathbb{R}^n} \frac{\Phi(\beta_n |f|^{\frac{n}{n-2}})}{(1+|f|)^{\frac{n}{n-2}}} dx \leq C(n) \|f\|_{\frac{n}{2}}^{\frac{n}{2}},$$

where $\Phi(t) = \exp(t) - \sum_{j=0}^{j_{\frac{n}{2}}-2} \frac{t^j}{j!}$, $j_{\frac{n}{2}} = \min\{j \in \mathbb{R} : j \geq \frac{n}{2}\} \geq n/2$ and $\beta_n = \beta(n,2) = \frac{n}{\omega_{n-1}}\left[\frac{\pi^{\frac{n}{2}} 4}{\Gamma(n/2-1)}\right]^{\frac{n}{n-2}}$.

We remark that both the power $\frac{n}{n-2}$ and the constant β_n are optimal. These can be justified by the following theorem.

Theorem 2.2 *If the power $\frac{n}{n-2}$ in the denominator is replaced by any $p < \frac{n}{n-2}$, there exists a sequence of functions $\{f_k\}$ such that $\|\Delta f_k\|_{\frac{n}{2}} \leq 1$, but*

$$\frac{1}{\|f_k\|_{\frac{n}{2}}^{\frac{n}{2}}} \int_{\mathbb{R}^n} \frac{\Phi(\beta_n (|f_k|)^{\frac{n}{n-2}})}{(1+|f_k|)^p} dx \to \infty.$$

Moreover, if $\alpha > \beta_n$, there exists a sequence of function $\{f_k\}$ such that $\|\Delta f_k\|_{\frac{n}{2}} \leq 1$, but

$$\frac{1}{\|f_k\|_{\frac{n}{2}}^{\frac{n}{2}}} \int_{\mathbb{R}^n} \frac{\Phi(\alpha (|f_k|)^{\frac{n}{n-2}})}{(1+|f_k|)^p} dx \to \infty,$$

for any $p \geq 0$.

To prove Theorem 2.1, we need to use Lemma 4.1. Similar to the argument in [27], to prove Lemma 4.1, we will use Talenti's rearrangement for solutions to elliptic equations [33] in $\mathbb{R}^n$ for all $n \geq 3$. To carry out that procedure, one needs Lemma 4.2 for all $p > 1$. The proof of Lemma 4.2 for the case $p = 2$ was given in [27] (corresponding to the case $n = 4$ for the Adams inequality of our Theorem 2.1). However, the extension for such a lemma to all $p > 1$ (corresponding to the case $n \geq 3$ for the Adams inequality of our Theorem 2.1) is highly nontrivial. The proof of Lemma 4.2 in the case $p \geq 2$ (corresponding to the case $n \geq 4$ for the Adams inequality of our Theorem 2.1) was given in [25]. But its proof does not work for $1 < p < 2$ (corresponding to the case $n = 3$ for the Adams inequality of our Theorem 2.1). The proof of Lemma 4.2 for the case $1 < p < 2$ is substantially different from that in the case $p \geq 2$.

As a consequence of Theorem 2.1, we can obtain the following Adachi-Tanaka type inequality in the space $W^{2,\frac{n}{2}}$, which give the best possible exponent for Ozawa's inequality.

Theorem 2.3 *For any $\alpha \in (0, \beta_n)$, there exists a constant $C(\alpha, n) > 0$ such that for all $f \in W^{2,\frac{n}{2}}(\mathbb{R}^n)$ $(n \geq 3)$ satisfying $\|\Delta f\|_{\frac{n}{2}} \leq 1$,*

$$\int_{\mathbb{R}^n} \Phi(\alpha |f|^{\frac{n}{n-2}}) dx \leq C(\alpha, n) \|f\|_{\frac{n}{2}}^{\frac{n}{2}}.$$

Moreover, the constant β_n is sharp in the sense that for any $\alpha \geq \beta_n$, there exists a sequence of function $\{f_k\}$ such that $\|\Delta f_k\|_{\frac{n}{2}} \leq 1$, but

$$\frac{1}{\|f_k\|_{\frac{n}{2}}^{\frac{n}{2}}} \int_{\mathbb{R}^n} \Phi(\alpha(|f_k|)^{\frac{n}{n-1}})dx \to \infty.$$

The organization of the paper is as follows. In Section 3, we introduce some preliminaries about the non-increasing rearrangement . In Section 4, we will establish an important lemma (Lemma 4.1) which plays a key role in the proof of our main result (Theorem 2.1). Section 5 will give Adams' inequality with exact growth condition in $\mathbb{R}^n (n \geq 3)$ (Theorem 2.1). In Section 6, we will prove the sharpness of the Adams inequality with the exact growth condition in Theorem 2.1 (Theorem 2.2). In Section 7, we obtain the best constant for Ozawa's inequality (Theorem 2.3).

3 Some preliminaries

In this section, we will introduce some properties about the rearrangement which will be used in the proof of the main theorem.

Let Ω be a measurable set in $\mathbb{R}^n$. Denote $\Omega^\sharp$ the open ball B_R centered at the origin with radius R such that

$$|\Omega| = |B_R|.$$

Let $f : \Omega \to \mathbb{R}^n$ be a real-valued measurable function in Ω. Then the distribution function of f is defined as

$$\mu_f(t) := |\{x \in \Omega : |f(x)| > t\} \textit{ for } t \geq 0.$$

Then its decreasing rearrangement f^* is defined by

$$f^* := \sup\{t \geq 0 : \mu_f(t) > s\}, \textit{ for } s \geq 0.$$

After that, define $f^\sharp : \Omega^\sharp \to [0, +\infty]$ by

$$f^\sharp = f^*(v_n|x|^n) \textit{ for } x \in \Omega^\sharp,$$

where v_n is the volume of the unit ball in $\mathbb{R}^n$. Then for every continuous increasing function $\Psi : [0, +\infty) \to [0, +\infty)$, we have that

$$\int_\Omega \Psi(f)dx = \int_{\Omega^\sharp} \Psi(f^\sharp)dx.$$

Since f^* is non-increasing, the maximal function f^{**} of the rearrangement of f^*, defined by

$$f^{**} := \frac{1}{s}\int_0^s f^* dt \textit{ for } s \geq 0,$$

is also nonincreasing and $f^* \leq f^{**}$. Moreover we have:

Lemma 3.1 *If $f \in L^p(\mathbb{R}^n)$ with $1 < p < +\infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$, then*

$$(\int_0^{+\infty} [f^{**}]^p ds)^{1/p} \le p'(\int_0^{+\infty} [f^*(s)]^p ds)^{1/p}.$$

In particular, if supp $f \subset \Omega$ with Ω a domain in $\mathbb{R}^n$, then

$$(\int_0^{|\Omega|} [f^{**}]^p ds)^{1/p} \le p'(\int_0^{|\Omega|} [f^*(s)]^p ds)^{1/p}.$$

Finally we recall some well-known inequalities due to G. Talenti [33]. Let $u \in L^{\frac{n}{2}}(\Omega)$ and Ω be a bounded domain in $\mathbb{R}^n$. We consider the following Dirichlet problem:

$$\begin{cases} -\triangle f = u \ in \ \Omega \\ u = 0 \ \ on \quad \partial\Omega. \end{cases} \tag{3.4}$$

Then we have the following lemma which can be found in [27]

Lemma 3.2 *Let $f \in L^{n/2}(\Omega)$ be the unique weak solution to (3.1), then*

$$f^*(t_1) - f^*(t_2) \le \frac{1}{[nv_n^{1/n}]^2} \int_{t_1}^{t_2} \frac{u^{**}(s)}{s^{1-\frac{2}{n}}} ds \, for \ 0 < t_1 \le t_2 \le |\Omega|.$$

4 Two crucial lemmas

In this section, we give two crucial lemmas which play an important role in the proof of Theorem 2.1.

Lemma 4.1 *Let $f \in W^{2,\frac{n}{2}}(\mathbb{R}^n)$, $n \ge 3$. If $f^*(v_n R^n) > 1$ and $u := -\Delta f$ in $\mathbb{R}^n$ satisfies*

$$\int_{v_n R^n}^{+\infty} [u^{**}(s)]^{\frac{n}{2}} ds \le (\frac{n}{n-2})^{\frac{n}{2}},$$

then

$$\frac{\exp[\beta_n f^*(v_n R^n)]^{\frac{n}{n-2}}}{[f^*(v_n R^n)]^{\frac{n}{n-2}}} R^n \le C \int_R^{+\infty} [f^*(v_n r^n)]^{n/2} r^{n-1} dr.$$

As we have pointed out earlier in Section 2, to prove Lemma 4.1, we will use Talenti's rearrangement for solutions to elliptic equations [33] in $\mathbb{R}^n$ for all $n \ge 3$. So the key is to establish Lemma 4.2 below for all $p > 1$ which is really the technical lemma in this paper.

Lemma 4.2 *Given any sequence $a = \{a_k\}_{k\ge 0}$, let $p > 1$, $||a||_1 = \sum_{k=0}^{+\infty} |a_k|$, $||a||_p = (\sum_{k=0}^{+\infty} |a_k|^p)^{1/p}$, $||a||_{(e)} = (\sum_{k=0}^{+\infty} |a_k|^p e^k)^{1/p}$ and $\mu(h) = \inf\{||a||_{(e)} : ||a||_1 = h, ||a||_p \le 1\}$. Then for $h > 1$, we have*

$$\mu(h) \sim \frac{e^{\frac{h^{\frac{p}{p-1}}}{p}}}{h^{\frac{1}{p-1}}}.$$

The proof for Lemma 4.2 in the case $p = 2$ was given in [11]. And in [25], the first and second author have proved the case $p \geq 2$. But the proof of the case $p \geq 2$ does not work for the case $1 < p < 2$. So we will need to give the proof of Lemma 4.2 in the case $1 < p < 2$ in this paper. For the sake of completeness, we include a proof for the case $p \geq 2$ as well.

Proof. Since $\mu(h)$ is increasing in h, it suffices to show that $\mu(N^{1-\frac{1}{p}}) \sim \frac{e^{\frac{N}{p}}}{N^{1/p}}$. Choose $a_k = \frac{1}{N^{1/p}}$ when $k \leq N - 1$ and $a_k = 0$, when $k \geq N$. Obviously,

$$\|a\|_p = 1, \|a\|_1 = N^{1-\frac{1}{p}} \textit{ and } \|a\|_{(e)} \lesssim \frac{e^{\frac{N}{p}}}{N^{1/p}}.$$

So

$$\mu(N^{1-\frac{1}{p}}) \lesssim \frac{e^{\frac{N}{p}}}{N^{1/p}}.$$

Now we only need to prove that $\mu(N^{1-\frac{1}{p}}) \gtrsim \frac{e^{\frac{N}{p}}}{N^{1/p}}$.

Case 1: $p \geq 2$**.**

By contradiction, suppose that for any $\varepsilon \ll 1$ and a sequence a, we have

$$\|a\|_p \leq 1,\ \|a\|_1 = N^{1-\frac{1}{p}},\ \|a\|_{(e)} \leq \varepsilon \frac{e^{\frac{N}{p}}}{N^{1/p}}.$$

From the last condition, we know that when $k \geq N$,

$$|a_k| \lesssim \frac{\varepsilon}{N^{1/p}} e^{\frac{N-k}{p}}.$$

Now set $a'_k = a_k$, for $k \leq N - 1$ and $a'_k = 0$ for $k \geq N$, then

$$\|a'\|_1 = \|a\|_1 - \sum_{k \geq N} |a_k| = N^{1-\frac{1}{p}} - \sum_{k \geq N} |a_k| \geq N^{1-\frac{1}{p}} - \frac{C\varepsilon}{N^{1/p}}. \tag{4.5}$$

Using the fundamental inequality: $(1 - x)^b \geq 1 - bx$, when $b > 1$ and $0 \leq x < 1$, we can obtain

$$\begin{aligned} &\|a'\|_1^{\frac{p}{p-1}} \geq (N^{1-1/p} - \frac{C\varepsilon}{N^{1/p}})^{\frac{p}{p-1}} \\ &= N(1 - \frac{C\varepsilon}{N})^{\frac{p}{p-1}} \\ &\geq N - C\varepsilon. \end{aligned} \tag{4.6}$$

On the other hand,

$$
\begin{aligned}
\|a'\|_1^{\frac{p}{p-1}} &= \Big(N\sum_{0\le j\le N-1}|a_j|^2-\sum_{0\le j,k\le N-1}\frac{(a_j-a_k)^2}{2}\Big)^{\frac{p}{2(p-1)}}\\
&= N^{\frac{p}{2(p-1)}}\Big(\sum_{0\le j\le N-1}|a_j|^2\Big)^{\frac{p}{2(p-1)}}\Big(1-\frac{1}{N}\frac{\sum_{0\le j,k\le N-1}\frac{(a_j-a_k)^2}{2}}{\sum_{0\le j\le N-1}|a_j|^2}\Big)^{\frac{p}{2(p-1)}}\\
&\le N^{\frac{p}{2(p-1)}}\Big(\sum_{0\le j\le N-1}|a_j|^2\Big)^{\frac{p}{2(p-1)}}\Big(1-\frac{p}{2(p-1)}\frac{\sum_{0\le j,k\le N-1}\frac{(a_j-a_k)^2}{2}}{N(\sum_{0\le j\le N-1}|a_j|^2)}\Big)\\
&\le N^{\frac{p}{2(p-1)}}\Big(N^{\frac{p}{2}-1}\Big(\sum_{0\le j\le N-1}|a_j|^p\Big)\Big)^{\frac{1}{p-1}}\Big(1-\frac{p}{2(p-1)}\frac{\sum_{0\le j,k\le N-1}\frac{(a_j-a_k)^2}{2}}{N(\sum_{0\le j\le N-1}|a_j|^2)}\Big)\\
&\le N\Big(1-\frac{p}{2(p-1)}\frac{\sum_{0\le j,k\le N-1}\frac{(a_j-a_k)^2}{2}}{N^{1+1-\frac{2}{p}}}\Big)\\
&= N-\frac{p}{2(p-1)}\frac{\sum_{0\le j,k\le N-1}\frac{(a_j-a_k)^2}{2}}{N^{1-\frac{2}{p}}},
\end{aligned}
\tag{4.7}
$$

where the first inequality uses the fundamental inequality:

$$(1-x)^q\le 1-qx,\ \textit{when}\ 0<q<1\ \textit{and}\ 0\le x<1,$$

and the second inequality uses the fundamental inequality:

$$\Big(\sum_{j=1}^{N}|c_j|\Big)^p\le N^{p-1}\sum_{j=1}^{N}|c_j|^p\ \textit{for}\ p\ge 1.$$

Then, by (4.6) and (4.7),

$$\sum_{0\le j,k\le N-1}\frac{(a_j-a_k)^2}{2}\le C\varepsilon N^{1-\frac{2}{p}}.$$

Now choose $m<N-1$ so that $\min_{0\le j\le N-1}|a_j|=a_m$. Then

$$
\begin{aligned}
&\|a'\|_1-N|a_m|\le\|a_j-a_m\|_{l^1(j\le N-1)}\\
&\le\sqrt{N}\|a_j-a_m\|_{l^2(j\le N-1)}\\
&\le C\varepsilon N^{1-\frac{1}{p}}.
\end{aligned}
$$

If ε is small enough, combining it with (4.5), we get

$$|a_m|\gtrsim\frac{1}{N^{1/p}}.$$

Hence

$$\|a\|_{(e)}\gtrsim\Big(\sum_{j\le N-1}\frac{e^j}{N}\Big)^{\frac{1}{p}}\gtrsim\frac{e^{N/p}}{N^{1/p}},$$

which yields a contradiction. Thus, we have completed the proof of Lemma 4.2 in the case $p \geq 2$.

Case 2: $1 < p < 2$**.**

Once again, we give the proof by contradiction. Suppose that for any $\varepsilon \ll 1$, there exists a sequence $a = \{a_k\}_0^\infty$ such that

$$\|a\|_p = 1, \ \|a\|_1 = N^{1-\frac{1}{p}}, \ \|a\|_{(e)} \leq \varepsilon \frac{e^{\frac{N}{p}}}{N^{1/p}}. \tag{4.8}$$

We know that when $k \geq N$,

$$a_k \leq \varepsilon \frac{e^{\frac{N-k}{p}}}{N^{\frac{1}{p}}},$$

thus,

$$\sum_{k=N}^{\infty} a_k \leq \sum_{k=N}^{\infty} \varepsilon \frac{e^{\frac{N-k}{p}}}{N^{\frac{1}{p}}} = O\left(\frac{\varepsilon}{N^{\frac{1}{p}}}\right),$$
$$\sum_{k=N}^{\infty} a_k^p \leq \sum_{k=N}^{\infty} \varepsilon^p \frac{e^{N-k}}{N} = O\left(\frac{\varepsilon^p}{N}\right).$$

Then we have

$$\sum_{k=0}^{N-1} a_k = N^{1-\frac{1}{p}} - O\left(\frac{\varepsilon}{N^{\frac{1}{p}}}\right),$$
$$\sum_{k=0}^{N-1} a_k^p = 1 - O\left(\frac{\varepsilon^p}{N}\right).$$

The above formulas can be rewritten as:

$$\sum_{k=0}^{N-1} \left(N^{-\frac{1}{p}} - a_k\right) = O\left(\frac{\varepsilon}{N^{\frac{1}{p}}}\right),$$
$$\sum_{k=0}^{N-1} \left(N^{-1} - a_k^p\right) = O\left(\frac{\varepsilon^p}{N}\right),$$

that is

$$\sum_{k=0}^{N-1} \left(1 - \frac{a_k}{N^{-\frac{1}{p}}}\right) = O\left(\varepsilon\right), \tag{4.9}$$

$$\sum_{k=0}^{N-1} \left(1 - \left(\frac{a_k}{N^{-\frac{1}{p}}}\right)^p\right) = O\left(\varepsilon^p\right). \tag{4.10}$$

Now, we estimate the following quantity

$$A_k(\varepsilon) = \sum_{k=0}^{N-1}\left(1-\left(\frac{a_k}{N^{-\frac{1}{p}}}\right)^2\right).$$

Let $1-\frac{a_k}{N^{-\frac{1}{p}}} = \varepsilon_k$, we rewrite (4.9),(4.10) as

$$\sum_{k=0}^{N-1}\varepsilon_k = O(\varepsilon), \tag{4.11}$$

$$\sum_{k=0}^{N-1}(1-(1-\varepsilon_k)^p) = O(\varepsilon^p) \tag{4.12}$$

and $A_k(\varepsilon)$ as

$$\left|\sum_{k=0}^{N-1}\varepsilon_k^2 - 2\sum_{k=0}^{N-1}\varepsilon_k\right|. \tag{4.13}$$

Using Jensen's inequality, we obtain

$$N - O(\varepsilon^p) = \sum_{k=0}^{N-1}(1-\varepsilon_k)^p \geq N\left(\frac{N-\sum\limits_{k=0}^{N-1}\varepsilon_k}{N}\right)^p = N\left(1-\frac{O(\varepsilon)}{N}\right)^p = N - O(\varepsilon),$$

which assures that $\varepsilon_k \to 0$, as $\varepsilon \to 0$. Then, by the Taylor expansion, we have

$$\begin{aligned}\sum_{k=0}^{N-1}(1-(1-\varepsilon_k)^p) &= \sum_{k=0}^{N-1}\left(1-\left(1-p\varepsilon_k+\frac{p(p-1)}{2}\varepsilon_k^2+o\left(\varepsilon_k^2\right)\right)\right)\\ &= \sum_{k=0}^{N-1}\left(p\varepsilon_k-\frac{p(p-1)}{2}\varepsilon_k^2+o\left(\varepsilon_k^2\right)\right)\\ &= p\sum_{k=0}^{N-1}\varepsilon_k-\frac{p(p-1)}{2}(1+o(1))\sum_{k=0}^{N-1}\varepsilon_k^2.\end{aligned}$$

It follows from (4.11) and (4.12) that

$$\begin{aligned}\sum_{k=0}^{N-1}\varepsilon_k^2 &\leq \frac{2(1+o(1))}{p(p-1)}\left(p\sum_{k=0}^{N-1}\varepsilon_k-\sum_{k=0}^{N-1}(1-(1-\varepsilon_k)^p)\right)\\ &= O(\varepsilon).\end{aligned} \tag{4.14}$$

Combining (4.11), (4.13) and (4.14), we have

$$\sum_{k=0}^{N-1}\left(1-\left(\frac{a_k}{N^{-\frac{1}{p}}}\right)^2\right) = O(\varepsilon).$$

From this, we have

$$\sum_{k=0}^{N-1} a_k^2 = N^{1-\frac{2}{p}} + O\left(\frac{\varepsilon}{N^{\frac{2}{p}}}\right). \tag{4.15}$$

Now, let's come back to (4.7). We have by (4.15) that

$$\begin{aligned}
\|a'\|_1^{\frac{p}{p-1}} &\le N^{\frac{p}{2(p-1)}}\Big(\sum_{0\le j\le N-1}|a_j|^2\Big)^{\frac{p}{2(p-1)}}\Big(1-\frac{\sum_{0\le j,k\le N-1}\frac{(a_j-a_k)^2}{2}}{N(\sum_{0\le j\le N-1}|a_j|^2)}\Big)\\
&\le N^{\frac{p}{2(p-1)}}\left(N^{1-\frac{2}{p}}+O\left(\frac{\varepsilon}{N^{\frac{2}{p}}}\right)\right)^{\frac{p}{2(p-1)}}\Big(1-\frac{\sum_{0\le j,k\le N-1}\frac{(a_j-a_k)^2}{2}}{N(\sum_{0\le j\le N-1}|a_j|^2)}\Big)\\
&\le N^{\frac{p}{2(p-1)}}\left(N^{1-\frac{2}{p}}\left(1+\frac{O(\varepsilon)}{N}\right)\right)^{\frac{p}{2(p-1)}}\Big(1-\frac{\sum_{0\le j,k\le N-1}\frac{(a_j-a_k)^2}{2}}{N(\sum_{0\le j\le N-1}|a_j|^2)}\Big)\\
&\le N^{\frac{p}{2(p-1)}}\left(N^{\frac{p-2}{2(p-1)}}\right)\Big(1-\frac{\sum_{0\le j,k\le N-1}\frac{(a_j-a_k)^2}{2}}{N(\sum_{0\le j\le N-1}|a_j|^2)}\Big)\left(1+\frac{O(\varepsilon)}{N}\right)^{\frac{p}{2(p-1)}}\\
&\le N\Big(1-\frac{\sum_{0\le j,k\le N-1}\frac{(a_j-a_k)^2}{2}}{N^{1+1-\frac{2}{p}}}\Big)\left(1+\frac{O(\varepsilon)}{N}\right)\\
&= N-\frac{\sum_{0\le j,k\le N-1}\frac{(a_j-a_k)^2}{2}}{N^{1-\frac{2}{p}}}\left(1+\frac{O(\varepsilon)}{N}\right)+O(\varepsilon).
\end{aligned}$$

Also, we can obtain (4.6), and then

$$N-\frac{\sum_{0\le j,k\le N-1}\frac{(a_j-a_k)^2}{2}}{N^{1-\frac{2}{p}}}\left(1+\frac{O(\varepsilon)}{N}\right)+O(\varepsilon) > N-C\varepsilon,$$

thus,

$$\sum_{0\le j,k\le N-1}\frac{(a_j-a_k)^2}{2} \le O(\varepsilon)\,N^{1-\frac{2}{p}}.$$

Now we choose $m < N-1$ so that $\min_{0\le j\le N-1}|a_j| = a_m$. Then

$$\begin{aligned}
\|a'\|_1 - N|a_m| &\le \|a_j-a_m\|_{l^1(j\le N-1)}\\
&\le \sqrt{N}\|a_j-a_m\|_{l^2(j\le N-1)}\\
&\le O(\varepsilon)\,N^{1-\frac{1}{p}}.
\end{aligned}$$

If ε is small enough, combining it with

$$\|a'\|_1 \ge N^{1-\frac{1}{p}} - \frac{C\varepsilon}{N^{1/p}},$$

we get

$$|a_m| \gtrsim \frac{1}{N^{1/p}}.$$

Hence

$$\|a\|_{(e)} \gtrsim (\sum_{j\le N-1} \frac{e^j}{N})^{\frac{1}{p}} \gtrsim \frac{e^{N/p}}{N^{1/p}}.$$

Thus, we have completed the proof of Lemma 4.2.

Once Lemma 4.2 is established, the proof of Lemma 4.1 for all $n \ge 3$ follows. Its proof uses a similar idea to that in [27] in the case of $n = 4$. Now, we are ready to give the proof of Lemma 4.1.

Proof. Let $h_k = c_n f^*(v_n R^n e^{\frac{k}{n}})$, where $c_n = [nv_n^{1/n}]^2 n^{1-\frac{2}{n}} \frac{n-2}{n}$. Define $a_k = h_k - h_{k+1}$ and $a = \{a_k\}$. Then $a_k \ge 0$ and

$$\sum_{k\ge 0} |a_k| = h_0 = c_n f^*(v_n R^n).$$

By Lemma 3.2 and Hölder's inequality

$$\begin{aligned} h_k - h_{k+1} &= c_n[f^*(v_n R^n e^{\frac{k}{n}}) - f^*(v_n R^n e^{\frac{k+1}{n}})] \\ &\le \frac{c_n}{[v_n^{1/n} n]^2}[\int_{v_n R^n e^{\frac{k}{n}}}^{v_n R^n e^{\frac{k+1}{n}}} |u^{**}(s)|^{\frac{n}{2}} ds]^{\frac{2}{n}} (\frac{1}{n})^{1-\frac{2}{n}}, \end{aligned}$$

then

$$\begin{aligned} \|a\|_{\frac{n}{2}} &= (\sum_{k\ge 1} |a_k|^{n/2})^{\frac{2}{n}} \\ &= (\sum_{k\ge 1} |h_k - h_{k+1}|^{n/2})^{\frac{2}{n}} \\ &\le \frac{n-2}{n}(\int_0^{+\infty} |u^{**}(s)|^{n/2} ds)^{2/n} \\ &\le 1. \end{aligned}$$

On the other hand,

$$\begin{aligned} &\frac{\int_R^{+\infty} |f^*(v_n r^n)|^{n/2} r^{n-1} dr}{R^n} \\ \ge\ & \sum_{k\ge 0} \frac{[f^*(v_n R^n e^{k+1})]^{n/2}}{R^n} \int_{Re^{\frac{k}{n}}}^{Re^{\frac{k+1}{n}}} r^{n-1} dr \\ \gtrsim\ & \sum_{k\ge 0} [f^*(v_n R^n e^{k+1})]^{n/2} e^{k+1} \\ \gtrsim\ & \sum_{k\ge 1} (h_k)^{n/2} e^k \ge \sum_{k\ge 1} (a_k)^{n/2} e^k. \end{aligned}$$

Therefore

$$\|a\|_{(e)}^{n/2} = a_0^{n/2} + \sum_{k\ge 1} a_k^{n/2} e^k \lesssim h_0^{n/2} + \frac{\int_R^{+\infty} |f^*(v_n r^n)|^{n/2} r^{n-1} dr}{R^n}. \tag{4.16}$$

Next, let's estimate h_0. Set $R < r < Re^{b/n}$, where $b = [\frac{[nv_n^{1/n}]^2}{2}(\frac{n}{n-2})^{-\frac{2}{n}}]^{\frac{n}{n-2}}$. Since

$$\begin{aligned} h_0 - c_n f^*(v_n r^n) &\le \frac{c_n}{[nv_n^{1/n}]^2}\int_{R^n v_n}^{r^n v_n} \frac{u^{**}(s)}{s^{1-\frac{2}{n}}}ds \\ \le \quad & \frac{c_n}{[nv_n^{1/n}]^2}[\int_{R^n v_n}^{r^n v_n}(u^{**}(s))^{n/2}ds]^{2/n} b^{1-\frac{2}{n}} \\ \le \quad & \frac{c_n}{2} \le \frac{c_n f^*(v_n R^n)}{2} = \frac{h_0}{2}. \end{aligned}$$

Then, when $R < r < Re^{b/n}$, $h_0 \lesssim f^*(v_n r^n)$. So

$$\frac{\int_R^{+\infty} |f^*(v_n r^n)|^{n/2} r^{n-1} dr}{R^n} \gtrsim \frac{\int_R^{Re^{b/n}} h_0^{n/2} r^{n-1} dr}{R^n} \gtrsim h_0^{n/2}. \tag{4.17}$$

Combining inequalities (4.16) and (4.17), we have

$$\|a\|_{(e)}^{n/2} \lesssim \frac{\int_R^{+\infty} |f^*(v_n r^n)|^{n/2} r^{n-1} dr}{R^n}.$$

By Lemma 4.2 and $\frac{(c_n)^{\frac{n}{n-2}}}{n} = \beta$

$$\|a\|_{(e)}^{n/2} \gtrsim \frac{\exp[\frac{(h_0)^{\frac{n}{n-2}}}{n}]}{(h_0)^{\frac{n}{n-2}}} \gtrsim \frac{\exp[\frac{(c_n f^*(v_n R^n))^{\frac{n}{n-2}}}{n}]}{(f^*(v_n R^n))^{\frac{n}{n-2}}} = \frac{\exp[\beta(f^*(v_n R^n))^{\frac{n}{n-2}}]}{(f^*(v_n R^n))^{\frac{n}{n-2}}}.$$

This completes the proof of Lemma 4.1.

5 Proof of Theorem 2.1

Using the density argument, we only need to prove the desired inequality for all $f \in C_0^\infty(\mathbb{R}^n)$. By the property of the rearrangement, we have

$$\int_{\mathbb{R}^n} \frac{\Phi(\beta_n |f|^{\frac{n}{n-2}})}{(1+|f|)^{\frac{n}{n-2}}} dx = \int_{\mathbb{R}^n} \frac{\Phi(\beta_n |f^\sharp|^{\frac{n}{n-2}})}{(1+|f^\sharp|)^{\frac{n}{n-2}}} dx,$$

and

$$\|f\|_{\frac{n}{2}}^{\frac{n}{2}} = \|f^\sharp\|_{\frac{n}{2}}^{\frac{n}{2}}.$$

Thus, it suffices to prove that

$$\int_{\mathbb{R}^n} \frac{\Phi(\beta_n |f^\sharp|^{\frac{n}{n-2}})}{(1+|f^\sharp|)^{\frac{n}{n-2}}} dx \le C\|f^\sharp\|_{\frac{n}{2}}^{\frac{n}{2}}.$$

Now, we split the integral into two parts as done in [27]

$$\int_{\mathbb{R}^n} \frac{\Phi(\beta_n |f^\sharp|^{\frac{n}{n-2}})}{(1+|f^\sharp|)^{\frac{n}{n-2}}} dx = \int_{B_{R_0}} \frac{\Phi(\beta_n |f^\sharp|^{\frac{n}{n-2}})}{(1+|f^\sharp|)^{\frac{n}{n-2}}} dx + \int_{\mathbb{R}^n \setminus B_{R_0}} \frac{\Phi(\beta_n |f^\sharp|^{\frac{n}{n-2}})}{(1+|f^\sharp|)^{\frac{n}{n-2}}} dx,$$

where $R_0 = \inf\{r \geq 0 : f^*(v_n r^n) \leq 1\} \in [0, +\infty)$. Then when $r \leq R_0$, $f^*(v_n r^n) > 1$, $f^*(v_n R_0^n) = 1$ and $f^*(v_n r^n) \geq 1$ for $r \leq R_0$.

Since when $0 < x \leq 1$, $\Phi(\beta_n x^{\frac{n}{n-2}}) \leq C_n x^{\frac{n}{2}}$, then

$$\int_{\mathbb{R}^n \setminus B_{R_0}} \frac{\Phi(\beta_n |f^\sharp|^{\frac{n}{n-2}})}{(1 + |f^\sharp|)^{\frac{n}{n-2}}} dx \leq C \int_{\mathbb{R}^n \setminus B_{R_0}} |f^\sharp|^{\frac{n}{2}} dx \leq \|f^\sharp\|_{\frac{n}{2}}^{\frac{n}{2}}. \tag{5.18}$$

Next, we consider the integral on B_{R_0}. Set $u = -\Delta f$ in $\mathbb{R}^n$ and $\alpha = \int_0^{+\infty} [u^{**}(s)]^{n/2} ds$, then by Lemma 3.1

$$\alpha \leq (\int_0^{+\infty} [u^*(s)]^{n/2} ds)(\frac{1}{1 - n/2})^{n/2} \leq \|\Delta f\|_{\frac{n}{2}}^{\frac{n}{2}} (\frac{n}{n-2})^{\frac{n}{2}} \leq (\frac{n}{n-2})^{\frac{n}{2}}.$$

Fix $0 < \varepsilon_0 < 1$ and define R_1 be such that

$$\int_0^{v_n R_1^n} [u^{**}(s)]^{n/2} ds \leq \alpha\varepsilon_0 \ \textit{and} \ \int_{v_n R_1^n}^{+\infty} [u^{**}(s)]^{n/2} ds \leq \alpha(1 - \varepsilon_0).$$

Applying Lemma 3.2 and Hölder's inequality, we have

$$f^*(t_1) - f^*(t_2) \leq \frac{1}{[n v_n^{1/n}]^2} (\int_{t_1}^{t_2} [u^{**}(s)]^{2/n} ds)^{n/2} (\ln \frac{t_2}{t_1})^{1 - \frac{2}{n}}.$$

Then

$$f^*(v_n r_1^n) - f^*(v_n r_2^n) \leq \frac{1}{[n v_n^{1/n}]^2} (\alpha\varepsilon_0)^{\frac{2}{n}} (\ln \frac{r_2}{r_1})^{1 - \frac{2}{n}}, \ \textit{when}\ 0 < r_1 \leq r_2 \leq R_1, \tag{5.19}$$

$$f^*(v_n r_1^n) - f^*(v_n r_2^n) \leq \frac{1}{[n v_n^{1/n}]^2} (\alpha(1 - \varepsilon_0))^{\frac{2}{n}} (\ln \frac{r_2}{r_1})^{1 - \frac{2}{n}}, \ \textit{when}\ R_1 \leq r_1 \leq r_2. \tag{5.20}$$

In order to estimate the integral B_{R_0}, we need to consider two cases: $R_1 \geq R_0$ and $R_1 < R_0$. First, we consider the case $R_1 \geq R_0$. By inequality (5.19), we have when $0 < r \leq R_0$,

$$f^*(v_n r^n) \leq 1 + \frac{1}{[n v_n^{\frac{1}{n}}]^2} (\alpha\varepsilon_0)^{\frac{2}{n}} (\ln (\frac{R_0}{r})^n)^{1 - \frac{2}{n}}.$$

It's well known that there exists a constant $C_\varepsilon = (1 - (1 + \varepsilon)^{-\frac{n-2}{2}})^{\frac{-2}{n-2}}$ such that

$$(1 + s^{\frac{n-2}{n}})^{\frac{n}{n-2}} \leq (1 + \varepsilon)s + C_\varepsilon, \ \textit{for}\ s > 0.$$

Thus

$$[f^*(v_n r^n)]^{\frac{n}{n-2}} \leq (1 + \varepsilon) \frac{(\alpha\varepsilon_0)^{\frac{2}{n-2}}}{[n v_n^{1/n}]^{\frac{2n}{n-2}}} \ln (\frac{R_0}{r})^n + C_\varepsilon. \tag{5.21}$$

Set $\varepsilon = 1 - \varepsilon_0^{\frac{n}{n-2}}$, then $(1+\varepsilon)\varepsilon_0^{\frac{n}{n-2}} < 1$. Since $\frac{\beta_n}{[nv_n^{1/n}]^{\frac{2n}{n-2}}} = (1-\frac{2}{n})^{\frac{n}{n-2}}$ and $\alpha \leq (\frac{n}{n-2})^{\frac{n}{2}}$, then

$$
\begin{aligned}
&\int_{B_{R_0}} \frac{\Phi(\beta_n|f^\sharp|^{\frac{n}{n-2}})}{(1+|f^\sharp|)^{\frac{n}{n-2}}}dx \leq \int_{B_{R_0}} \exp(\beta_n|f^\sharp|^{\frac{n}{n-2}})dx \\
=\ & \omega_{n-1}\int_0^{R_0} \exp(\beta_n[f^*(v_n r^n)]^{\frac{n}{n-2}})r^{n-1}dr \\
\leq\ & C\int_0^{R_0} \exp(\beta_n((1+\varepsilon)\frac{(\alpha\varepsilon_0)^{\frac{2}{n-2}}}{[nv_n^{1/n}]^{\frac{2n}{n-2}}}\ln(\frac{R_0}{r})^n + C_\varepsilon)r^{n-1}dr \\
=\ & C\int_0^{R_0} R_0^{n(1+\varepsilon)\varepsilon_0^{\frac{2}{n-2}}} r^{n-1-n(1+\varepsilon)\varepsilon_0^{\frac{2}{n-2}}}dr \\
=\ & CR_0^n \leq C\int_0^{R_0} f^*(v_n r^n)r^{n-1}dr \\
\leq\ & C\|f^\sharp\|_{\frac{n}{2}}^{\frac{n}{2}}.
\end{aligned}
\tag{5.22}
$$

Then by inequalities (5.18) and (5.22), we get the desired inequality when $R_1 \geq R_0$.

Next, let's consider the case $R_1 < R_0$. We split

$$
\int_{B_{R_0}} \frac{\Phi(\beta|f^\sharp|^{\frac{n}{n-2}})}{(1+|f^\sharp|)^{\frac{n}{n-2}}}dx = \int_{B_{R_0}\setminus B_{R_1}} \frac{\Phi(\beta_n|f^\sharp|^{\frac{n}{n-2}})}{(1+|f^\sharp|)^{\frac{n}{n-2}}}dx + \int_{B_{R_1}} \frac{\Phi(\beta_n|f^\sharp|^{\frac{n}{n-2}})}{(1+|f^\sharp|)^{\frac{n}{n-2}}}dx.
$$

The estimate of the integral on $B_{R_0}\setminus B_{R_1}$ is much easier. In fact, by inequality (5.20),

$$
f^*(v_n r^n) \leq 1 + \frac{(1-\varepsilon_0)^{\frac{2}{n}}\alpha^{\frac{2}{n}}}{[nv_n^{1/n}]^2}[\ln(\frac{R_0}{r})^n]^{1-\frac{2}{n}},\ \textit{when } R_1 < r < R_0.
$$

Set $\varepsilon_1 = 1-(1-\varepsilon_0)^{\frac{2}{n-2}}$, then $(1+\varepsilon_1)(1-\varepsilon_0)^{\frac{2}{n-2}} < 1$. By inequality (5.21), we have

$$
[f^*(v_n r^n)]^{\frac{n}{n-2}} \leq (1+\varepsilon_1)\frac{[\alpha(1-\varepsilon_0)]^{\frac{2}{n-2}}}{[nv_n^{1/n}]^{\frac{2n}{n-2}}}\ln(\frac{R_0}{r})^n + C_\varepsilon.
$$

Thus

$$
\begin{aligned}
&\int_{B_{R_0}\setminus B_{R_1}} \frac{\Phi(\beta_n|f^\sharp|^{\frac{n}{n-2}})}{(1+|f^\sharp|)^{\frac{n}{n-2}}}dx \leq \int_{B_{R_0}\setminus B_{R_1}} \exp(\beta_n|f^\sharp|^{\frac{n}{n-2}})dx \\
\leq\ & C\int_{R_1}^{R_0} (\frac{R_0}{r})^{n(1+\varepsilon_1)(1-\varepsilon_0)^{\frac{2}{n-2}}} r^{n-1}dr \\
\leq\ & CR_0^n \leq C\|f^\sharp\|_{n/2}^{n/2}.
\end{aligned}
$$

So we only need to consider the integral on B_{R_1}. Since when $0 < r < R_1$,

$$
[f^*(v_n r^n)]^{\frac{n}{n-2}} \leq (1+\varepsilon_2)[f^*(v_n r^n) - f^*(v_n R_1^n)]^{\frac{n}{n-2}} + C_{\varepsilon_2}[f^*(v_n R_1^n)]^{\frac{n}{n-2}},
$$

then

$$\begin{aligned}
&\int_{B_{R_1}} \frac{\Phi(\beta_n|f^\sharp|^{\frac{n}{n-2}})}{(1+|f^\sharp|)^{\frac{n}{n-2}}}dx \\
= \quad & \omega_{n-1}\int_0^{R_1} \frac{\Phi(\beta_n[f^*(v_n r^n)]^{\frac{n}{n-2}})}{[1+f^*(v_n r^n)]^{\frac{n}{n-2}}} r^{n-1}dr \\
\leq \quad & \frac{\omega_{n-1}}{[f^*(v_n r^n)]^{\frac{n}{n-2}}}\int_0^{R_1} \exp(\beta_n[f^*(v_n r^n)]^{\frac{n}{n-2}})r^{n-1}dr \\
\leq \quad & \frac{\omega_{n-1}\exp(C_{\varepsilon_2}\beta_n[f^*(v_n R_1^n)]^{\frac{n}{n-2}})}{[f^*(v_n R_1^n)]^{\frac{n}{n-2}}} \\
\times \quad & \int_0^{R_1} \exp(\beta_n(1+\varepsilon_2)[f^*(v_n r^n) - f^*(v_n R_1^n)]^{\frac{n}{n-2}})r^{n-1}dr.
\end{aligned}$$

Since

$$\frac{\beta_n}{[nv_n^{1/n}]^{\frac{2n}{n-2}}} = (1-\frac{2}{n})^{\frac{n}{n-2}}$$

and

$$0 < f^*(v_n r^n) - f^*(v_n R_1^n) \leq \frac{1}{[nv_n^{1/n}]^2}\int_{v_n r^n}^{v_n R^n} \frac{u^{**}(s)}{s^{1-\frac{2}{n}}}ds,$$

so

$$\begin{aligned}
&\int_{B_{R_1}} \frac{\Phi(\beta_n|f^\sharp|^{\frac{n}{n-2}})}{(1+|f^\sharp|)^{\frac{n}{n-2}}}dx \\
\leq \quad & \frac{\omega_{n-1}\exp(C_{\varepsilon_2}\beta_n[f^*(v_n R_1^n)]^{\frac{n}{n-2}})}{[f^*(v_n R_1^n)]^{\frac{n}{n-2}}} \\
\times \quad & \int_0^{R_1} \exp[(1+\varepsilon_2)^{\frac{n-2}{n}}(1-\frac{2}{n})\int_{v_n r^n}^{v_n R^n} \frac{u^{**}(s)}{s^{1-\frac{2}{n}}}ds]^{\frac{n}{n-2}} r^{n-1}dr \\
= \quad & R_1^n \frac{\omega_{n-1}\exp(C_{\varepsilon_2}\beta_n[f^*(v_n R_1^n)]^{\frac{n}{n-2}})}{n[f^*(v_n R_1^n)]^{\frac{n}{n-2}}} \\
\times \quad & \int_0^{+\infty} \exp[(1+\varepsilon_2)^{\frac{n-2}{n}}(1-\frac{2}{n})\int_{v_n R_1^n e^{-t}}^{v_n R_1^n} \frac{u^{**}(s)}{s^{1-\frac{2}{n}}}ds]^{\frac{n}{n-2}} e^{-t}dt, \qquad (5.23)
\end{aligned}$$

where the last equation we make the change of variable $r = R_1 e^{-\frac{t}{n}}$.

Now set

$$\phi(t) = v_n^{\frac{2}{n}} R_1^2 (1+\varepsilon_2)^{\frac{n-2}{n}}(1-\frac{2}{n})u^{**}(v_n R_1^n e^{-t})e^{-\frac{2}{n}t}.$$

Then

$$
\begin{aligned}
& \int_{-\infty}^{+\infty} [\phi(t)]^{\frac{n}{2}} dt \\
= \quad & \int_{0}^{+\infty} v_n R_1^n (1+\varepsilon_2)^{\frac{n-2}{n}} (1-\frac{2}{n})^{\frac{n}{2}} [u^{**}(v_n R_1^n e^{-t})] e^{-t} dt \\
= \quad & (1+\varepsilon_2)^{\frac{n-2}{2}} (1-\frac{2}{n})^{\frac{n}{2}} \int_0^{v_n R_1^n} [u^{**}(s)]^{n/2} ds \\
\leq \quad & \alpha\varepsilon_0 (1+\varepsilon_2)^{\frac{n-2}{2}} (1-\frac{2}{n})^{n/2} \leq 1,
\end{aligned}
$$

provided that $\varepsilon_0(1+\varepsilon_2)^{\frac{n-2}{2}} = 1$. In particular, choosing $\varepsilon_2 = (\frac{1}{\varepsilon_0})^{\frac{2}{n-2}} - 1$, we have that

$$\int_{-\infty}^{+\infty} [\phi(t)]^{\frac{n}{2}} dt \leq 1.$$

Choose $a(s,t) = \chi_{\{0,t\}}(s)$, we need the following lemma established by Adams [2]

Lemma 5.1 *Let $a(s,t)$ be a nonnegative measurable function on $(-\infty,+\infty)\times[0,+\infty)$ such that (a.e)*

$$a(s,t) \leq 1, \ for\ 0 < s < t,$$

$$\sup_{t>0}(\int_{-\infty}^{0} + \int_t^{\infty} a(s,t)^{p'} ds)^{1/p'} = b < \infty.$$

If there is a constant $c_0 = c_0(p,b)$ such that if for $\phi \geq 0$,

$$\int_{-\infty}^{+\infty} \phi(s)^p ds \leq 1,$$

then

$$\int_0^{+\infty} e^{-F(t)} dt \leq c_0,$$

where

$$F(t) = t - (\int_{\infty}^{+\infty} a(s,t)\phi(s) ds)^{p'}.$$

So by the lemma, we have

$$\int_0^{\infty} \exp(\int_0^t \phi(s) ds)^{\frac{n}{n-2}} e^{-t} dt \leq c_0.$$

Since

$$
\begin{aligned}
& \int_0^{\infty} \exp(\int_0^t \phi(s) ds)^{\frac{n}{n-2}} e^{-t} dt \\
= \quad & \int_0^{\infty} \exp[\int_0^t v_n^{2/n} R_1^2 (1+\varepsilon_2)^{\frac{n-2}{n}} (1-\frac{2}{n}) u^{**}(v_n R_1^n e^{-s}) e^{-\frac{2}{n}s} ds]^{\frac{n}{n-2}} e^{-t} dt \\
= \quad & \int_0^{+\infty} \exp[(1+\varepsilon_2)^{\frac{n-2}{n}} (1-\frac{2}{n}) \int_{v_n R_1^n e^{-t}}^{v_n R_1^n} \frac{u^{**}(r)}{r^{1-\frac{2}{n}}} dr]^{\frac{n}{n-2}} e^{-t} dt,
\end{aligned}
$$

where we make the change of variable $r = v_n R_1^n e^{-s}$. So

$$\int_0^{+\infty} \exp[(1+\varepsilon_2)^{\frac{n-2}{n}}(1-\frac{2}{n})\int_{v_n R_1^n e^{-t}}^{v_n R_1^n} \frac{u^{**}(r)}{r^{1-\frac{2}{n}}} dr]^{\frac{n}{n-2}} e^{-t} dt \le C. \tag{5.24}$$

Since $\varepsilon_2 = (\frac{1}{\varepsilon_0})^{\frac{2}{n-2}} - 1$ then $C_{\varepsilon_2} = (1-\varepsilon_0)^{-\frac{2}{n-2}}$. Thus by inequalities (5.23) and (5.24), we obtain

$$\begin{aligned} & \int_{B_{R_1}} \frac{\Phi(\beta_n |f^\sharp|^{\frac{n}{n-2}})}{(1+|f^\sharp|)^{\frac{n}{n-2}}} dx \\ \le \quad & C R_1^n \frac{\omega_{n-1}\exp(C_{\varepsilon_2}\beta_n [f^*(v_n R_1^n)]^{\frac{n}{n-2}})}{n[f^*(v_n R_1^n)]^{\frac{n}{n-2}}} \\ = \quad & C R_1^n \frac{\exp((1-\varepsilon_0)^{-\frac{2}{n-2}}\beta_n [f^*(v_n R_1^n)]^{\frac{n}{n-2}})}{[f^*(v_n R_1^n)]^{\frac{n}{n-2}}}. \end{aligned}$$

Since $\int_{v_n R_1^n}^{+\infty} [u^{**}(s)]^{n/2} ds \le (\frac{n}{n-2})^{n/2}(1-\varepsilon_0)$, by scaling then using Lemma 4.1, we have

$$R_1^n \frac{\exp((1-\varepsilon_0)^{-\frac{2}{n-2}}\beta_n [f^*(v_n R_1^n)]^{\frac{n}{n-2}})}{[f^*(v_n R_1^n)]^{\frac{n}{n-2}}} \le C \frac{\int_{R_1}^{+\infty} [f^*(v_n R_1^n)]^{n/2} r^{n-1} dr}{(1-\varepsilon_0)^{\frac{n}{n-2}}}.$$

Therefore

$$\int_{B_{R_1}} \frac{\Phi(\beta_n |f^\sharp|^{\frac{n}{n-2}})}{(1+|f^\sharp|)^{\frac{n}{n-2}}} dx \le C \|f^\sharp\|_{\frac{n}{2}}^{\frac{n}{2}},$$

that completes the proof of the theorem.

6 Sharpness of Theorem 2.1: Proof of Theorem 2.2

In this section, we will establish Theorem 2.2. Namely, we will give the proof of the sharpness of Theorem 2.1. First, we will show that the inequality in Theorem 2.1 does not hold if the power $\frac{n}{n-2}$ in the denominator is replaced by any $p < \frac{n}{n-2}$.

For fixed $p < \frac{n}{n-2}$. We choose $\{f_k\}_{k=1}^{\infty}$ as follows:

$$f_k(x) = \begin{cases} [\frac{1}{\beta}\ln\frac{1}{R_k}]^{1-\frac{2}{n}} - \frac{|x|^2}{(R_k \ln\frac{1}{R_k})^{2/n}} + \frac{1}{(R_k \ln\frac{1}{R_k})^{2/n}}, & \text{if } |x| \le R_k^{\frac{1}{n}}, \\ n\beta^{2/n-1}[\ln\frac{1}{R_k}]^{-\frac{2}{n}} \ln\frac{1}{|x|}, & \text{if } R_k^{1/n} \le |x| < 1, \\ 0, & \text{if } 1 < |x|, \end{cases}$$

where $R_k \ge 0$, $R_k \to 0$, and $\beta = \beta(n,2) = \frac{n}{\omega_{n-1}}[\frac{\pi^{\frac{n}{2}} 4}{\Gamma(n/2-1)}]^{\frac{n}{n-2}}$. The choice of this sequence is inspired by a similar sequence in [26] in dimension four case. Then by calculation, we have

$$\|f_k\|_{\frac{n}{2}}^{\frac{n}{2}} = O(\frac{1}{\ln\frac{1}{R_k}}),$$

and

$$1 \le \|\Delta f_k\|_{\frac{n}{2}}^{\frac{n}{2}} \le 1 + O\Big(\frac{1}{\ln\frac{1}{R_k}}\Big).$$

Now let $\tilde{f}_k = \frac{f_k}{\|\Delta f_k\|_{\frac{n}{2}}}$, then

$$\|\tilde{f}_k\|_{\frac{n}{2}}^{\frac{n}{2}} = O\Big(\frac{1}{\ln\frac{1}{R_k}}\Big)$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^n} \frac{\Phi(\beta|\tilde{f}_k|^{\frac{n}{n-2}})}{(1+|\tilde{f}_k|)^p}dx \\
\ge\ & \int_{|x|\le R_k^{\frac{1}{n}}} \frac{\Phi(\beta|\tilde{f}_k|^{\frac{n}{n-2}})}{(1+|\tilde{f}_k|)^p}dx \\
\gtrsim\ & \int_{|x|\le R_k^{\frac{1}{n}}} \frac{\exp(\beta|\tilde{f}_k|^{\frac{n}{n-2}})}{(|\tilde{f}_k|)^p}dx \\
\gtrsim\ & \exp\Big[\Big(\frac{1}{\|\Delta f_k\|_{\frac{n}{2}}^{\frac{n}{2}}} - 1\Big)\ln\frac{1}{R_k}\Big]\Big(\ln\frac{1}{R_k}\Big)^{-(1-\frac{2}{n})p} \\
\gtrsim\ & \Big(\ln\frac{1}{R_k}\Big)^{-(1-\frac{2}{n})p}.
\end{aligned}$$

So

$$\frac{1}{\|\tilde{f}_k\|_{\frac{n}{2}}^{\frac{n}{2}}}\int_{\mathbb{R}^n} \frac{\Phi(\beta|\tilde{f}_k|^{\frac{n}{n-2}})}{(1+|\tilde{f}_k|)^p}dx \gtrsim \Big(\ln\frac{1}{R_k}\Big)^{1-(1-\frac{2}{n})p} \to +\infty,$$

as $k \to +\infty$, since $p < \frac{n}{n-2}$. This explains why the power $\frac{n}{n-2}$ in the denominator can not be replaced by any $p < \frac{n}{n-2}$.

Next, let's show that the constant β_n is optimal. We only need to find a sequence of function $\{f_k\}$ such that $\|\Delta f_k\|_{\frac{n}{2}} \le 1$, but for any $p \ge 0$ and $\alpha > \beta_n$,

$$\frac{1}{\|f_k\|_{\frac{n}{2}}^{\frac{n}{2}}}\int_{\mathbb{R}^n} \frac{\Phi(\alpha(|f_k|)^{\frac{n}{n-2}})}{(1+|f_k|)^p}dx \to \infty.$$

In fact, we can choose the same sequence of function $\{\tilde{f}_k\}$ as we did above. With similar computations, we can prove that the constant β is optimal.

7 Proof of Theorem 2.3

In this section, we will prove Theorem 2.3. Namely, we will establish the optimal version of Theorem F by Ozawa [29] by finding the best constant in the Adams inequality when only the restriction on the norm $\|\nabla^m u\|_{L^{\frac{n}{m}}(\mathbb{R}^n)} \le 1$ is given.

Proof. Since, when $\alpha < \beta_n$,

$$\Phi(\alpha|t|^{\frac{n}{n-2}}) \leq C\frac{\Phi(\beta_n|t|^{\frac{n}{n-2}})}{(1+|t|)^{\frac{n}{n-2}}},$$

for any $t \in \mathbb{R}$, thus

$$\int_{\mathbb{R}^n} \Phi(\alpha|f|^{\frac{n}{n-2}})dx \leq C\int_{\mathbb{R}^n} \frac{\Phi(\beta_n|f|^{\frac{n}{n-2}})}{(1+|f|)^{\frac{n}{n-2}}}dx \leq C\|f\|_{\frac{n}{2}}^{\frac{n}{2}}.$$

To prove that the constant β_n is sharp, we choose $\{f_k\}_{k=1}^{\infty}$ as follows:

$$f_k(x) = \begin{cases} [\frac{1}{\beta}\ln\frac{1}{R_k}]^{1-\frac{2}{n}} - \frac{|x|^2}{(R_k\ln\frac{1}{R_k})^{2/n}} + \frac{1}{(R_k\ln\frac{1}{R_k})^{2/n}}, & \text{if } |x| \leq R_k^{\frac{1}{n}}, \\ n\beta^{2/n-1}[\ln\frac{1}{R_k}]^{-\frac{2}{n}}\ln\frac{1}{|x|}, & \text{if } R_k^{1/n} \leq |x| < 1, \\ 0, & \text{if } 1 < |x|, \end{cases}$$

where $R_k \geq 0$, $R_k \to 0$, and $\beta_n = \beta(n,2) = \frac{n}{\omega_{n-1}}[\frac{\pi^{\frac{n}{2}}4}{\Gamma(n/2-1)}]^{\frac{n}{n-2}}$. Then by calculation, we have

$$\|f_k\|_{\frac{n}{2}}^{\frac{n}{2}} = O(\frac{1}{\ln\frac{1}{R_k}}),$$

and

$$1 \leq \|\triangle f_k\|_{\frac{n}{2}}^{\frac{n}{2}} \leq 1 + O(\frac{1}{\ln\frac{1}{R_k}}).$$

Now let $\tilde{f}_k = \frac{f_k}{\|\Delta f_k\|_{\frac{n}{2}}}$. Then

$$\|\tilde{f}_k\|_{\frac{n}{2}}^{\frac{n}{2}} = O(\frac{1}{\ln\frac{1}{R_k}})$$

and

$$\begin{aligned} & \int_{\mathbb{R}^n} \Phi(\beta_n|\tilde{f}_k|^{\frac{n}{n-2}})dx \\ \geq\ & \int_{|x|\leq R_k^{\frac{1}{n}}} \Phi(\beta_n|\tilde{f}_k|^{\frac{n}{n-2}})dx \\ \gtrsim\ & \int_{|x|\leq R_k^{\frac{1}{n}}} \exp(\alpha|\tilde{f}_k|^{\frac{n}{n-2}})dx \\ \gtrsim\ & \exp[(\frac{1}{\|\Delta f_k\|_{\frac{n}{2}}^{\frac{n}{2}}} - 1)\ln\frac{1}{R_k}] \\ \geq\ & C. \end{aligned}$$

But

$$\frac{1}{\|\tilde{f}_k\|_{\frac{n}{2}}^{\frac{n}{2}}}\int_{\mathbb{R}^n} \Phi(\beta_n|\tilde{f}_k|^{\frac{n}{n-2}})dx \gtrsim \ln\frac{1}{R_k} \to +\infty,$$

This completes the proof.

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